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by

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ABSTRACT. – Global strong solutions of the Navier-Stokes equations in 4 and 5 dimensional domains with *non-compact* boundaries are constructed. We prove also their asymptotic behaviour as $t \to \infty$. © Elsevier, Paris

INTRODUCTION

The purpose of this paper is to construct global strong solutions with small initial data of the Navier-Stokes equations in 4 and 5 dimensional unbounded domains. We are mainly interested in unbounded domains with *non-compact boundaries*. Let Ω be a domain in $\mathbf{R}^n (n = 4, 5)$ with uniformly C^3-boundary $\partial\Omega$. Our result covers the case when $\partial\Omega$ is non-compact. The motion of the incompressible fluid occupied in Ω is governed by the

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following Navier-Stokes equations:

$$(\text{N-S}) \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } x \in \Omega, t > 0, \\[2mm] \operatorname{div} u = 0 & \text{in } x \in \Omega,\ t > 0, \\[1mm] u = 0 & \text{on } \partial\Omega,\ t > 0, \\[1mm] u|_{t=0} = a, \end{cases}$$

where $u = u(x,t) = (u^1(x,t), \cdots, u^n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the pressure of the fluid at point $(x,t) \in \Omega \times (0, \infty)$, while $a = a(x) = (a^1(x), \cdots, a^n(x))$ is a given initial velocity vector field. For simplicity, we assume that the external force is equal to zero.

Since the pioneer work of Leray [20], energy decay of weak solutions to (N-S) in unbounded domains has been discussed by many authors and it is now clarified that the asymptotic behavior of *strong solutions* $u(t)$ as $t \to \infty$ plays an important role in such a decay problem. On account of lack compactness, the problem of existence of global strong solutions in *unbounded* domains seems to be more difficult than that in bounded domains; we need to pay a more exquisite attention to the analysis near $\lambda = 0$ of the resolvent $(A + \lambda)^{-1}$ for the Stokes operator A. In recent years, a number of skillful technique such as method of compact perturbation from the whole space $\mathbf{R}^n$ were developed and applied to the exterior problem. $L^p - L^q$ estimates for the Stokes semigroup $\{e^{-tA}\}_{t \geq 0}$ in exterior domains are established by Giga-Sohr [11], Iwashita [14] and Borchers-Miyakawa [2] and the decay properties of weak and strong solutions are investigated to a considerable extent ([3], [17]). In unbounded domains with non-compact boundaries, however, there is no L^p-theory for the Stokes operator and the only L^2-theory is available. Based on the linear analysis of the Stokes operator, Kato [16] and Giga-Miyakawa [10] obtained a priori L^p-bounds for $p > n$ of solutions to (N-S), which yields global existence of strong solutions. On the other hand, L^2-method enables us only to get the energy inequality which bounds $\|\nabla u(t)\|_2$ for all $t \geq 0$ (see (E.I.) below). By the Sobolev embedding $H^1(\Omega) \subset L^{2n/(n-2)}(\Omega)$, we can dominate the L^p-norm for $p > n$ of solutions if the spatial dimension $n \leq 3$. Galdi-Maremonti [8] first established apriori and decay estimates in H^1 in case $n = 3$, and then Maremonti [21] improved the decay rates of weak solutions. After their results, making use of the identity $\|\nabla u\|_2 = \|A^{1/2}u\|_2$, Kozono-Ogawa [18], [19] derived a sharp estimate of the nonlinear term $u \cdot \nabla u$ in terms of the fractional power A^α of the Stokes operator and proved the existence of global strong solutions to (N-S) in general unbounded domains in case $n = 2$ and $n = 3$ for large and small data, respectively.

To treat a higher dimension, one needs to get more a priori estimates than those in $H^1(\Omega)$. In the present paper, we shall establish H^2-a priori bounds which yield L^p-estimates for $p > n$ of solutions even if the dimension $n = 4$ and $n = 5$. To this end, we shall make use of the estimate given by Heywood [12]

$$\|D^2 u\|_2 \leq C(\|\nabla u\|_2 + \|Au\|_2).$$

Recently, Borchers-Miyakawa [3] showed the existence of weak solutions u of (N-S) with $\lim_{t\to\infty} \|u(t)\|_2 = 0$ for $n \leq 4$. This restriction on n stems from validity of the energy inequality of the *strong form*

(E.I.) $$\|u(t)\|_2^2 + 2 \int_s^t \|\nabla u(\tau)\|_2^2 d\tau \leq \|u(s)\|_2^2$$

for all s and t such that $0 \leq s \leq t$. The importance of the above inequality (E.I.) in unbounded domains was pointed out by Masuda [23] and Kato [16]. Leray [20] called a weak solution u satisfying (E.I.) a *turbulent solution*, the existence of which was shown for $n \leq 4$ up to the present(see Kato [16] and Miyakawa-Sohr [25]). Our global strong solution satisfies (E.I.) and hence we can construct the solution decaying in L^2 even for $n = 5$. Moreover, we shall show a sharp decay of the strong solution as $t \to \infty$, which seems to be optimal. The decay rates will be furnished in terms of the fractional powers A^α for $0 \leq \alpha \leq 1$. Since we need the additional term $\|\nabla u\|_2$ on the right hand side of the above Heywood's estimate, the decay $\|A^\alpha u(t)\|_2$ for $1/2 < \alpha \leq 1$ cannot be directly obtained by investigating the linearized equation of (N-S). To get around this difficulty, we shall first establish the decay of $\frac{\partial u}{\partial t}(t)$ as $t \to \infty$. Such a method was first introduced by Masuda [22]. To get sharper decay , however, we need more calculations than [22].

In Section 1 we shall state our main theorem. Section 2 is devoted to the estimate of the nonlinear term $u \cdot \nabla u$ by $(A+\varepsilon)^{-1}$ which holds independently of $\varepsilon > 0$. To this end, we shall first introduce the Hilbert scale via fractional powers A^α and define a family of certain homogeneous Sobolev spaces by complex interpolation. This procedure is done by Miyakawa [24] in 3-dimensional exterior domains. Then we shall apply Heywood's estimate to obtain bounds of higher derivatives. In Section 3, an explicit representation of the time interval where the strong solutions exist will be given in terms of the Hilbert scale of the initial data and we shall show the a priori estimate of solutions in such a scale. As a result the global strong solution will be constructed provided the initial data a is sufficiently small. Finally in Section 4, we shall show the decay property of the strong solution.

1. RESULT

Throughout this paper, we impose the following assumption on the domain Ω.

ASSUMPTION. – Ω ($\subset \mathbf{R}^n, n = 4, 5$) *is of class uniformly C^3 (for the definition, see Tanabe [28, Definition 1.2.2]) and regular, i.e., the boundary $\partial\Omega$ consists of finitely many, disjoint simple C^3 surfaces.*

We may treat the domain Ω with a *non-compact* boundary $\partial\Omega$.

Before stating our results, we introduce some notations and function spaces. Let $C^\infty_{0,\sigma}$ denote the set of all C^∞ vector functions $\phi = (\phi^1, \cdots, \phi^n)$ with compact support in Ω, such that div $\phi = 0$. L^2_σ is the closure of $C^\infty_{0,\sigma}$ with respect to the L^2-norm $\| \ \|_2$; $(\cdot, \cdot)$ denotes the L^2-inner product. $\| \cdot \|_p$ is the L^p-norm on Ω, $1 \leq p \leq \infty$. For simplicity, we abbreviate $\| \cdot \|_2 = \| \cdot \|$. $H^1_{0,\sigma}$ denotes the closure of $C^\infty_{0,\sigma}$ with respect to the norm

$$\|\phi\|_{H^1} = \|\phi\| + \|\nabla\phi\|,$$

where $\nabla\phi = (\partial\phi^i/\partial x_j; \ \ i,j = 1, \cdots, n)$. When X is a Banach space, we denote by $\| \cdot \|_X$ the norm on X. Then $C^m((t_1, t_2); X)$ and $L^p((t_1, t_2); X)$ are the usual Banach spaces, where $m = 0, 1, 2, \cdots$ and t_1 and t_2 are real numbers such that $t_1 < t_2 \leq \infty$. Further, $BC((t_1, t_2); X)$ is the set of all functions u in $C^0((t_1, t_2); X)$ such that $\sup_{t_1 < t < t_2} \|u(t)\|_X < \infty$.

We denote by P the orthogonal projection from $L^2(\Omega)$ onto L^2_σ. Then the Stokes operator A is defined by $A = -P\Delta$ with domain $D(A) = \{u \in H^2(\Omega); u|_{\partial\Omega} = 0\} \cap L^2_\sigma$. More precisely, A coincides with the non-negative self-adjoint operator defined by the quadratic form $a(u, v) = (\nabla u, \nabla v)$, $u, v \in H^1_{0,\sigma}$, that is, $Au = f$ ($u \in D(A), f \in L^2_\sigma$) is equivalent to the relation $a(u, v) = (f, v)$ for all $v \in H^1_{0,\sigma}$. Therefore we have $D(A^{1/2}) = H^1_{0,\sigma}$ and

$$(1.1) \qquad \|A^{1/2}u\| = \|\nabla u\|, \qquad u \in D(A^{1/2}).$$

Let $0 < \alpha \leq 1$. $D(A^\alpha)$ is a Banach space equipped with the graph norm

$$\|u\|_{D(A^\alpha)} = \|u\| + \|A^\alpha u\|.$$

Then it follows from Fujita-Morimoto [6] that

$$(1.2) \qquad D(A^\alpha) \subset H^{2\alpha,2} \qquad \text{(continuous embedding)},$$

where $H^{2\alpha,2}$ denotes the Sobolev space over Ω.

