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## A remark on multiplicity of solutions for the Ginzburg-Landau equation

by

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**ABSTRACT.** – In this paper we study the structure of certain level set of the Ginzburg-Landau functional which has similar topology with the configuration space. As an application, we generalize Almeida-Bethuel's result on multiplicity of solutions for the Ginzburg-Landau equation. © Elsevier, Paris

*Key words:* Ginzburg-Landau equation, Ljusternik-Schnirelman theory, renormalized energy

**RÉSUMÉ.** – On étudie la structure de certains ensembles de niveau de la fonctionnelle du type Ginzburg-Landau qui ont des topologies similaires à celles de l'espace de configuration. Comme application, on généralise le résultat d'Almeida-Bethuel sur la multiplicité des solutions des équations de G-L. © Elsevier, Paris

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### 1. INTRODUCTION

Let  $\Omega \subset \mathbb{C}$  be a smooth, bounded and simply connected domain. Let  $g : \partial\Omega \rightarrow \mathbb{C}$  be a prescribed smooth map with  $|g(x)| = 1$ , for all  $x \in \partial\Omega$ .

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The Ginzburg-Landau functional, for any  $\varepsilon > 0$ , is given by

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 \right\} \quad (1.1)$$

which is defined on the Hilbert space

$$H_g^1(\Omega, \mathbb{C}) = \{u \in H^1(\Omega, \mathbb{C}); u = g \text{ on } \partial\Omega\}.$$

It is easy to verify that  $E_\varepsilon$  is a positive,  $C^2$ -functional satisfying the Palais-Smale condition. So

$$\mu_\varepsilon = \min_{u \in H_g^1(\Omega, \mathbb{C})} E_\varepsilon(u)$$

is achieved by some  $u_\varepsilon \in H_g^1(\Omega, \mathbb{C})$  and these minimizers satisfy the following Ginzburg-Landau equation:

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

The Ginzburg-Landau equation (1.2) has been extensively studied by F. Bethuel, H. Brezis and F. Hélein [BBH1, 2] and many others. A complete characterization of asymptotic behavior (as  $\varepsilon \rightarrow 0^+$ ) for minimizing solutions of (1.2) is given. It has been shown that the degree of  $g$ , denoted by  $k = \deg(g, \partial\Omega)$ , plays a crucial role in the asymptotic analysis of the minimizers. Without loss of generality, we will always assume  $k \geq 0$  throughout this paper.

In this paper, we will study the multiplicity of the solutions for the Ginzburg-Landau equation (1.2), many such results have been given for special domains and/or boundary values (see for instance Almeida and Bethuel [AB1], Felmer and Del Pino [FP], F.H. Lin [Li]). The motivation of our paper comes from the recent work of Almeida-Bethuel [AB2, 3] concerning the existence of non-minimizing solutions of (1.2). They showed that if  $k \geq 2$ , the Ginzburg-Landau equation (1.2) has at least three distinct solutions, among which at least one is not minimizing. Based on topological arguments directly inspired by Almeida-Bethuel's work, we obtain our main result as follows

**THEOREM 1.** — *Assume that  $k \geq 2$ , there is some  $\varepsilon_0 > 0$  (depending on  $\Omega$  and  $g$  only) such that if  $\varepsilon < \varepsilon_0$ , the equation (1.2) has at least  $k + 1$  distinct solutions.*

To prove Theorem 1, we will apply the standard Ljusternik-Schnirelman theory to a suitable covering space of a level set

$$E_\varepsilon^a = \{u \in H_g^1(\Omega, \mathbb{C}); E_\varepsilon(u) < a\},$$

for an  $a$  of the form

$$a = \mu_\varepsilon + \lambda \tag{1.3}$$

where  $\lambda$  is a fixed positive constant to be determined later. The proof is strongly related to the topological similarities between  $E_\varepsilon^a$  and the configuration space  $\Sigma_k(\Omega)$  of  $k$  distinct points in  $\Omega$ . As in [AB3], we need to use a map  $\tilde{\Phi}$  from  $E_\varepsilon^a$  into  $\Sigma_k(\Omega)$ . More precisely, We may assign to each function  $u$  in  $E_\varepsilon^a$ , a set of  $k$  distinct points  $\{a_1, \dots, a_k\}$ , called the vortices of  $u$ , where each vortex has the topological degree  $+1$ . The map  $\tilde{\Phi} : E_\varepsilon^a \rightarrow \Sigma_k(\Omega)$  is not continuous. However this difficulty can be overcome by applying the notion of  $\eta$ -almost continuity given in [AB3]. The topological similarity between  $E_\varepsilon^a$  and  $\Sigma_k(\Omega)$  allows us to define a covering space  $\tilde{E}_\varepsilon^a$  of  $E_\varepsilon^a$  corresponding to the covering  $F_k(\Omega) \rightarrow \Sigma_k(\Omega)$ , where  $F_k(\Omega)$  is the configuration space of ordered  $k$  distinct points in  $\Omega$ . Again we have topological similarity between these two spaces, and we than can prove that the category of  $\tilde{E}_\varepsilon^a$  is at least  $k$ . The Ljusternik-Schnirelman minimax theorem concludes that the functional  $\tilde{E}_\varepsilon$  on  $\tilde{E}_\varepsilon^a$ , which is the composition of  $E_\varepsilon$  and the covering projection either has at least  $k$  distinct critical values or the dimension of the critical set is at least 1. These imply that  $E_\varepsilon$  has at least  $k$  critical points on  $E_\varepsilon^a$ . Finally, the fact that  $E_\varepsilon^\infty = H_g^1(\Omega, \mathbb{C})$  is an affine space guarantee that  $E_\varepsilon^a$  has at least another critical point outside of  $E_\varepsilon^a$ , if  $k \geq 2$ .

