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Periodic solutions for singular Hamiltonian systems and closed geodesics on non-compact Riemannian manifolds

by

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ABSTRACT. – We study the existence of periodic solutions of singular Hamiltonian systems as well as closed geodesics on non-compact Riemannian manifolds via variational methods.

For Hamiltonian systems, we show the existence of a periodic solution of prescribed-energy problem:

$$\ddot{q} + \nabla V(q) = 0,$$

$$\frac{1}{2}|\dot{q}|^2 + V(q) = 0$$

under the conditions: (i) $V(q) < 0$ for all $q \in \mathbb{R}^N \setminus \{0\}$; (ii) $V(q) \sim -1/|q|^2$ as $|q| \sim 0$ and $|q| \sim \infty$.

For closed geodesics, we show the existence of a non-constant closed geodesic on $(\mathbb{R} \times S^{N-1}, g)$ under the condition:

$$g_{(s,x)} \sim ds^2 + h_0 \quad \text{as } s \sim \pm\infty,$$

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where h_0 is the standard metric on S^{N-1}. © 2000 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Nous étudions l'existence de solutions p'eriodiques pour des systèmes Hamiltoniens singuliers, et de géodésiques fermées sur des variétes Riemanniennes non-compactes par des méthodes variationnelles.

Pour les systèmes Hamiltoniens, nouns montrons l'existence d'une solution périodique pour un probl'eme à énergie prescrite :

$$\ddot{q} + \nabla V(q) = 0,$$
$$\frac{1}{2}|\dot{q}|^2 + V(q) = 0$$

sous les conditions : (i) $V(q) < 0$ pour tout $q \in \mathbb{R}^N \setminus \{0\}$; (ii) $V(q) \sim -1/|q|^2$ quand $|q| \sim 0$ et $|q| \sim \infty$.

Pour les géodésiques fermées, nouns montrons l'existence d'une géodésique fermée non-constante sur $(\mathbb{R} \times S^{N-1}, g)$ sous la condition :

$$g_{(s,x)} \sim ds^2 + h_0 \quad \text{quand } s \sim \pm\infty,$$

où h_0 est la métrique standard sur S^{N-1}. © 2000 Éditions scientifiques et médicales Elsevier SAS

0. INTRODUCTION

In this paper we study the existence of periodic solutions of singular Hamiltonian systems as well as the existence of closed geodesics on non-compact Riemannian manifolds in a related situation.

As to periodic solutions of Hamiltonian systems, we consider the existence of periodic solutions of the so-called prescribed energy problem:

$$\ddot{q} + \nabla V(q) = 0, \tag{HS.1}$$
$$\frac{1}{2}|\dot{q}|^2 + V(q) = H, \tag{HS.2}$$

where $q(t): \mathbb{R} \to \mathbb{R}^N \setminus \{0\}$ $(N \geqslant 2)$, $V: \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ and $H \in \mathbb{R}$. We consider the situation where $V(q)$ has a singularity at 0;

(V0) $V(q) \in C^2(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$.

(V1) $V(q) < 0$ for all $q \in \mathbb{R}^N \setminus \{0\}$.

(V2) There exists an $\alpha > 0$ such that

$$V(q) \sim -\frac{1}{|q|^\alpha} \quad \text{near } q = 0;$$

more precisely, for $W(q) = V(q) + (1/|q|^\alpha)$

$$|q|^\alpha W(q),\ |q|^{\alpha+1}\nabla W(q),\ |q|^{\alpha+2}\nabla^2 W(q) \to 0 \quad \text{as } |q| \to 0.$$

The order α of the singularity 0 plays an important role for the existence of periodic solutions. For example, for $V(q) = -1/|q|^\alpha$, (HS.1)–(HS.2) has a periodic solution if and only if

$$H > 0 \quad \text{for } \alpha > 2, \tag{0.1}$$

$$H = 0 \quad \text{for } \alpha = 2, \tag{0.2}$$

$$H < 0 \quad \text{for } \alpha \in (0, 2). \tag{0.3}$$

The situation which generalizes the case (0.1)—which is called strong force—is considered by [2,10,15,18] and the existence of a periodic solution is obtained via minimax methods. The situation which generalizes the case (0.3)—which is called weak force—is also studied by [2,12,13, 19,22,23]. We also refer to Ambrosetti and Coti Zelati [3] and references therein. See also [4] for generalization for the first order Hamiltonian systems. However, it seems that the situation related to the border case $\alpha = 2$ is not well studied; The only work, we know, is Ambrosetti and Bessi [1]. They considered potentials $V(q) \sim -(1/|q|^2) - (1/|q|)$ and proved the existence of multiple periodic solutions of (HS.1)–(HS.2) for suitable range of $H < 0$. See also [2,22,23] in which periodic solutions are constructed for $H < 0$ and $V(q) \sim -(\varepsilon/|q|^2) - (1/|q|^\alpha)$ where $\varepsilon > 0$ is sufficiently small and $\alpha \in (0, 2)$. We remark that a perturbation of weak force case is studied in these works and the case $V(q) = -1/|q|^2$, $H = 0$ is excluded.

In this paper we study a class of perturbations of $-1/|q|^2$ and we look for periodic solutions of (HS.1)–(HS.2) for $H = 0$. Our result does not exclude the case $V(q) = -1/|q|^2$, $H = 0$.

Since (HS.1)–(HS.2) with $V(q) = -1/|q|^2$ has a periodic solution if and only if $H = 0$, it seems that the situation is rather delicate and the problem (HS.1)–(HS.2) accepts only very restricted class of perturbations. However, we have the following existence result which ensures the existence for rather wide class of $V(q)$'s.

THEOREM 0.1. – *Assume* (V0)–(V2) *with* $\alpha = 2$ *and*
(V3) *Set* $W(q) = V(q) + (1/|q|^2)$, *then* $W(q)$ *satisfies*

$$|q|^2 W(q),\ |q|^3 \nabla W(q),\ |q|^4 \nabla^2 W(q) \to 0 \quad \text{as } |q| \to \infty.$$

Then (HS.1)–(HS.2) *with* $H = 0$ *has at least one periodic solution.*

The conditions (V2) and (V3) request

$$V(q) \sim -\frac{1}{|q|^2} \quad \text{as } |q| \sim 0 \text{ and } |q| \sim \infty.$$

This condition is necessary for the existence of periodic solutions of (HS.1)–(HS.2) with $H = 0$ in the following sense; if

$$V(q) \sim -\frac{a}{|q|^2} \quad \text{as } |q| \sim 0,$$
$$V(q) \sim -\frac{b}{|q|^2} \quad \text{as } |q| \sim \infty$$

and $a \neq b$, then (HS.1)–(HS.2) with $H = 0$ does not have periodic solutions in general. (Of course, if $a = b > 0$, the existence of periodic solutions is ensured by Theorem 0.1.) More precisely, we have the following

THEOREM 0.2. – *Suppose* $\varphi(r) \in C^2([0, \infty), \mathbb{R})$ *satisfies*

$$\varphi'(r) \neq 0 \quad \text{for all } r > 0, \tag{0.4}$$
$$\varphi(r) \to a > 0 \quad \text{as } r \to 0, \tag{0.5}$$
$$\varphi(r) \to b > 0 \quad \text{as } r \to \infty \tag{0.6}$$

and let

$$V(q) = -\frac{\varphi(|q|)}{|q|^2}. \tag{0.7}$$

Then (HS.1)–(HS.2) *with* $H = 0$ *does not have periodic solutions.*

Since $\mathbb{R} \times S^{N-1}$ and $\mathbb{R}^N \setminus \{0\}$ are diffeomorphic through a mapping

$$\mathbb{R} \times S^{N-1} \to \mathbb{R}^N \setminus \{0\}; \quad (s, x) \mapsto e^s x,$$

we can reduce (HS.1)–(HS.2) to the existence problem for closed geodesics on non-compact Riemannian manifold $\mathbb{R}\times S^{N-1}$ with a metric g^V defined by

$$g^V_{(s,x)}=e^{2s}\big(H-V(e^s x)\big)g^0_{(s,x)}. \tag{0.8}$$

Here g^0 is the standard product metric on $\mathbb{R}\times S^{N-1}$;

$$g^0_{(s,x)}\big((\xi,\eta),(\xi,\eta)\big)=|\xi|^2+|\eta|^2 \tag{0.9}$$

for $(s,x)\in\mathbb{R}\times S^{N-1}$ and $(\xi,\eta)\in T_{(s,x)}(\mathbb{R}\times S^{N-1})=\mathbb{R}\times T_xS^{N-1}$. Here we identify

$$T_xS^{N-1}=\{\eta\in\mathbb{R}^N;\ x\cdot\eta=0\}. \tag{0.10}$$

We will give ono-to-one correspondence between periodic solutions of (HS.1)–(HS.2) and non-constant closed geodesics on $(\mathbb{R}\times S^{N-1},g^V)$ in Section 1.

We study the existence of non-constant closed geodesics on $(\mathbb{R}\times S^{N-1},g)$ in more general situation. Our main result for closed geodesics is the following

THEOREM 0.3. – *Let g be a Riemannian metric on $\mathbb{R}\times S^{N-1}$ and suppose that g satisfies $g\sim g^0$ as $s\sim\pm\infty$. More precisely,*

(g0) *g is a C^2-Riemannian metric on $\mathbb{R}\times S^{N-1}$.*

(g1) *$g\sim g^0$ as $s\sim\pm\infty$ in the following sense; let $(\xi^1,\dots,\xi^{N-1})$ be a local coordinate of S^{N-1} in an open set $U\subset S^{N-1}$ and set $\xi^0=s$. We write*

$$g=\sum_{i,j=0}^{N-1}g_{ij}(\xi^0,\xi^1,\dots,\xi^{N-1})\,d\xi^i\otimes d\xi^j.$$

We also write $g^0=\sum g^0_{ij}(\xi^0,\xi^1,\dots,\xi^{N-1})\,d\xi^i\otimes d\xi^j$, where g^0 is the standard product Riemannian metric on $\mathbb{R}\times S^{N-1}$. We remark that $g^0_{ij}(\xi^0,\xi^1,\dots,\xi^{N-1})$ is independent of $\xi^0=s$. We assume

$$g_{ij}(s,\xi^1,\dots,\xi^{N-1})\to g^0_{ij}(\xi^1,\dots,\xi^{N-1})\quad \text{in } C^2(U,\mathbb{R}),$$
$$\frac{\partial g_{ij}}{\partial s}(s,\xi^1,\dots,\xi^{N-1})\to 0\quad \text{in } C^1(U,\mathbb{R}) \text{ as } |s|\to\infty.$$

Then $(\mathbb{R}\times S^{N-1},g)$ has at least one non-constant closed geodesic.

Remark 0.4. – We have a non-existence result for non-constant closed geodesics on $(\mathbb{R} \times S^{N-1}, g)$ which is related to Theorem 0.2. See Section 1.2.

We remark that if $V(q) = -1/|q|^2$ then the corresponding metric g^V is the standard product metric, that is, $g^V = g^0$. We can derive our Theorem 0.1 from Theorem 0.3.