Applying the orthogonal projection P to both sides of (N-S), we get the following differential equation in L_σ^2:

(E)
$$\begin{cases} \dfrac{du}{dt} + Au + P(u \cdot \nabla u) = 0, \quad 0 < t < \infty, \\ u(0) = a. \end{cases}$$

Our result on global strong solutions now reads:

THEOREM. – *Let $n = 4$ and 5. There is a constant $\mu = \mu(n) > 0$ such that if $a \in D(A^{\frac{n}{4}-\frac{1}{2}})$ satisfies $\|a\|_{D(A^{\frac{n}{4}-\frac{1}{2}})} \leq \mu$, then we have a unique strong solution u of (E) with the following properties:*

(1.3) $u \in BC([0, \infty); D(A^{\frac{n}{4}-\frac{1}{2}})) \cap C((0, \infty); D(A));$

(1.4) $\dfrac{du}{dt} \in C((0, \infty); D(A^{\frac{1}{2}}));$

the energy identity

(1.5) $\|u(t)\|^2 + 2 \displaystyle\int_0^t \|\nabla u(\tau)\|^2 d\tau = \|a\|^2$

holds for all $t \geq 0$. Moreover, such a solution u decays like

(1.6) $\|A^\alpha u(t)\| = o(t^{-\alpha}) \quad for\ 0 \leq \alpha \leq 1,$

(1.7) $\|\dfrac{du(t)}{dt}\| = o(t^{-1})$

as $t \to \infty$.

Remark. – In case when $\Omega = \mathbf{R}^n$, the half space $\mathbf{R}_+^n$, a bounded domain or an exterior domain with compact boundaries, the global strong solution of (E) was obtained provided a is sufficiently small in $L^n(\Omega)$. See Kato [16], Ukai [29], Giga-Miyakawa [10] and Iwashita [14]. It should be noted that, by (1.2) the space $D(A^{\frac{n}{4}-\frac{1}{2}})$ is continuously embedded into $L^n(\Omega)$. In general unbounded domains $\Omega \subset \mathbf{R}^3$ with non-compact boundaries $\partial\Omega$, Kozono-Ogawa [19] obtained the same result as in the above theorem under the weaker assumption that $\|A^{\frac{1}{4}}a\|$ is small; $\|a\|$ itself need not be small. In our theorem, however, $\|a\|$ must be also small, which stems from the fact that $\|a\|_p$ for $p > n$ does not seem to be dominated only by the homogeneous norm $\|A^\alpha a\|$ with $\alpha = \frac{n}{2}(\frac{1}{2} - \frac{1}{p})$(see [19, Lemma 2.1]).

2. PRELIMINARIES

In this section, we shall estimate the nonlinear term $u \cdot \nabla u$ in terms of the fractional power A^α. In what follows we shall denote by C various constants. In particular, $C = C(*, \cdots, *)$ denotes constants depending only on the quantities appearing in the parenthesis. Let us first introduce the following estimate given by Heywood [12, Lemma 6]:

LEMMA 2.1 (Heywood). – *Under our assumption on the domain Ω, there is a constant $C = C(\Omega)$ such that*

$$(2.1) \qquad \|D^2 u\| \leq C(\|\nabla u\| + \|Au\|) \qquad \text{for all } u \in D(A),$$

where $D^2 u = \left(\dfrac{\partial^2 u}{\partial x^i \partial x^j} \right)_{i,j=1,\cdots,n}$.

It should be noted that in general we cannot avoid to add $\|\nabla u\|$ on the right hand side of (2.1)(see Borchers-Miyakawa [4] and Kozono-Ogawa [17]). We shall next define the space $\widehat{D}(A^\alpha)$ for $0 \leq \alpha \leq 1$ as follows.

For $0 \leq \alpha \leq 1/2$

(2.2) $\widehat{D}(A^\alpha) = $ the completion of $D(A)$ with respect to the norm $\|A^\alpha u\|$.

For $1/2 \leq \alpha \leq 1$

$\widehat{D}(A^\alpha)$
$(2.3) =$ the completion of $D(A)$ with respect to the norm $\|A^{1/2} u\| + \|A^\alpha u\|$.

Note that $\widehat{D}(A^\alpha)$ is larger than $D(A^\alpha)$. Then it follows from Miyakawa [24, Theorems 2.5-2.5] that

LEMMA 2.2 (Miyakawa).

$$(2.4) \qquad\qquad [L_\sigma^2, \widehat{D}(A^{\frac{1}{2}})]_\alpha = \widehat{D}(A^{\frac{\alpha}{2}}),$$
$$(2.5) \qquad\qquad [\widehat{D}(A^{\frac{1}{2}}), \widehat{D}(A)]_\alpha = \widehat{D}(A^{\frac{1}{2}+\frac{\alpha}{2}})$$

for $0 \leq \alpha \leq 1$, where $[X, Y]_\alpha$ denotes the space of complex interpolation between X and Y.

By Lemmata 2.1 and 2.2, we can establish the following estimate of the nonlinear term $u \cdot \nabla u$:

LEMMA 2.3. – *Let $n = 4$ and $n = 5$. For $n/2 - 2 \leq \theta < 1$, there is a constant $C = C(n, \theta)$ such that*

$$(2.6) \quad \|P(u \cdot \nabla v)\| \leq C(\|A^{\frac{1}{2}}u\| + \|A^{\frac{1}{2}+\frac{\theta}{2}}u\|)(\|A^{\frac{1}{2}}v\| + \|A^{\frac{n}{4}-\frac{\theta}{2}}v\|)$$

$$(2.7) \quad \|(A + \varepsilon)^{-\frac{\theta}{2}}P(u \cdot \nabla v)\| \leq C\|A^{\frac{1}{2}}u\|(\|A^{\frac{1}{2}}v\| + \|A^{\frac{n}{4}-\frac{\theta}{2}}v\|)$$

for all $u \in D(A^{\frac{1}{2}+\frac{\theta}{2}}), v \in D(A^{\frac{n}{4}-\frac{\theta}{2}})$ and all $\varepsilon > 0$.

Proof. – Let us first show that $u \cdot \nabla v \in L^2$ provided $u \in D(A^{\frac{1}{2}+\frac{\theta}{2}})$ and $v \in D(A^{\frac{n}{4}-\frac{\theta}{2}})$. Indeed, by (1.2) and the Sobolev inequality we have the following continuous embeddings:

$$D(A^{\frac{1}{2}+\frac{\theta}{2}}) \subset H^{1+\theta,2} \subset L^p \quad \text{for } \frac{1}{p} = \frac{1}{2} - \frac{1+\theta}{n},$$

$$D(A^{\frac{n}{4}-\frac{\theta}{2}}) \subset H^{\frac{n}{2}-\theta,2} \subset H^{1,q} \quad \text{for } \frac{1}{q} = \frac{1+\theta}{n}.$$

Since $1/p + 1/q = 1/2$, the Hölder inequality yields $u \cdot \nabla v \in L^2$ and hence $P(u \cdot \nabla v) \in L^2_\sigma$ is well-defined. We shall next show the continuous embedding

$$(2.8) \quad \widehat{D}(A^{\frac{1}{2}+\frac{\alpha}{2}}) \subset L^p, \quad 0 \leq \alpha < 1 \quad \text{for } \frac{1}{p} = \frac{1}{2} - \frac{1+\alpha}{n}.$$

By the Sobolev inequality, (1.1) and Lemma 2.1, there holds

$$(2.9) \quad \|u\|_{\frac{2n}{n-2}} \leq C\|\nabla u\| = C\|A^{\frac{1}{2}}u\| \quad \text{for } n = 4, 5$$

$$\|u\|_{BMO} \leq C\|\nabla u\|_4 \leq C(\|\nabla u\| + \|\nabla(\nabla u)\|)$$
$$(2.10) \quad \leq C(\|A^{\frac{1}{2}}u\| + \|Au\|) \quad \text{for } n = 4$$

$$\|u\|_{10} \leq C\|\nabla u\|_{\frac{10}{3}} \leq C(\|\nabla u\| + \|\nabla(\nabla u)\|)$$
$$(2.11) \quad \leq C(\|A^{\frac{1}{2}}u\| + \|Au\|) \quad \text{for } n = 5$$

for all $u \in D(A)$ with $C = C(n)$, where BMO denotes the space of bounded mean oscillation over Ω. It should be noted that we do not need to add $\|u\|_4$ and $\|u\|_{\frac{10}{3}}$ on the right hand side of the above estimates (2.9) and (2.10), respectively, since $u = 0$ on $\partial\Omega$. This implies that the injection i is a continuous operator as

$$i : \widehat{D}(A^{\frac{1}{2}}) \hookrightarrow L^{\frac{2n}{n-2}}$$

$$i : \widehat{D}(A) \hookrightarrow \begin{cases} BMO, & \text{for } n = 4 \\ L^{10}, & \text{for } n = 5 \end{cases}$$

Since $[L^p, BMO]_\alpha = L^{p/(1-\alpha)}$ for $0 \le \alpha < 1$(see e.g., Janson-Jones [15]), Lemma 2.2 and the interpolation theory yield the continuous embedding (2.8). Moreover, (2.9), (2.10) and (2.11) show that the gradient ∇ defines a bounded operator as

$$\nabla : \widehat{D}(A^{\frac{1}{2}}) \to L^2,$$
$$\nabla : \widehat{D}(A) \to L^{\frac{2n}{n-2}},$$

from which and (2.5) we obtain

$$(2.12) \qquad \nabla : \widehat{D}(A^{\frac{1}{2}+\frac{\alpha}{2}}) \to L^q, \quad 0 \le \alpha \le 1 \quad \text{for } \frac{1}{q} = \frac{1}{2} - \frac{\alpha}{n} .$$

Now let us take p and q in such a way that

$$(2.13) \qquad \frac{1}{p} = \frac{1}{2} - \frac{1+\theta}{n}, \quad \frac{1}{q} = \frac{1+\theta}{n}.$$

