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A logarithmic Gauss curvature flow and the Minkowski problem

by

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ABSTRACT. – Let X_0 be a smooth uniformly convex hypersurface and f a positive smooth function in S^n . We study the motion of convex hypersurfaces $X(\cdot, t)$ with initial $X(\cdot, 0) = \theta X_0$ along its inner normal at a rate equal to $\log(K/f)$ where K is the Gauss curvature of $X(\cdot, t)$. We show that the hypersurfaces remain smooth and uniformly convex, and there exists $\theta^* > 0$ such that if $\theta < \theta^*$, they shrink to a point in finite time and, if $\theta > \theta^*$, they expand to an asymptotic sphere. Finally, when $\theta = \theta^*$, they converge to a convex hypersurface of which Gauss curvature is given explicitly by a function depending on $f(x)$. © 2000 Éditions scientifiques et médicales Elsevier SAS

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INTRODUCTION

Let f be a positive smooth function defined in the n -dimensional sphere S^n and let $X_0: S^n \rightarrow \mathbf{R}^{n+1}$ be a parametrization of a smooth, uniformly convex hypersurface M_0 . In this paper we are concerned with the motion of the convex hypersurfaces $M(t)$ satisfying the equation

$$\frac{\partial X}{\partial t} = -\log \frac{K(v)}{f(v)} v, \quad (0.1)$$

with $X(p, 0) = X_0(p)$. Here for each t $X(\cdot, t)$ parametrizes $M(t)$, $K(v(p, t))$ is the Gauss curvature of $M(t)$ and $v(p, t)$ is the unit outer normal at $X(p, t)$. Notice that by strict convexity the Gauss curvature can be regarded as a function of the normal. Recall that a uniformly convex hypersurface is a hypersurface with positive Gaussian curvature and hence it is strictly convex.

Our study on (0.1) is motivated by the search for a variational proof of the classical Minkowski problem in the smooth category. Recall that for a convex hypersurface the inverse of its Gauss map induces a Borel measure on the unit sphere called the area measure of the hypersurface. Naturally one asks when a given Borel measure on S^n is the area measure of some convex hypersurface. This problem was formulated and solved by Minkowski [13] for polytopes in 1897 by a variational argument. Later he extended his result to cover all Borel measures which are of the form $1/f d\sigma$ where f is continuous and $d\sigma$ is the standard Lebesgue measure on S^n [14]. The regularity of the convex hypersurface realizing the area measure was not considered by Minkowski. Thus it led to the Minkowski problem in the smooth category, namely, when is a positive, smooth function in S^n the Gauss curvature of a smooth convex hypersurface? There are two approaches for this problem. On one hand, the method of continuity was used by Lewy [12], Miranda [15], Nirenberg [16], and Cheng and Yau [3]. On the other hand, a regularity theory was developed for the generalized solution (see Pogorelov [17]).

Let M be a convex hypersurface and $V(M)$ its enclosed volume. We have

$$V(M) = \frac{1}{n+1} \int_{S^n} \frac{H(x)}{K(x)} d\sigma(x),$$

where H and K are respectively the support function and Gauss curvature of M . When expressed in the smooth category, Minkowski's

original proof is to show that the solution is the convex hypersurface which minimizes the functional $\int H(x)/f(x) d\sigma(x)$ over all convex hypersurfaces of the same enclosed volume. In view of this we may consider the functional

$$J(M) = -V(M) + \int_{S^n} \frac{H}{f} d\sigma.$$

It is not hard to see that (0.1) is a negative gradient flow for J . By a careful study of this flow, we shall give another proof of the Minkowski problem in the smooth category.

THEOREM A. – *Let X_0 be a smooth uniformly convex hypersurface. For $\theta > 0$, consider (0.1) subject to*

$$X(\cdot, 0) = \theta X_0. \quad (0.2)$$

There exists $\theta^ > 0$ such that the flow $X(\cdot, t)$ beginning at $\theta^* X_0$ tends to a smooth uniformly convex hypersurface X^* in the sense that*

$$X(\cdot, t) - \xi t \rightarrow X^*,$$

smoothly as $t \rightarrow \infty$ where ξ is uniquely determined by

$$\int_{S^n} \frac{x_i}{e^{\xi \cdot x} f(x)} d\sigma(x) = 0, \quad i = 1, \dots, n+1.$$

Furthermore, the Gauss curvature of X^ , when regarded as a function of the normal, is equal to $e^{\xi \cdot x} f(x)$.*

THEOREM B. – *Let θ^* be as in Theorem A. If $\theta \in (0, \theta^*)$, the solution of (0.1), (0.2) shrinks to a point in finite time. If $\theta \in (\theta^*, \infty)$, the solution expands to infinity as t goes to infinity. In the latter case, the hypersurface $X(\cdot, t)/r(t)$ where $r(t)$ is the inner radius of $X(\cdot, t)$ converges to a unit sphere uniformly.*

As a direct consequence of Theorem A we have

COROLLARY (Minkowski problem). – *A positive, smooth function f in S^n is the Gauss curvature of a uniformly convex hypersurface if and only if it satisfies*

$$\int_{S^n} \frac{x_i}{f(x)} d\sigma(x) = 0, \quad i = 1, \dots, n+1.$$

Theorems A and B will be proved in the following sections by an approach similar to that used in [4], namely, by introducing the support function of $X(\cdot, t)$ and reducing (0.1) to a single parabolic equation of Monge–Ampère type for its support function. In Section 1 we collect some facts on the support function of a convex hypersurface. In Section 2 *a priori* estimates for the support function, in particular upper and lower bounds for the second derivatives, will be derived. They are used in Section 3 to establish Theorems A and B.