Proof of Theorem 0.1. – Under the conditions (V0)–(V3), we can see that $(\mathbb{R} \times S^{N-1}, g^V)$ is a Riemannian manifold and satisfies the assumptions of Theorem 0.3. Thus $(\mathbb{R} \times S^{N-1}, g^V)$ has at least one closed geodesic by Theorem 0.3. As we stated before, non-constant closed geodesics on $(\mathbb{R} \times S^{N-1}, g^V)$ are corresponding to periodic solutions of (HS.1)–(HS.2). □

The existence of closed geodesics on compact Riemannian manifolds is rather well studied (see for example [16] and references therein). For non-compact manifolds, the existence of closed geodesics is studied only in a few papers. Thorbergsson [25] obtains the existence of a closed geodesic when M is complete, non-contractible and its sectional curvature is non-negative outside some compact set. Benci and Giannoni [11] also shows the existence of a closed geodesic for non-compact complete Riemannian manifolds M with asymptotically non-positive sectional curvature. We remark that our Theorem 0.3 ensures the existence of a closed geodesic in a situation different from [25,11]. See Section 1.2 below.

This paper is organized as follows: In Section 1, we study the relation between periodic solutions of (HS.1)–(HS.2) and non-constant closed geodesics on $(\mathbb{R} \times S^{N-1}, g^V)$ where g^V is defined in (0.8). We also give a proof to Theorem 0.2. Sections 2–5 are devoted to the proof of Theorem 0.3. Here we use an idea form Bahri and Li [5] and our proof uses the structure of closed geodesics on the standard sphere S^{N-1}; closed geodesics on the standard sphere S^{N-1} are great circles on S^{N-1}.

1. PRELIMINARIES

In this section, we first study the relation between periodic solutions of (HS.1)–(HS.2) and non-constant closed geodesics on $\mathbb{R} \times S^{N-1}$ with a suitable metric g and we reduce our Theorem 0.1 to our Theorem 0.3. Second, we give some reviews on the results due to Thorbergsson [25] and Benci and Giannoni [11].

1.1. Periodic solutions of (HS.1)–(HS.2) and closed geodesics on $(\mathbb{R} \times S^{N-1}, g)$

In this section, we assume that $V(q) \in C^2(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ and H satisfies

$$H - V(q) > 0 \quad \text{for all } q \in \mathbb{R}^N \setminus \{0\}. \tag{1.1}$$

We introduce a metric h^V on $\mathbb{R}^N \setminus \{0\}$ by

$$h_q^V(v, v) = \big(H - V(q)\big)|v|^2 \quad \text{for } v \in \mathbb{R}^N = T_q\big(\mathbb{R}^N \setminus \{0\}\big).$$

Suppose that $u(\tau)$ is a non-constant closed geodesic on $(\mathbb{R}^N \setminus \{0\}, h^V)$, that is, $u(\tau)$ is a non-constant critical point of the energy functional:

$$E(u) = \frac{1}{2}\int_0^1 \big(H - V(u(\tau))\big)|\dot{u}(\tau)|^2\, d\tau \tag{1.2}$$

acting on 1-periodic functions. Then $u(\tau)$ satisfies for some constant $E_0 > 0$

$$\frac{1}{2}\big(H - V(u(\tau))\big)|\dot{u}(\tau)|^2 = E_0, \tag{1.3}$$

$$\frac{d}{d\tau}\big(\big(H - V(u)\big)\dot{u}\big) + \frac{1}{2}|\dot{u}|^2 \nabla V(u) = 0 \quad \text{for all } \tau. \tag{1.4}$$

By (1.3) and (1.4), we get

$$\frac{1}{2}\left|\frac{H - V(u)}{\sqrt{E_0}}\frac{du}{d\tau}\right|^2 + V(u) = H,$$

$$\frac{H - V(u)}{\sqrt{E_0}}\frac{d}{d\tau}\left(\frac{H - V(u)}{\sqrt{E_0}}\frac{du}{d\tau}\right) + \nabla V(u) = 0.$$

Now we define

$$t(\tau) = \int_0^\tau \frac{\sqrt{E_0}}{H - V(u(\eta))}\, d\eta$$

and let $\tau = \tau(t)$ be the inverse of $t = t(\tau)$. Then $q(t) = u(\tau(t))$ is $t(1)$-periodic and satisfies (HS.1)–(HS.2). Conversely, if $q(t)$ satisfies (HS.1)–(HS.2), we remark that $q(t)$ is not constant in t because of (1.1). We can also see $u(\tau) = q(t(\tau))$ is a non-constant closed geodesic on $(\mathbb{R}^N \setminus \{0\}, h^V)$ after a suitable change of variable $t = t(\tau)$.

Since $\mathbb{R}^N \setminus \{0\} \simeq \mathbb{R} \times S^{N-1}$, we can reduce our problem to the existence problem for closed geodesics on $\mathbb{R} \times S^{N-1}$ with a suitable metric.

Let g^V be a metric on $\mathbb{R} \times S^{N-1}$ induced from h^V by a mapping

$$\mathbb{R} \times S^{N-1} \to \mathbb{R}^N \setminus \{0\}; \quad (s, x) \mapsto e^s x.$$

That is,

$$g^V_{(s,x)}\big((\xi, \eta), (\xi, \eta)\big) = e^{2s}\big(H - V(e^s x)\big) g^0_{(s,x)}\big((\xi, \eta), (\xi, \eta)\big), \qquad (1.5)$$

where g^0 is the standard product metric on $\mathbb{R} \times S^{N-1}$ defined in (0.9).

Therefore there is a one-to-one correspondence between periodic solutions of (HS.1)–(HS.2) and non-constant closed geodesics on $(\mathbb{R} \times S^{N-1}, g^V)$.

We also remark that $(\mathbb{R} \times S^{N-1}, g^V)$ is complete under the condition:

$$\inf_{(s,x) \in \mathbb{R} \times S^{N-1}} e^{2s}\big(H - V(e^s x)\big) > 0.$$

Especially $(\mathbb{R} \times S^{N-1}, g^V)$ is complete if there exists a constant $C > 0$ such that

$$H \geqslant 0 \quad \text{and} \quad -V(q) \geqslant \frac{C}{|q|^2} \quad \text{for all } q \in \mathbb{R}^N \setminus \{0\}. \qquad (1.6)$$

1.2. Variational characterization of closed geodesics on $(\mathbb{R} \times S^{N-1}, g)$ and non-existence result

From now on, we consider a complete Riemannian metric g on $\mathbb{R} \times S^{N-1}$. Closed geodesics on $(\mathbb{R} \times S^{N-1}, g)$ can be characterized as critical points of the following functional:

$$E(u) = \frac{1}{2} \int_0^1 g_u(\dot{u}, \dot{u})\, dt : \Lambda \to \mathbb{R},$$

where Λ is a space of 1-periodic curves on $\mathbb{R} \times S^{N-1}$, i.e.,

$$\Lambda = \big\{u \in H^1(0, 1; \mathbb{R} \times S^{N-1});\ u(0) = u(1)\big\}.$$

It is known that Λ is a C^∞ Hilbert manifold and its tangent space at $u(t) = (s(t), x(t)) \in \Lambda$ is given by

$$T_u\Lambda = \{(\xi(t),\eta(t)) \in H^1(0,1;\mathbb{R}\times\mathbb{R}^N);\ \xi(0)=\xi(1),\ \eta(0)=\eta(1),$$
$$(\xi(t),\eta(t)) \in \mathbb{R}\times T_{u(t)}S^{N-1} \text{ for all } t\in[0,1]\}.$$

We will give a precise Hilbert structure to Λ later in Section 1.3.

Using this variational formulation, we can give a proof of Theorem 0.2.

Proof of Theorem 0.2. – The corresponding metric g^V and functional $E(u)$ for $V(q) = -\phi(|q|)/|q|^2$, $H=0$ are given by

$$g^V_{(s,x)} = \phi(e^s)g^0_{(s,x)},$$
$$E(u) = \frac{1}{2}\int_0^1 \phi(e^{s(t)})\big(|\dot{s}(t)|^2 + |\dot{x}(t)|^2\big)\,dt \quad \text{for } u(t) = (s(t),x(t)) \in \Lambda.$$

We set $u_\tau(t) = (s(t)+\tau, x(t))$ and we see

$$E'(u)(1,0) = \frac{d}{d\tau}\Big|_{\tau=0} E(u_\tau) = \int_0^1 e^{s(t)}\phi'(e^{s(t)})\big(|\dot{s}(t)|^2 + |\dot{x}(t)|^2\big)\,dt.$$

Thus $E'(u) \neq 0$ for all non-constant curve $u \in \Lambda$ under the condition (0.4). □

Remark 1.1. – We have a similar non-existence result for closed geodesics on $(\mathbb{R}\times S^{N-1}, \phi(e^s)g)$ if $\phi(s)$ satisfies the condition (0.4)–(0.6).

In [11,25], the existence of closed geodesics on non-compact manifolds is studied. In Section 2 of [25] and Section 2 of [11], they consider the case of a "warped product" Riemannian manifold; let (M_0, h_0) be a compact Riemannian manifold and let $M = \mathbb{R}\times M_0$. We consider the warped product metric on M:

$$g_{(s,x)}\big((\xi,\eta),(\xi,\eta)\big) = \xi^2 + \beta(s)h_{0x}(\eta,\eta)$$
$$\text{for } (s,x)\in\mathbb{R}\times M_0,\ (\xi,\eta)\in\mathbb{R}\times T_xM_0.$$

Here $\beta:\mathbb{R}\to(0,\infty)$ is a smooth positive function. They showed if $\beta'(s)\neq 0$ for all $s\in\mathbb{R}$, then (M,g) does not have non-constant closed geodesics (see Proposition 2.2 in [11]). We can also modify their arguments to prove our Theorem 0.2.

Their arguments and Remark 1.1 show that if a metric $g_{(s,x)}$ on $\mathbb{R}\times M_0$ satisfies

$$g_{(s,x)} \sim a(ds^2 + h_0) \quad \text{as } s \sim \infty, \tag{1.7}$$

$$g_{(s,x)} \sim b(ds^2 + h_0) \quad \text{as } s \sim -\infty \tag{1.8}$$

and $a \neq b$, then $(\mathbb{R} \times M_0, g)$ does not have non-constant closed geodesics in general. Conversely, when $(M_0, h_0) =$ the standard shere S^{N-1}, our Theorem 0.3 ensures the existence of a non-constant closed geodesic under the condition $a = b$ in (1.7)–(1.8). It seems that for general compact Riemannian manifolds (M_0, h_0) the existence of non-constant closed geodesics is not known under the condition $a = b$ in (1.7)–(1.8).

