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## **Geometric restrictions for the existence of viscosity solutions**

by

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ABSTRACT. – We study the Hamilton-Jacobi equation

$$\begin{cases} F(Du) &= 0 & \text{a.e. in } \Omega \\ u &= \varphi & \text{on } \partial\Omega \end{cases} \quad (0.1)$$

where  $F : \mathbb{R}^N \longrightarrow \mathbb{R}$  is not necessarily convex. When  $\Omega$  is a convex set, under technical assumptions our first main result gives a necessary and

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sufficient condition on the geometry of  $\Omega$  and on  $D\varphi$  for (0.1) to admit a Lipschitz *viscosity solution*. When we drop the convexity assumption on  $\Omega$ , and relax technical assumptions our second main result uses the viability theory to give a necessary condition on the geometry of  $\Omega$  and on  $D\varphi$  for (0.1) to admit a Lipschitz *viscosity solution*. © Elsevier, Paris

RÉSUMÉ. – Nous étudions l'équation de Hamilton-Jacobi suivante

$$\begin{cases} F(Du) = 0 & \text{p.p. dans } \Omega \\ u = \varphi & \text{sur } \partial\Omega \end{cases} \quad (0.2)$$

où  $F : \mathbb{R}^N \longrightarrow \mathbb{R}$  n'est pas nécessairement convexe. Lorsque  $\Omega$  est un ensemble convexe, notre premier résultat donne une condition nécessaire et suffisante sur la géométrie du domaine  $\Omega$  et sur  $D\varphi$  afin que (0.2) admette une *solution de viscosité lipschitzienne*. Si on enlève la condition de convexité du domaine  $\Omega$ , notre second résultat permet, à l'aide du théorème de viabilité, de donner une condition nécessaire sur la géométrie du domaine  $\Omega$  et sur  $D\varphi$  afin que (0.2) admette une *solution de viscosité lipschitzienne*. © Elsevier, Paris

## 1. INTRODUCTION

In this article we give a necessary and sufficient geometric condition for the following Hamilton-Jacobi equation

$$\begin{cases} F(Du) = 0 & \text{a.e. in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

to admit a  $W^{1,\infty}(\Omega)$  *viscosity solution*. Here,  $\Omega \subset \mathbb{R}^N$  is a bounded, open set,  $F : \mathbb{R}^N \longrightarrow \mathbb{R}$  is continuous and  $\varphi \in C^1(\overline{\Omega})$ . We prove that existence of *viscosity solutions*<sup>1</sup> depends strongly on geometric compatibilities of the set of zeroes of  $F$ , of  $\varphi$  and of  $\Omega$ , however it does not depend on the smoothness of the data.

The Hamilton-Jacobi equations are classically derived from the calculus of variations, and the interest of finding *viscosity solutions* (notion introduced by M.G. Crandall-P.L. Lions [8]) of problem (1.1) is well-known

<sup>1</sup> Equation (1.1) may admit only continuous or even discontinuous *viscosity solutions* (see [4]). We are here interested only in  $W^{1,\infty}$  solutions.

in optimal control and differential games theory (c.f. M. Bardi-I. Capuzzo Dolcetta [3], G. Barles [4], W.H. Fleming-H.M. Soner [13] and P.L. Lions [17]).

It has recently been shown by B. Dacorogna-P. Marcellini in [9], [10] and [11] (cf. also A. Bressan and F. Flores [6]) that (1.1) has infinitely (even  $G_\delta$  dense) many solutions  $u \in W^{1,\infty}(\Omega)$  provided the compatibility condition

$$D\varphi(x) \in \text{int}(\text{conv}(Z_F)) \cup Z_F, \quad \text{for every } x \in \Omega \quad (1.2)$$

holds, where

$$Z_F = \{\xi \in \mathbb{R}^N : F(\xi) = 0\}, \quad (1.3)$$

and  $\text{conv}(Z_F)$  denotes the convex hull of  $Z_F$  and  $\text{int}(\text{conv}(Z_F))$  its interior. In fact (1.2) is, in some sense, almost a necessary condition for the existence of  $W^{1,\infty}(\Omega)$  solution of (1.1). The classical existence results on  $W^{1,\infty}(\Omega)$  viscosity solution of (1.1) require stronger assumptions than (1.2) (see M. Bardi-I. Capuzzo Dolcetta, [3], G. Barles [4], W.H. Fleming-H.M. Soner [13] and P.L. Lions [17]).

Here we wish to investigate the question of existence of  $W^{1,\infty}(\Omega)$  viscosity solution under the sole assumption (1.2). As mentioned above, the answer will be, in general, that such solutions do not exist unless strong geometric restrictions on the set  $Z_F$ , on  $\Omega$  and on  $\varphi$  are assumed.