This paper is organized as follows: In the next section we will recall some preliminary results about the configuration space and the construction of the map  $\tilde{\Phi}$  in [AB3] and Theorem 1 will be proved in Section 3.

## 2. PRELIMINARIES

Our proof of Theorem 1 relies essentially on the properties of the map  $\tilde{\Phi} : E_\varepsilon^a \rightarrow \Sigma_k(\Omega)$  described by Almeida and Bethuel [AB3]. With a such map, they showed that the fundamental group  $\pi_1(E_\varepsilon^a)$  is non trivial for some suitable value  $a$  of the form (1.3) when  $\varepsilon$  is sufficiently small. We review here some basic facts about the configuration space and the construction of the map  $\tilde{\Phi}$ .

We study the configuration space and renormalized energy first. Let the metric on  $\mathbb{C}^k$  be defined by the following norm

$$\|(z_1, \dots, z_k)\| = \sum_{i=1}^k |z_i|. \tag{2.1}$$

The configuration space of the ordered  $k$  distinct points in  $\Omega$

$$F_k(\Omega) = \{(a_1, \dots, a_k) \in \Omega^k; a_i \neq a_j \text{ for all } i \neq j\} \subset \mathbb{C}^k$$

with the inherited metric (2.1) on  $\mathbb{C}^k$  is a smooth manifold. The cohomology ring  $H^*(F_k(\Omega)) = H^*(F_k(\Omega), \mathbb{R})$  of the space  $F_k(\Omega)$  has been determined by Arnol'd in 1969 (see [Ar]), which is generated by elements  $\omega_{ij} \in H^1(F_k(\Omega)), 1 \leq i < j \leq k$  and subject to the following defining relations

$$\omega_{ij}\omega_{jl} + \omega_{jl}\omega_{il} + \omega_{il}\omega_{ij} = 0.$$

Arnol'd also showed that the  $p$ th Betti number  $B_p$  of  $F_k(\Omega)$  is the coefficient of  $t^p$  in the polynomial

$$(1 + t)(1 + 2t) \cdots (1 + (k - 1)t).$$

In particular,  $B_{k-1} = (k - 1)! \neq 0$ , and this concludes that

LEMMA 2. – *The cuplength of  $F_k(\Omega)$  is  $k - 1$ .*

The cuplength of a space  $X$  is the largest integer  $n$  such that there are  $n$  elements  $\varphi_j \in H^{p_j}(X), p_j > 0, 1 \leq j \leq n$  and  $\varphi_1 \cup \cdots \cup \varphi_n \neq 0$ .

The symmetric group  $S_k$  on  $\{1, \dots, k\}$  acts isometrically on  $F_k(\Omega)$  by permuting coordinates, i.e., for all  $\sigma \in S_k$ ,

$$\sigma(a_1, \dots, a_k) = (a_{\sigma(1)}, \dots, a_{\sigma(k)}).$$

This action is free, and the quotient space  $F_k(\Omega)/S_k$  is called the configuration space of  $k$  distinct point in  $\Omega$  and it will be denoted by  $\Sigma_k(\Omega)$ .

On  $\Sigma_k(\Omega)$ , we have a natural metric such that the quotient map  $\pi : F_k(\Omega) \rightarrow \Sigma_k(\Omega)$  is a Riemannian regular covering. This metric on  $\Sigma_k(\Omega)$  is the same as the length of minimal connection introduced by Brezis, Coron and Lieb in [BCL], i.e., for  $a = \{a_1, \dots, a_k\}, a' = \{a'_1, \dots, a'_k\} \in \Sigma_k(\Omega)$ ,

$$\|a - a'\| = L(a, a') = \inf_{\sigma \in S_k} \sum_{i=1}^k |a_i - a'_{\sigma(i)}|.$$

We now define the renormalized energy  $W_g$  on  $\Sigma_k(\Omega)$  which is introduced by Bethuel-Brezis-Hélein in [BBH2] as follows, for  $a = \{a_1, \dots, a_k\} \in \Sigma_k(\Omega)$ ,

$$W_g(a_1, \dots, a_k) = -\pi \sum_{i \neq j} \log |a_i - a_j| + \frac{1}{2} \int_{\partial\Omega} \phi \cdot (g \times g_\tau) - \pi \sum_{j=1}^k R(a_j)$$

where  $\phi$  is the solution of

$$\begin{cases} \Delta\phi = 2\pi \sum_{i=1}^k \delta_{a_i} & \text{in } \Omega \\ \frac{\partial\phi}{\partial\nu} = g \times g_\tau & \text{on } \partial\Omega \\ \int_{\partial\Omega} \phi = 0. \end{cases}$$