Besides non-existence result for a "warped-product" Riemannian manifolds, [11,25] study the existence of non-constant closed geodesics on non-compact complete Riemannian manifolds. In [25], the existence of non-constant closed geodesics is proved for non-compact complete manifolds whose sectional curvature is non-negative outside some compact sets. For $N = 2$, [25] also proves the existence for non-compact complete surfaces which are neither homeomorphic to $\mathbb{R}^2$ nor $\mathbb{R} \times S^1$. (See also Bangert [8].) Benci and Giannoni [11] proves the existence for non-compact complete Riemannian manifolds M with asymptotically non-positive sectional curvature under a suitable condition on the topology of the free loop space $\Lambda(M)$ on M.

1.3. A Hilbert structure on Λ

For later use, we define the space Λ and fix a Hilbert structure on Λ precisely.

We embed $\mathbb{R} \times S^{N-1}$ into $\mathbb{R}^{N+1}$ in a standard way:

$$\mathbb{R} \times S^{N-1} = \{(s, x) \in \mathbb{R} \times \mathbb{R}^N;\ |x| = 1\} \subset \mathbb{R}^{N+1}.$$

We also identify

$$T_{(s,x)}(\mathbb{R} \times S^{N-1}) = \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^N;\ x \cdot \eta = 0\}.$$

We introduce the free loop space Λ on $\mathbb{R} \times S^{N-1}$ by

$$\begin{aligned}\Lambda = \{u(t) = (s(t), x(t)) \in H^1(0, 1; \mathbb{R} \times \mathbb{R}^N);\\ (s(0), x(0)) = (s(1), x(1)) \text{ and}\\ |x(t)| = 1 \text{ for all } t \in [0, 1]\}.\end{aligned}$$

We equip Λ with a Riemannian structure

$$\langle(\xi_1,\eta_1),(\xi_2,\eta_2)\rangle_{T_{(s,x)}\Lambda} = \int_0^1 \dot{\xi}_1\dot{\xi}_2 + D_t\eta_1\cdot D_t\eta_2\,dt + \xi_1(0)\xi_2(0) + \eta_1(0)\cdot\eta_2(0)$$

for (ξ_1,η_1), $(\xi_2,\eta_2)\in T_{(s,x)}\Lambda$ and $(s,x)\in\Lambda$. Here $D_t\eta$ is the covariant derivative of $\eta(t)$, i.e., denoting by $P(x(t))$ the projection from $\mathbb{R}^N$ onto $T_{x(t)}S^{N-1}$, $D_t\eta = P(x(t))\,\dot{\eta}(t)$. We also denote by $\mathrm{dist}_\Lambda(\cdot,\cdot)$ the distance on Λ induced by the Riemannian structure $\langle\cdot,\cdot\rangle_{T\Lambda}$. We have

(i) For $u_j,\ u_0\in\Lambda$, $\mathrm{dist}_\Lambda(u_j,u_0)\to 0$ if and only if

$$\|u_j-u_0\|_{H^1(0,1;\mathbb{R}\times\mathbb{R}^N)}\to 0.$$

(ii) For $(u_j)_{j=1}^\infty\subset\Lambda$, $(u_j)_{j=1}^\infty$ is a Cauchy sequence in $(\Lambda,\mathrm{dist}_\Lambda(\cdot,\cdot))$ if and only if

$$\|u_i-u_j\|_{H^1(0,1;\mathbb{R}\times\mathbb{R}^N)}\to 0 \quad \text{as } i,j\to\infty.$$

(iii) $(\Lambda,\mathrm{dist}_\Lambda(\cdot,\cdot))$ is a complete metric space.

Similarly for $S^{N-1}=\{x\in\mathbb{R}^N;\ |x|=1\}$, we define

$$\Lambda_{S^{N-1}} = \{x(t)\in H^1(0,1;\mathbb{R}^N);\ x(0)=x(1),\ |x(t)|=1 \text{ for all } t\in[0,1]\},$$

$$T_x\Lambda_{S^{N-1}} = \{\eta(t)\in H^1(0,1;\mathbb{R}^N);\ \eta(0)=\eta(1),\ \eta(t)\cdot x(t)=0 \text{ for all } t\in[0,1]\} \quad \text{for } x(t)\in\Lambda_{S^{N-1}},$$

$$\langle\eta_1,\eta_2\rangle = \int_0^1 D_t\eta_1\cdot D_t\eta_2\,dt + \eta_1(0)\cdot\eta_2(0) \quad \text{for } \eta_1(t),\eta_2(t)\in T_x\Lambda_{S^{N-1}}.$$

$(\Lambda_{S^{N-1}},\langle\cdot,\cdot\rangle_{T\Lambda_{S^{N-1}}})$ has properties similar to the above (i)–(iii).

Finally in this section, we give a minimax characterization of closed geodesics on S^{N-1} with the standard metric. Closed geodesics on the standard sphere S^{N-1} can be characterized as critical points of

$$E_{S^{N-1}}(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\,dt \in C^2(\Lambda_{S^{N-1}},\mathbb{R}) \tag{1.9}$$

and they are great circles, that is,

$$y_k(t) = e_1\cos 2\pi kt + e_2\sin 2\pi kt, \tag{1.10}$$

where $k \in \mathbb{Z}$ and $e_1,\ e_2 \in \mathbb{R}^N$ are vectors such that $e_i \cdot e_j = \delta_{ij}$. Their critical values are

$$E_{S^{N-1}}(y_k) = 2\pi^2 k^2 \quad (k \in \mathbb{Z}). \tag{1.11}$$

When $N = 2$, we set

$$\Sigma_{S^1} = \{u \in \Lambda_{S^{N-1}};\ \text{the winding number of } u = 1\}. \tag{1.12}$$

Then it is clear that

$$\inf_{u \in \Sigma_{S^1}} E_{S^1}(u) = 2\pi^2 \tag{1.13}$$

and it corresponds to the prime closed geodesic $y_1(t)$.

When $N \geqslant 3$, we set

$$\Sigma_{S^{N-1}} = \left\{\sigma \in C(S^{N-2}, \Lambda_{S^{N-1}});\ \deg\tilde{\sigma} = 1\right\}, \tag{1.14}$$

where

$$\tilde{\sigma} : S^{N-2} \times \left([0,1]/\{0,1\}\right) \simeq S^{N-2} \times S^1 \to S^{N-1}$$

is defined by

$$\tilde{\sigma}(z,t) = \sigma(z)(t) \quad \text{for } \sigma \in C(S^{N-2}, \Lambda_{S^{N-1}})$$

and $\deg\tilde{\sigma}$ is its Brouwer degree.

We have

$$\inf_{\sigma \in \Sigma_{S^{N-1}}} \max_{z \in S^{N-2}} E_{S^{N-1}}\left(\sigma(z)\right) = 2\pi^2 \tag{1.15}$$

and it corresponds to the prime closed geodesic $y_1(t)$. In fact, it is well known that the minimax value

$$c = \inf_{\sigma \in \Sigma_{S^{N-1}}} \max_{z \in S^{N-2}} E_{S^{N-1}}\left(\sigma(z)\right)$$

gives a non-zero critical value of $E_{S^{N-1}}(u)$. Thus $c = 2\pi^2 k^2$ for some $k \in \mathbb{N}$. On the other hand, we find for a suitable $\sigma_0(z) \in \Sigma_{S^{N-1}}$

$$\max_{z \in S^{N-2}} E_{S^{N-1}}\left(\sigma_0(z)\right) = 2\pi^2. \tag{1.16}$$

Thus we have (1.15). An example of $\sigma_0(z)$ is

$$\sigma_0(z)(t)$$
$$= \begin{cases} (2z_1, \ldots, 2z_{N-2}, \sqrt{4z_{N-1}^2 - 3}\cos 2\pi t, & \\ \sqrt{4z_{N-1}^2 - 3}\sin 2\pi t) & \text{if } |z_{N-1}| \geqslant \sqrt{3}/2, \\ (\frac{2|z_{N-1}|}{\sqrt{3}} \frac{z_1}{\sqrt{1-z_{N-1}^2}}, \ldots, \frac{2|z_{N-1}|}{\sqrt{3}} \frac{z_{N-2}}{\sqrt{1-z_{N-1}^2}}, & \\ \sqrt{\frac{3-4z_{N-1}^2}{3}}, 0) & \text{if } |z_{N-1}| < \sqrt{3}/2. \end{cases} \tag{1.17}$$

Here we use notation $z = (z_1, \ldots, z_{N-1}) \in S^{N-2} = \{z \in \mathbb{R}^{N-1};\ |z| = 1\}$.

2. BREAK DOWN OF THE PALAIS–SMALE CONDITION AND MINIMAX METHODS FOR $E(u)$

In what follows, we will give a proof of Theorem 0.3. We assume a Riemannian metric g on $\mathbb{R} \times S^{N-1}$ satisfies (g0)–(g1) and we are going to prove the existence of a critical point $u \in \Lambda$ of

$$E(u) = \frac{1}{2} \int_0^1 g_u(\dot{u}, \dot{u})\, dt \in C^2(\Lambda, \mathbb{R}).$$

First, we study break down of the Palais–Smale condition for $E(u)$.

PROPOSITION 2.1. – *Suppose that $(u_j)_{j=1}^{\infty} \subset \Lambda$ satisfies for some $c > 0$*

$$E(u_j) \to c, \tag{2.1}$$
$$\|E'(u_j)\|_{(T_{u_j}\Lambda)^*} \to 0. \tag{2.2}$$

Then there is a subsequence—we still denote it by u_j—such that one of the following two statements holds:

(i) *There is a non-constant closed geodesic $u_0 \in \Lambda$ on $(\mathbb{R} \times S^{N-1}, g)$ such that*

$$u_j \to u_0 \text{ in } \Lambda, \quad \text{equivalently}, \quad \|u_j - u_0\|_{H^1(0,1;\mathbb{R}\times\mathbb{R}^N)} \to 0.$$

(ii) *There is a closed geodesic $x_0(t) \in \Lambda_{S^{N-1}}$ on the standard sphere S^{N-1} such that if we write $u_j(t) = (s_j(t), x_j(t))$, then*

(1) $s_j(0) \to \infty$ *or* $s_j(0) \to -\infty$.

(2) $\tilde{u}_j(t) \equiv (s_j(t) - s_j(0), x_j(t)) \to (0, x_0(t))$ *in* Λ, *equivalently,* $\|\dot{s}_j\|_{L^2(0,1)} \to 0$ *and* $\|x_j - x_0\|_{H^1(0,1;\mathbb{R}^N)} \to 0$.