Motion of convex hypersurfaces driven by functions of Gauss curvature of the form

$$\frac{\partial X}{\partial t} = \Phi(v, K)v$$

has been studied by several authors including Andrews [1], Chou [4], Chow [7], Frey [8], Gerhardt [10] and Urbas [18]. When $\Phi = -K^\sigma$, $\sigma > 0$, it was proved in [7] that $M(t)$ exists and shrinks to a point in finite time. Moreover, it becomes asymptotically round when σ is equal to $1/n$. In [1] it was shown that $M(t)$ becomes an asymptotic ellipsoid when σ is equal to $1/(n+2)$. Expanding flows rather than contracting ones were studied in [10] and [18]. For a class of curvature functions including $\Phi = K^{-1/n}$ it was proved that $M(t)$ expands to infinity like a sphere in infinite time. In all these results Φ is independent of v . For anisotropic flows very little is known. We mention the works Andrew [2], Chou and Zhu [6], and Gage and Li [9].

1. THE SUPPORT FUNCTION

In this section we collect some basic facts concerning a convex hypersurface and its support function. Details can be found in Cheng and Yau [3] and Pogorelov [17].

Let M be a closed convex hypersurface in \mathbf{R}^{n+1} . Its support function H is defined on S^n by

$$H(x) = \sup\{x \cdot p : p \in M\},$$

where $x \cdot p$ is the inner product in \mathbf{R}^{n+1} . We extend H to a homogeneous function of degree 1 in \mathbf{R}^{n+1} . So H is convex and satisfies

$$\sup_{S^n} |\nabla H| \leq \sup_{S^n} |H|, \quad (1.1)$$

since it is the supremum of linear functions. If M is strictly convex, that is, for each x in S^n there is a unique point p on M whose unit outer normal is x , H is differentiable at x and

$$p_i = \frac{\partial H}{\partial x_i}, \quad i = 1, \dots, n + 1.$$

Thus the map $x \mapsto p(x)$ gives a parametrization of M by its normal. In fact, it is nothing but the inverse of the Gauss map.

Geometric quantities of M can now be expressed through H . Let e_1, \dots, e_n be an orthonormal frame fields on S^n . By a direct computation one sees that the principal radii of curvature at $p(x)$ are precisely the eigenvalues of the matrix $(\nabla_\beta \nabla_\alpha H + H \delta_{\alpha\beta})_{\alpha, \beta=1, \dots, n}$, where ∇_α is the covariant differentiation with respect to e_α . In particular, the Gauss curvature at $p(x)$ is given by

$$K(x) = 1/\det(\nabla_\beta \nabla_\alpha H + H \delta_{\alpha\beta}). \tag{1.2}$$

When H is viewed as a homogeneous function over \mathbf{R}^{n+1} , the principal radii of curvature of M are also equal to the non-zero eigenvalues of the Hessian matrix $(\partial^2 H/\partial x_i \partial x_j)_{i, j=1, \dots, n+1}$.

Now we can reduce the problem (0.1), (0.2) to an initial value problem for the support function. In fact, let $H(x, t)$ be the support function of $M(t)$. By definition we have

$$x \cdot \frac{\partial X}{\partial t}(p(x), t) = -\frac{\partial H}{\partial t}(x, t).$$

From (0.1) and (0.2) it follows that H satisfies

$$\frac{\partial H}{\partial t} = \log \det(\nabla_\beta \nabla_\alpha H + H \delta_{\alpha\beta}) f, \tag{1.3}$$

$$H(x, 0) = \theta H_0(x), \tag{1.4}$$

where H_0 is the support function for M_0 . Conversely, if $X(\cdot, t)$ is a family of convex hypersurfaces determined by a solution of (1.3) and (1.4), it is not hard to see that $X(\cdot, t)$ does solve (0.1) and (0.2). See, for instance, [4] for details. Notice from (1.3) $H(x, t)$ must determine a uniformly convex hypersurface.