Then by (2.8), we have

$$(2.14) \qquad \|u\|_p \le C(\|A^{\frac{1}{2}}u\| + \|A^{\frac{1}{2}+\frac{\theta}{2}}u\|) \quad \text{for all } u \in D(A^{\frac{1}{2}+\frac{\theta}{2}}).$$

Taking $\alpha = n/2 - 1 - \theta$, we have by the assumption on θ and (2.13) that $0 < \alpha \le 1$ and $1/q = 1/2 - \alpha/n$. Hence (2.12) yields

$$(2.15) \qquad \|\nabla v\|_q \le C(\|A^{\frac{1}{2}}v\| + \|A^{\frac{1}{2}+\frac{\alpha}{2}}v\|) \le C(\|A^{\frac{1}{2}}v\| + \|A^{\frac{n}{4}-\frac{\theta}{2}}v\|)$$

for all $v \in D(A^{\frac{n}{4}-\frac{\theta}{2}})$. Since $1/p + 1/q = 1/2$, we have by (2.14), (2.15) and the Hölder inequality that

$$\|P(u \cdot \nabla v)\| \le \|u\|_p \|\nabla v\|_q$$
$$\le C(\|A^{\frac{1}{2}}u\| + \|A^{\frac{1}{2}+\frac{\theta}{2}}u\|)(\|A^{\frac{1}{2}}v\| + \|A^{\frac{n}{4}-\frac{\theta}{2}}v\|)$$

for all $u \in D(A^{\frac{1}{2}+\frac{\theta}{2}})$ and all $v \in D(A^{\frac{n}{4}-\frac{\theta}{2}})$. This implies (2.6).

We shall next prove (2.7). By (2.4), (2.9) and the interpolation inequality, there holds a continuous inclusion

$$(2.16) \qquad \widehat{D}(A^{\frac{\theta}{2}}) \subset L^p \quad \text{for } \frac{1}{p} = \frac{1}{2} - \frac{\theta}{n}.$$

Hence it follows from (2.9), (2.15) and (2.16) that

$$|((A+\varepsilon)^{-\frac{\theta}{2}}P(u \cdot \nabla v), \phi)| = |(P(u \cdot \nabla v), (A+\varepsilon)^{-\frac{\theta}{2}}\phi)|$$
$$\le \|u \cdot \nabla v\|_{p'} \|(A+\varepsilon)^{-\frac{\theta}{2}}\phi\|_p$$
$$\le C\|u\|_{\frac{2n}{n-2}} \|\nabla v\|_{\frac{n}{1+\theta}} \|A^{\frac{\theta}{2}}(A+\varepsilon)^{-\frac{\theta}{2}}\phi\|$$
$$\le C\|A^{\frac{1}{2}}u\|(\|A^{\frac{1}{2}}v\| + \|A^{\frac{n}{4}-\frac{\theta}{2}}v\|)\|\phi\|$$

for all $\phi \in C_{0,\sigma}^{\infty}$ and all $\varepsilon > 0$ with $C = C(n, \theta)$. Now (2.7) follows from the duality argument. This completes the proof of Lemma 2.3. ∎

By Lemma 2.3 we can define the weak limit $(A + \varepsilon)^{-\frac{\theta}{2}} P(u \cdot \nabla v)$ in L_{σ}^2 as $\varepsilon \downarrow +0$.

LEMMA 2.4. – *Let* $n = 4, 5$ *and* $n/2 - 2 \leq \theta < 1$.

(1) For $u, v \in D(A^{\frac{1}{2} + \frac{\theta}{2}}) \cap D(A^{\frac{n}{4} - \frac{\theta}{2}})$, *there exists a weak limit*

$$w - \lim_{\varepsilon \downarrow +0} (A + \varepsilon)^{-\frac{\theta}{2}} P(u \cdot \nabla v) \quad in \ L_{\sigma}^2.$$

(2) Let us denote by $\widetilde{F}_{\theta}(u, v)$ *the weak limit:*

$$\widetilde{F}_{\theta}(u, v) \equiv w - \lim_{\varepsilon \downarrow +0} (A + \varepsilon)^{-\frac{\theta}{2}} P(u \cdot \nabla v).$$

Then there holds

$$(2.17) \qquad \|\widetilde{F}_{\theta}(u, v)\| \leq C \|A^{\frac{1}{2}} u\| (\|A^{\frac{1}{2}} v\| + \|A^{\frac{n}{4} - \frac{\theta}{2}} v\|)$$

with $C = C(n, \theta)$ *independent of* u *and* v.

Proof. – By (1.1) zero is not an eigenvalue of A. Hence the range $R(A^{\frac{\theta}{2}})$ of $A^{\frac{\theta}{2}}$ is dense in L_{σ}^2 and we have $(A + \varepsilon)^{-\frac{\theta}{2}} A^{\frac{\theta}{2}} \phi \to \phi$ in L_{σ}^2 for all $\phi \in D(A^{\frac{\theta}{2}})$ as $\varepsilon \downarrow +0$. Then for every $u, v \in D(A^{\frac{1}{2} + \frac{\theta}{2}}) \cap D(A^{\frac{n}{4} - \frac{\theta}{2}})$ and every $\phi \in D(A^{\frac{\theta}{2}})$, there holds

$$((A + \varepsilon)^{-\frac{\theta}{2}} P(u \cdot \nabla v), A^{\frac{\theta}{2}} \phi) = (P(u \cdot \nabla v), ((A + \varepsilon)^{-\frac{\theta}{2}} A^{\frac{\theta}{2}} \phi)$$

$$(2.18) \qquad\qquad\qquad \to (P(u \cdot \nabla v), \phi) \qquad \text{as } \varepsilon \downarrow +0.$$

Since $R(A^{\frac{\theta}{2}})$ is dense in L_{σ}^2, this convergence together with (2.7) yields a weak limit $w - \lim_{\varepsilon \downarrow +0} (A + \varepsilon)^{-\frac{\theta}{2}} P(u \cdot \nabla v)$ in L_{σ}^2. Then the assertion (2) is an immediate consequence of (2.7) and the resonance theorem. ∎

Let us now define an operator $F_{\theta}(u, v)$ in a rigorous way which is formally written as $A^{-\frac{\theta}{2}} P(u \cdot \nabla v)$. We shall follow the procedure of Masuda [22]. Let θ be as in the hypotheses of Lemma 2.4. Since $D(A^{\frac{1}{2} + \frac{\theta}{2}}) \cap D(A^{\frac{n}{4} - \frac{\theta}{2}})$ is dense in $D(A^{\frac{n}{4} - \frac{\theta}{2}})$, we can define a bilinear operator $F_{\theta}(u, v)$ for $u, v \in D(A^{\frac{n}{4} - \frac{\theta}{2}})$ by

$$F_{\theta}(u, v) = \text{s}-\lim_{j \to \infty} \widetilde{F}_{\theta}(u_j, v_j) \quad \text{(strong limit in } L_{\sigma}^2\text{)},$$

where $\{u_j\}_{j=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ are sequences in $D(A^{\frac{1}{2} + \frac{\theta}{2}}) \cap D(A^{\frac{n}{4} - \frac{\theta}{2}})$ such that $u_j \to u, v_j \to v$ in $D(A^{\frac{n}{4} - \frac{\theta}{2}})$. Then we have

LEMMA 2.5. – *Let $n = 4, 5$ and $n/2 - 2 \le \theta < 1$.*

(1) The estimate

$$\|F_\theta(u,v)\| \le C\|A^{\frac{1}{2}}u\|(\|A^{\frac{1}{2}}v\| + \|A^{\frac{n}{4}-\frac{\theta}{2}}v\|)$$

holds for all $u, v \in D(A^{\frac{n}{4}-\frac{\theta}{2}})$ with the same constant $C = C(n, \theta)$ as in (2.17).

(2) If $u, v \in D(A^{\frac{1}{2}+\frac{\theta}{2}}) \cap D(A^{\frac{n}{4}-\frac{\theta}{2}})$, then we have $F_\theta(u,v) \in D(A^{\frac{\theta}{2}})$ with $A^{\frac{\theta}{2}}F_\theta(u,v) = P(u \cdot \nabla v)$.

Proof. – The assertion (1) can be obtained directly from (2.17) and the definition of $F_\theta(u,v)$. Let us prove the assertion (2). For $u, v \in D(A^{\frac{1}{2}+\frac{\theta}{2}}) \cap D(A^{\frac{n}{4}-\frac{\theta}{2}})$, we have $F_\theta(u,v) = \widetilde{F}_\theta(u,v)$. Hence by (2.18),

$$\begin{aligned}
(F_\theta(u,v), A^{\frac{\theta}{2}}\phi) &= (\widetilde{F}_\theta(u,v), A^{\frac{\theta}{2}}\phi) \\
&= \lim_{\varepsilon \downarrow +0}((A+\varepsilon)^{-\frac{\theta}{2}}P(u \cdot \nabla v), A^{\frac{\theta}{2}}\phi) \\
&= (P(u \cdot \nabla v), \phi)
\end{aligned}$$

holds for all $\phi \in D(A^{\frac{\theta}{2}})$, which implies the desired assertion. ∎

Remark 2.6. – *If $n/2 - 2 \le \theta \le n/4 - 1/2$, then we have $1/2 + \theta/2 \le n/4 - \theta/2$ and the inclusion $D(A^{\frac{n}{4}-\frac{\theta}{2}}) \subset D(A^{\frac{1}{2}+\frac{\theta}{2}})$ holds. In such case, the above lemma states $F_\theta(u,v) \in D(A^{\frac{\theta}{2}})$ with $A^{\frac{\theta}{2}}F_\theta(u,v) = P(u \cdot \nabla v)$ for $u, v \in D(A^{\frac{n}{4}-\frac{\theta}{2}})$.*

3. LOCAL AND GLOBAL STRONG SOLUTIONS

In this section, we shall give an explicit characterization of time interval $(0, T)$ while the strong solution exists, and then establish a priori estimates to show $T = \infty$ provided the initial data is small.