We use the following property frequently in what follows:

DEFINITION. – *For* $c \in \mathbb{R}$ *we say that* $E(u)$ *satisfies* $(PS)_c$ *in* Λ *if and only if any sequence* $(u_j)_{j=1}^{\infty} \subset \Lambda$ *satisfying* (2.1) *and* (2.2) *has a strongly convergent subsequence.*

Recalling (1.10) and (1.11), we have

COROLLARY 2.2. – $(PS)_c$ *holds for* $E(u)$ *for* $c \in (0, \infty) \setminus \{2\pi^2 k^2;\ k \in \mathbb{N}\}$. *Moreover non-convergent sequence* $(u_j)_{j=1}^{\infty} = (s_j, x_j)_{j=1}^{\infty} \subset \Lambda$ *satisfying* (2.1) *and* (2.2) *with* $c = 2\pi^2 k^2$ *has a subsequence—still denoted by* u_j*—such that*

(1) $s_j(0) \to \infty$ *or* $s_j(0) \to -\infty$.
(2) $\tilde{u}_j(t) \equiv (s_j(t) - s_j(0), x_j(t)) \to (0, y_k(t))$ *in* Λ *as* $j \to \infty$, *where* $y_k(t)$ *is given in* (1.10).
(3) $E(u_j) \to 2\pi^2 k^2$.
(4) $\liminf_{j\to\infty} \operatorname{index} E''(u_j) \geqslant (N-2)(2k-1)$, *where* $\operatorname{index} E''(u_j)$ *denotes the Morse index of* $E''(u_j)$.

To prove Proposition 2.1, we first observe

LEMMA 2.3. – *Under the assumption* (g1), *there are constants* m_1, $m_2 > 0$ *such that*

$$m_1(\xi^2 + |\eta|^2) \leqslant g_u\big((\xi, \eta), (\xi, \eta)\big) \leqslant m_2(\xi^2 + |\eta|^2) \tag{2.3}$$

for all $u = (s, x) \in \mathbb{R} \times S^{N-1}$ *and* $(\xi, \eta) \in T_u(\mathbb{R} \times S^{N-1}) = \mathbb{R} \times T_x S^{N-1} = \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^N;\ x \cdot \eta = 0\}$.

Proof of Proposition 2.1. – Assume that $(u_j)_{j=1}^{\infty} = (s_j, x_j)_{j=1}^{\infty} \subset \Lambda$ satisfies (2.1) and (2.2). By Lemma 2.3, we see for some constant $C > 0$ independent of j

$$\|\dot{s}_j\|_{L^2(0,1)}^2 + \|\dot{x}_j\|_{L^2(0,1)}^2 \leqslant C \quad \text{for all } j. \tag{2.4}$$

Thus we have

$$\|s_j(t) - s_j(0)\|_{L^\infty(0,1)} \leqslant \|\dot{s}_j\|_{L^2(0,1)} \leqslant C' \quad \text{for all } j. \tag{2.5}$$

We may assume that $\lim_{j\to\infty} s_j(0) \in [-\infty, \infty]$ exists and consider two cases:

$$\text{Case 1: } \lim_{j\to\infty} s_j(0) \in (-\infty, \infty),$$
$$\text{Case 2: } \lim_{j\to\infty} s_j(0) = \pm\infty.$$

Case 1: $\lim_{j\to\infty} s_j(0) \in (-\infty, \infty)$.

In this case, (s_j, x_j) stays bounded as $j \to \infty$ by (2.4), (2.5). Thus we can show the statement (i) of Proposition 2.1 in a standard way.

Case 2: $\lim_{j\to\infty} s_j(0) = \pm\infty$.

Setting $\tilde{u}_j(t) = (s_j(t) - s_j(0), x_j(t)) \in \Lambda$, we get from (2.5) and (g1)

(1) $\tilde{u}_j$ stays bounded in Λ as $j \to \infty$,

(2) $\|E^{0\prime}(\tilde{u}_j)\|_{(T_{\tilde{u}_j}\Lambda)^*} \to 0$ as $j \to \infty$. Here g^0 is the standard product metric defined in (0.9) and $E^0(u)$ is a functional corresponding to closed geodesics on $(\mathbb{R} \times S^{N-1}, g^0)$:

$$E^0(u) = \frac{1}{2}\int_0^1 g_u^0(\dot{u}, \dot{u})\,dt = \frac{1}{2}\int_0^1 |\dot{s}|^2 + |\dot{x}|^2\,dt$$

for $u = (s, x) \in \Lambda$.

Thus, we can extract a subsequence—still we denote it by $\tilde{u}_j$—such that for some $\tilde{u}_0 = (\tilde{s}_0, \tilde{x}_0) \in \Lambda$

$$\tilde{u}_j \to \tilde{u}_0 \quad \text{in } \Lambda.$$

Clearly $\tilde{u}_0$ is a critical point of $E^0(u)$, that is, $\tilde{u}_0$ is a closed geodesic on $(\mathbb{R} \times S^{N-1}, g^0)$. Thus we have

(1) $\tilde{s}_0(t) \equiv p$ is a constant,

(2) $\tilde{x}_0(t)$ is a closed geodesic on the standard sphere S^{N-1},

(3) $x_j \to \tilde{x}_0$ in Λ and $\tilde{s}_j(t) \equiv s_j(t) - s_j(0) \to p$ in $H^1(0, 1; \mathbb{R})$. Since $\tilde{s}_j(0) = 0$, p must be 0.

Therefore we get the statement (ii). □

Proof of Corollary 2.2. – It suffices to show (4). Since $\tilde{u}_j(t) = (s_j(t) - s_j(0), x_j(t)) \to (0, y_k(t))$ and $E''(u_j) \to E^{0\prime\prime}(0, y_k)$, it suffices to show index $E^{0\prime\prime}(0, y_k) \geqslant (N-2)(2k-1)$. Let $e_1, \ldots, e_N \in \mathbb{R}^N$ be an orthonormal basis of $\mathbb{R}^N$ and assume $y_k(t) = e_1 \cos 2\pi kt + e_2 \sin 2\pi kt$. Then we can easily see

$$E^{0\prime\prime}(0, y_k)\big((0, v), (0, v)\big) < 0 \quad \text{for all } v \in V \setminus \{0\},$$

where $V = \text{span}\{e_i \cos 2\pi jt, e_i \sin 2\pi jt;\ i = 3, 4, \ldots, N,\ j = 0, 1, \ldots, k-1\}$. Thus index $E^{0''}(0, y_k) \geqslant \dim V = (N-2)(2k-1)$. □

Next we define two minimax values to find a critical point of $E(u)$. Our methods are inspired by the argument of Bahri and Li [5] in which the existence of positive solutions of semilinear inhomogeneous elliptic equations in $\mathbb{R}^N$ is studied. See also Bahri and Lions [7]. In what follows, we mainly deal with the case $N \geqslant 3$. The case $N = 2$ will be studied in Section 5.

To define our first minimax value, we need the following definitions: for a $(N-2)$-dimensional compact manifold M, we set

$$\Gamma(M) = \{\gamma \in C(M, \Lambda);\ \gamma(M) \text{ is NOT contractible in } \Lambda\}. \tag{2.6}$$

We consider the following class of compact manifolds:

$$\mathcal{M}_{N-2} = \{M;\ M \text{ is a } (N-2)\text{-dimensional compact connected manifold such that } \Gamma(M) \neq \emptyset\}. \tag{2.7}$$

We remark that $S^{N-2} \in \mathcal{M}_{N-2}$ and $\mathcal{M}_{N-2} \neq \emptyset$ because of the existence of $\sigma_0(z)$ given in (1.17). For $M \in \mathcal{M}_{N-2}$, we set

$$b(M) = \inf_{\gamma \in \Gamma(M)} \max_{u \in \gamma(M)} E(u) \tag{2.8}$$

and

$$\underline{b} = \inf_{M \in \mathcal{M}_{N-2}} b(M). \tag{2.9}$$

To define our second minimax value, we consider the following class of mappings:

$$\overline{\Gamma} = \{\gamma \in C(\mathbb{R} \times S^{N-2}, \Lambda);\ \gamma(r, z)(t) = (r, \sigma_0(z)(t)) \text{ for sufficiently large } |r|\}, \tag{2.10}$$

where $\sigma_0(z)$ is defined in (1.17). We remark $\gamma_0(r, z)(t) = (r, \sigma_0(z)(t)) \in \overline{\Gamma}$ and $\overline{\Gamma} \neq \emptyset$. We define

$$\overline{b} = \inf_{\gamma \in \overline{\Gamma}} \sup_{u \in \gamma(\mathbb{R} \times S^{N-2})} E(u). \tag{2.11}$$

Two values $\underline{b}$ and $\overline{b}$ play an important role to show the existence of critical point of $E(u)$. $\underline{b}$ and $\overline{b}$ have the following properties:

PROPOSITION 2.4. –

(i) $0 < \underline{b} \leqslant 2\pi^2$.
(ii) $2\pi^2 \leqslant \overline{b}$.

To prove Proposition 2.4, we need

LEMMA 2.5. – *Let M be a compact $(N-2)$-dimensional manifold such that*

$$\Gamma_{S^{N-1}}(M) = \{\sigma \in C(M, \Lambda_{S^{N-1}});\ \sigma(M) \text{ is not contractible in } \Lambda_{S^{N-1}}\}$$

is not empty. Then

$$\inf_{\sigma \in \Gamma_{S^{N-1}}(M)} \max_{x \in \sigma(M)} E_{S^{N-1}}(x) \geqslant 2\pi^2, \tag{2.12}$$

where $E_{S^{N-1}}(x) \in C^2(\Lambda_{S^{N-1}}, \mathbb{R})$ is defined in (1.9).

Proof. – First we show that

$$\inf_{\sigma \in \Gamma_{S^{N-1}}(M)} \max_{x \in \sigma(M)} E_{S^{N-1}}(x) > 0.$$

If not, for any $\varepsilon > 0$ we can find $\sigma \in \Gamma_{S^{N-1}}(M)$ such that

$$E_{S^{N-1}}(x) \leqslant \varepsilon \quad \text{for all } x \in \sigma(M).$$

Thus we have for $x = \sigma(z)$, $z \in M$

$$\max_{t_1, t_2 \in [0,1]} \left|x(t_1) - x(t_2)\right| \leqslant \int_0^1 |\dot{x}|\, dt \leqslant \sqrt{2E_{S^{N-1}}(x)} \leqslant \sqrt{2\varepsilon}.$$

Therefore for $\varepsilon \in (0, \frac{1}{2})$

$$\left|(1-\tau)x(t) + \tau x(0)\right| \geqslant |x(t)| - \tau\left|x(t) - x(0)\right| > 0$$

for all $\tau \in [0,1]$, $x \in \sigma(M)$, $t \in [0,1]$. Thus

$$\sigma_\tau(z)(t) = \frac{(1-\tau)\sigma(z)(t) + \tau\sigma(z)(0)}{|(1-\tau)\sigma(z)(t) + \tau\sigma(z)(0)|} : [0,1] \times M \to \Lambda_{S^{N-1}}$$

is well-defined. We remark that $\sigma_1(z)(t)$ is independent of t and it can be regarded as a map from M to S^{N-1}. Since $\dim M = N-2$, σ_1 is not onto and $\sigma_1(M)$ is contractible to a point in S^{N-1}. Thus $\sigma(M)$ is contractible in $\Lambda_{S^{N-1}}$ and this contradicts $\sigma \in \Gamma_{S^{N-1}}(M)$. Therefore

$$\inf_{\sigma\in\Gamma_{S^{N-1}}(M)} \max_{x\in\sigma(M)} E_{S^{N-1}}(x) > 0.$$

Since the Palais–Smale condition holds for $E_{S^{N-1}}(u) \in C^2(\Lambda_{S^{N-1}}, \mathbb{R})$, we can see that $\inf_{\sigma\in\Gamma_{S^{N-1}}(M)} \max_{x\in\sigma(M)} E_{S^{N-1}}(x)$ is a positive critical value of $E_{S^{N-1}}(x)$. (For a similar argument, see the proof of Proposition 2.6 below.) Since critical points of $E_{S^{N-1}}(x)$ correspond to closed geodesics on the standard sphere S^{N-1}, we can see that the least positive critical value is $2\pi^2$. Thus we get (2.12). □