Eq. (1.3) has a variational structure. Consider the enclosed volume of a uniformly convex hypersurface M ,

$$\begin{aligned} V(M) &= \frac{1}{n+1} \int_{S^n} \frac{H(x)}{K(x)} d\sigma(x) \\ &= \frac{1}{n+1} \int_{S^n} H \det(\nabla_\beta \nabla_\alpha H + H \delta_{\alpha\beta}) d\sigma. \end{aligned}$$

Regarding V as a functional on support functions, we find that the first variation of V is

$$\delta V(H)h = \int_{S^n} h \det(\nabla_\beta \nabla_\alpha H + H \delta_{\alpha\beta}) d\sigma,$$

where h is any smooth function. Let's consider the functional J defined on all uniformly convex hypersurfaces

$$J(H) = -V(H) + \int_{S^n} \frac{H}{f} d\sigma,$$

where f is positive. When H is a solution of (1.3),

$$\begin{aligned} \frac{d}{dt} J(H(\cdot, t)) &= - \int_{S^n} \left[\det(\nabla_\beta \nabla_\alpha H + H \delta_{\alpha\beta}) - \frac{1}{f} \right] \frac{\partial H}{\partial t} d\sigma \\ &= - \int_{S^n} \frac{1}{f} (e^{H_t} - 1) H_t d\sigma \\ &\leq 0. \end{aligned} \tag{1.5}$$

Hence (1.3) is a negative gradient flow for J . (1.5) will be used in the proof of Theorem A. This variational approach to the problem of prescribed Gauss curvature was first adopted in Chou [5].

To obtain a priori estimates for the higher derivatives for H it is convenient to express Eq. (1.3) locally in the Euclidean space. Thus let $u(y, t)$ be the restriction of $H(x, t)$ to the hypersurface $x_{n+1} = -1$, i.e., $u(y, t) = H(y, -1, t)$. Then u is convex in \mathbf{R}^n and we have

$$\det \nabla^2 u(y, t) = (1 + |y|^2)^{-\frac{n+2}{2}} \det(\nabla_\beta \nabla_\alpha H + H \delta_{\alpha\beta})(x, t)$$

and

$$\frac{\partial u}{\partial t}(y, t) = \sqrt{1 + |y|^2} \frac{\partial H}{\partial t}(x, t)$$

for $x = (y, -1)/\sqrt{1 + |y|^2}$. Extend f to be a homogenous function of degree 0 in \mathbf{R}^{n+1} . We get

$$\frac{\partial u}{\partial t} = \sqrt{1 + |y|^2} \log \det \nabla^2 u + g(y), \quad y \in \mathbf{R}^n, \tag{1.6}$$

where

$$g(y) = \sqrt{1 + |y|^2} \left[\frac{n + 2}{2} \log(1 + |y|^2) + \log f(y, -1) \right].$$

2. A PRIORI ESTIMATION

First of all we note that the uniqueness of solution to (1.3), (1.4) follows from the following comparison principle which is a direct consequence of the maximum principle.

LEMMA 2.1. – For $i = 1, 2$, let f_i be two positive C^2 -functions on S^n and H_i $C^{2,1}$ -solutions of

$$\frac{\partial H}{\partial t} = \log \det(\nabla_\beta \nabla_\alpha + H \delta_{\alpha\beta}) f_i.$$

Suppose that $H_1(x, 0) \leq H_2(x, 0)$ and $f_1(x) \leq f_2(x)$ on S^n . Then $H_1 \leq H_2$ for all $t > 0$ and $H_1 < H_2$ unless $H_1 \equiv H_2$.

In the following we shall always assume $H \in C^{4,2}(S^n \times [0, T])$ is a solution of (1.3), (1.4). Let $R(t)$ and $r(t)$ be the outer and inner radii of the hypersurface $X(\cdot, t)$ determined by $H(x, t)$ respectively. We set

$$R_0 = \sup\{R(t): t \in [0, T]\}$$

and

$$r_0 = \inf\{r(t): t \in [0, T]\}.$$

We shall estimate the principal radii of curvatures of $X(\cdot, t)$ from both side in terms of r_0^{-1} , R_0 , and initial data.

LEMMA 2.2. – Let r and R be the inner and outer radii of a uniformly convex hypersurface X respectively. Then there exists a dimensional constant C such that

$$\frac{R^2}{r} \leq C \sup\{R(x, \xi): x, \xi \in S^n\},$$

where $R(x, \xi)$ is the principal radius of curvature of X at the point with normal x and along the direction ξ .

Proof. – For any given $t > 0$, let

$$h = \inf\{H(x) + H(-x) : x \in S^n\}.$$

Then X is pinched between two parallel hyperplanes with distance h . Suppose the infimum is attained at $x = (1, 0, \dots, 0)$. By convexity we can choose a direction perpendicular to the x_1 -axis, say, the x_2 -axis such that

$$H(0, 1, 0, \dots, 0) + H(0, -1, 0, \dots, 0) \geq \frac{1}{2}R.$$

Let F be the projection of X on the plane $x_3 = \dots = x_{n+1} = 0$. Then F is a convex set and its diameter is larger than $\frac{1}{2}R$. By a proper choice of the origin we may assume F is contained in $\{-h < x_1 < h\}$ and $\{0, \pm\frac{1}{8}R\}$ belongs to F . By projection we see that the supremum of the principal radii of curvatures of the boundary of F cannot exceed that of X .