To understand better our results one should keep in mind the following example.

EXAMPLE 1.1. – *Let*

$$F(\xi_1, \xi_2) = -(\xi_1^2 - 1)^2 - (\xi_2^2 - 1)^2 \quad (1.4)$$

(Note that  $F$  is a polynomial of degree 4). Clearly,

$$\begin{cases} Z_F = \{\xi \in \mathbb{R}^2 : \xi_1^2 = \xi_2^2 = 1\} \\ \text{conv}(Z_F) = \{\xi \in \mathbb{R}^2 : |\xi_1| \leq 1, |\xi_2| \leq 1\} \\ \quad = \{\xi \in \mathbb{R}^2 : |\xi|_\infty = \max\{|\xi_1|, |\xi_2|\} \leq 1\} \\ Z_F \subset \partial(\text{conv}(Z_F)) \text{ and } Z_F \neq \partial(\text{conv}(Z_F)). \end{cases} \quad (1.5)$$

Our article will be divided into two parts, obtaining essentially the same results. The first one (c.f. Section 2) will compare the Dirichlet problem (1.1)

with an appropriate problem involving a certain gauge. The second one (c.f. Section 3) will use the viability approach.

We start by describing the first approach. We will assume there that  $\Omega$  is convex. To the set  $\text{conv}(Z_F)$  we associate its gauge, i.e.

$$\rho(\xi) = \inf \{ \lambda > 0 : \xi \in \lambda \text{conv}(Z_F) \}. \quad (1.6)$$

(In the example  $\rho(\xi) = |\xi|_\infty$ ).

The  $W^{1,\infty}(\Omega)$  viscosity solutions of (1.1) will then be compared to those of

$$\begin{cases} \rho(Du) &= 1 & \text{a.e. in } \Omega \\ u &= \varphi & \text{on } \partial\Omega. \end{cases} \quad (1.7)$$

The compatibility condition on  $\varphi$  will then be

$$D\varphi(x) \in \text{int}(\text{conv}(Z_F)) , \forall x \in \overline{\Omega} \Leftrightarrow \rho(D\varphi(x)) < 1 , \forall x \in \overline{\Omega}.$$

We will first show (c.f. Theorem 2.2) that if  $Z_F \subset \partial(\text{conv}(Z_F))$  and  $Z_F$  is bounded, then any  $W^{1,\infty}(\Omega)$  viscosity solution of (1.1) is a viscosity solution of (1.7). However by classical results (c.f. S.H. Benton [5], A. Douglis [12], S.N. Kruzkov [16], P.L. Lions [17] and the bibliography there) we know that the viscosity solution of (1.7) is given by

$$u(x) = \inf_{y \in \partial\Omega} \{ \varphi(y) + \rho^o(x - y) \}, \quad (1.8)$$

where  $\rho^o$  is the polar of  $\rho$ , i.e.

$$\rho^o(\xi^*) = \sup_{\rho(\xi) \neq 0} \left\{ \frac{\langle \xi^*, \xi \rangle}{\rho(\xi)} \right\}. \quad (1.9)$$

(In the example  $\rho^o(\xi^*) = |\xi^*|_1 = |\xi_1^*| + |\xi_2^*|$ .)

The main result of Section 2 (c.f. Theorem 2.6, c.f. also Theorem 3.2) uses the above representation formula to give a *necessary and sufficient condition* for existence of  $W^{1,\infty}(\Omega)$  viscosity solutions of (1.1). This geometrical condition can be roughly stated as  $\forall y \in \partial\Omega$  where the inward unit normal,  $\nu(y)$ , is uniquely defined (recall that here  $\Omega$  is convex and therefore this is the case for almost every  $y \in \partial\Omega$ ) there exists  $\lambda(y) > 0$  such that

$$D\varphi(y) + \lambda(y)\nu(y) \in Z_F. \quad (1.10)$$

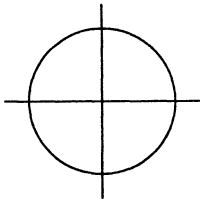
In particular if  $\varphi \equiv 0$ , we find that  $\lambda(y) = \frac{1}{\rho(\nu(y))}$  and therefore the necessary and sufficient condition reads as

$$\frac{\nu(y)}{\rho(\nu(y))} \in Z_F. \quad (1.11)$$

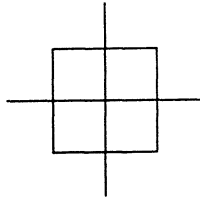
In the above example  $Z_F = \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$ , therefore the only convex  $\Omega$ , which allows for  $W^{1,\infty}(\Omega)$  viscosity solution of