THEOREM 3.1. – *(1) Let $n = 4, 5$ and let $n/4 - 1/2 < \gamma \le (n+2)/8$. Then for every $a \in D(A^\gamma)$, there exist $0 < T \le 1$ and a unique solution u of (E) on $(0, T)$ with*

(3.1) $u \in BC([0, T); D(A^\gamma)) \cap C((0, T); D(A))$;

(3.2) $\dfrac{du}{dt} \in C((0, T); D(A^{\frac{1}{2}}))$;

the energy identity

$$(3.3) \qquad \|u(t)\|^2 + 2\int_0^t \|\nabla u(\tau)\|^2 d\tau = \|a\|^2$$

holds for all $t \in [0, T)$. *Here* T *is estimated from below as*

$$(3.4) \qquad T \geq C(\|a\| + \|A^\gamma a\|)^{-\frac{1}{\gamma - (\frac{n}{4} - \frac{1}{2})}}$$

with $C = C(n, \gamma)$ *independent of* a.

(2) *For* n, γ *as above (1) and any* $\varepsilon > 0$, *there is a constant* $\mu = \mu(n, \gamma, \varepsilon) > 0$ *such that if* $a \in D(A^{\frac{n}{4} - \frac{1}{2}})$ *satisfies*

$$(3.5) \qquad \|a\| + \|A^{\frac{n}{4} - \frac{1}{2}} a\| \leq \mu,$$

then we have a unique solution u *of (E) on* $(0, 1)$ *with (3.1) replaced by*

$$(3.6) \qquad u \in BC([0, 1); D(A^{\frac{n}{4} - \frac{1}{2}})) \cap C((0, 1); D(A))$$

and (3.2), (3.3) for $T = 1$ *such that*

$$(3.6') \qquad \sup_{0<t<1} \|u(t)\| + \sup_{0<t<1} t^{\gamma - (\frac{n}{4} - \frac{1}{2})} \|A^\gamma u(t)\| < \varepsilon$$

Proof. – (1) Since this theorem is concerned with the local existence of strong solutions, the proof can be done in the same away as Fujita-Kato [5] and Giga-Miyakawa [10]. However, we give the outline of the proof for completeness. Let us transform the unknown function $u(t)$ of (E) into $v(t)$ by $u(t) = e^t v(t)$, which yields

$$(3.7) \qquad \begin{cases} \dfrac{dv}{dt} + \widetilde{A}v + e^t P(v \cdot \nabla v) = 0, & t > 0, \\ v(0) = a, \end{cases}$$

where $\widetilde{A} = A + 1$. By Duhamel's principle we may solve the following integral equation

$$(3.8) \qquad v(t) = e^{-t\widetilde{A}} a - \int_0^t e^{-(t-s)\widetilde{A}} e^s P(v \cdot \nabla v)(s) ds.$$

The solution $v(t)$ of (3.8) is constructed by the successive approximation:

$$(3.9) \qquad \begin{cases} v_0(t) = e^{-t\widetilde{A}} a, \\ v_{m+1}(t) = v_0(t) - \int_0^t e^{-(t-s)\widetilde{A}} e^s P(v_m \cdot \nabla v_m)(s) ds, & m = 0, 1, \cdots. \end{cases}$$

Let us recall the estimate given by Inoue-Wakimoto [13, Lemma 4.2] and Giga-Miyakawa[10, Lemma 2.2]:

$$(3.10) \qquad \|P(v \cdot \nabla w)\| \leq C \|\widetilde{A}^\theta v\| \|\widetilde{A}^\gamma w\|, \quad v \in D(A^\theta), w \in D(A^\gamma)$$

for $\theta = (n+2)/4 - \gamma$ with $C = C(n, \gamma)$. This estimate can be proved in a similar way as for Lemma 2.3. Since $n/4 - 1/2 < \gamma \leq (n+2)/8$, we have

$$(3.11) \qquad \frac{n}{4} - \frac{1}{2} < \theta < 1.$$

Then for $n/4 - 1/2 \leq \alpha < 1$ and $T > 0$, there holds

$$(3.12) \quad K_{\alpha,m}(T) \equiv \sup_{0 < t < T} t^{\alpha - (\frac{n}{4} - \frac{1}{2})} \|\widetilde{A}^\alpha v_m(t)\| < \infty, \quad m = 0, 1, \cdots.$$

Indeed, for $m = 0$, we have

$$(3.13) \quad t^{\alpha - (\frac{n}{4} - \frac{1}{2})} \|\widetilde{A}^\alpha v_0(t)\| = t^{\alpha - (\frac{n}{4} - \frac{1}{2})} \|\widetilde{A}^\alpha e^{-t\widetilde{A}} a\|$$
$$= t^{\alpha - (\frac{n}{4} - \frac{1}{2})} \|\widetilde{A}^{\alpha - (\frac{n}{4} - \frac{1}{2})} e^{-t\widetilde{A}} \widetilde{A}^{\frac{n}{4} - \frac{1}{2}} a\|$$

for all $t > 0$ and $n/4 - 1/2 \leq \alpha < 1$. $K_{\alpha,0}(T)$ can be defined as

$$K_{\alpha,0}(T) \equiv \sup_{0 < t < T} t^{\alpha - (\frac{n}{4} - \frac{1}{2})} \|\widetilde{A}^\alpha e^{-t\widetilde{A}} a\|, \quad \frac{n}{4} - \frac{1}{2} \leq \alpha < 1.$$

Suppose that (3.12) is true for m. Then it follows from (3.10) that

$$\|\widetilde{A}^\alpha v_{m+1}(t)\|$$
$$\leq \|\widetilde{A}^\alpha v_0(t)\| + \int_0^t \|\widetilde{A}^\alpha e^{-(t-s)\widetilde{A}} e^s P(v_m \cdot \nabla v_m)(s)\| ds$$
$$\leq K_{\alpha,0}(T) t^{\frac{n}{4} - \frac{1}{2} - \alpha} + Ce \int_0^t (t - s)^{-\alpha} \|\widetilde{A}^\theta v_m(s)\| \|\widetilde{A}^\gamma v_m(s)\| ds$$
$$\leq K_{\alpha,0}(T) t^{\frac{n}{4} - \frac{1}{2} - \alpha} + Ce K_{\theta,m}(T) K_{\gamma,m}(T) \int_0^t (t - s)^{-\alpha} s^{2(\frac{n}{4} - \frac{1}{2}) - \theta - \gamma} ds$$
$$= \{K_{\alpha,0}(T) + Ce B(1 - \alpha, \tfrac{n}{4} - \tfrac{1}{2}) K_{\theta,m}(T) K_{\gamma,m}(T)\} t^{\frac{n}{4} - \frac{1}{2} - \alpha}$$

for all $0 < t \leq T \leq 1$. Hence (3.12) is true for $m + 1$ if we take $K_{\alpha,m+1}(T)$ as

$$(3.14) \quad K_{\alpha,m+1}(T) \equiv K_{\alpha,0}(T) + Ce B(1 - \alpha, \tfrac{n}{4} - \tfrac{1}{2}) K_{\theta,m}(T) K_{\gamma,m}(T).$$

Defining $K_m(T) \equiv \max\{K_{\theta,m}(T), K_{\gamma,m}(T)\}$, we have by (3.14) that

$$(3.15) \qquad K_{m+1}(T) \leq K_0(T) + C_* K_m(T)^2,$$

where $C_* = C_*(n, \theta, \gamma)$ is independent of $T \in (0, 1]$. Hence if

$$(3.16) \qquad K_0(T) < \frac{1}{4C_*},$$

then the sequence $\{K_m(T)\}_{m=0}^{\infty}$ is bounded as

$$(3.17) \qquad K_m(T) \leq \frac{1 - \sqrt{1 - 4C_* K_0(T)}}{2C_*} \quad \text{for all } m = 0, 1, \cdots.$$

It is easy to see as in the proof of Fujita-Kato [5] and Giga-Miyakawa [GM] that such a uniform bound yields a limit $\lim_{m \to \infty} v_m(t) = v(t)$ which is a unique solution of (3.7) with properties (3.1), (3.2) and (3.3). It remains therefore to show that we can take $T > 0$ so that (3.16) is fulfilled. Indeed, since $a \in D(A^\gamma)$, there holds

$$(3.18) \qquad K_{\alpha,0}(T) = \sup_{0 < t < T} t^{\alpha - (\frac{n}{4} - \frac{1}{2})} \|\widetilde{A}^\alpha e^{-t\widetilde{A}} a\|$$

$$= \sup_{0 < t < T} t^{\alpha - (\frac{n}{4} - \frac{1}{2})} \|\widetilde{A}^{\alpha - \gamma} e^{-t\widetilde{A}} \widetilde{A}^\gamma a\|$$

$$\leq T^{\gamma - (\frac{n}{4} - \frac{1}{2})} \|\widetilde{A}^\gamma a\|$$

for all $\gamma \leq \alpha < 1$ and all $0 < T \leq 1$. Since $\gamma \leq (n+2)/8$, we have $\gamma \leq \theta$ and hence (3.18) shows that the condition (3.16) can be achieved provided

$$(3.19) \qquad T = (8C_* \|\widetilde{A}^\gamma a\|)^{-\frac{1}{\gamma - (\frac{n}{4} - \frac{1}{2})}}.$$

This implies (3.4).