Let E be the ellipse given by

$$\frac{x_1^2}{b^2} + \frac{x_2^2}{(R/16)^2} = 1$$

where b is chosen so that $E \subset F$ and $\partial E \cap \partial F$ is non-empty. Then $h/4 \leq b \leq h/2$ provided $R \gg r$. For any $(\bar{x}_1, \bar{x}_2) \in \partial E \cap \partial F$, since $(0, \pm\frac{1}{8}R) \in F$, we have $|\bar{x}_1| \geq b/2$. Hence $|\bar{x}_2| \leq \sqrt{3}R/32$. Simple computation shows that the principal radius of curvature of the boundary of F at (\bar{x}_1, \bar{x}_2) is larger than $R^2/8^3b$. Hence by noticing $b \leq r$ we obtain

$$\frac{R^2}{r} \leq C \frac{R^2}{b} \leq C \sup_{x, \xi} R(x, \xi). \quad \square$$

LEMMA 2.3. – Suppose that $a(t), b(t) \in C^1([0, T])$ and $a(t) < b(t)$ for all t . Then there exists $h(t) \in C^{0,1}([0, T])$ such that

- (1) $a(t) - 2M \leq h(t) \leq b(t) + 2M$;
- (2) $\sup\{\frac{|h(t_1) - h(t_2)|}{|t_1 - t_2|} : t_1, t_2 \in [0, T]\} \leq 2 \max\{\sup_t b'(t), \sup_t (-a'(t))\}$,

where $M = \sup_t (b(t) - a(t))$.

Proof. – We define $h(t)$ step by step. Let $t_0 = 0$, and $h_0 = (a(0) + b(0))/2$. For $j \geq 1$, let

$$t_j = \sup\{\tau \in (t_{j-1}, T): a(t) \geq h_{j-1} - M, b(t) \leq h_{j-1} + M, \forall t \in (t_{j-1}, \tau)\},$$

$$h_j = \frac{1}{2}(a(t_j) + b(t_j)),$$

and

$$h(t) = h_{j-1} + \frac{h_j - h_{j-1}}{t_j - t_{j-1}}(t - t_{j-1}) \quad \text{for } t \in (t_{j-1}, t_j).$$

Then $h(t)$ is the desired function. \square

Now we give an upper estimate for the principal radii of curvature.

LEMMA 2.4. – *For any $\gamma \in (1, 2]$ there exists a constant C_γ , which may depend on initial data, such that*

$$\sup\{H_{\xi\xi}(x, t): \xi \text{ tangential to } S^n\} \leq C_\gamma(1 + D^\gamma),$$

where $D = \sup\{d(t): t \in [0, T]\}$ and $d(t)$ is the diameter of $X(\cdot, t)$.

Proof. – Applying Lemma 2.3 to the functions $-H(-e_i, t)$ and $H(e_i, t)$ where $\pm e_i$ are the intersection points of S^n with the x_i -axis, $i = 1, \dots, n + 1$, we obtain $p_i(t)$ so that

$$-H(-e_i, t) - 2D \leq p_i(t) \leq H(e_i, t) + 2D$$

and

$$\begin{aligned} &\sup\left\{\frac{|p_i(t_1) - p_i(t_2)|}{|t_1 - t_2|}: t_1, t_2 \in [0, T]\right\} \\ &\leq 2 \sup\{H_i(x, t): (x, t) \in S^n \times [0, T]\}. \end{aligned} \tag{2.1}$$

Henceforth

$$\left|H(x, t) - \sum_{i=1}^{n+1} p_i(t)x_i\right| \leq 2D \quad \text{for } (x, t) \in S^n \times [0, T], \tag{2.2}$$

and by (1.1)

$$\sum_{i=1}^{n+1} |H_i(x, t) - p_i|^2 \leq 4D^2. \tag{2.3}$$

Let

$$\Phi(x, t) = H_{\xi\xi}(x, t) + \left[1 + \sum_{i=1}^{n+1} |H_i(x, t) - p_i(t)|^2 \right]^{\gamma/2}$$

where $\gamma \in (1, 2]$. Suppose that the supremum

$$\sup\{\Phi(x, t) : (x, t) \in S^n \times [0, T], \xi \text{ tangential to } S^n, |\xi| = 1\}$$

is attained at the south pole $x = (0, \dots, 0, -1)$ at $t = \bar{t} > 0$ and in the direction $\xi = e_1$. For any x on the south hemisphere, let

$$\xi(x) = \left(\sqrt{1 - x_1^2}, -\frac{x_1 x_2}{\sqrt{1 - x_1^2}}, \dots, -\frac{x_1 x_{n+1}}{\sqrt{1 - x_1^2}} \right).$$