$$\begin{cases} F(Du) = 0 & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

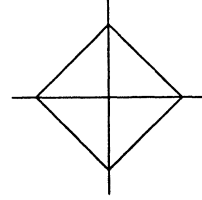
are rectangles whose normals are in  $Z_F$ . In particular for any smooth domain (such as the unit disk), (1.1) has no  $W^{1,\infty}(\Omega)$  viscosity solution, while by the result of B. Dacorogna-P. Marcellini in [9], [10] and [11], (since  $0 \in \text{int}(\text{conv}(Z_F))$ ) the existence of general  $W^{1,\infty}(\Omega)$  solutions is guaranteed. Note that in the above example with  $\Omega$  the unit disk,  $F$  and  $\varphi$  are analytic and therefore existence of  $W^{1,\infty}(\Omega)$  viscosity solutions do not depend on the smoothness of the data.



No viscosity solution



No viscosity solution



One viscosity solution

It is interesting to note that if  $F : \mathbb{R}^N \rightarrow \mathbb{R}$  is convex and coercive (such as the eikonal equation), as in the classical literature, then  $\partial(\text{conv}(Z_F)) \subset Z_F$ . Therefore the above necessary and sufficient condition does not impose any restriction on the set  $\Omega$ . However as soon as non convex  $F$  are considered, such as in the example, (1.10) drastically restricts the geometry of the set  $\Omega$ , if existence of  $W^{1,\infty}(\Omega)$  viscosity solution is to be ensured.

In Section 3 the basic ingredient for proving such a result is the viability Theorem (Theorem 3.3.2 of [2]). This Theorem gives an equivalence between the geometry of a closed set and the existence of solutions of some differential inclusion remaining in this set. The idea of putting together viscosity solutions and the viability Theorem is due to H. Frankowska in [15].

The main result of this section (c.f. Theorem 3.1, c.f. also Corollary 2.8) will show that if

$$\partial(\text{conv}(Z_F)) \setminus Z_F \neq \emptyset \quad (1.12)$$

then we can always find an affine function  $\varphi$  with  $D\varphi \in \text{int}(\text{conv}(Z_F))$  so that (1.1) has no  $W^{1,\infty}(\Omega)$  viscosity solution.

The advantage of the second approach is that it will require weaker assumptions on  $F$  and on  $\Omega$  than the first one. However the first approach will give more precise information since we will use the explicit formula for the viscosity solution of (1.7).

Some technical results are gathered in two appendixes.

## 2. COMPARISON WITH THE SOLUTION ASSOCIATED TO THE GAUGE

Throughout this section we assume that  $F : \mathbb{R}^N \longrightarrow \mathbb{R}$  is continuous and that

- (H1)  $Z_F \subset \partial(\text{conv}(Z_F))$ .  
We recall that  $Z_F = \{\xi \in \mathbb{R}^N : F(\xi) = 0\}$ .
- (H2)  $Z_F$  is bounded.
- (H3)  $D\varphi(x) \in \text{int}(\text{conv}(Z_F))$ ,  $\forall x \in \overline{\Omega}$ .

In addition we assume that the interior of the convex hull of  $Z_F$  is nonempty, i.e.

$$\text{int}(\text{conv}(Z_F)) \neq \emptyset \quad (2.1)$$

REMARKS 2.1.

(i) In light of (2.1) we may assume without loss of generality that  $0 \in \text{int}(\text{conv}(Z_F))$ , since up to a translation this always holds.

(ii) Observe that  $\text{int}(\text{conv}(Z_F)) \neq \emptyset$  is necessary for (H3) to make sense.

(iii) Recall that (H3) (without the interior) is, in some sense, necessary for existence of  $W^{1,\infty}(\Omega)$  solutions (c.f. P.L. Lions [17]).

(iv) It is well-known (c.f. [18]) that the following properties hold:

- $\rho$  is convex, homogeneous of degree one and  $\rho^{\circ\circ} = \rho$ .
- $\text{conv}(Z_F) = \{z \in \mathbb{R}^N : \rho(z) \leq 1\}$ .
- $\partial(\text{conv}(Z_F)) = \{z \in \mathbb{R}^N : \rho(z) = 1\}$ .

- $\rho(z) > 0$  for every  $z \neq 0$ .

(v) Since  $Z_F \subset \partial(\text{conv}(Z_F))$ , the function  $F$  has a definite sign in  $\text{int}(\text{conv}(Z_F))$ . We will assume, without loss of generality, that

$$F(\xi) < 0, \quad (2.2)$$

for every  $\xi \in \text{int}(\text{conv}(Z_F))$ . Otherwise in the following analysis we should replace  $F$  by  $-F$ .