Here  $\nu$  denotes the unit outer normal to  $\partial\Omega$  and  $\tau$  is unit tangent to  $\partial\Omega$  oriented so that  $\nu \times \tau = 1$ . And the function  $R$  is the regular part of  $\phi$ , i.e.,

$$R(z) = \phi(z) - \sum_{i=1}^k \log |z - a_i|.$$

It is clear that  $W_g(a) \rightarrow +\infty$  if  $\text{dist}(a_j, \partial\Omega) \rightarrow 0$  for some  $i$  or if  $|a_i - a_j| \rightarrow 0$  for some  $i \neq j$ . It has been proved in [BBH2] that, as  $\varepsilon \rightarrow 0$ , we have

$$\mu_\varepsilon = k\pi |\log \varepsilon| + W_g(a_1^*, \dots, a_k^*) + k\nu_0 + o(1),$$

where  $o(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ,  $\nu_0$  is a universal constant, and  $(a_1^*, \dots, a_k^*)$  is a global minimum of the function  $W_g$ .

Next we will turn to the construction of the map  $\tilde{\Phi}$ . We will use a regularization technique, that is, for any  $u \in E_\varepsilon^a$ , we can associate a map  $u^h$ , which is a minimizer (not necessarily to be unique) of the following minimization problem

$$\inf_{v \in H_g^1(\Omega, \mathbb{C})} \left\{ E_\varepsilon(v) + \int_{\Omega} \frac{|u - v|^2}{2h^2} \right\} \tag{2.2}$$

where  $h = \varepsilon^{\frac{2}{4k+1}} > 0$ . We denote  $u^h = T(u)$  where  $T : H_g^1(\Omega, \mathbb{C}) \rightarrow H_g^1(\Omega, \mathbb{C})$ . Clearly we have  $u^h \in E_\varepsilon^a$  and it satisfies an equation similar to the Ginzburg-Landau equation (1.2). One of the main observations in [AB3]

is that we can describe the “vortex structure” not only for the solutions of the Ginzburg-Landau equation, but also for such maps  $u^h$ . To be more precise, let us collect some of results of [AB3].

**THEOREM 3 [AB3].** – *Assume that  $a$  is of the form (1.3) for some constant  $\lambda > 0$ . Then there is a constant  $0 < \varepsilon'_0 < 1$  depending only on  $\Omega$ ,  $g$  and  $\lambda$ , such that if  $\varepsilon < \varepsilon'_0$ , then for  $u \in E_\varepsilon^a$ ,  $|u| \leq 1$  on  $\Omega$ , there is a point  $a = \{a_1, \dots, a_k\}$  in  $\Sigma_k(\Omega)$  such that*

$$|u^h(x)| \geq \frac{1}{2}, \quad \forall x \in \Omega \setminus \bigcup_{i=1}^k B(a_i, \rho)$$

where  $\rho$  satisfies  $\varepsilon^\chi \leq \rho \leq \varepsilon^{\bar{\chi}}$ , for some constants  $\chi, \bar{\chi} \in ]0, 1[$  independent of  $\varepsilon$ .

$$\deg(u^h, a_i) = \deg\left(\frac{u^h}{|u^h|}, \partial B(a_i, \rho)\right) = +1, \text{ for all } 1 \leq i \leq k.$$

Moreover, there exists some constant  $\beta > 0$  depending only on  $\Omega, g$  and  $\lambda$  such that  $\text{dist}(a_i, \partial\Omega) \geq \beta$ , for all  $1 \leq i \leq k$  and  $|a_i - a_j| \geq \beta$ , for all  $1 \leq i \neq j \leq k$ .

Thus we can see that the properties of maps  $u^h$  are very close to that of minimizers of (1.2) as in [BBH], and it allows us to define vortices  $\{a_1, \dots, a_k\}$  for  $u^h$  and each of the vortices has topological degree +1. That defines a map  $\Psi$  from  $\text{Im}(T(P(E_\varepsilon^a)))$  to  $\Sigma_k(\Omega)$ , by  $\Psi(u^h) = \{a_1, \dots, a_k\}$ , where the map  $P : H_g^1(\Omega, \mathbb{C}) \rightarrow H_g^1(\Omega, \mathbb{C})$  defined by

$$\begin{cases} Pu(x) = u(x) & \text{if } |u(x)| \leq 1 \\ Pu(x) = \frac{u(x)}{|u(x)|} & \text{if } |u(x)| \geq 1 \end{cases}$$

is continuous. Composing  $P, T$  and  $\Psi$ , we define  $\tilde{\Phi} : E_\varepsilon^a \rightarrow \Sigma_k(\Omega)$ ;

$$\tilde{\Phi}(u) = \Psi(T(Pu)).$$

As already noticed in [AB3], the minimizer  $u^h$  to the problem (2.2) may not be unique and moving slightly the points  $a_i$ 's, the new positions would still match the requirements of Theorem 2. Hence the assignment of  $u^h$  and the vortices for  $u^h$  require some choices, so we can not expect the map  $\tilde{\Phi}$  to be continuous. However the freedom in these choices are not too wild, and we can say that  $\tilde{\Phi}$  is “almost” a continuous map from  $E_\varepsilon^a$  to  $\Sigma_k(\Omega)$ . More precisely, we have