Let u be the restriction of H on $x_{n+1} = -1$. Using the homogeneity of H we obtain, after a direct computation,

$$\begin{aligned} & \sum_{i=1}^{n+1} (H_i - p_i)^2(x, t) \\ &= \sum_{i=1}^n (u_i(y, t) - p_i(t))^2 + \left| u(y, t) + p_{n+1} - \sum_{i=1}^n y_i u_i(y, t) \right|^2 \end{aligned}$$

and

$$H_{\xi\xi}(x, t) = u_{11}(y, t) \frac{(1 + y_1^2 + \dots + y_n^2)^{3/2}}{1 + y_2^2 + \dots + y_n^2},$$

where $y = -(x_1, \dots, x_n)/x_{n+1}$ in \mathbf{R}^n . Thus the function

$$\begin{aligned} \varphi(y, t) &= u_{11} \frac{(1 + y_1^2 + \dots + y_n^2)^{3/2}}{1 + y_2^2 + \dots + y_n^2} \\ &+ \left[1 + \sum (u_i - p_i)^2 + \left| u + p_{n+1} - \sum y_i u_i \right|^2 \right]^{\gamma/2} \end{aligned}$$

attains its maximum at $(y, t) = (0, \bar{t})$. Without loss of generality we may further assume that the Hessian of u at $(0, \bar{t})$ is diagonal. Hence at $(0, \bar{t})$ we have, for each k ,

$$\begin{aligned} 0 &\leq \varphi_t = u_{11t} + \gamma [(u_i - p_i)(u_{it} - p_{i,t}) \\ &\quad + (u + p_{n+1})(u_t + p_{n+1,t})] Q^{(\gamma-2)/2}, \\ 0 &= \varphi_k = u_{11k} + \gamma (u_i - p_i) u_{ik} Q^{(\gamma-2)/2}, \end{aligned}$$

and

$$0 \geq \varphi_{kk} = u_{kk11} + \tau_k u_{11} + \gamma [u_{kk}^2 + (u_i - p_i) u_{ikk} - (u + p_{n+1}) u_{kk}] Q^{(\gamma-2)/2} + \gamma(\gamma-2)(u_i - p_i)^2 u_{ik}^2 Q^{(\gamma-4)/2},$$

where $Q = 1 + \sum (u_i - p_i)^2 + (u + p_{n+1})^2$, $\tau_k = 3$ if $k > 1$ and $\tau_1 = 1$, and $p_{i,t} = dp_i/dt$. On the other hand, differentiating Eq. (1.6) gives

$$u_{kt} = \sum_i u^{ii} u_{iik} + g_k,$$

$$u_{kkt} = \sum_i u^{ii} u_{iikk} - \sum_{i,j} u^{ii} u^{jj} u_{ijk}^2 + \log \det \nabla^2 u + g_{kk},$$

where $\{u^{ij}\}$ is the inverse matrix of $\{u_{ij}\}$. Hence at $(0, \bar{t})$ we have

$$\begin{aligned} 0 &\geq \sum_k u^{kk} \varphi_{kk} - \varphi_t \\ &\geq \sum_k u^{kk} u_{kk11} - u_{11t} + u_{11} u^{kk} \\ &\quad + \gamma \left\{ \sum_k u_{kk} \left[1 + \frac{(\gamma-2)(u_k - p_k)^2}{1 + \sum (u_i - p_i)^2 + (u + p_{n+1})^2} \right] \right. \\ &\quad + (u_i - p_i) \left(\sum_k u^{kk} u_{ikk} - u_{it} \right) - n(u + p_{n+1}) \\ &\quad \left. - (u + p_{n+1})(u_t + p_{n+1,t}) + (u_i - p_i) p_{i,t} \right\} Q^{(\gamma-2)/2} \\ &\geq u_{11} u^{kk} - \log \det \nabla^2 u - g_{11} + \gamma [(\gamma-1)u_{kk} - (u_i - p_i)g_i \\ &\quad - n(u + p_{n+1}) - (u + p_{n+1})(u_t + p_{n+1,t}) \\ &\quad + (u_i - p_i) p_{i,t}] Q^{(\gamma-2)/2}. \end{aligned}$$

To proceed further let's assume $u_{11} > 1$. By (2.2) we have $|u + p_{n+1}| \leq 2D$ and $|u_i - p_i| \leq 2D$. From the inequality above we therefore obtain, in view of (2.1),

$$\begin{aligned} u_{kk} + u^{kk} &\leq C(1 + |u_t|) Q^{(2-\gamma)/2} + C(1 + |u + p_{n+1}|)(1 + |u_t| + |p_{n+1,t}|) \\ &\leq C \left[1 + D \log(u_{kk} + u^{kk}) + D \sup_{t \leq T} H_t(x, t) \right]. \end{aligned}$$

From Eq. (1.3),

$$\sup_{t \leq T} H_t(x, t) \leq C + \log \left[\sup_{t < T} \{ H_{\xi\xi}^n(x, t); x \in S^n, \xi \text{ tangential to } S^n \} \right].$$

It follows

$$u_{kk} + u^{kk} \leq C(1 + D \log(u_{kk} + u^{kk})).$$

Hence $u_{11} \leq C(1 + D|\log^2 D|)$. This completes the proof of the lemma.

□

By combining Lemmas 2.2 and 2.4 we deduce the following important corollary.

LEMMA 2.5. – *For any given $\gamma \in (1, 2]$, there exists $\delta = \delta(\gamma) > 0$ such that*

$$r(t) \geq \frac{\delta R^2(t)}{1 + \sup_{\tau \leq t} R^\gamma(\tau)}.$$

Next we give a positive lower bound for the principal radii of the curvature. In view of Lemma 2.4 and Eq. (1.3) it suffices to give a lower bound on H_t .