Our first result compares viscosity solutions of (1.1) and those of (1.7).

**THEOREM 2.2.** – Let  $\Omega \subset \mathbb{R}^N$  be a bounded open set, let  $F$  and  $\varphi$  satisfy (H1), (H2), (H3) and (2.2). Then any  $W^{1,\infty}(\Omega)$  viscosity solution of (1.1) is also a  $W^{1,\infty}(\Omega)$  viscosity solution of (1.7). Conversely if, in addition  $F > 0$  outside  $\text{conv}(Z_F)$  then a  $W^{1,\infty}(\Omega)$  viscosity solution of (1.7) is also a  $W^{1,\infty}(\Omega)$  viscosity solution of (1.1).

**REMARK 2.3.** – In the converse part of the above theorem the facts that  $F$  is continuous,  $F < 0$  in  $\text{int}(\text{conv}(Z_F))$ , and  $F > 0$  outside  $\text{conv}(Z_F)$  implies that

$$\partial(\text{conv}(Z_F)) = Z_F.$$

We recall the definition of subdifferential and superdifferential of functions (c.f. M. Bardi-I. Capuzzo Dolcetta [3], G. Barles [4] or W.H. Fleming-H.M. Soner [13]).

**DEFINITION 2.4.** – Let  $u \in C(\Omega)$ , we define for  $x \in \Omega$  the following sets,

$$D^+u(x) = \left\{ p \in \mathbb{R}^N : \limsup_{y \rightarrow x, y \in \Omega} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|x - y|} \leq 0 \right\},$$

$$D^-u(x) = \left\{ p \in \mathbb{R}^N : \liminf_{y \rightarrow x, y \in \Omega} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|x - y|} \geq 0 \right\}.$$

$D^+u(x)$  ( $D^-u(x)$ ) is called superdifferential (subdifferential) of  $u$  at  $x$ .

We recall a useful lemma stated in G. Barles [4].

**LEMMA 2.5.**

(i)  $u \in C(\Omega)$  is a viscosity subsolution of  $F(D(u(x))) = 0$  in  $\Omega$  if and only if,  $F(p) \leq 0$  for every  $x \in \Omega$ ,  $\forall p \in D^+u(x)$ .

(ii)  $u \in C(\Omega)$  is a viscosity supersolution of  $F(D(u(x))) = 0$  in  $\Omega$  if and only if,  $F(p) \geq 0$  for every  $x \in \Omega$ ,  $\forall p \in D^-u(x)$ .



We now give the proof of our first theorem.

*Proof of Theorem 2.2.*

1. Let  $u \in W^{1,\infty}(\Omega)$  be a *viscosity* solution of (1.1).

(i) We first show that  $u$  is a *viscosity* supersolution of (1.7). Since  $u$  is a *viscosity* supersolution of (1.1), then in light of Lemma 4.2 and 2.5 we have for every  $x \in \Omega$ , and every  $p \in D^-u(x)$ ,

$$p \in \text{conv}(Z_F) \text{ and } F(p) \geq 0. \quad (2.3)$$

Combining (2.2), (2.3) and (H1), we obtain that  $p \in \partial(\text{conv}(Z_F))$ , and so,  $\rho(p) - 1 = 0$ . Hence, by Lemma 2.5,  $u$  is a *viscosity* supersolution of (1.7).

(ii) We next show that  $u$  is a *viscosity* subsolution of (1.7). Since  $u$  is a *viscosity* subsolution of (1.1), then for every  $x \in \Omega$ , and  $p \in D^+u(x)$ , we have by Lemma 4.2,  $p \in \text{conv}(Z_F)$  and so,  $\rho(p) - 1 \leq 0$ . We therefore deduce that  $u$  is a *viscosity* subsolution of (1.7).

Combining (i) and (ii) we have that  $u \in W^{1,\infty}(\Omega)$ , is a *viscosity* solution of (1.7).

2. We show that  $u \in W^{1,\infty}(\Omega)$ , the *viscosity* solution of (1.7) defined by (1.8), is also a *viscosity* solution of (1.1).

(iii) We recall that

$$F(\xi) > 0, \quad (2.4)$$

for all  $\xi \in \mathbb{R}^N \setminus \text{conv}(Z_F)$ . Since  $u$  is a *viscosity* supersolution of (1.7), then for every  $x \in \Omega$ , and  $p \in D^-u(x)$ , we have that  $\rho(p) - 1 \geq 0$ , i.e.  $p \in \mathbb{R}^N \setminus \text{int}(\text{conv}(Z_F))$ . From (2.4), it follows that  $F(p) \geq 0$  and thus  $u$  is a *viscosity* supersolution of (1.1).