PROPOSITION 4 [AB3]. – Assume that  $a, \varepsilon'_0, \bar{\chi}$  are as in Theorem 3. Then for all  $\varepsilon < \varepsilon'_0, u, v \in E_\varepsilon^a$  we have

$$\|\tilde{\Phi}(u) - \tilde{\Phi}(v)\| \leq C_1 \left( |\log \varepsilon| \varepsilon^{\frac{2}{4k+1}} + \varepsilon^{\bar{\chi}} + \|u - v\|_{H_0^1(\Omega, \mathbb{C})} \right)$$

where  $C_1$  is a constant depending only on  $\Omega$  and  $g$ .

Remark. – In [AB3], Almeida-Bethuel studied the more general configuration space corresponding to the “vortices” of the map  $u^h$  for  $u \in E_\varepsilon^a$ , where  $a$  is of the form

$$\mu_\varepsilon \leq a \leq K_1(|\log \varepsilon| + 1),$$

and the map  $\tilde{\Phi}$  from  $E_\varepsilon^a$  to the configuration space. We refer reader to [AB3] for the details.

Here is the notion of  $\eta$ -almost continuity introduced in [AB3]: A map  $\Phi : X \rightarrow Y$  from a metric space  $X$  to a metric space  $Y$  is said to be  $\eta$ -almost continuous, if for all  $x \in X$  and  $\varepsilon > 0$ , there is a  $\delta$ , such that for all  $x'$  with  $d_X(x, x') < \delta$ , we have  $d_Y(\Phi(x), \Phi(x')) \leq \eta + \varepsilon$ . Proposition 4 says that the map  $\tilde{\Phi}$  is actually  $\eta$ -almost equi-continuous for  $\eta = C_1(|\log \varepsilon| \varepsilon^{\frac{2}{4k+1}} + \varepsilon^{\bar{\chi}})$ .

By Theorem 3, the image of  $\tilde{\Phi}$  lies in the set

$$\begin{aligned} &\Sigma_{k,\beta}(\Omega) \\ &= \{ \{a_1, \dots, a_k\} \in \Sigma_k(\Omega); \text{dist}(a_i, \partial\Omega) \geq \beta, \text{ and } |a_i - a_j| \geq \beta \text{ for } i \neq j \} \end{aligned}$$

which is compact in  $\Sigma_k$ . So we have

PROPOSITION 5 [AB3]. – We have an  $\eta_0$  which only depends on  $\beta$ , such that for any  $\eta \leq \eta_0$  and compact set  $W \in H_g^1(\Omega, \mathbb{C})$ , if  $\tilde{\Phi}$  is  $\eta$ -almost continuous and  $\tilde{\Phi}(W) \subset \Sigma_{k,\beta}(\Omega)$ , then there exists a continuous map  $\Phi : W \rightarrow \Sigma_k(\Omega)$  such that

$$\|\tilde{\Phi}(u) - \Phi(u)\| \leq 3\eta \quad \text{for all } u \in W.$$

### 3. PROOF OF THEOREM 1

In this section, we are going to prove Theorem 1 which is stated in §1.

Let  $K \subset \Sigma_k(\Omega)$  be a compact core, i.e.,  $K$  is compact and the natural inclusion  $i : K \rightarrow \Sigma_k(\Omega)$  is a homotopy equivalence. Actually,  $\Sigma_{k,\beta}(\Omega)$

is a compact core for sufficiently small  $\beta$ . We start with a construction of maps  $f_\varepsilon : K \rightarrow E_\varepsilon^a$ .

LEMMA 6. – *There are constants  $\varepsilon_0'' > 0$ ,  $\lambda$  and  $C_2$  such that for all  $\varepsilon \leq \varepsilon_0''$ , we can define  $f_\varepsilon : K \rightarrow E_\varepsilon^a$ , where  $a = k\pi |\log \varepsilon| + \lambda$  such that*

$$\|\tilde{\Phi} \cdot f_\varepsilon - \text{id}\| \leq \eta$$

on  $K$ , where  $\eta$  is given by

$$\eta = C_2 \left( |\log \varepsilon| \varepsilon^{\frac{2}{4k+1}} + \varepsilon^{\bar{\lambda}} \right).$$