Proof of Proposition 2.4. – (i) For $M \in \mathcal{M}_{N-2}$ and $\gamma(z) = (s(z), x(z)) \in \Gamma(M)$ we set $\tilde{\gamma} \in C(M, \Lambda_{S^{N-1}})$ by

$$\tilde{\gamma}(z) = x(z) \quad \text{for } z \in M.$$

We can easily see that $\tilde{\gamma} \in \Gamma_{S^{N-1}}(M)$. Thus by Lemma 2.5

$$\max_{x\in\tilde{\gamma}(M)} E_{S^{N-1}}(x) \geqslant 2\pi^2.$$

On the other hand, by (2.3) we have for $u = (s, x) \in \Lambda$

$$E(u) = \frac{1}{2}\int_0^1 g_u(\dot{u}, \dot{u})\,dt \geqslant \frac{m_1}{2}\int_0^1 |\dot{s}|^2 + |\dot{x}|^2\,dt$$
$$\geqslant \frac{m_1}{2}\int_0^1 |\dot{x}|^2\,dt = m_1 E_{S^{N-1}}(x).$$

Thus we have

$$\max_{u\in\gamma(M)} E(u) \geqslant m_1 \max_{x\in\tilde{\gamma}(M)} E_{S^{N-1}}(x) \geqslant 2\pi^2 m_1.$$

Therefore

$$\underline{b} = \inf_{M\in\mathcal{M}_{N-2}} \inf_{\gamma\in\Gamma(M)} \max_{u\in\gamma(M)} E(u) \geqslant 2\pi^2 m_1 > 0.$$

To show $\underline{b} \leqslant 2\pi^2$, we recall $S^{N-2} \in \mathcal{M}_{N-2}$ and we set $\gamma_\ell(z)(t) = (\ell, \sigma_0(z)(t))$ for $\ell \in \mathbb{R}$ and $z \in S^{N-2}$. Then $\gamma_\ell(S^{N-2})$ is not contractible in Λ and

$$b(S^{N-2}) \leqslant \max_{z\in S^{N-2}} E\big(\gamma_\ell(z)\big) = \max_{z\in S^{N-2}} E\big(\ell,\sigma_0(z)\big).$$

Letting $\ell\to\infty$, we get by (g1) and (1.16)

$$b(S^{N-2}) \leqslant 2\pi^2.$$

Thus $\underline{b} \leqslant b(S^{N-2}) \leqslant 2\pi^2$. Thus (i) is proved.

To prove (ii), we remark that for any $\gamma(r,z)\in\overline{\Gamma}$ there exists a $R>0$ such that

$$\gamma(r,z)(t) = \big(r,\sigma_0(z)(t)\big) \quad \text{for } |r|\geqslant R.$$

Thus,

$$\sup_{(r,z)\in\mathbb{R}\times S^{N-2}} E\big(\gamma(r,z)\big) \geqslant \limsup_{r\to\pm\infty}\ \max_{z\in S^{N-2}} E\big(r,\sigma_0(z)\big) = 2\pi^2.$$

Therefore we obtain $\overline{b}\geqslant 2\pi^2$. □

By the above Proposition 2.4, it occurs one of the following four cases:

Case A: $0<\underline{b}<2\pi^2$.
Case B: $\overline{b}\notin\{2\pi^2k^2;\ k\in\mathbb{N}\}$.
Case C: $\overline{b}\in\{2\pi^2k^2;\ k\in\mathbb{N}\setminus\{1\}\}$.
Case D: $\underline{b}=\overline{b}=2\pi^2$.

In each case, we will show that $E(u)$ has a critical point. Actually $\underline{b}$ is a critical value of $E(u)$ in cases A, D and $\overline{b}$ is a critical value in cases B, C.

The cases A, B are easy to deal with and we can see that $\underline{b}$ or $\overline{b}$ is a critical value of $E(u)$ rather in a standard way.

PROPOSITION 2.6. –

(i) *If* $\underline{b}\in(0,2\pi^2)$, *then* $\underline{b}$ *is a critical value of* $E(u)$.
(ii) *If* $\overline{b}\notin\{2\pi^2k^2;\ k\in\mathbb{N}\}$, *then* $\overline{b}$ *is a critical value of* $E(u)$.

To prove the above proposition, we need the following deformation lemma.

LEMMA 2.7. – *Assume that* $(PS)_c$ *holds at level* c *for* $E(u)\in C^2(\Lambda,\mathbb{R})$ *and* c *is not a critical value of* $E(u)$. *Then for any* $\overline{\varepsilon}>0$ *there exist* $\varepsilon_0\in(0,\overline{\varepsilon})$ *and* $\eta(\tau,u)\in C([0,1]\times\Lambda,\Lambda)$ *such that*

(1) $\eta(0,u)=u$ *for all* $u\in\Lambda$.
(2) $\eta(\tau,u)=u$ *for all* $\tau\in[0,1]$ *if* $E(u)\notin[c-\overline{\varepsilon},c+\overline{\varepsilon}]$.
(3) $E(\eta(\tau,u))\leqslant E(u)$ *for all* $\tau\in[0,1]$ *and* $u\in\Lambda$.
(4) $E(\eta(1,u))\leqslant c-\varepsilon_0$ *if* $E(u)\leqslant c+\varepsilon_0$.

Proof. – See Appendix A of Rabinowitz [20]. □

Proof of Proposition 2.6. – (i) By the assumption $\underline{b}\in(0,2\pi^2)$, there exists a sequence $(M_j)_{j=1}^{\infty}\subset\mathcal{M}_{N-2}$ such that

$$b(M_j)\in(0,2\pi^2)\quad\text{for all } j,$$
$$b(M_j)\to\underline{b}\qquad\text{as } j\to\infty.$$

First we show that $b(M_j)$ is a critical value of $E(u)$. If not, we choose $\overline{\varepsilon}>0$ so that $(b(M_j)-\overline{\varepsilon},b(M_j)+\overline{\varepsilon})\subset(0,2\pi^2)$ and apply Lemma 2.7 to obtain $\varepsilon_0\in(0,\overline{\varepsilon})$ and $\eta(\tau,u)\in C([0,1]\times\Lambda,\Lambda)$. We choose $\gamma\in\Gamma(M_j)$ such that

$$\max_{u\in\gamma(M_j)}E(u)\leqslant b(M_j)+\varepsilon_0.$$

We can easily see that $\tilde{\gamma}(z)\equiv\eta(1,\gamma(z))$ belongs to $\Gamma(M_j)$ and by (4) of Lemma 2.7

$$\max_{u\in\tilde{\gamma}(M_j)}E(u)\leqslant b(M_j)-\varepsilon_0.$$

This is a contradiction and $b(M_j)$ is a critical value of $E(u)$. Since $(PS)_c$ holds in $(0,2\pi^2)$ and $\underline{b}=\lim_{j\to\infty}b(M_j)\in(0,2\pi^2)$ is an accumulation point of critical values, $\underline{b}$ is also a critical value of $E(u)$.

(ii) By the assumption $\overline{b}\notin\{2\pi^2k^2;\ k\in\mathbb{N}\}$, $\overline{b}>2\pi^2$ and $(PS)_{\overline{b}}$ follow from (ii) of Proposition 2.4 and Corollary 2.2. Thus we can prove (ii) in a similar way to (i). □

Thus we can find at least one critical point in cases A, B. The following two sections will be devoted to study cases C, D.

3. CASE C: $\overline{b}\in\{2\pi^2k^2;\ k\in\mathbb{N}\setminus\{1\}\}$

Here we suppose $\overline{b}=2\pi^2k_0^2$ $(k_0=2,3,\ldots)$. We use the Morse indices to deal with this case. We refer to Fang and Ghoussoub [14] for a related argument.

Since $E(u)$ does not satisfy the Palais–Smale compactness condition, we introduce a perturbed functional $E_\nu(u): \Lambda \to \mathbb{R}$ ($\nu \in [0,1]$) by

$$E_\nu(u) = \frac{1}{2}\int_0^1 \left(g_u(\dot{u},\dot{u}) + \nu(e^{2s}+e^{-2s})\left(|\dot{s}|^2+|\dot{x}|^2\right)\right) dt \tag{3.1}$$

for $u(t) = (s(t), x(t)) \in \Lambda$. This perturbation is introduced to obtain the Palais–Smale condition $(PS)_c$ for all $c > 0$. The corresponding argument for singular Hamiltonian systems is developed in [22].

First we have

PROPOSITION 3.1. – *For $\nu \in (0,1]$, the functional $E_\nu(u)$ satisfies $(PS)_c$ for all $c > 0$. That is, if $(u_j)_{j=1}^\infty \subset \Lambda$ satisfies*

$$E_\nu(u_j) \to c > 0, \tag{3.2}$$

$$E'_\nu(u_j) \to 0 \quad \text{as } j \to \infty. \tag{3.3}$$

Then $(u_j)_{j=1}^\infty$ has a strongly convergent subsequence.

Proof. – By (2.3), we have

$$\frac{m_1}{2}\int_0^1 |\dot{s}|^2 + |\dot{x}|^2\, dt \leqslant E_\nu(u) \quad \text{for all } u \in \Lambda. \tag{3.4}$$

Thus under the condition (3.2) we can see $\dot{u}_j = (\dot{s}_j, \dot{x}_j)$ is bounded in $L^2(0,1)$ and there exists a constant $C > 0$ such that

$$\max_{t\in[0,1]} \left|s_j(t) - s_j(0)\right| \leqslant C \quad \text{for all } j \in \mathbb{N}. \tag{3.5}$$

Remark that $|x_j(t)| = 1$ for all t. To obtain boundedness of $u_j = (s_j, x_j)$ in Λ, we show that $|s_j(0)|$ remains bounded as $j \to \infty$.