(iv) Since  $u$  is a *viscosity* subsolution of (1.7), we have for every  $x \in \Omega$ , and  $p \in D^+u(x)$ , that  $\rho(p) - 1 \leq 0$ , i.e.  $p \in \text{conv}(Z_F)$  and then  $F(p) \leq 0$ . Thus  $u$  is a *viscosity* subsolution of (1.1).

Combining (iii) and (iv) we conclude that  $u$  is a *viscosity* solution of (1.1).  $\square$

We now state the main result of this section (see also Theorem 3.4).

**THEOREM 2.6.** – *Let  $F$  and  $\varphi$  satisfy (H1), (H2), (H3) and (2.2). If  $\Omega$  is bounded, open and convex and  $\varphi \in C^1(\overline{\Omega})$ , then the two following conditions are equivalent*

1. *There exists  $u \in W^{1,\infty}(\Omega)$  viscosity solution of (1.1).*

2. For every  $y \in \partial\Omega$ , where the unit inward normal in  $y$  (denoted  $\nu(y)$ ) exists, there exists a unique  $\lambda_0(y) > 0$  such that

$$\begin{cases} D\varphi(y) + \lambda_0(y)\nu(y) \in Z_F \\ \rho(D\varphi(y) + \lambda_0(y)\nu(y)) = 1. \end{cases} \quad (2.5)$$

Before proving Theorem 2.6, we make few remarks, mention an immediate corollary and prove a lemma.

REMARKS 2.7. – (i) By  $\nu(y)$ , the unit inward normal at  $y$ , exists we mean that it is uniquely defined there. Since  $\Omega$  is convex, then this is the case for almost every  $y \in \partial\Omega$ .

- (ii) In particular if  $\varphi \equiv 0$ , then

$$\lambda_0(y) = \frac{1}{\rho(\nu(y))}$$

and so, the necessary and sufficient condition becomes

$$\frac{\nu(y)}{\rho(\nu(y))} \in Z_F.$$

(iii) If  $F$  is convex and coercive, then (2.5) is always satisfied and therefore no restriction on the geometry of  $\Omega$  is imposed by our theorem (as in the classical theory of M.G. Crandall-P.L. Lions [8]).

COROLLARY 2.8. – Let  $\Omega \subset \mathbb{R}^N$  be a bounded open convex set, let  $F : \mathbb{R}^N \rightarrow \mathbb{R}$  be continuous and such that

$$Z_F \subset \partial(\text{conv}(Z_F)) \text{ and } Z_F \neq \partial(\text{conv}(Z_F)).$$

Then there exists  $\varphi$  affine with  $D\varphi(x) \in \text{int}(\text{conv}(Z_F))$ ,  $\forall x \in \overline{\Omega}$  such that (1.1) has no  $W^{1,\infty}(\Omega)$  viscosity solutions.

In section 3 we will strengthen this corollary by assuming only that  $\partial(\text{conv}(Z_F)) \setminus Z_F \neq \emptyset$ .

We next state a lemma which plays a crucial role in the proof of Theorem 2.6.

LEMMA 2.9. – Let  $\Omega$  be bounded open and convex and  $\varphi \in C^1(\overline{\Omega})$  with  $\rho(D\varphi(x)) < 1$ ,  $\forall x \in \overline{\Omega}$ . Let  $u$  be defined by

$$u(x) = \inf_{y \in \partial\Omega} \{\varphi(y) + \rho^o(x - y)\}, \quad x \in \overline{\Omega}.$$

Let  $y(x) \in \partial\Omega$  be such that  $u(x) = \varphi(y(x)) + \rho^o(x - y(x))$ . The two following properties then hold

(i) If  $D^-u(x)$  is nonempty then the inward unit normal  $\nu(y(x))$  at  $y(x)$  exists (i.e. is uniquely defined).

(ii) Furthermore if  $p \in D^-u(x)$  then there exists  $\lambda_0(y(x)) > 0$  such that,  $p = D\varphi(y(x)) + \lambda_0(y(x))\nu(y(x))$ , where  $\nu(y(x))$  is the unit inward normal to  $\partial\Omega$  at  $y$ .

*Proof.*

1. Let

$$I(x) = \{z \in \partial\Omega : u(x) = \varphi(z) + \rho^o(x - z)\}.$$

If  $p \in D^-u(x)$  then for every compact set  $K \subset \mathbb{R}^N$  and  $h > 0$ , we have

$$u(x + h\omega) - u(x) \geq \langle p, h\omega \rangle + \epsilon(h), \quad \omega \in K \quad (2.6)$$

where  $\epsilon$  satisfies  $\liminf_{h \rightarrow 0} \frac{\epsilon(h)}{h} = 0$ .