*Proof.* – Since  $K$  is compact, we can pick  $\eta_K > 0$  such that for any  $\{a_1, \dots, a_k\} \in K$ , the balls  $B(a_i, 4\eta_K) \subset \Omega$  and are pairwise disjoint. Now once  $\varepsilon \leq 4\eta_K$ , we can construct a map  $f_\varepsilon : \Sigma_k(\Omega) \rightarrow H_g^1(\Omega, \mathbb{C})$  as follows: for any  $a = \{a_1, \dots, a_k\} \in \Sigma_k(\Omega)$ , let

$$\Omega_{\varepsilon,a} = \Omega \setminus \bigcup_{i=1}^k B(a_i, \varepsilon),$$

then on  $\Omega_{\varepsilon,a}$ ,  $f_\varepsilon(a)$  is defined by

$$f_\varepsilon(a)(z) = e^{i\varphi_{\varepsilon,a}(z)} \prod_{j=1}^k \frac{z - a_j}{|z - a_j|}$$

where the function  $\varphi_{\varepsilon,a}$  is defined on  $\Omega$  by the following equation

$$\begin{cases} \Delta \varphi_{\varepsilon,a}(z) = 0 & \text{in } \Omega \\ e^{i\varphi_{\varepsilon,a}(z)} \prod_{j=1}^k \frac{z - a_j}{|z - a_j|} = g & \text{on } \partial\Omega. \end{cases}$$

Notice that for a given  $a$  the map  $\varphi_{\varepsilon,a}$  is uniquely defined, up to an integer multiple of  $2\pi$ . In fact, we can choose this constant such that the map  $a \rightarrow e^{i\varphi_{\varepsilon,a}}$  is continuous by the standard lifting argument. On each  $B(a_i, \varepsilon)$ ,  $f_\varepsilon(a)$  is defined by

$$\begin{cases} \Delta f_\varepsilon(a) = 0 & \text{in } B(a_i, \varepsilon) \\ f_\varepsilon(a)(z) = e^{i\varphi_{\varepsilon,a}(z)} \prod_{j=1}^k \frac{z - a_j}{|z - a_j|} & \text{on } \partial B(a_i, \varepsilon). \end{cases}$$

It is then easy to check that  $f_\varepsilon$  is a continuous map from  $\Sigma_k(\Omega)$  to  $H_g^1(\Omega, \mathbb{C})$ .

Moreover we can estimate the energy  $E_\varepsilon(f_\varepsilon(a))$ . Using the same analysis as in [Section I, BBH2], we have a constant  $C$  which depends on  $\Omega$  and  $g$  only, such that

$$E_\varepsilon(f_\varepsilon(a)) \leq W_g(a_1, \dots, a_k) + k\pi|\log \varepsilon| + C.$$

Let

$$\lambda' = \sup_{a \in K} W_g(a_1, \dots, a_k),$$

it is finite by the compactness of  $K$ . So

$$E_\varepsilon(f_\varepsilon(a)) \leq k\pi|\log \varepsilon| + \lambda' + C.$$

Hence there is an  $\varepsilon_1 > 0$  such that for all  $\varepsilon \leq \varepsilon_1$ ,  $f_\varepsilon(a) \in E_\varepsilon^a$ , for  $a = \mu_\varepsilon + \lambda$  provided  $\lambda$  is chosen large enough (but independent of  $\varepsilon$ ).

Now suppose that  $\varepsilon \leq \varepsilon'_0 = \min\{\varepsilon'_0, \varepsilon_1, 4\eta_K\}$ , and denote  $f_\varepsilon(a)$  by  $f_{\varepsilon,a}$  for simplicity. Let  $a = \{a_1, \dots, a_k\}$  be given in  $K$ , and  $a' = \{a'_1, \dots, a'_k\}$  be the vortices for  $(f_{\varepsilon,a})^h$ , i.e.,  $\tilde{\Phi}(f_{\varepsilon,a}) = \{a'_1, \dots, a'_k\}$ . According to Theorem 3, on  $\Omega_{\rho,a'} = \Omega \setminus \bigcup_{i=1}^k B(a'_i, \rho)$ , we have

$$|f_{\varepsilon,a}^h(x)| \geq \frac{1}{2}, \text{ for all } x \in \Omega_{\rho,a'},$$

where  $\varepsilon^x \leq \rho \leq \varepsilon^{\bar{x}}$ . We may therefore consider on  $\tilde{\Omega} = \Omega_{\rho,a'} \setminus \bigcup_{i=1}^k B(a_i, \varepsilon)$ , the map  $\xi = \frac{f_{\varepsilon,a}^h}{|f_{\varepsilon,a}^h|} f_{\varepsilon,a}^{-1}$ .  $\xi$  takes its values in  $S^1$  and satisfies  $\xi \equiv 1$  on  $\partial\Omega$ . Moreover we have

$$|\xi - 1| \leq 4|f_{\varepsilon,a}^h - f_{\varepsilon,a}|.$$

This yields

$$\begin{aligned} \int_{\tilde{\Omega}} |\xi - 1|^2 &\leq 16 \int_{\tilde{\Omega}} |f_{\varepsilon,a}^h - f_{\varepsilon,a}|^2 \leq 32\varepsilon^{\frac{4}{4k+1}} (E_\varepsilon(f_{\varepsilon,a}) - E_\varepsilon(f_{\varepsilon,a}^h)) \\ &\leq C|\log \varepsilon| \varepsilon^{\frac{4}{4k+1}} \end{aligned} \tag{3.1}$$

for some constant  $C$  depending only on  $g, K$  and  $\Omega$ .