Arguing indirectly, we assume that $s_j(0) \to \infty$. The case $s_j(0) \to -\infty$ can be treated similarly. By (3.5), $\min_{t\in[0,1]} s_j(t) \to \infty$. Thus by (3.2) and the definition of $E_\nu(u)$, we have

$$(\dot{s}_j, \dot{x}_j) \to (0,0) \quad \text{strongly in } L^2(0,1). \tag{3.6}$$

We remark that (3.6) and (2.3) imply

$$E(u_j) = \frac{1}{2}\int_0^1 g_{u_j}(\dot{u}_j, \dot{u}_j)\, dt \to 0.$$

Thus

$$c = \frac{\nu}{2} \lim_{j\to\infty} \int_0^1 e^{2s_j}\left(|\dot{s}_j|^2 + |\dot{x}_j|^2\right) dt. \tag{3.7}$$

On the other hand, we have from (3.3) that $E'_\nu(u_j)(1,0) \to 0$, i.e.,

$$E'(u_j)(1,0) + \nu \int_0^1 \left(e^{2s_j} - e^{-2s_j}\right)\left(|\dot{s}_j|^2 + |\dot{x}_j|^2\right) dt \to 0.$$

Remarking that (3.6) implies $E'(u_j)(1,0) \to 0$, we have

$$\int_0^1 e^{2s_j}\left(|\dot{s}_j|^2 + |\dot{x}_j|^2\right) dt \to 0. \tag{3.8}$$

Combining (3.7) and (3.8), we get $c = 0$. But this contradicts the assumption (3.2). Thus $u_j = (s_j, x_j)$ remains bounded as $j \to \infty$ and we can show the existence of a strongly convergent subsequence in a standard way. □

Next we study an asymptotic behavior of critical points $(u_\nu)_{\nu\in(0,1]}$ of $E_\nu(u)$ as $\nu \to 0$.

PROPOSITION 3.2. – *Suppose that for $\nu \in (0,1]$ there exists a critical point $u_\nu \in \Lambda$ of $E_\nu(u)$ such that for some $c > 0$*

$$E_\nu(u_\nu) \to c > 0 \quad \text{as } \nu \to 0. \tag{3.9}$$

Then it occurs one of the following two cases:

(i) *There exists a strongly convergent subsequence (u_{ν_j}) $(\nu_j \to 0)$ in Λ.*

(ii) *There exist a subsequence $u_{\nu_j} = (s_{\nu_j}, x_{\nu_j})$ and $k \in \mathbb{N}$ such that*

(1) $s_{\nu_j}(0) \to \pm\infty$.

(2) *Set $\tilde{u}_{\nu_j}(t) = (s_{\nu_j}(t) - s_{\nu_j}(0), x_{\nu_j}(t))$, then*

$$\tilde{u}_{\nu_j}(t) \to \big(0, y_k(t)\big) \quad \text{in } \Lambda, \tag{3.10}$$

where $y_k(t) = e_1 \cos 2\pi kt + e_2 \sin 2\pi kt$ $(k \in \mathbb{N},\ e_i \cdot e_j = \delta_{ij})$ is a closed geodesic on the standard sphere.

(3) $E_{\nu_j}(u_{\nu_j}) \to 2\pi^2 k^2$.

(4) $\liminf_{j\to\infty} \operatorname{index} E''_{\nu_j}(u_{\nu_j}) \geqslant (N-2)(2k-1)$.

Proof. – Suppose that a sequence of critical points $(u_\nu)_{\nu>0}$ satisfies (3.9). As in the proof of Proposition 3.1, we can see that

$$\dot{u}_\nu = (\dot{s}_\nu, \dot{x}_\nu) \text{ is bounded in } L^2(0,1). \tag{3.11}$$

Thus,

$$\max_{t\in[0,1]} \left|s_\nu(t) - s_\nu(0)\right| \leqslant C \quad \text{for all } \nu \in (0,1] \tag{3.12}$$

and we can see that $(u_\nu)_{\nu\in(0,1]}$ has a strongly convergent subsequence if $|s_\nu(0)|$ remains bounded as $\nu \to 0$.

We suppose that

$$s_\nu(0) \to \infty \quad \text{as } \nu \to 0. \tag{3.13}$$

The case $s_\nu(0) \to -\infty$ can be treated in a similar way. By (3.11)–(3.13) and (g1), we have

$$E'(u_\nu)(1,0) = \int_0^1 \frac{d}{d\eta}\bigg|_{\eta=0} g_{(s_\nu(t)+\eta, x_\nu(t))}\big((\dot{s}_\nu, \dot{x}_\nu), (\dot{s}_\nu, \dot{x}_\nu)\big)\,dt \to 0 \quad \text{as } \nu \to 0.$$

Thus we have from $E'_\nu(u_\nu) = 0$ that

$$\nu \int_0^1 e^{2s_\nu(t)}\big(|\dot{s}_\nu|^2 + |\dot{x}_\nu|^2\big)\,dt = E'_\nu(u_\nu)(1,0) - E'(u_\nu)(1,0) - \nu\int_0^1 e^{-2s_\nu(t)}\big(|\dot{s}_\nu|^2 + |\dot{x}_\nu|^2\big)\,dt \to 0 \quad \text{as } \nu \to 0. \tag{3.14}$$

Thus we see

$$E(u_\nu) = E_\nu(u_\nu) - \frac{\nu}{2}\int_0^1 \big(e^{2s_\nu(t)} + e^{-2s_\nu(t)}\big)\big(|\dot{s}_\nu|^2 + |\dot{x}_\nu|^2\big)\,dt \to \lim_{\nu\to 0} E_\nu(u_\nu) = c \quad \text{as } \nu \to 0, \tag{3.15}$$

Therefore by (3.13)

$$\lim_{\nu\to 0} E(u_\nu) = \lim_{\nu\to 0} \frac{1}{2}\int_0^1 \left(|\dot{s}_\nu|^2 + |\dot{x}_\nu|^2\right) dt = c > 0. \tag{3.16}$$

Using (3.14) again, we get from (3.12), (3.16) that

$$\nu e^{2s_\nu(0)} \to 0 \quad \text{as } \nu \to 0. \tag{3.17}$$

We also have from (3.11), (3.12), (3.17)

$$\begin{aligned} &\left\|E'(u_\nu) - E'_\nu(u_\nu)\right\|_{(T_{u_\nu}\Lambda)^*} \\ &= \sup_{(\xi,\eta)\in T_{u_\nu}\Lambda,\ \|\xi\|^2+\|\eta\|^2\leqslant 1} \left(\nu \int_0^1 \xi(t)\left(e^{2s_\nu(t)} - e^{-2s_\nu(t)}\right)\right. \\ &\quad \times \left(|\dot{s}_\nu|^2 + |\dot{x}_\nu|^2\right) dt + \nu \int_0^1 \left(e^{2s_\nu(t)} + e^{-2s_\nu(t)}\right) \\ &\quad \left. \times \left(\dot{s}_\nu \cdot \dot{\xi} + \dot{x}_\nu \cdot \dot{\eta}\right) dt \right) \to 0 \quad \text{as } \nu \to 0. \end{aligned} \tag{3.18}$$

Similarly we can see also from (3.11), (3.12), (3.17) that

$$\left\|E''(u_\nu) - E''_\nu(u_\nu)\right\| \to 0 \quad \text{as } \nu \to 0. \tag{3.19}$$

By (3.15), (3.18), we can see that (u_ν) satisfies

$$E(u_\nu) \to c > 0 \quad \text{and} \quad E'(u_\nu) \to 0.$$

Thus we can apply Proposition 2.1, Corollary 2.2 and we have (3.10) for a suitable $k \in \mathbb{N}$. Now statements (3) and (4) follow from (3.15), (3.19) and (3)–(4) of Corollary 2.2. □

Now we use the above Propositions 3.1 and 3.2 to deal with the case C: $\overline{b} = 2\pi^2 k_0^2$ ($k_0 = 2, 3, \ldots$).

We choose $L_0 \geqslant 2$ such that

$$E\left(r, \sigma_0(z)\right) \leqslant 3\pi^2 \quad \text{for all } |r| \geqslant L_0 \text{ and } z \in S^{N-2} \tag{3.20}$$

and we set

$$\overline{\Gamma}_0=\{\gamma\in C([-L_0,L_0]\times S^{N-2},\Lambda);\ \gamma(\pm L_0,z)=(\pm L_0,\sigma_0(z))\},$$
$$\overline{b}_0=\inf_{\gamma\in\overline{\Gamma}_0}\max_{(r,z)\in[-L_0,L_0]\times S^{N-2}}E(\gamma(r,z)).$$

We have the following

LEMMA 3.3. – $\overline{b}_0=\overline{b}$.

Proof. – For any $\gamma\in\overline{\Gamma}_0$, we set

$$\overline{\gamma}(r,z)=\begin{cases}\gamma(r,z) & \text{for } |r|\leqslant L_0,\\ (r,\sigma_0(z)) & \text{for } |r|>L_0.\end{cases}$$

Then we see $\overline{\gamma}\in\overline{\Gamma}$. By (3.20) and $\overline{b}=2\pi^2k_0^2\geqslant 8\pi^2$, we have

$$\sup_{(r,z)\in\mathbb{R}\times S^{N-2}}E(\overline{\gamma}(r,z))=\max_{(r,z)\in[-L_0,L_0]\times S^{N-2}}E(\gamma(r,z)).$$

Thus we get $\overline{b}\leqslant\overline{b}_0$.

Conversely, for any $\overline{\gamma}\in\overline{\Gamma}$ we can find $L\geqslant L_0$ such that

$$\overline{\gamma}(r,z)=(r,\sigma_0(z))\quad\text{for } |r|\geqslant L.$$

We set $\gamma_0\in\overline{\Gamma}_0$ by

$$\gamma_0(r,z)=\begin{cases}(-(|r|-L_0+1)L_0\\ \quad-(L_0-|r|)L,\sigma_0(z)) & \text{for } r\in[-L_0,-L_0+1],\\ \overline{\gamma}(\frac{L}{L_0-1}r,z) & \text{for } |r|<L_0-1,\\ ((|r|-L_0+1)L_0\\ \quad+(L_0-|r|)L,\sigma_0(z)) & \text{for } r\in[L_0-1,L_0].\end{cases}$$

Then we have

$$\max_{(r,z)\in[-L_0,L_0]\times S^{N-2}}E(\gamma_0(r,z))=\sup_{(r,z)\in\mathbb{R}\times S^{N-2}}E(\overline{\gamma}(r,z))$$

and we obtain $\overline{b}_0\leqslant\overline{b}$. □

Next we set for $\nu \in (0, 1]$

$$\overline{b}_\nu = \inf_{\gamma \in \overline{\Gamma}_0} \max_{(r,z)\in[-L_0,L_0]\times S^{N-2}} E_\nu\big(\gamma(r,z)\big).$$

Then we can easily see

$$\overline{b} = \overline{b}_0 \leqslant \overline{b}_\nu \quad \text{for all } \nu \in (0, 1], \tag{3.21}$$

$$\overline{b}_\nu \to \overline{b}_0 = \overline{b} \quad \text{as } \nu \to 0. \tag{3.22}$$

By Proposition 3.1, we can see

PROPOSITION 3.4. – *For any $\nu \in (0, 1]$, $\overline{b}_\nu$ is a critical value of $E_\nu(u)$ and there exists a critical point $u_\nu \in \Lambda$ such that*

$$E_\nu(u_\nu) = \overline{b}_\nu, \tag{3.23}$$

$$E'_\nu(u_\nu) = 0, \tag{3.24}$$

$$\text{index } E_{\nu''}(u_\nu) \leqslant N - 1. \tag{3.25}$$

Proof. – Since $\overline{b}_\nu \geqslant \overline{b} \geqslant 8\pi^2$ and $(PS)_c$ holds for $E_\nu(u)$ at the level $c = \overline{b}_\nu$, we can see that $\overline{b}_\nu$ is a critical value of $E_\nu(u)$. Thus there exists a critical point $u_\nu \in \Lambda$ such that (3.23) and (3.24) hold. We can get (3.25) as in [21] (see also [6,9,17,24]). □

Proof of Theorem 0.3 in case C. – By Proposition 3.4, we can find a sequence $(u_\nu)_{\nu\in(0,1]} \subset \Lambda$ such that (3.21)–(3.22) and (3.23)–(3.25) hold. Applying Proposition 3.2, we can extract a subsequence u_{ν_j} ($\nu_j \to 0$) such that either the statement (i) or (ii) of Proposition 3.2 occurs.