In the sequel we assume without loss of generality that

$$0 \in \text{int}(\Omega), \quad (2.7)$$

since, by a change of variables (2.7) holds. Let  $\rho_\Omega$  be the gauge associated to  $\Omega$  i.e.

$$\rho_\Omega(z) = \inf \{\lambda > 0 : z \in \lambda\Omega\}.$$

We recall that

$$\partial\Omega = \{z \in \mathbb{R}^N : \rho_\Omega(z) = 1\}, \quad (2.8)$$

and

$$\Omega = \{z \in \mathbb{R}^N : \rho_\Omega(z) < 1\}. \quad (2.9)$$

Now, let  $x_0 \in \Omega$ , let  $y_0 \in I(x_0)$  and let  $q_0 \in \partial\rho_\Omega(y_0)$  (the subdifferential of  $\rho_\Omega$  at  $y_0$ , in the sense of convex analysis, see R.T. Rockafellar [18]). Since  $\rho_\Omega$  is a convex function, we have  $\partial\rho_\Omega(y_0) = D^-\rho_\Omega(y_0)$  (see [4]). We have

$$\rho_\Omega(z) \geq \rho_\Omega(y_0) + \langle q_0, z - y_0 \rangle, \quad z \in \mathbb{R}^N. \quad (2.10)$$

Note that  $q_0 \neq 0$  since otherwise we would have  $0 \in \partial\rho_\Omega(y_0)$  and so,  $y_0$  would be a minimizer for  $\rho_\Omega$  whereas  $\rho_\Omega(y_0) > \rho_\Omega(0) = 0$ . Define the hyperplane touching  $\partial\Omega$  at  $y_0$  and normal to  $q_0$ ,

$$P_0 = \{z \in \mathbb{R}^N : \langle q_0, z - y_0 \rangle = 0\},$$

and the barrier function

$$v(z) = \inf_{y \in P_0} \{\varphi(y) + \rho^o(x - y)\}.$$

**2. Claim 1.** – We have  $u \leq v$  on  $\Omega$  and  $u(x_0) = v(x_0)$ .

Indeed, for  $x \in \Omega$ , let  $y_1(x) \in P_0$  be such that

$$v(x) = \varphi(y_1(x)) + \rho^o(x - y_1(x)),$$

and let

$$z_t = (1 - t)x + ty_1(x), \quad t \in [0, 1].$$

In light of (2.8), (2.9), (2.10), and the fact that  $y_1(x) \in P_0$ , we have

$$\rho_\Omega(z_0) = \rho_\Omega(x) < 1 \quad (2.11)$$

and

$$\rho_\Omega(z_1) = \rho_\Omega(y_1(x)) \geq 1. \quad (2.12)$$

Using (2.8), (2.11), and (2.12) we conclude that there exists  $\mu \in (0, 1]$  such that

$$z_\mu \in \partial\Omega.$$

Using the homogeneity of  $\rho^o$  we obtain that

$$\begin{aligned} \rho^o(x - y_1(x)) &= \mu\rho^o(x - y_1(x)) + (1 - \mu)\rho^o(x - y_1(x)) \\ &= \rho^o(x - z_\mu) + \rho^o(z_\mu - y_1(x)). \end{aligned}$$

We therefore deduce that

$$\begin{aligned} v(x) &= \varphi(y_1(x)) + \rho^o(x - y_1(x)) \\ &= \varphi(y_1(x)) + \rho^o(x - z_\mu) + \rho^o(z_\mu - y_1(x)). \end{aligned} \quad (2.13)$$

As  $\rho(D\varphi) \leq 1$  we have (see Lemma 4.1)

$$\varphi(z_\mu) - \varphi(y_1(x)) \leq \rho^o(z_\mu - y_1(x)). \quad (2.14)$$

From (2.14) and the definition of  $u$ , we obtain

$$v(x) \geq \varphi(z_\mu) + \rho^o(x - z_\mu) \geq u(x).$$

So we have  $v(x) \geq u(x)$ . Observe also that  $v(x_0) \leq u(x_0)$  and so,  $v(x_0) = u(x_0)$ . This concludes the proof of Claim 1.