On the other hand, for any  $1 \leq i \leq k$ , we have

$$\deg(\xi, \partial B(a'_i, \rho)) = -\deg(\xi, \partial B(a_i, \varepsilon)) = 1.$$

So for any regular value  $y \in S^1$  of  $\xi$  and  $y \neq 1$ ,  $\xi^{-1}(y)$  is a connection between balls  $B(a_i, \varepsilon)$  and  $B(a'_i, \rho)$ . By the definition of length of minimal connection  $L$  given in (2.2), we get

$$L(a', a) - k(\rho + \varepsilon) \leq \mathcal{H}^1(\xi^{-1}(y)) \text{ for almost every } y \in S^1.$$

Let

$$N = \left\{ y \in S^1, \frac{1}{8} \leq |y - 1| \leq \frac{1}{4} \right\}$$

and take  $A = \xi^{-1}(N)$ , using the coarea formula of Federer-Fleming, we obtain

$$\begin{aligned} \int_N \mathcal{H}^1(\xi^{-1}(y)) dy &= \int_A |\nabla \xi| \\ &\leq \left( \int_A |\nabla \xi|^2 \right)^{1/2} (\text{meas } A)^{1/2}. \end{aligned} \tag{3.2}$$

By (3.1), we have

$$(\text{meas } A) \leq 64 \int_{\tilde{\Omega}} |\xi - 1|^2 \leq C |\log \varepsilon| \varepsilon^{\frac{4}{4k+1}};$$

On the other hand

$$\int_{\tilde{\Omega}} |\nabla \xi|^2 \leq 8 \left( \int_{\Omega} |\nabla f_{\varepsilon,a}^h|^2 + |\nabla f_{\varepsilon,a}|^2 \right) \leq C |\log \varepsilon|.$$

Together with (3.2) we get that

$$L(a, a') - k(\rho + \varepsilon) \leq \frac{1}{(\text{meas } N)} \int_N \mathcal{H}^1(\xi^{-1}(y)) dy \leq C |\log \varepsilon| \varepsilon^{\frac{2}{4k+1}},$$

that is the conclusion we required. □

For any  $a \in K$ , the ball  $B(a, 4\eta_K) \subset \Sigma_k(\Omega)$  with radius  $4\eta_K$ , where  $\eta_K$  is the constant in the proof of Lemma 6, is in fact isometric to a standard ball in  $\mathbb{C}^k$ . To see this, let  $\tilde{K} = \pi^{-1}(K) \subset F_k(\Omega)$ , which is also a compact core of  $F_k(\Omega)$ , and for any  $\tilde{a} \in \pi^{-1}(a)$ , the condition that  $B(a_i, 4\eta_K)$ 's are pairwise disjoint implies that the ball  $B(\tilde{a}, 4\eta_K) \subset \mathbb{C}^k$  is contained in  $F_k(\Omega)$  entirely, and  $B(a, 4\eta_K)$  is isometric to  $B(\tilde{a}, 4\eta_K)$ .

LEMMA 7. – *There is an  $\varepsilon_o$ , such that for any  $\varepsilon \leq \varepsilon_o$ , the map  $f_\varepsilon$  induces an injection*

$$f_{\varepsilon*} : \pi_1(K) \rightarrow \pi_1(E_\varepsilon^a),$$

where  $a$  is chosen by Lemma 6.

*Proof.* – The constant  $\varepsilon_0 \leq \varepsilon_0''$  is chosen such that

$$\max(C_1, C_2) \left( |\log \varepsilon_0| \varepsilon_0^{\frac{2}{4k+1}} + \varepsilon_0^{\tilde{\lambda}} \right) \leq \min\{\eta_0, \eta_K\},$$

where  $C_1$  is as in Proposition 4,  $\eta_0$  as in Proposition 5 and  $\varepsilon''_0, C_2$  and  $\eta_K$  as in Lemma 6.

For each element  $\alpha \in \pi_1(K)$ , we can choose a closed path  $c : S^1 \rightarrow K$  which representing  $\alpha$ . Now for  $\varepsilon \leq \varepsilon_0$ , if  $f_\varepsilon \cdot c : S^1 \rightarrow E_\varepsilon^a$  is null homotopic, we get a map  $\tilde{f} : D^2 \rightarrow E_\varepsilon^a$ , such that  $\tilde{f}|_{\partial D^2} = f_\varepsilon \cdot c$ . By Proposition 5, on the compact set  $\tilde{f}(D^2) \subset E_\varepsilon^a$ , we can define a continuous map  $\Phi : \tilde{f}(D^2) \rightarrow \Sigma_k(\Omega)$ , such that for any  $u \in \tilde{f}(D^2)$ ,  $\|\Phi(u) - \tilde{\Phi}(u)\| < 3\eta_K$ . The map  $\Phi \cdot \tilde{f}|_{\partial D^2} = \Phi \cdot f_\varepsilon \cdot c : S^1 \rightarrow \Sigma_k(\Omega)$  is null homotopic. On the other hand, by Lemma 6,

$$\|\Phi \cdot f_\varepsilon \cdot c(t) - c(t)\| \leq \|\Phi \cdot f_\varepsilon \cdot c(t) - \tilde{\Phi} \cdot f_\varepsilon \cdot c(t)\| + \|\tilde{\Phi} \cdot f_\varepsilon \cdot c(t) - c(t)\| < 4\eta_K.$$