Suppose that (ii) occurs. Then by (ii)(3), (4), we have

$$\liminf_{j\to\infty} \text{index } E''_{\nu_j}(u_{\nu_j}) \geqslant (N-2)(2k_0 - 1).$$

Since $k_0 \geqslant 2$ and $N \geqslant 3$, this contradicts (3.25). Thus (i) takes a place and $u = \lim_{j\to\infty} u_{\nu_j}$ satisfies $E(u) = 2\pi^2 k_0^2$ and $E'(u) = 0$. □

4. CASE D: $b = \overline{b} = 2\pi^2$

Here we suppose $\underline{b} = \overline{b} = 2\pi^2$ and we show that $2\pi^2$ is a critical value of $E(u)$.

First we assume $\overline{b} = 2\pi^2$ and we find $M \in \mathcal{M}_{N-2}$ and $\gamma \in \Gamma(M)$ with special properties.

PROPOSITION 4.1. – *Assume $\overline{b} = 2\pi^2$. Then for any $\varepsilon > 0$ there exist $\hat{M} \in \mathcal{M}_{N-2}$ and $\hat{\gamma} \in \Gamma(\hat{M})$ such that*

$$\max_{u \in \gamma(\hat{M})} E(u) \leqslant 2\pi^2 + \varepsilon, \tag{4.1}$$

$$s(0) \in [0, 1] \quad \textit{for all } u(t) = \big(s(t), x(t)\big) \in \hat{\gamma}(\hat{M}). \tag{4.2}$$

Proof. – Since $\overline{b} = 2\pi^2$, for any $\varepsilon > 0$ there exists a $\gamma \in \overline{\Gamma}$ such that

$$\sup_{(r,z) \in \mathbb{R} \times S^{N-2}} E\big(\gamma(r, z)\big) \leqslant 2\pi^2 + \varepsilon.$$

Approximating γ by a C^∞-mapping, we may assume that $\gamma \in C^\infty(\mathbb{R} \times S^{N-2}, \Lambda) \cap \overline{\Gamma}$. We write $\gamma(r, z)(t) = (s(r, z)(t), x(r, z)(t))$ and consider a C^∞-mapping

$$f(r, z) = s(r, z)(0) : \mathbb{R} \times S^{N-2} \to \mathbb{R}.$$

By the Sard's theorem, we can find $\beta \in [0, 1]$ such that

$$f(r, z) = \beta \quad \text{implies} \quad \big(f_r(r, z), f_z(r, z)\big) \neq (0, 0). \tag{4.3}$$

By (4.3), $f^{-1}(\beta)$ is a $(N-2)$-dimensional submanifold of $\mathbb{R} \times S^{N-1}$. Since $f(r, z) = r$ for sufficiently large $|r|$, $f^{-1}(\beta)$ is compact and we can write

$$f^{-1}(\beta) = M_1 \cup M_2 \cup \cdots \cup M_n,$$

where $M_1, M_2, \ldots, M_n$ are $(N-2)$-dimensional compact connected submanifolds of $\mathbb{R} \times S^{N-2}$. Later we show that

$$\begin{aligned} &\text{there exists a } j_0 \in \{1, \ldots, n\} \text{ such that } M_{j_0} \in \mathcal{M}_{N-2} \\ &\text{and } \gamma|_{M_{j_0}} \in \Gamma(M_{j_0}). \end{aligned} \tag{4.4}$$

We set $\hat{M} = M_{j_0}$ and $\hat{\gamma} = \gamma|_{M_{j_0}}$. Then we have (4.1) and (4.2). □

To prove (4.4) we need

LEMMA 4.2. – *For any $\gamma \in \overline{\Gamma}$, a mapping*

$$\tilde{\gamma}(r, z, t) = \gamma(r, z)(t) : \mathbb{R} \times S^{N-2} \times [0, 1] \to \mathbb{R} \times S^{N-1}$$

is onto.

Proof. – We identify $[0,1]/\{0,1\} \simeq S^1$ and we compute the mapping degree of

$$\tilde{\gamma} : \big([-R,R] \times S^{N-2} \times S^1, \{-R,R\} \times S^{N-2} \times S^1\big) \\ \to \big([-R,R] \times S^{N-1}, \{-R,R\} \times S^{N-1}\big).$$

We can see easily that $\deg \tilde{\gamma} = \pm 1$ for large R. Thus $[-R,R] \times S^{N-1} \subset \tilde{\gamma}([-R,R] \times S^{N-2} \times [0,1])$ for all $R > 0$. Therefore $\tilde{\gamma}$ is onto. □

Proof of (4.4). – It suffices to show that $\gamma(M_j)$ is not contractible in Λ at least for one $j \in \{1,2,\ldots,n\}$. Arguing indirectly, we suppose that $\gamma(M_j)$ is contractible in Λ for all j.

By (4.3) for some $\delta > 0$, there exist neighborhoods $N_\delta(M_j)$ of M_j and diffeomorphisms

$$\phi_j : N_\delta(M_j) \to (-\delta,\delta) \times M_j$$

such that $N_\delta(M_i) \cap N_\delta(M_j) \neq \emptyset$ $(i \neq j)$. We may assume that $\gamma(N_\delta(M_j))$ is also contractible in Λ. We write $\gamma(r,z)(t) = (s(r,z)(t), x(r,z)(t))$. Then $x(N_\delta(M_j))$ is contractible in $\Lambda_{S^{N-1}}$. Thus there exists a contraction:

$$\eta_j : x\big(N_\delta(M_j)\big) \times [0,1] \to \Lambda$$

such that

$$\eta_j(y,0)(t) = y(t),$$
$$\eta_j(y,1)(t) = pt \in S^{N-1} \quad \text{for all } y \in x(N_\delta(M_j)) \text{ and } t \in [0,1].$$

We define for $(r,z) \in N_\delta(M_j) = \phi_j((-\delta,\delta) \times M_j)$ and $\tau \in [0,1]$

$$f_\tau(r,z,t) = \Bigg((1-\tau)s(r,z)(t) + \tau s(r,z)(0), \\ \eta_j\bigg(x(r,z), \tau\frac{\delta - |a(r,z)|}{\delta}\bigg)(t)\Bigg),$$

where $a(r,z) \in (-\delta,\delta)$ is a unique number such that $(r,z) = \phi_j(a(r,z), m)$ for some $m \in M_j$.

For $(r,z) \in (\mathbb{R} \times S^{N-1}) \setminus \bigcup_{j=1}^n N_\delta(M_j)$, we set

$$f_\tau(r,z,t) = \big((1-\tau)s(r,z)(t) + \tau s(r,z)(0), x(r,z)(t)\big).$$

We can see

$$f_\tau(r,z,t) : [0,1] \times \mathbb{R} \times S^{N-2} \times [0,1] \to \mathbb{R} \times S^{N-1}$$

is well-defined and $f_\tau \in \overline{\Gamma}$ for all $\tau \in [0,1]$. Moreover

$$f_1\big(\mathbb{R}\times S^{N-2}\times[0,1]\big)\cap\big(\{\beta\}\times S^{N-1}\big) = f_1\Big(\Big(\bigcup M_j\Big)\times[0,1]\Big) \\ = (\beta, pt).$$

Thus $f_1:\mathbb{R}\times S^{N-2}\times[0,1]\to\mathbb{R}\times S^{N-1}$ is not onto. This contradicts Lemma 4.2 and at least one $\gamma(M_j)$ is not contractible in Λ. □

To obtain the existence of a critical point, we need the following version of Ekeland's principle.

LEMMA 4.3. – *Let $M\in\mathcal{M}_{N-2}$ and suppose that $\gamma\in\Gamma(M)$ satisfies for some $\varepsilon>0$*

$$b(M)\leqslant \max_{u\in\gamma(M)} E(u)\leqslant b(M)+\varepsilon.$$

Then there exists $v\in\Lambda$ such that

$$\operatorname{dist}_\Lambda\big(v,\gamma(M)\big)\leqslant 2\sqrt{\varepsilon}, \tag{4.5}$$

$$\|E'(v)\|_{(T_v\Lambda)^*}\leqslant\sqrt{\varepsilon}, \tag{4.6}$$

$$E(v)\in\big[b(M)-\varepsilon, b(M)+\varepsilon\big]. \tag{4.7}$$

Proof. – Arguing indirectly, we assume that

$$\operatorname{dist}_\Lambda\big(u,\gamma(M)\big)\leqslant 2\sqrt{\varepsilon}\quad\text{and}\quad E(u)\in\big[b(M)-\varepsilon, b(M)+\varepsilon\big] \\ \text{implies}\quad \|E'(u)\|_{(T_u\Lambda)^*}\geqslant\sqrt{\varepsilon}.$$

Choose a smooth function $\varphi:\mathbb{R}\to\mathbb{R}$ such that

$$\varphi(\tau)=0\quad\text{for } \tau\in\mathbb{R}\setminus\big[b(M)-\varepsilon, b(M)+\varepsilon\big],$$
$$\varphi(\tau)=1\quad\text{for } \tau\in\big(b(M)-\varepsilon/2, b(M)+\varepsilon/2\big),$$
$$\varphi(\tau)\in[0,1]\quad\text{for all } \tau\in\mathbb{R}.$$

We consider the flow $\eta:\mathbb{R}\times\Lambda\to\Lambda$ defined by

$$\frac{d\eta}{d\tau} = -\frac{\varphi(E(\eta))}{\|E'(\eta)\|_{(T_\eta\Lambda)^*}}E'(\eta), \tag{4.8}$$

$$\eta(0,u)=u. \tag{4.9}$$

We can see that

(1) for each $u\in\gamma(M)$ the solution $\eta(\tau,u)$ of (4.8)–(4.9) exists for $\tau\in[0,2\sqrt{\varepsilon}]$,

(2) $\tilde{\gamma}(z) \equiv \eta(2\sqrt{\varepsilon}, \gamma(z)) \in \Gamma(M)$,
(3) $E(\tilde{\gamma}(z)) = E(\eta(2\sqrt{\varepsilon}, \gamma(z))) \leqslant b(M) - (\varepsilon/2)$ for all $z \in M$.