(2) Let us take $T = 1$ in the above argument. By (3.13) we have

$$(3.20) \qquad K_0(1) \leq \|\widetilde{A}^{\frac{n}{4} - \frac{1}{2}} a\|.$$

On the other hand, (3.17) shows that, according to the size of $K_0(1)$, we can control $\sup_{0 < t < 1} t^{\gamma - (\frac{n}{4} - \frac{1}{2})} \|\widetilde{A}^\gamma u(t)\| (\leq e \sup_{0 < t < 1} t^{\gamma - (\frac{n}{4} - \frac{1}{2})} \|\widetilde{A}^\gamma v(t)\|)$ arbitrarily small. Since $\sup_{0 < t < 1} \|u(t)\| \leq \|a\|$, (3.20) guarantees the existence of such a constant μ as (3.5). This completes the proof of Theorem 3.1. ∎

Based on (3.4), we can construct a global strong solution for small initial data.

THEOREM 3.2. – *Let $n = 4, 5$ and let $\gamma_* = (n + 2)/8$. There is a constant $\delta = \delta(n) > 0$ such that if $a \in D(A^{\gamma_*})$ satisfies*

$$(3.21) \qquad\qquad \|a\| + \|A^{\gamma_*} a\| \leq \delta,$$

then the solution u in Theorem 3.1 is global, and we may set $T = \infty$ in (3.1), (3.2) and (3.3) with $\gamma = \gamma_*$.

Proof. – Let u be the solution on $(0, T)$ in Theorem 3.1 (1). On account of concrete characterization of the time interval for existence of the local solution such as (3.4), it suffices to show the a priori bounds of $\sup_{0<t<T} \|u(t)\|$ and $\sup_{0<t<T} \|A^{\gamma_*} u(t)\|$ which hold *independently* of T. Then by the usual argument of continuation, we may set $T = \infty$.

(3.3) yields that $\sup_{0<t<T} \|u(t)\| \leq \|a\|$ and hence we have only to obtain an a priori estimate of $\sup_{0<t<T} \|A^{\gamma_*} u(t)\|$. To this end, we shall make use of the following representation

$$(3.23) \qquad u(t) = e^{-t(A+1)} a + \int_0^t e^{-(t-s)(A+1)} u(s) ds$$

$$- \int_0^t e^{-(t-s)(A+1)} P(u \cdot \nabla u)(s) ds$$

$$\equiv u_0(t) + u_1(t) + u_2(t), \qquad 0 < t < T.$$

Since A is a non-negative self-adjoint operator in L^2_σ, there holds

$$(3.23) \qquad \|A^\alpha e^{-t(A+1)} b\| \leq t^{-\alpha} e^{-t} \|b\|, \quad 0 \leq \alpha \leq 1$$

for all $b \in L^2_\sigma$ and all $t > 0$. Hence we have

$$(3.24) \qquad \sup_{0<t<T} \|A^\alpha u_0(t)\| \leq \|A^\alpha a\| \quad \text{for } 0 \leq \alpha \leq \gamma_*.$$

By (3.3) and (3.23) there holds

$$(3.25) \qquad \|A^\alpha u_1(t)\| \leq \int_0^t \|A^\alpha e^{-(t-s)(A+1)} u(s)\| ds$$

$$\leq \int_0^t (t-s)^{-\alpha} e^{-(t-s)} \|u(s)\| ds$$

$$\leq \sup_{0<s<T} \|u(s)\| \int_0^t (t-s)^{-\alpha} e^{-(t-s)} ds$$

$$\leq \Gamma(1-\alpha) \|a\|, \qquad 0 \leq \alpha < 1$$

for all $0 < t < T$. Moreover, it follows from (2.6) with $\theta = n/4 - 1/2$ that

$$(3.26) \quad \|A^\alpha u_2(t)\| \leq \int_0^t \|A^\alpha e^{-(t-s)(A+1)} P(u \cdot \nabla u)(s)\| ds$$

$$\leq \int_0^t (t-s)^{-\alpha} e^{-(t-s)} \|P(u \cdot \nabla u)(s)\| ds$$

$$\leq C_* \int_0^t (t-s)^{-\alpha} e^{-(t-s)} (\|A^{\frac{1}{2}} u(s)\|^2 + \|A^{\gamma_*} u(s)\|^2) ds$$

$$\leq C_* \Gamma(1-\alpha) (\sup_{0<t<T} \|A^{\frac{1}{2}} u(t)\|^2 + \sup_{0<t<T} \|A^{\gamma_*} u(t)\|^2)$$

for $0 < t < T$, $0 \leq \alpha < 1$ with $C_* = C_*(n)$ independent of T.

Defining $K(T) \equiv \max\{\sup_{0<t<T} \|A^{\frac{1}{2}} u(t)\|, \sup_{0<t<T} \|A^{\gamma_*} u(t)\|\}$, we have by (3.24), (3.25) and (3.26) that

$$(3.27) \qquad K(T) \leq \eta(\|a\| + \|A^{\gamma_*} a\|) + 2\eta C_* K(T)^2,$$

where $\eta = \Gamma(1 - \gamma_*)(> \Gamma(1/2))$. Now we take such δ in (3.21) as

$$0 < \delta < \frac{1}{8\eta^2 C_*}.$$

Then under the assumption (3.21), we have by (3.27) that

$$K(T) \leq \frac{1 - \sqrt{1 - 8\eta^2 C_*(\|a\| + \|A^{\gamma_*} a\|)}}{4\eta C_*} < \frac{1}{4\eta C_*}.$$

Since the right hand side of the above inequality is independent of T, we get the desired a priori estimate. This proves Theorem 3.2. ∎

4. DECAY OF STRONG SOLUTION

In this section, we shall first investigate the asymptotic behavior of the global solution $u(t)$ given by Theorem 3.2, and then prove the main theorem by making use of smoothing effect for $t > 0$ of the solution with $a \in D(A^{\frac{n}{4} - \frac{1}{2}})$.

THEOREM 4.1. – *The global solution u in Theorem 3.2 has the following decay properties:*

$$(4.1) \qquad \|A^\alpha u(t)\| = o(t^{-\alpha}) \quad for\ 0 \leq \alpha \leq 1,$$

$$(4.2) \qquad \|\frac{du(t)}{dt}\| = o(t^{-1})$$

as $t \to \infty$.

This theorem will be proved in the series of the subsequent lemmata; (4.1) and (4.2) will be obtained from Lemmata 4.9 and 4.8 below, respectively. We shall follow the method established by Masuda [22]. To get sharper rates of decay as $t \to \infty$, however, we need the continuous embedding as in the proof of Lemma 2.3 and make deeper investigation into $du(t)/dt$.

PROPOSITION 4.2 (Masuda). – *Let* $0 \le \alpha < 1$ *and let* $g \in L^p(0, \infty; L_\sigma^2)$ *for some p with* $(1 - \alpha)^{-1} < p < \infty$. *Then the function* $f_\alpha(t)$ *defined by*

$$f_\alpha(t) \equiv \int_0^t A^\alpha e^{-(t-s)(A+1)} g(s) ds$$

converges to zero as $t \to \infty$ *in* L_σ^2.

For the proof, see Masuda [22, Lemma 21].

LEMMA 4.3. – *For the global solution* $u(t)$ *given by Theorem 3.2 we have*

$$(4.3) \qquad \qquad \|A^{\frac{1}{2}} u(t)\| \to 0 \qquad as \ t \to \infty.$$

Proof. – By (3.1) for $\gamma = \gamma_* (\equiv (n+2)/8)$ and (3.3) with $T = \infty$, there holds

$$(4.4) \qquad A^\alpha u \in BC([0, \infty); L_\sigma^2) \quad \text{for } 0 \le \alpha \le \gamma_*;$$
$$(4.5) \qquad A^{\frac{1}{2}} u \in L^p(0, \infty; L_\sigma^2) \quad \text{for } 2 \le p \le \infty.$$

Let us recall the representation of $u(t)$ such as (3.22). Clearly by (3.23), we have

$$(4.6) \qquad \|A^{\frac{1}{2}} u_0(t)\| = \|A^{\frac{1}{2}} e^{-t(A+1)} a\| \le t^{-\frac{1}{2}} e^{-t} \|a\| \to 0 \quad \text{as } t \to .\infty.$$

By (4.4), (4.5) and Proposition 4.2 with $p = 2$,

$$(4.7) \quad \|A^{\frac{1}{2}} u_1(t)\| = \left\| \int_0^t e^{-(t-s)(A+1)} A^{\frac{1}{2}} u(s) ds \right\| \to 0 \quad \text{as } t \to \infty.$$

Taking $\theta = n/4 - 1/2$, we have by Remark 2.6 that $F_\theta(u, u)(t) \in D(A^{\frac{\theta}{2}})$ with $A^{\frac{\theta}{2}} F_\theta(u, u)(t) = P(u \cdot \nabla u)(t)$ for all $t \ge 0$. Hence it follows that

$$A^{\frac{1}{2}} u_2(t) = \int_0^t A^{\frac{1}{2} + \frac{\theta}{2}} e^{-(t-s)(A+1)} F_\theta(u, u)(s) ds.$$

Since $n/4 - \theta/2 = \gamma_*$, we have by Lemma 2.5, (4.4) and (4.5)

$$(4.8) \qquad F_\theta(u, u) \in L^p(0, \infty; L^2_\sigma) \quad \text{for } 2 \le p \le \infty.$$

Since $\{1 - (1/2 + \theta/2)\}^{-1} = 8/(6 - n) \ge 2$, Proposition 4.2 and (4.8) yield

$$(4.9) \qquad \|A^{\frac{1}{2}} u_2(t)\| \to 0 \quad \text{as } t \to \infty.$$

Now (4.3) follows from (4.6), (4.7) and (4.9). ∎

LEMMA 4.4. – *For the global solution u in Theorem 3.2, there exist $T_0 > 0$ and a family $\{f_\alpha(t)\}_{0 \le \alpha \le 1/2}$ of continuous functions on $[T_0, \infty)$ such that*

$$\|A^\alpha u_h(t)\| \le f_\alpha(t), \quad T_0 \le t < \infty$$

for all sufficiently small h, where $u_h(t) \equiv \dfrac{u(t+h) - u(t)}{h}$.