Then we can find a unique minimum geodesic in  $\Sigma_k(\Omega)$  connecting  $\Phi \cdot f_\varepsilon \cdot c(t)$  and  $c(t)$ . This implies that  $\Phi \cdot f_\varepsilon \cdot c$  is homotopic to  $c$ . So  $\alpha$  is a trivial element in  $\pi_1(K)$ , and this means that  $f_{\varepsilon^*}$  is injective.  $\square$

Since  $\pi_* : \pi_1(\tilde{K}) \rightarrow \pi_1(K)$  and  $f_{\varepsilon^*} : \pi_1(K) \rightarrow \pi_1(E_\varepsilon^a)$  are injective, so is  $f_{\varepsilon^*} \cdot \pi_* : \pi_1(\tilde{K}) \rightarrow \pi_1(E_\varepsilon^a)$ . Consider a covering space  $p : \tilde{E}_\varepsilon^a \rightarrow E_\varepsilon^a$  corresponding to the group  $f_{\varepsilon^*} \cdot \pi_*(\pi_1(\tilde{K})) \subset \pi_1(E_\varepsilon^a)$ , the map  $f_\varepsilon \cdot \pi : \tilde{K} \rightarrow E_\varepsilon^a$  can be lift to a map  $\tilde{f} : \tilde{K} \rightarrow \tilde{E}_\varepsilon^a$  such that the following diagram commutes

$$\begin{array}{ccc} \tilde{K} & \longrightarrow & \tilde{E}_\varepsilon^a \\ \downarrow & & \downarrow \\ K & \longrightarrow & E_\varepsilon^a. \end{array}$$

LEMMA 8. – *The map  $\tilde{f}$  induces maps  $\tilde{f}_* : H_p(\tilde{K}) \rightarrow H_p(\tilde{E}_\varepsilon^a)$  on the homology groups which are injective for all  $p$ .*

*Proof.* – The argument here goes in the same fashion as the proof of Lemma 7. Consider a singular cycle  $c \in Z_p(\tilde{K})$  such that  $\tilde{f}_*([c]) = 0$  in  $H_p(\tilde{E}_\varepsilon^a)$ . This means that we have a  $p + 1$ -chain  $c' \in C_{p+1}(\tilde{E}_\varepsilon^a)$  and  $\partial c' = \tilde{f}_*(c)$ . The set  $W = \tilde{f}(\tilde{K}) \cup \text{support}(c')$  is compact in  $\tilde{E}_\varepsilon^a$ . Then we define a continues map  $\Phi_1 : p(W) \rightarrow \Sigma_k(\Omega)$  such that for any  $u \in p(W)$ ,  $\|\Phi_1(u) - \tilde{\Phi}(u)\| < 3\eta_K$ .

Notice that  $\|\Phi_1 \cdot f_\varepsilon - \text{id}\| < 4\eta_K$ , as before, we have  $\Phi_{1*} \cdot f_{\varepsilon^*} = \text{id}$ . This implies that  $\Phi_{1*} \cdot \pi_*(\pi_1(W)) \subset \Phi_{1*} \cdot f_{\varepsilon^*} \cdot \pi_*(\pi_1(\tilde{K})) = \pi_*(\pi_1(\tilde{K}))$ . So we can lift  $\Phi_1 \cdot p : W \rightarrow \Sigma_k(\Omega)$  to  $\tilde{\Phi}_1 : W \rightarrow \tilde{F}_k(\Omega)$ .

In fact, we can make  $\|\tilde{\Phi}_1 \cdot \tilde{f} - \text{id}\| < 4\eta_K$ . Since  $\|\Phi_1 \cdot p \cdot \tilde{f} - \pi\| < 4\eta_K$ , there is a homotopy  $H_t$  such that  $H_0 = \pi$  and  $H_1 = \Phi_1 \cdot p \cdot \tilde{f}$ . Lift this homotopy to a homotopy  $\tilde{H}_t$  with  $\|\tilde{H}_0 - \tilde{H}_1\| < 4\eta_K$  and  $\tilde{H}_0 = \text{id}_{\tilde{K}}$ . Define  $\tilde{\Phi}_2 : \tilde{f}(\tilde{K}) \rightarrow \tilde{F}_k(\Omega)$  by  $\tilde{\Phi}_2(\tilde{f}(a)) = \tilde{H}_1(a)$ . Note that

$$\pi \cdot \tilde{\Phi}_2 = \Phi_1 \cdot p = \pi \cdot \tilde{\Phi}_1|_{\tilde{f}(\tilde{K})}.$$

$\tilde{\Phi}_2$  and  $\tilde{\Phi}_1$  differ by a deck transformation, i.e., there is an elements  $\sigma \in S^k$ , such that

$$\tilde{\Phi}_2 = \sigma \cdot \tilde{\Phi}_1|_{\tilde{f}(\tilde{K})}.$$

Replace  $\tilde{\Phi}_1$  by  $\sigma \cdot \tilde{\Phi}_1$ , which is also a lifting of  $\Phi_1 \cdot p : W \rightarrow \Sigma_k(\Omega)$  and  $\|\sigma \cdot \tilde{\Phi}_1 \cdot \tilde{f} - \text{id}\| < 4\eta_K$ . The new lifting will still denoted by  $\tilde{\Phi}_1$ .