This contradicts the definition of $b(M)$. □

Proof of Theorem 0.3 in case D. – By Proposition 4.1, under the assumption $\overline{b} = 2\pi^2$, for any $\varepsilon > 0$ there exist a $M_\varepsilon \in \mathcal{M}_{N-2}$ and $\gamma_\varepsilon \in \Gamma(M_\varepsilon)$ satisfying

$$\max_{u \in \gamma_\varepsilon(M_\varepsilon)} E(u) \leqslant 2\pi^2 + \varepsilon, \tag{4.10}$$

$$s(0) \in [0, 1] \quad \text{for all } u(t) = \big(s(t), x(t)\big) \in \gamma_\varepsilon(M_\varepsilon). \tag{4.11}$$

Since $\underline{b} = 2\pi^2$, we have

$$2\pi^2 = \underline{b} \leqslant b(M_\varepsilon) \leqslant \max_{u \in \gamma_\varepsilon(M_\varepsilon)} E(u) \leqslant 2\pi^2 + \varepsilon \leqslant b(M_\varepsilon) + \varepsilon.$$

Applying Lemma 4.3, there exists $u_\varepsilon = (s_\varepsilon, x_\varepsilon) \in \Lambda$ such that

$$\operatorname{dist}_\Lambda\big(u_\varepsilon, \gamma_\varepsilon(M_\varepsilon)\big) \leqslant 2\sqrt{\varepsilon}, \tag{4.12}$$

$$\big\|E'(u_\varepsilon)\big\|_{(T_{u_\varepsilon}\Lambda)^*} \leqslant \sqrt{\varepsilon}, \tag{4.13}$$

$$E(u_\varepsilon) \in \big[b(M_\varepsilon) - \varepsilon, b(M_\varepsilon) + \varepsilon\big] \subset \big[2\pi^2 - \varepsilon, 2\pi^2 + 2\varepsilon\big]. \tag{4.14}$$

By (4.12), we find for some $v = (\overline{s}, \overline{x}) \in \gamma_\varepsilon(M_\varepsilon)$

$$\big|s_\varepsilon(0) - \overline{s}(0)\big| \leqslant \operatorname{dist}_\Lambda\big(u_\varepsilon, \gamma_\varepsilon(M_\varepsilon)\big) \leqslant 2\sqrt{\varepsilon}.$$

It follows from (4.11) that $|\overline{s}(0)| \in [0, 1]$. Thus,

$$|s_\varepsilon(0)| \leqslant 2\sqrt{\varepsilon} + 1. \tag{4.15}$$

Since (4.13) and (4.14) hold, we have

$$E(u_\varepsilon) \to 2\pi^2, \quad E'(u_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0$$

and we can apply Proposition 2.1. By (4.15), the statement (ii) of Proposition 2.1 cannot take a place. Therefore there exists a strongly convergent subsequence $u_{\varepsilon_j}(t)$ ($\varepsilon_j \to 0$) and the limit $u_0 = \lim_{j\to\infty} u_{\varepsilon_j}$ is a critical point of $E(u)$ with $E(u_0) = 2\pi^2$. □

5. THE CASE $N=2$

We give an outline of a proof in case of $N=2$. We study the existence of closed geodesics on $(\mathbb{R}\times S^1, g)$. We use the winding number of $u\in\Lambda$ in an essential way.

We denote the winding number of $u:[0,1]/\{0,1\}\simeq S^1\to S^1$ by wind(u) and set

$$\Lambda_1=\{u\in\Lambda;\ \mathrm{wind}(\pi\circ u)=1\},$$

where $\pi:\mathbb{R}\times S^1\to S^1;\ (r,z)\mapsto z$ is the projection.

As to the break down of the Palais–Smale condition for the restricted functional $E(u):\Lambda_1\to\mathbb{R}$, we have

PROPOSITION 5.1. – *Suppose that $(u_j)_{j=1}^{\infty}\subset\Lambda_1$ satisfies for some $c>0$*

$$E(u_j)\to c>0,$$
$$\|E'(u_j)\|_{(T_{u_j}\Lambda_1)^*}\to 0\quad \text{as } j\to\infty$$

for some $c>0$. Then there is a subsequence—still denoted by u_j—such that one of the following two statements holds:

(i) *There is a non-constant closed geodesic $u_0\in\Lambda_1$ on $(\mathbb{R}\times S^1,g)$ such that*

$$u_j\to u_0\quad \text{in } \Lambda_1.$$

(ii) *We write $u_j(t)=(s_j(t),x_j(t))$. Then we have*
   (1) $s_j(0)\to\infty$ *or* $s_j(0)\to-\infty$;
   (2) $\dot{s}_j(t)\to 0$ *in* $L^2(0,1)$;
   (3) $x_j(t)\to(\cos 2\pi(t-\theta),\sin 2\pi(t-\theta))$ *in* $H^1(0,1)$ *for some* θ;
   (4) $E(u_j)\to 2\pi^2$.

In particular, $(PS)_c$ holds in Λ_1 for $c\in(0,\infty)\setminus\{2\pi^2\}$.

Proof. – We remark that $(\cos 2\pi k(t-\theta),\sin 2\pi k(t-\theta))\notin\Lambda_1$ for $k\in\mathbb{Z}\setminus\{1\}$. The proof can be given just as in Proposition 2.1. □

We define $\underline{b}$ and $\overline{b}$ as follows:

$$\underline{b}=\inf_{u\in\Lambda_1}E(u),\qquad \overline{b}=\inf_{\gamma\in\overline{\Gamma}}\sup_{u\in\gamma(\mathbb{R})}E(u),$$

where

$$\overline{\Gamma} = \{\gamma \in C(\mathbb{R}, \Lambda_1);\ \gamma(r)(t) = (r, (\cos 2\pi t, \sin 2\pi t)) \text{ for large } |r|\}.$$

Then we have

(i) $0 < \underline{b} \leqslant 2\pi^2 \leqslant \overline{b}$.
(ii) If $\underline{b} < 2\pi^2$, then $\underline{b}$ is a critical value of $E(u)$.
(iii) If $\overline{b} > 2\pi^2$, then $\overline{b}$ is a critical value of $E(u)$.

Lastly we can also show that $2\pi^2$ is a critical value in case $\underline{b} = \overline{b} = 2\pi^2$ as in Section 4. ($\mathcal{M} = \{pt\}$ in this case.) We remark that the case C does not need to study for $N = 2$. $\square$

REFERENCES

[1] A. AMBROSETTI and U. BESSI, Multiple periodic trajectories in a relativistic gravitational field, in: H. Berestycki, J.-M. Coron and I. Ekeland (Eds.), *Variational Methods*, Birkhäuser, 1990, pp. 373–381.

[2] A. AMBROSETTI and V. COTI ZELATI, Closed orbits of fixed energy for singular Hamiltonian systems, *Arch. Rat. Mech. Anal.* 112 (1990) 339–362.

[3] A. AMBROSETTI and V. COTI ZELATI, *Periodic Solutions of Singular Lagrangian Systems,* Birkhäuser, Boston, 1993.

[4] A. AMBROSETTI and M. STRUWE, Periodic motions for conservative systems with singular potentials, *NoDEA Nonlinear Differential Equations Appl.* 1 (1994) 179–202.

[5] A. BAHRI and Y.Y. LI, On a min-max procedure for the existence of a positive solution for certain scalar field equations in $\mathbb{R}^N$, *Revista Mat. Iberoamericana* 6 (1990) 1–15.

[6] A. BAHRI and P.L. LIONS, Morse index of some min-max critical points. I. Application to multiplicity results, *Comm. Pure Appl. Math.* 41 (1988) 1027–1037.

[7] A. BAHRI and P.L. LIONS, On the existence of a positive solution of semilinear elliptic equations in unbounded domains, *Ann. Inst. Henri Poincaré, Analyse Non Linéaire* 14 (1997) 365–413.

[8] V. BANGERT, Closed geodesics on complete surfaces, *Math. Ann.* 251 (1980) 83–96.

[9] V. BENCI and D. FORTUNATO, Subharmonic solutions of prescribed minimal period for autonomous differential equations, in: Dell'Antonio and D'Onofrio (Eds.), *Recent Advances in Hamiltonian Systems*, World Scientific, Singapore, 1986.

[10] V. BENCI and F. GIANNONI, Periodic solutions of prescribed energy for a class of Hamiltonian systems with singular potentials, *J. Differential Equations* 82 (1989) 60–70.

[11] V. BENCI and F. GIANNONI, On the existence of closed geodesics on noncompact Riemannian manifolds, *Duke Math. J.* 68 (1992) 195–215.

[12] V. COTI ZELATI, Periodic solutions for a class of planar, singular dynamical systems, *J. Math. Pure Appl.* 68 (1989) 109–119.

[13] V. COTI ZELATI and E. SERRA, Collisions and non-collisions solutions for a class of Keplerian-like dynamical systems, *Ann. Mat. Pura Appl.* 166 (4) (1994) 343–362.

[14] G. FANG and N. GHOUSSOUB, Morse-type information on Palais–Smale sequences obtained by min-max principles, *Comm. Pure Appl. Math.* 47 (1994) 1595–1653.

[15] C. GRECO, Remarks on periodic solutions, with prescribed energy, for singular Hamiltonian systems, *Comment. Math. Univ. Carolin.* 28 (1987) 661–672.

[16] W. KLINGENBERG, *Lectures on Closed Geodesics*, Grundlehren der Math. Wiss. 230, Springer, Berlin, 1978.

[17] A.C. LAZER and S. SOLIMINI, Nontrivial solutions of operator equations and Morse indices of critical points of min-max type, *Nonlinear Analysis: T.M.A.* 12 (1988) 761–775.

[18] L. PISANI, Periodic solutions with prescribed energy for singular conservative systems involving strong force, *Nonlinear Analysis: T.M.A.* 21 (1993) 167–179.

[19] E. SERRA and S. TERRACINI, Noncollision solutions to some singular minimization problems with Keplerian-like potentials, *Nonlinear Analysis: T.M.A.* 22 (1994) 45–62.

[20] P.H. RABINOWITZ, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Ser. in Math., Vol. 65, Amer. Math. Soc., Providence, RI, 1986.

[21] K. TANAKA, Morse indices at critical points related to the symmetric mountain pass theorem and applications, *Comm. Partial Differential Equations* 14 (1989) 99–128.

[22] K. TANAKA, A prescribed energy problem for a singular Hamiltonian system with a weak force, *J. Funct. Anal.* 113 (1993) 351–390.

[23] K. TANAKA, A prescribed-energy problem for a conservative singular Hamiltonian system, *Arch. Rational Mech. Anal.* 128 (1994) 127–164.

[24] C. VITERBO, Indice de Morse des points critiques obtenus par minimax, *Ann. Inst. Henri Poincaré, Analyse non Linéaire* 5 (1988) 221–225.

[25] G. THORBERGSSON, Closed geodesics on non-compact Riemannian manifold, *Math. Z.* 159 (1978) 249–258.