Proof. – Let us first note the identity

$$(4.10) \qquad (u \cdot \nabla v, w) = -(u \cdot \nabla w, v) \quad \text{for all } u, v, w \in D(A^{\frac{1+\alpha}{2}}),$$

where $\alpha \equiv n/6 - 2/3$. From this we have $(u \cdot \nabla v, v) = 0$ for all $u, v \in D(A)$. For $u, v, w \in C^\infty_{0,\sigma}$, (4.10) is an immediate consequence of integration by parts. By (1.2), there holds the continuous embedding

$$D(A^{\frac{1+\alpha}{2}}) \subset H^{1+\alpha,2} \subset H^{1,p} \subset L^q,$$

where $1/p = 1/2 - \alpha/n$ and $1/q = 1/2 - (1+\alpha)/n$. Since $1/p + 2/q = 1$, we have by the Hölder inequality and the above embedding that

$$|(u \cdot \nabla v, w)| \le \|u\|_q \|\nabla v\|_p \|w\|_q \le C\|u\|_{D(A^{\frac{1+\alpha}{2}})} \|v\|_{D(A^{\frac{1+\alpha}{2}})} \|w\|_{D(A^{\frac{1+\alpha}{2}})}$$

with $C = C(n)$, which implies that $(u \cdot \nabla v, w)$ is a trilinear continuous form on $D(A^{\frac{1+\alpha}{2}})$. On the other hand since $n = 4, 5$, we have $0 \le \alpha < 1/2$ and hence $C^\infty_{0,\sigma}$ is dense in $D(A^{\frac{1+\alpha}{2}})$ (see Fujiwara [7] and Fujita-Morimoto [6]). Now (4.10) follows from the density argument.

Since u satisfies (E),
$$(4.11)$$
$$\dot{u}_h(t) + A u_h(t) + P(u_h(t) \cdot \nabla u(t)) + P(u(t+h) \cdot \nabla u_h(t)) = 0, \quad t > |h|,$$

where $\dot{u}_h = du_h/dt$. Taking the inner product in L^2_σ with u_h and (4.11), we obtain from (4.10)

$$(4.12) \qquad \frac{d}{dt}\|u_h(t)\|^2 + 2\|A^{\frac{1}{2}} u_h(t)\|^2 = 2(u_h(t) \cdot \nabla u_h(t), u(t)).$$

By the Sobolev inequality and (2.8) with $\alpha = n/2 - 2$, there holds

(4.13) $\quad |(u_h(t) \cdot \nabla u_h(t), u(t))|$

$$\leq \|u_h(t)\|_{\frac{2n}{n-2}} \|\nabla u_h(t)\| \|u(t)\|_n$$

$$\leq C \|\nabla u_h(t)\|^2 (\|A^{\frac{1}{2}} u(t)\| + \|A^{\frac{n}{4} - \frac{1}{2}} u(t)\|)$$

$$\leq C \|A^{\frac{1}{2}} u_h(t)\|^2 (\|A^{\frac{1}{2}} u(t)\| + \|A^{\frac{1}{2}} u(t)\|^{\frac{6-n}{n-2}} \|A^{\gamma_*} u(t)\|^{\frac{2(n-4)}{n-2}})$$

with $C = C(n)$ independent of t and h. By (4.4) and Lemma 4.3, there exists $T_0 > 0$ such that

$$\|A^{\frac{1}{2}} u(t)\| + \|A^{\frac{1}{2}} u(t)\|^{\frac{6-n}{n-2}} \|A^{\gamma_*} u(t)\|^{\frac{2(n-4)}{n-2}} \leq \frac{1}{2C} \quad \text{for all } t \geq T_0.$$

Hence by (4.12) and (4.13) we have

(4.14) $\qquad \dfrac{d}{dt} \|u_h(t)\|^2 + \|A^{\frac{1}{2}} u_h(t)\|^2 \leq 0 \quad \text{for all } t \geq T_0,$

which yields

$$\|u_h(t)\|^2 \leq \|u_h(T_0)\|^2 \leq \|\overset{\bullet}{u}(T_0)\|^2 + 1, \quad t \geq T_0$$

for all sufficiently small h, where $\overset{\bullet}{u} = du/dt$. Then $f_0(t)$ is defined by the constant function

$$f_0(t) \equiv (\|\overset{\bullet}{u}(T_0)\|^2 + 1)^{\frac{1}{2}}, \quad t \geq T_0.$$

We shall next consider for $\alpha = 1/2$. Taking the inner product in L_σ^2 with $Au_h(t)$ and (4.11), we have

(4.15) $\quad \dfrac{1}{2} \dfrac{d}{dt} \|A^{\frac{1}{2}} u_h(t)\|^2 + \|Au_h(t)\|^2$

$$\leq |(P(u_h(t) \cdot \nabla u(t)), Au_h(t))| + |(P(u(t+h) \cdot \nabla u_h(t)), Au_h(t))|$$

$$\equiv g_1^{(h)}(t) + g_2^{(h)}(t).$$

Take p and q as $1/p = 1/4 - 1/2n$ and $1/q = 1/4 + 1/2n$, respectively. Then we have by (2.8), (2.12) with $\alpha = n/4 - 1/2$ and (4.4) that

$$
\begin{aligned}
g_1^{(h)}(t) & \\
& \leq \|u_h(t)\|_p \|\nabla u(t)\|_q \|Au_h(t)\| \\
& \leq C(\|A^{\frac{1}{2}}u_h(t)\| + \|A^{\gamma_*}u_h(t)\|)(\|A^{\frac{1}{2}}u(t)\| + \|A^{\gamma_*}u(t)\|)\|Au_h(t)\| \\
& \leq C(\|A^{\frac{1}{2}}u_h(t)\| + \|A^{\frac{1}{2}}u_h(t)\|^{\frac{3}{2}-\frac{n}{4}}\|Au_h(t)\|^{\frac{n}{4}-\frac{1}{2}})\|Au_h(t)\| \\
& \leq \frac{1}{4}\|Au_h(t)\|^2 + C\|A^{\frac{1}{2}}u_h(t)\|^2; \\
g_2^{(h)}(t) & \\
& \leq \|u(t+h)\|_p \|\nabla u_h(t)\|_q \|Au_h(t)\| \\
& \leq C(\|A^{\frac{1}{2}}u(t+h)\| + \|A^{\gamma_*}u(t+h)\|) \\
& \qquad (\|A^{\frac{1}{2}}u_h(t)\| + \|A^{\gamma_*}u_h(t)\|)\|Au_h(t)\| \\
& \leq C(\|A^{\frac{1}{2}}u_h(t)\| + \|A^{\frac{1}{2}}u_h(t)\|^{\frac{3}{2}-\frac{n}{4}}\|Au_h(t)\|^{\frac{n}{4}-\frac{1}{2}})\|Au_h(t)\| \\
& \leq \frac{1}{4}\|Au_h(t)\|^2 + C\|A^{\frac{1}{2}}u_h(t)\|^2
\end{aligned}
$$

with $C = C(n)$ independent of t and h. Hence it follows from (4.15) that

$$
\frac{d}{dt}\|A^{\frac{1}{2}}u_h(t)\|^2 + \|Au_h(t)\|^2 \leq C\|A^{\frac{1}{2}}u_h(t)\|^2, \quad t > |h|.
$$

By the Gronwall inequality, we have

$$
\begin{aligned}
\|A^{\frac{1}{2}}u_h(t)\|^2 & \leq \|A^{\frac{1}{2}}u_h(T_0)\|^2 e^{C(t-T_0)} \\
& \leq (\|A^{\frac{1}{2}}\dot{u}(T_0)\|^2 + 1)e^{C(t-T_0)}, \quad t \geq T_0
\end{aligned}
$$

for all sufficiently small h with $C = C(n)$. Now the function $f_{1/2}(t)$ is defined by

$$
f_{\frac{1}{2}}(t) \equiv (\|A^{\frac{1}{2}}\dot{u}(T_0)\|^2 + 1)^{\frac{1}{2}} e^{C(t-T_0)}, \quad t \geq T_0.
$$

For $0 < \alpha < 1/2$, we have $\|A^\alpha u\| \leq \|u\|^{1-2\alpha}\|A^{1/2}u\|^{2\alpha}$ and hence $f_\alpha(t)$ can be defined by $f_\alpha(t) = f_0(t)^{1-2\alpha}f_{1/2}(t)^{2\alpha}$. This completes the proof of Lemma 4.4. ∎

LEMMA 4.5. – *Let $T_0 > 0$ be as in Lemma 4.4. Then we have*

$$
(4.16) \qquad \|\dot{u}(t)\|^2 + \int_s^t \|\nabla \dot{u}(\tau)\|^2 d\tau \leq \|\dot{u}(s)\|^2
$$

for all s and t such that $T_0 \leq s \leq t < \infty$ and

$$(4.17) \qquad\qquad \int_{T_0}^{\infty} \| \overset{\bullet}{u}(\tau) \|^2 d\tau < \infty.$$

Proof. – By (4.14), there holds

$$\|u_h(t)\|^2 + \int_s^t \|\nabla u_h(\tau)\|^2 d\tau \leq \|u_h(s)\|^2 \quad \text{for } T_0 \leq s \leq t < \infty.$$

Letting $h \to 0$ in the above inequality, from Lemma 4.4 and the Lebesgue convergence theorem we obtain (4.16).