Now  $\tilde{\Phi}_1$  maps the chain  $c'$  into a chain in  $C_{p+1}(F_k(\Omega))$ , and  $\partial\tilde{\Phi}_1(c') = \tilde{\Phi}_1(\partial c') = \tilde{\Phi}_1 \cdot \tilde{f}_*(c)$ . We get that  $\tilde{\Phi}_1 \cdot \tilde{f}_*(c)$  is a boundary in  $C_p(F_k(\Omega))$ . On the other hand,  $\tilde{\Phi}_1 \cdot \tilde{f}$  is homotopic to the natural inclusion  $i : \tilde{K} \rightarrow F_k(\Omega)$ . So  $c$  is homologous to  $\tilde{\Phi}_1 \cdot \tilde{f}_*(c)$ , and  $c$  is null homologous as well. This shows that  $\tilde{f}_*$  is injective.  $\square$

The lemma allows us to estimate the category of  $\tilde{E}_\varepsilon^a$ .

COROLLARY 9. – *The category  $\text{cat}(\tilde{E}_\varepsilon^a)$  of  $\tilde{E}_\varepsilon^a$  is at least  $k$ .*

*Proof.* – By Lemma 8, the map  $f^* : H^*(\tilde{E}_\varepsilon^a) \rightarrow H^*(\tilde{K})$  between cohomology rings are surjective, and this implies that the cuplength of  $\tilde{E}_\varepsilon^a$  is at least the cuplength of  $\tilde{K}$ , which is the same as the cuplength of  $F_k(\Omega)$ . By Lemma 2, the cuplength of  $\tilde{E}_\varepsilon^a$  is at least  $k - 1$ . Finally, according to [BG], the category  $\text{cat}(\tilde{E}_\varepsilon^a)$  of  $\tilde{E}_\varepsilon^a$  is at least the cuplength of  $\tilde{E}_\varepsilon^a$  plus one. This completes the proof.  $\square$

Now we are in the position to complete the proof of Theorem 1. The Lusternik-Schnirelman minimax theorem we will use is the following

THEOREM 10. – *Suppose  $F$  is a  $C^2$  non-negative functional defined on a smooth Hilbert manifold  $M$  such that*

- i) the backwards gradient flow is complete;*
- ii)  $F$  satisfies the following weak Palais-Smale condition: if we have a sequence  $\{u_n\}$  in  $M$  such that  $F(u_n) \rightarrow c$  and  $\|\nabla F(u_n)\| \rightarrow 0$  as  $n \rightarrow \infty$ , then  $c$  is a critical value;*
- iii)  $\text{cat}M = k$ .*

*Then we have either  $F$  has at least  $k$  distinct critical values in  $[0, a]$  or the dimension of the critical set of  $F$  is at least 1.*

The proof is standard, we refer reader to [Pa].

*Proof of Theorem 1.* – Now we want to apply Theorem 10 to the positive functional  $\tilde{E}_\varepsilon = E_\varepsilon \cdot p : \tilde{E}_\varepsilon^a \rightarrow \mathbb{R}$ . Notice that  $\tilde{E}_\varepsilon$  and  $E_\varepsilon$  have the same critical values and critical sets of the two functionals have the same dimension. If all three conditions in the theorem hold, both conclusions will imply that  $E_\varepsilon$  has at least  $k$  critical points on  $E_\varepsilon^a$ .

We now check the three conditions in Theorem 10. First, the backwards gradient flow of  $\tilde{E}_\varepsilon$  is a lift of the backwards flow of  $E_\varepsilon$ , so it is

complete. Second, let  $\{u_n\}$  be a sequence in  $\tilde{E}_\varepsilon^a$  such that  $\tilde{E}_\varepsilon(u_n) \rightarrow c$  and  $\|\nabla \tilde{E}_\varepsilon(u_n)\| \rightarrow 0$  as  $n \rightarrow \infty$ , then  $E_\varepsilon(p(u_n)) \rightarrow c$  and  $\|\nabla E_\varepsilon(p(u_n))\| \rightarrow 0$ . We know that  $E_\varepsilon$  satisfies Palais-Smale condition, so  $p(u_n)$  has a subsequence converges to a critical point. This shows that  $c$  is a critical value of  $E_\varepsilon$  and then it is a critical value of  $\tilde{E}_\varepsilon$  as well. Finally,  $\text{cat} \tilde{E}_\varepsilon^a \geq k$  is the conclusion of Corollary 9. So we now can conclude that  $E_\varepsilon$  has at least  $k$  critical points on  $E_\varepsilon^a$ .

Outside of  $E_\varepsilon^a$ ,  $E_\varepsilon$  has at least another critical point, since  $H_g^1(\Omega, \mathbb{C})$  is contractible, but  $E_\varepsilon^a$  is not (if  $k \geq 2$ ). So totally  $E_\varepsilon$  will have at least  $k + 1$  critical points on  $H_g^1(\Omega, \mathbb{C})$ .  $\square$

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