**3. Claim 2.** – We have  $p \in D^-v(x_0)$ .

Indeed, in light of Claim 1 and (2.6) we have

$$v(x_0 + hd) - v(x_0) - \langle p, hd \rangle \geq u(x_0 + hd) - u(x_0) - \langle p, hd \rangle \geq \epsilon(h), \quad (2.15)$$

for every  $d$  in a compact set, and so,

$$p \in D^-v(x_0).$$

**4. Claim 3.** –  $p - D\varphi(y_0)$  is parallel to  $q_0$  (recall that  $q_0 \neq 0$ ).

Let  $q_1, \dots, q_{N-1}$  be such that  $\{q_0, \dots, q_{N-1}\}$  is a set of orthogonal vectors. Using the definition of  $v$ , Claim 1 and the fact that

$$y_0 + hq_i \in P_0, \quad i = 1, \dots, N-1, \quad (2.16)$$

we obtain

$$\begin{aligned} v(x_0 + hq_i) &\leq \varphi(y_0 + hq_i) + \rho^o(x_0 + hq_i - y_0 - hq_i) \\ &= \varphi(y_0 + hq_i) + \rho^o(x_0 - y_0) \\ &= \varphi(y_0 + hq_i) - \varphi(y_0) + v(x_0). \end{aligned} \quad (2.17)$$

Combining (2.15) and (2.17) we deduce that

$$h \langle p, q_i \rangle \leq h \langle D\varphi(y_0), q_i \rangle + \epsilon(h). \quad (2.18)$$

When we divide both sides of (2.18) by  $h > 0$  and let  $h$  tend to 0 we obtain

$$\langle p, q_i \rangle \leq \langle D\varphi(y_0), q_i \rangle. \quad (2.19)$$

Similarly, when we divide both sides of (2.18) by  $h < 0$  and let  $h$  tend to 0 we obtain

$$\langle p, q_i \rangle \geq \langle D\varphi(y_0), q_i \rangle. \quad (2.20)$$

Using (2.19) and (2.20) we conclude that

$$\langle p - D\varphi(y_0), q_i \rangle = 0, \quad i = 1, \dots, N-1,$$

thus,

$$p - D\varphi(y_0) = \lambda q_0, \quad (2.21)$$

for some  $\lambda \in \mathbb{R}$ . It is clear that  $\lambda \neq 0$ , since  $\rho(p) = 1$  (by the fact that  $u$  is a supersolution of (1.7) and by Lemma 4.2) and  $\rho(D\varphi(y_0)) < 1$ .

**5. Claim 4.** –  $\rho_\Omega$  is differentiable at  $y_0$  (so  $\nu(y_0)$  exists and  $\nu(y_0) = q_0$  by definition of  $q_0$ ).

Suppose there exists  $q \in \partial\rho_\Omega(y_0)$  with  $q \neq q_0$ . We obtain repeating the same development as before, that

$$p - D\varphi(y_0) = \mu q, \quad (2.22)$$

for some  $\mu \neq 0$ . So

$$q = \alpha q_0 \quad (2.23)$$

with  $\alpha = \frac{\lambda}{\mu} \neq 0$ . If  $\alpha < 0$ , then any convex combination of  $q$  and  $q_0$  is in  $\partial\rho_\Omega(y_0)$  and thus  $0 \in \partial\rho_\Omega(y_0)$  which yields that  $y_0$  is a minimizer for  $\rho_\Omega$  which, as already seen, is absurd. So we have  $\alpha > 0$ .

We will next prove that

$$\rho_\Omega^o(q) = 1, \quad (2.24)$$

for every  $q \in \partial\rho_\Omega(y_0)$ .

Assume for the moment that (2.24) holds and assume that  $q \in \partial\rho_\Omega(y_0)$  satisfies (2.23). Then,

$$1 = \rho_\Omega^o(\alpha q_0) = \alpha \rho_\Omega^o(q_0) = \alpha.$$

Consequently,  $\alpha = 1$ ,  $q = q_0$  and so,

$$\partial\rho_\Omega(y_0) = \{q_0\}. \quad (2.25)$$

By (2.25) we deduce that  $\rho_\Omega$  is differentiable at  $y_0$  (see [18] Theorem 25.1).

We now prove (2.24). Denoting by  $\rho_\Omega^*$  the Legendre transform of  $\rho_\Omega$ , one can readily check that

$$\rho_\Omega^*(x^*) = \begin{cases} 0 & \text{if } \rho_\Omega^o(x^*) \leq 1 \\ +\infty & \text{if } \rho_\Omega^o(x^*) > 1. \end{cases} \quad (2.26)$$