Taking $\theta = n/4 - 1/2$ in Lemma 2.5, we have by (4.4) and Remark 2.6 that $A^{\frac{\theta}{2}} F_\theta(u, u)(t) = P(u \cdot \nabla u)(t)$ for all $t \geq 0$. Hence the identity (E) yields

$$\begin{aligned}
\| \overset{\bullet}{u}(t) \|^2 &= -(Au, \overset{\bullet}{u}) - (P(u \cdot \nabla u), \overset{\bullet}{u}) \\
&= -(A^{\frac{1}{2}} u, A^{\frac{1}{2}} \overset{\bullet}{u}) - (F_\theta(u, u), A^{\frac{\theta}{2}} \overset{\bullet}{u}) \\
&\leq \|A^{\frac{1}{2}} u\| \|A^{\frac{1}{2}} \overset{\bullet}{u}\| + \|F_\theta(u, u)\| \|A^{\frac{\theta}{2}} \overset{\bullet}{u}\| \\
&\leq \|A^{\frac{1}{2}} u\|^2 + \|A^{\frac{1}{2}} \overset{\bullet}{u}\|^2 \\
&\quad + C\|A^{\frac{1}{2}} u\| (\|A^{\frac{1}{2}} u\| + \|A^{\gamma_*} u\|) \| \overset{\bullet}{u}\|^{1-\theta} \|A^{\frac{1}{2}} \overset{\bullet}{u}\|^{\theta} \\
&\leq \|A^{\frac{1}{2}} u\|^2 + \|A^{\frac{1}{2}} \overset{\bullet}{u}\|^2 + \frac{1}{2} \| \overset{\bullet}{u}\|^2 \\
&\quad + C\Big\{ \|A^{\frac{1}{2}} u\| (\|A^{\frac{1}{2}} u\| + \|A^{\gamma_*} u\|) \Big\}^{\frac{2}{1+\theta}} \|A^{\frac{1}{2}} \overset{\bullet}{u}\|^{\frac{2\theta}{1+\theta}} \\
&\leq \|A^{\frac{1}{2}} u\|^2 + \|A^{\frac{1}{2}} \overset{\bullet}{u}\|^2 + \frac{1}{2} \| \overset{\bullet}{u}\|^2 \\
&\quad + C\Big\{ \|A^{\frac{1}{2}} u\| (\|A^{\frac{1}{2}} u\| + \|A^{\gamma_*} u\|) \Big\}^2 + \|A^{\frac{1}{2}} \overset{\bullet}{u}\|^2 \\
&\leq \|A^{\frac{1}{2}} u\|^2 + 2\|A^{\frac{1}{2}} \overset{\bullet}{u}\|^2 + \frac{1}{2} \| \overset{\bullet}{u}\|^2 \\
&\quad + C\|A^{\frac{1}{2}} u\|^4 + C\|A^{\gamma_*} u\|^2 \|A^{\frac{1}{2}} u\|^2,
\end{aligned}$$

for all $t > 0$ with $C = C(n)$, from which and (4.4) we obtain

$$\|\overset{\bullet}{u}(t)\|^2 \leq C \Big\{ \|A^{\frac{1}{2}} u(t)\|^4 + (1 + \sup_{\tau \geq 0} \|A^{\gamma_*} u(\tau)\|^2) \|A^{\frac{1}{2}} u(t)\|^2 + \|A^{\frac{1}{2}} \overset{\bullet}{u}(t)\|^2 \Big\}$$

for all $t > 0$. Then (4.17) follows from (4.5) and (4.16). This completes the proof of Lemma 4.5. ∎

LEMMA 4.6. – *The global solution u in Theorem 3.2 decays like*

$$(4.18) \qquad \|\dot{u}(t)\| \le C(t - T_0)^{-\frac{1}{2}},$$

$$(4.19) \qquad \|A^{\frac{1}{2}}u(t)\| \le C(t - T_0)^{-\frac{1}{4}},$$

$$(4.20) \qquad \|Au(t)\| \le C(t - T_0)^{-\frac{1}{2}},$$

for all $t > T_0$ with $C = C(n, T_0)$, where T_0 is chosen as in Lemma 4.4.

Proof. – By (4.16) we have

$$\|\dot{u}(t)\|^2 \le \|\dot{u}(s)\|^2 \quad \text{for } T_0 \le s \le t.$$

Integrating this inequality on the interval (T_0, t) with respect to s, we obtain from (4.17)

$$(t - T_0)\|\dot{u}(t)\|^2 \le \int_{T_0}^t \|\dot{u}(s)\|^2 ds \le \int_{T_0}^\infty \|\dot{u}(s)\|^2 ds < \infty$$

for all $t > T_0$, which implies (4.18).

By (4.10) and (E) there holds $(\dot{u}(t), u(t)) + (Au(t), u(t)) = 0$ for all $t > 0$ and hence (3.3) and (4.18) yield

$$\begin{aligned}
\|A^{\frac{1}{2}}u(t)\|^2 &= (Au(t), u(t)) \\
&\le |(\dot{u}(t), u(t))| \\
&\le \|\dot{u}(t)\|\|u(t)\| \\
&\le C\|a\|(t - T_0)^{-\frac{1}{2}} \quad \text{for all } t \ge T_0
\end{aligned}$$

with $C = C(n, T_0)$, which shows (4.19).

To prove (4.20) notice that there is a constant $C_* = C_*(T_0)$ such that

$$(4.21) \qquad \|Au(t)\| \le C_* \quad \text{for all } t \ge T_0.$$

Indeed, by (2.6) with $\theta = n/4 - 1/2$, we have

$$(4.22) \quad \begin{aligned}
\|Au(t)\| &\le \|\dot{u}(t)\| + \|P(u \cdot \nabla u)(t)\| \\
&\le \|\dot{u}(t)\| + C(\|A^{\frac{1}{2}}u(t)\| + \|A^{\gamma_*}u(t)\|)^2 \quad \text{for all } t > 0
\end{aligned}$$

Hence (4.21) follows from (4.4) and (4.18). Moreover, by (4.21) and (4.22) there holds

$$\begin{aligned}
\|Au(t)\| &\le \|\dot{u}(t)\| + C\|A^{\frac{1}{2}}u(t)\|^2 + C\|A^{\gamma_*}u(t)\|^2 \\
&\le \|\dot{u}(t)\| + C\|A^{\frac{1}{2}}u(t)\|^2 + C(\|A^{\frac{1}{2}}u(t)\|^{2-2\gamma_*}\|Au(t)\|^{2\gamma_* - 1})^2 \\
&\le \|\dot{u}(t)\| + C_\varepsilon\|A^{\frac{1}{2}}u(t)\|^2 + \varepsilon\|Au(t)\|^2 \\
&\le \|\dot{u}(t)\| + C_\varepsilon\|A^{\frac{1}{2}}u(t)\|^2 + \varepsilon C_*\|Au(t)\|, \qquad t \ge T_0
\end{aligned}$$

for all $\varepsilon >$ with C_ε depending only on ε. Taking $\varepsilon = 1/2C_*$ in the above estimate, we have

$$(4.23) \qquad \|Au(t)\| \le C(\|\dot{u}(t)\| + \|A^{\frac{1}{2}}u(t)\|^2) \quad \text{for all } t \ge T_0$$

with $C = C(n, T_0)$. Now (4.20) follows from (4.18) and (4.19). This proves Lemma 4.6. ∎

Really, the global solution u in Theorem 3.2 decays more rapidly than in Lemma 4.6. To obtain sharper rates of decay, we need to estimate the nonlinear term $u \cdot \nabla u$ in L^2 by means of $\|A^\alpha u\|$ with $\alpha \ge 1/2$, which differs essentially from the result of Masuda [22].

LEMMA 4.7. – *The global solution u in Theorem 3.2 decays, in fact, like*

$$\|A^{\frac{1}{2}}u(t)\| = o(t^{-\frac{1}{2}}) \quad \text{as } t \to \infty.$$

Proof. – We shall make use of the representation

$$(4.24) \qquad u(t) = e^{-(t-T)A}u(T) - \int_T^t e^{-(t-s)A}P(u \cdot \nabla u)(s)ds$$
$$\equiv I_0(t) + I_1(t)$$

for all $0 \le T \le t$. By (4.19) and (4.20), there is $T_1 > T_0$ such that

$$(4.25) \qquad \|A^{\frac{1}{2}}u(t)\| \le Ct^{-\frac{1}{4}}, \quad \|Au(t)\| \le Ct^{-\frac{1}{2}} \quad \text{for all } t \ge T_1$$

with $C = C(T_0, T_1)$ independent of t. By (3.1) and Lemma 2.5 (2), we have $P(u \cdot \nabla u)(t) = A^{\theta/2}F_\theta(u, u)(t)$ for all θ with $n/2 - 2 \le \theta < 1$ and all $t > 0$. Let us take θ as $2/3 < \theta < 1$. Then we obtain from Lemma 2.5 and (4.25)

$$\|A^{\frac{1}{2}}I_1(t)\| \le \int_T^t \|A^{\frac{1}{2}}e^{-(t-s)A}P(u \cdot \nabla u)(s)\|ds$$

$$\le \int_T^t \|A^{\frac{1}{2}+\frac{\theta}{2}}e^{-(t-s)A}F_\theta(u, u)(s)\|ds$$

$$\le C\int_T^t (t-s)^{-\frac{1}{2}-\frac{\theta}{2}}\|A^{\frac{1}{2}}u(s)\|(\|A^{\frac{1}{2}}u(s)\| + \|A^{\frac{n}{4}-\frac{\theta}{2}}u(s)\|)ds$$

$$(4.26) \qquad \le C\int_T^t (t-s)^{-\frac{1}{2}-\frac{\theta}{2}}(\|A^{\frac{1}{2}}u(s)\|^2 + \|A^{\frac{1}{2}}u(s)\|^{2-\alpha}\|Au(s)\|^\alpha)ds$$

$$\le C\int_T^t (t-s)^{-\frac{1}{2}-\frac{\theta}{2}}(s^{-\frac{1}{2}} + s^{-\frac{1}{2}-\frac{\alpha}{4}})ds$$

$$\le CT^{-\frac{\theta}{2}}$$

for all $t \geq T \geq T_1$ with $C = C(T_0, T_1)$, where $\alpha \equiv n/2 - 1 - \theta$. Hence by (4.24) and (4.26), there holds

$$\|A^{\frac{1}{2}} u(t)\| \leq \|A^{\frac{1}{2}} I_0(t)\| + \|A^{\frac{1}{2}} I_1(t)\|$$
$$\leq (t - T_0)^{-\frac{1}{2}} \|a\| + C T^{-\frac{\theta}{2}}, \qquad T_1 < T \leq t$$

with C independent of T and t. Taking $T = t/2$ in the above estimate, we have

(4.27) $$\|A^{\frac{1}{2}} u(t)\| \leq C t^{-\frac{\theta}{2}} \quad \text{for all } t \geq 2T_1.$$