LEMMA 2.6. – *There exists a constant C depending only on n, r_0, R_0, f , and initial data such that*

$$\inf \{ H_t(x, t): (x, t) \in S^n \times [0, T] \} \geq -C.$$

Proof. – Let

$$q(t) = \frac{1}{|S^n|} \int_{S^n} x H(x, t) d\sigma(x)$$

be the Steiner point of $X(\cdot, t)$. Then there exists a positive δ which depends only on n, r_0 , and R_0 so that $H(x, t) - q(t) \cdot x \geq 2\delta$. Let us consider consider the function

$$\Psi(x, t) = \frac{H_t(x, t)}{H(x, t) - x \cdot q(t) - \delta}.$$

Suppose the (negative) infimum of Ψ attains at $x = (0, \dots, 0, -1)$ and $\bar{t} > 0$. Let u be the restriction of H to $x_{n+1} = -1$ as before. Then

$$\psi(y, t) = \frac{u_t(y, t)}{u(y, t) - q(t) \cdot (y, -1) - \delta \sqrt{1 + |y|^2}}$$

attains its negative minimum at $(0, \bar{t})$. Hence

$$0 \geq \psi_t = \frac{u_{tt}}{u + q_{n+1}(t) - \delta} - \frac{u_t(u_t + dq_{n+1}/dt)}{(u + q_{n+1}(t) - \delta)^2},$$

$$0 = \psi_k = \frac{u_{tk}}{u + q_{n+1}(t) - \delta} - \frac{u_t(u_k - q_k(t))}{(u + q_{n+1}(t) - \delta)^2},$$

and

$$0 \leq \psi_{kk} = \frac{u_{tkk}}{u + q_{n+1}(t) - \delta} - \frac{u_t u_{kk}}{(u + q_{n+1}(t) - \delta)^2} + \frac{\delta u_t}{(u + q_{n+1}(t) - \delta)^2}.$$

On the other hand, we differentiate (1.3) to get

$$u_{tt} = u^{ij} u_{ijt}.$$

Rotate the axes so that $\{u^{ij}\}$ is diagonal at $(0, \bar{t})$. Then

$$0 \leq \sum u^{kk} \psi_{kk} - \psi_t$$

$$\leq \frac{\delta u_t \sum u^{kk} - n u_t + u_t (u_t + dq_{n+1}/dt)}{(u + q_{n+1} - \delta)^2}.$$

Since u_t is negative at $(0, \bar{t})$, it follows from Lemma 2.4 that

$$\sum u^{kk} \leq \frac{n}{\delta} \left(1 + |u_t| + \left| \frac{dq_{n+1}}{dt} \right| \right)$$

$$\leq C \frac{n}{\delta} (1 + |u_t| + R_0)$$

$$\leq C \frac{n}{\delta} (1 + \log \sum u^{kk} + R_0).$$

We therefore conclude $\sum u^{kk} \leq C \delta^{-2} (1 + R_0)^2$. Hence

$$u_t \geq -C - C \log \sum u^{kk}$$

$$\geq -C (1 + \log(1 + R_0) - \log r_0)$$

and the lemma follows. \square

Finally by comparing (1.3), (1.4) with the problem

$$\frac{d\rho}{dt} = \log \rho^n M, \quad \rho(0) = \rho_0$$

where $M = \max\{f(x) : x \in S^n\}$ and ρ_0 is sufficiently large, we see that $H(x, t)$ is always bounded in any finite time interval. Furthermore, its gradient is also bounded by (1.1). It follows from the regularity property of fully nonlinear parabolic equations [11] that a $C^{4+\alpha, 2+\alpha/2}$ -estimate holds for H , provided $H_0 \in C^{4+\alpha}(S^n)$, $0 < \alpha < 1$. By a continuity argument we arrive at

THEOREM 2.1. – *The problem (1.3), (1.4) with $H_0 \in C^{4+\alpha}(S^n)$ admits a unique $C^{4+\alpha, 2+\alpha/2}$ solution in a maximal interval $[0, T^*)$, $T^* \leq \infty$. Moreover, $\lim_{t \uparrow T^*} R(t) = 0$ if T^* is finite.*

Notice that the last assertion follows from Lemma 2.5.

3. PROOFS OF THEOREMS A AND B

We first prove Theorem A. Let $m = \inf f$ and $M = \sup f$ on S^n . It is readily seen that if the initial hypersurface X_0 is a sphere of radius $\rho_0 > m^{-1/n}$, the solution $X(\cdot, t)$ to the equation

$$\frac{\partial X}{\partial t} = -\log \frac{K}{m} \nu, \quad X(\cdot, 0) = X_0,$$

remains to be spheres and the flow expands to infinity as $t \rightarrow \infty$. On the other hand, if X_0 is a sphere of radius less than $M^{-1/n}$, the solution to

$$\frac{\partial X}{\partial t} = -\log \frac{K}{M} \nu, \quad X(\cdot, 0) = X_0$$

is a family of spheres which shrinks to a point in finite time. Henceforth by the comparison principle the solution $X(x, t)$ of (1.3), (1.4) will shrink to a point if θ is small enough, and will expand to infinity if $\theta > 0$ is large. We put

$$\theta_* = \sup\{\theta > 0 : X(\cdot, t) \text{ shrinks to a point in finite time}\}$$

and

$$\theta^* = \inf\{\theta > 0 : X(\cdot, t) \text{ expands to infinity as } t \rightarrow \infty\}.$$

By the results in Section 2, it is easy to see that $X(\cdot, t)$ continuously depends on θ . Hence by the comparison principle $\theta_* \leq \theta^*$.

By Lemma 2.5 we know that for any $\theta \in [\theta_*, \theta^*]$ the inner radii of $X(\cdot, t)$ have a uniform positive lower bound and the outer radii are

uniformly bound from above. Hence (1.3) is uniformly parabolic and we have $C^{4+\alpha, 2+\alpha/2}$ -bound on the solution in $S^n \times [0, \infty)$.

In the following we fix $\theta \in [\theta_*, \theta^*]$. Let $\xi \in \mathbf{R}^{n+1}$ be the point uniquely determined by

$$\int_{S^n} \frac{x_i}{e^{\xi \cdot x} f(x)} d\sigma(x) = 0, \quad i = 1, \dots, n + 1. \tag{3.1}$$

Write $\tilde{X}(x, t) = X(x, t) + \xi \cdot t$. So \tilde{X} is X translated in $\xi/|\xi|$ with speed $|\xi|$. \tilde{X} satisfies

$$\frac{\partial \tilde{X}}{\partial t} = -\log \frac{K}{f} \nu + \xi$$

and the corresponding support function $\tilde{H} = H + \xi \cdot xt$ satisfies

$$\tilde{H}_t = \log \det(\nabla_\beta \nabla_\alpha \tilde{H} + \tilde{H} \delta_{\alpha\beta}) + \log f e^{\xi \cdot x}.$$

The enclosed volumes of \tilde{X} and X are equal to

$$V(t) = \frac{1}{n + 1} \int \tilde{H} \det(\nabla_\beta \nabla_\alpha \tilde{H} + \tilde{H} \delta_{\alpha\beta})$$

and is uniformly bounded. On the other hand, by (3.1)

$$\int \frac{\tilde{H}}{e^{\xi \cdot x} f} = \int \frac{H - q(t) \cdot x}{f e^{\xi \cdot x}}$$

is also uniformly bounded for all t . Hence the functional $\tilde{J}(t) = J(\tilde{H}(\cdot, t))$ is uniformly bounded. Moreover, from (1.5) it is non-increasing. By the $C^{4+\alpha, 2+\alpha/2}$ -regularity of \tilde{H} we also have that

$$|\tilde{J}'(t)| \leq C$$

and

$$\sup \frac{|\tilde{J}'(t + \tau) - \tilde{J}'(t)|}{\tau^{\alpha/2}} \leq C.$$

Therefore, we conclude that $\lim_{t \rightarrow \infty} \tilde{J}'(t) = 0$.

We claim that \tilde{H} is bounded for all t . In fact, it is sufficient to show that $\int x \frac{\tilde{H}}{f e^{\xi \cdot x}} d\sigma$ is bounded. For, assume \tilde{H} is unbounded. Then we can

find $\{t_j\}$, $t_j \rightarrow \infty$, such that $\tilde{X}(x, t_j)/d(t_j)$, where $d(t_j)$ is the distance from the origin to $\tilde{X}(\cdot, t_j)$, converges to a point on S^n . Without loss of generality we take this point to be e_{n+1} . Then the characteristic functions of $A_j = \{x \in S^n: x_{n+1} > 0, H(x, t_j) > 0\}$ and $B_j = \{x \in S^n: x_{n+1} < 0, H(x, t_j) < 0\}$ converges pointwisely to the upper and lower hemispheres. We may also assume that $\tilde{H}(x, t_j)/(fe^{\xi \cdot x} d(t_j))$ converges uniformly to some function g which is positive on the upper hemi-sphere S^+ . Therefore, we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \int \frac{x_{n+1} H(x, t_j)}{d(t_j) f e^{\xi \cdot x}} &= \lim_{j \rightarrow \infty} \left[\mathcal{X}_{A_j \cup B_j} \frac{|x_{n+1} H(x, t_j)|}{d(t_j) f e^{\xi \cdot x}} \right] \\ &\geq \int_{S^+} x_{n+1} g(x) \\ &> 0. \end{aligned}$$

Hence $\int \frac{x_{n+1} H(x, t_j)}{f e^{\xi \cdot x}}$ can be arbitrarily large for large t_j .