Now, substituting (4.25) and (4.27) again into (4.26) we get

$$\|A^{\frac{1}{2}} I_1(t)\| \leq C \int_T^t (t - s)^{-\frac{1}{2} - \frac{\theta}{2}} (s^{-\theta} + s^{-\frac{\theta}{2}(2-\alpha)} s^{-\frac{\alpha}{2}}) ds$$
$$\leq C T^{-\frac{1}{2}(3\theta - 1)}$$

for all $t \geq T \geq 2T_1$ which yields

$$\|A^{\frac{1}{2}} u(t)\| \leq C(t - T)^{-\frac{1}{2}} \|u(T)\| + C T^{-\frac{1}{2}(3\theta - 1)} \quad \text{for all } t \geq T \geq 2T_1$$

with $C = C(T_0, T_1)$. Taking $T = t/2$ in the above estimate, we have

(4.28) $$\|A^{\frac{1}{2}} u(t)\| \leq C(t^{-\frac{1}{2}} \|u(t/2)\| + t^{-\frac{1}{2}(3\theta - 1)}) \quad \text{for all } t \geq 4T_1.$$

Since u satisfies (3.3) with $T = \infty$, we know $\lim_{t \to \infty} \|u(t)\| = 0$ (see e.g., Masuda [23, Corollary 2]). Hence this estimate together with the relation $(3\theta - 1)/2 > 1/2$ yields the desired result. This proves Lemma 4.7. ∎

Let us improve the decay rate for $\overset{\bullet}{u}$.

LEMMA 4.8. – *For the global solution u in Theorem 3.2 we have*

$$\| \overset{\bullet}{u}(t) \| = o(t^{-1}) \qquad \text{as } t \to \infty.$$

Proof. – We shall make use of (4.14). To this end, we need to estimate $\|A^{\frac{1}{2}} u_h(t)\|$ from below. Since u satisfies (E), by (4.10) there holds

(4.29) $$\|u_h\|^2 = (u_h, u_h - \overset{\bullet}{u}) + (u_h, \overset{\bullet}{u})$$
$$= (u_h, u_h - \overset{\bullet}{u}) + (u_h, -Au - P(u \cdot \nabla u))$$
$$= (u_h, u_h - \overset{\bullet}{u}) - (A^{1/2} u_h, A^{1/2} u) + (u \cdot \nabla u_h, u)$$
$$\leq \|u_h\| \|u_h - \overset{\bullet}{u}\| + \|A^{1/2} u_h\| \|A^{1/2} u\| + (u \cdot \nabla u_h, u).$$

By (2.8) with $\alpha = n/4 - 1$ we have

$$
\begin{aligned}
|(u \cdot \nabla u_h, u)| &\leq \|u\|_4^2 \|\nabla u_h\| \\
&\leq C(\|A^{\frac{1}{2}}u\|^2 + \|A^{\frac{n}{8}}u\|^2)\|A^{\frac{1}{2}}u_h\| \\
&\leq C\left(\|A^{\frac{1}{2}}u\|^2 + (\|A^{\frac{1}{2}}u\|^{2-\frac{n}{4}}\|Au\|^{\frac{n}{4}-1})^2\right)\|A^{\frac{1}{2}}u_h\| \\
&\leq C(\|A^{\frac{1}{2}}u\| + \|A^{\frac{1}{2}}u\|^{3-\frac{n}{2}}\|Au\|^{\frac{n}{2}-2})\|A^{\frac{1}{2}}u\|\|A^{\frac{1}{2}}u_h\|.
\end{aligned}
$$

By Lemma 4.6, there exists $T_* > T_0$ such that

$$
\|A^{\frac{1}{2}}u(t)\| + \|A^{\frac{1}{2}}u(t)\|^{3-\frac{n}{2}}\|Au(t)\|^{\frac{n}{2}-2} \leq \frac{1}{C} \quad \text{for all } t > T_*
$$

and hence we have

$$(4.30) \quad |(u(t) \cdot \nabla u_h(t), u(t))| \leq \|A^{\frac{1}{2}}u(t)\|\|A^{\frac{1}{2}}u_h(t)\| \quad \text{for all } t > T_*.$$

(4.29) and (4.30) yield

$$
\|A^{\frac{1}{2}}u_h(t)\|^2 \geq \frac{\|u_h(t)\|^4}{8\|A^{\frac{1}{2}}u(t)\|^2} - \frac{\|u_h(t)\|^2\|u_h(t) - \dot{u}(t)\|^2}{4\|A^{\frac{1}{2}}u(t)\|^2}
$$

for all $t \geq T_*$. Substituting this estimate into (4.14), we have

$$
\frac{d\|u_h(t)\|^2}{dt} + \frac{\|u_h(t)\|^4}{8\|A^{\frac{1}{2}}u(t)\|^2} \leq \|u_h(t)\|^2 \frac{\|u_h(t) - \dot{u}(t)\|^2}{4\|A^{\frac{1}{2}}u(t)\|^2}
$$

for all $t \geq T_*$ and all $h > 0$. Set $y(t) \equiv \|u_h(t)\|^2$, $p(t) \equiv 8\|A^{1/2}u(t)\|^2$ and $\varepsilon_h(t) \equiv 2\|u_h(t) - \dot{u}(t)\|^2$. Then we have from above

$$
\frac{dy(t)}{dt} + \frac{y(t)^2}{p(t)} \leq \frac{\varepsilon_h(t)}{p(t)} \cdot y(t), \qquad t \geq T_*
$$

which may be regarded as a differential inequality of the Bernoulli type with respect to $y(t)$. A standard calculation yields the following estimate:

$$
y(t) \leq e^{\int_s^t \frac{\varepsilon_h(\tau)}{p(\tau)}d\tau} \left(\frac{1}{y(s)} + \int_s^t \frac{d\tau}{p(\tau)}\right)^{-1}, \quad T_* \leq s \leq t.
$$

This implies that

$$
\|u_h(t)\|^2 \leq e^{\int_s^t \frac{\varepsilon_h(\tau)}{8\|A^{1/2}u(\tau)\|^2})d\tau} \left(\frac{1}{\|u_h(s)\|^2} + \frac{1}{8}\int_s^t \frac{d\tau}{\|A^{\frac{1}{2}}u(\tau)\|^2}\right)^{-1}
$$

for all $t > s \geq T_0$. Letting $h \to 0$ in the above, we have by Lemma 4.4 and the Lebesgue dominated convergence theorem that

$$\| \dot{u}(t) \|^2 \leq \left(\frac{1}{\| \dot{u}(s) \|^2} + \frac{1}{8} \int_s^t \frac{d\tau}{\| A^{\frac{1}{2}} u(\tau) \|^2} \right)^{-1}$$

for all $t > s \geq T_0$. Now the desired result follows from Lemma 4.7. This completes the proof. ■

We shall next show (4.1).

LEMMA 4.9. – *For the global solution u in Theorem 3.2 we have*

$$\| A^\alpha u(t) \| = o(t^{-\alpha}) \quad \text{for } 0 \leq \alpha \leq 1$$

as $t \to \infty$.

Proof. – For $\alpha = 0$, the assertion follows from energy decay of solutions with (3.3) (see e.g., Masuda [23, Corollary 2]). Since $\| A^\alpha u \| \leq \| u \|^{1-\alpha} \| Au \|^\alpha$ for $0 < \alpha < 1$, it suffices to show for $\alpha = 1$. By Lemmata 4.7 and 4.8 we have

$$\| \dot{u}(t) \| = o(t^{-1}), \quad \| A^{\frac{1}{2}} u(t) \| = o(t^{-\frac{1}{2}}) \quad \text{as } t \to \infty.$$

Applying this decay property to (4.23), we can deduce $\| Au(t) \| = o(t^{-1})$ as $t \to \infty$. ■

Now Theorem 4.1 follows form Lemmata 4.8 and 4.9.

Proof of the Main Theorem. – Let γ_* and $\delta = \delta = \delta(n)$ be as in Theorem 3.2.

By Theorem 3.1 (2), there is a constant $\mu = \mu(n)$ such that if

$$\| a \|_{D(A^{\frac{n}{4} - \frac{1}{2}})} \leq \mu,$$

then we have a unique solution v of (E) on $(0, 1)$ with (3.1') and (3.2) for $T = 1$ satisfying

$$\| v(1/2) \| + \| A^{\gamma_*} v(1/2) \| \leq \delta.$$

On the other hand, by Theorems 3.2 and 4.1, there exists a solution $w(t)$ of (E) on $[1/2, \infty)$ with $w(t)|_{t=1/2} = v(1/2)$ satisfying (4.1) and (4.2) with $u(t)$ replaced by $v(t)$. Let us define a function $u(t)$ on $(0, \infty)$ by

$$u(t) = \begin{cases} v(t) & \text{for } 0 \leq t \leq 1/2, \\ w(t) & \text{for } 1/2 \leq t < \infty. \end{cases}$$

Then it is easy to verify that u is a solution of (E) on $(0, \infty)$ with the desired properties. By (1.2) and (1.3) we have

$$u \in BC([0, \infty); L^n).$$

and the assertion on uniqueness follows from Serrin [26] and Sohr-von Wahl [27].

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