Now we have, by (1.5),

$$\tilde{J}(0) - \tilde{J}(\infty) \geq \int_0^t |\tilde{J}'(t)| dt \geq \int_0^t \int_{S^n} \tilde{H}_t^2 d\sigma dt.$$

On the other hand, by the necessary condition for the Minkowski problem, we have

$$\begin{aligned} 0 &= \int x \frac{1}{K} d\sigma = \int x \frac{1}{f e^{\xi \cdot x}} (1 + \tilde{H}_t + O(\tilde{H}_t^2)) \\ &= \int x \frac{1}{f e^{\xi \cdot x}} (\tilde{H}_t + O(\tilde{H}_t^2)) \end{aligned}$$

as \tilde{H}_t is uniformly small for large t . Therefore,

$$\begin{aligned} \left| \int_0^t \frac{d}{dt} \left(\int x \frac{\tilde{H}}{f e^{\xi \cdot x}} d\sigma \right) dt \right| &\leq C \int_0^t \int_{S^n} \tilde{H}_t^2 d\sigma dt \\ &\leq C(\tilde{J}(0) - \tilde{J}(\infty)). \end{aligned}$$

Hence $\int x \frac{\tilde{H}}{f e^{\xi \cdot x}}$ is uniformly bounded for all time. Consequently by the Blaschke selection theorem for any sequence $\{t_j\}$, $t_j \rightarrow \infty$, we can extract a subsequence $\{t_{j_k}\}$ such that $\{\tilde{H}(x, t_{j_k})\}$ converges uniformly to some $H(x)$ on S^n . Clearly H is a solution of $K = f e^{\xi \cdot x}$. To show the convergence is actually uniform let's consider another limit H' . Since the

curvature of H' is also given by $f e^{\xi \cdot x}$, H and H' differ by a translation. Let $H - H' = l \cdot x$ for some $l \in \mathbf{R}^{n+1}$. Since

$$\left| \int_s^t \frac{d}{dt} \int x \frac{\tilde{H}}{f e^{\xi \cdot x}} d\sigma dt \right| \leq C(\tilde{J}(t) - \tilde{J}(s)) \rightarrow 0$$

as $t, s \rightarrow \infty$. So $l = 0$ and $H = H'$.

Finally let's show $\theta_* = \theta^*$. First we observe that by the comparison principle one must have $H_* = H^*$, where H_* (respectively H^*) is the solution of $K = f e^{\xi \cdot x}$ starting from $\theta_* H_0$ (respectively $\theta^* H_0$). However, consider the equation obtained by differentiating (1.3) and (1.4) in θ :

$$\begin{cases} \frac{\partial H'}{\partial t} = A^{\alpha\beta} (\nabla_\beta \nabla_\alpha H' + H' \delta_{\alpha\beta}), \\ H'(0) = H_0(x), \end{cases}$$

where $(A^{\alpha\beta})$ is the inverse of $(\nabla_\beta \nabla_\alpha H + H \delta_{\alpha\beta})$. By the maximum principle $H'(x, t) \geq \min H_0 > 0$. Thus

$$\begin{aligned} 0 &= H^*(\cdot) - H_*(\cdot) \\ &= \lim_{t \rightarrow \infty} (H_{\theta^*}(\cdot, t) - H_{\theta_*}(\cdot, t)) \\ &\geq (\min H_0)(\theta^* - \theta_*) \\ &> 0. \end{aligned}$$

So $\theta^* = \theta_*$. The proof of Theorem A is finished.

Proof of Theorem B. – It remains to show that the normalized hypersurface $X(\cdot, t)/r(t)$ converges to a unit sphere in case $\theta > \theta^*$. Let's denote the solution of (1.3), (1.4) by $H(\cdot, t)$ and its hypersurface by $X(\cdot, t)$. Since X is expanding, we may simply assume that it contains the ball $B_{R_1}(0)$ where $R_1 > 1 + m^{-1/n}$ at $t = 0$. On the other hand, we fix R_2 so large that $X(\cdot, 0)$ is contained in $B_{R_2}(0)$.

For $i = 1, 2$, let $X_i(\cdot, t)$ be the solution of (1.3), (1.4) where f is replaced by m and M respectively and $X_i(\cdot, 0) = \partial B_{R_i}$. Clearly $X_i(\cdot, t)$ are spheres whose radii $R_i(t)$ satisfy

$$\begin{aligned} C^{-1}(1+t) \log(1+t) &\leq R_1(t) \\ &\leq R_2(t) \\ &\leq C[1 + (1+t) \log^2(1+t)] \end{aligned}$$

for some $C > 0$. Hence

$$\begin{aligned} \frac{d}{dt}(R_2(t) - R_1(t)) &\leq n \log \frac{R_2(t)}{R_1(t)} + C \\ &\leq \log \log(1+t) + C \end{aligned}$$

and so

$$R_2(t) - R_1(t) \leq C[1 + t \log \log(1+t)].$$

Consequently $\lim_{t \rightarrow 0} \frac{R_2(t) - R_1(t)}{R_1(t)} = 0$. By the comparison principle $X(\cdot, t)$ is pinched between $X_2(\cdot, t)$ and $X_1(\cdot, t)$. So $X(\cdot, t)/r(t)$ must tend to the unit sphere uniformly.

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