We recall the following well known facts:

$$\rho_\Omega(y_0) + \rho_\Omega^*(q) = \langle y_0, q \rangle, \quad (2.27)$$

for every  $q \in \partial\rho_\Omega(y_0)$ , (see [18] Theorem 23.5) and

$$\langle y_0, q \rangle \leq \rho_\Omega(y_0) \rho_\Omega^o(q). \quad (2.28)$$

Since  $y_0 \in \partial\Omega$ , we have  $\rho_\Omega(y_0) = 1$ , which, together with (2.27) and (2.28) implies that

$$\rho_\Omega^o(q) \geq 1 + \rho_\Omega^*(q). \quad (2.29)$$

Hence,  $\rho_\Omega^o(q)$  being finite, we deduce  $\rho_\Omega^*(q) = 0$ . Using (2.26) and (2.29) we obtain that

$$\rho_\Omega^o(q) = 1. \quad (2.30)$$

**6. Claim 5.** – We have  $p = D\varphi(y_0) + \lambda_0\nu(y_0)$ , where  $\nu(y_0)$  is the unit inward normal at  $y_0$ .

By Claim 3 and Claim 4, there exists  $\lambda_0 \in \mathbb{R}$  such that

$$p = D\varphi(y_0) + \lambda_0\nu(y_0). \quad (2.31)$$

The task will be to show that  $\lambda_0 > 0$ . Let

$$x_h = (1 - h)x_0 + hy_0, \quad h \in (0, 1).$$

We have

$$u(x_h) = \inf_{y \in \partial\Omega} \{\varphi(y) + \rho^o(x_h - y)\} \leq \varphi(y_0) + \rho^o(x_h - y_0). \quad (2.32)$$

Using the definition of  $x_h$  and the homogeneity of  $\rho^o$  we get

$$\rho^o(x_h - y_0) = \rho^o((1 - h)(x_0 - y_0)) = (1 - h)\rho^o(x_0 - y_0),$$

which, along with (2.32) implies

$$u(x_h) \leq \varphi(y_0) + \rho^o(x_0 - y_0) - h\rho^o(x_0 - y_0) = u(x_0) - h\rho^o(x_0 - y_0). \quad (2.33)$$

In light of (2.6) and (2.33), we have

$$h < p, y_0 - x_0 > + \epsilon(h) \leq -h\rho^o(x_0 - y_0),$$

which yields,

$$< p, y_0 - x_0 > \leq -\rho^o(x_0 - y_0). \quad (2.34)$$

Using the definition of  $\rho^o$  (see (1.9)) we have

$$- < D\varphi(y_0), y_0 - x_0 > = < D\varphi(y_0), x_0 - y_0 > \leq \rho(D\varphi(y_0))\rho^o(x_0 - y_0).$$

Also, by (H3), there exists  $\delta > 0$  such that

$$\rho(D\varphi(z)) \leq 1 - \delta, \quad z \in \overline{\Omega}. \quad (2.35)$$

Combining (2.34) and (2.35) we obtain

$$\langle p - D\varphi(y_0), y_0 - x_0 \rangle \leq -\delta\rho^o(x_0 - y_0). \quad (2.36)$$

Moreover, since we can express  $y_0 - x_0$  as a linear combination of the normal  $\nu(y_0)$  and the tangential vectors  $\{q_i\}_{i=1}^{N-1}$  at  $\partial\Omega$  in  $y_0$ , there exist  $\alpha$  and  $\mu_i$  with  $i = 1, \dots, N-1$  such that

$$y_0 - x_0 = \alpha\nu(y_0) + \sum_{i=1}^{N-1} \mu_i q_i.$$

As  $x_0 \in \Omega$  and  $\Omega$  is convex,  $\alpha < 0$ . Using (2.31), and (2.36) we obtain

$$\alpha\lambda_0 = \alpha \langle p - D\varphi(y_0), \nu(y_0) \rangle \leq -\delta\rho^o(x_0 - y_0).$$

Thus,  $\lambda_0 > 0$ . □

We now give the proof of the main theorem

*Proof of Theorem 2.6.*

1. (1)  $\Rightarrow$  (2) We assume that  $u$  is a *viscosity* solution of (1.1).

From Theorem 2.2, we have that every *viscosity* solution of (1.1) is a *viscosity* solution of (1.7) and therefore by (1.8)  $u$  can be written as

$$u(x) = \inf_{y \in \partial\Omega} \{\varphi(y) + \rho^o(x - y)\}. \quad (2.37)$$

Let  $y_0 \in \partial\Omega$  be a point where  $\partial\rho_\Omega(y_0) = \{\nu(y_0)\}$  (see the notations of the proof of Lemma 2.9). Let  $x \in \Omega$  be such that  $u$  is differentiable at  $x$  and  $x$  sufficiently close from  $y_0$ . Moreover the minimum in (2.37) is attained, at some  $y(x) \in \partial\Omega$  close to  $y_0$ . In light of Lemma 2.9 there exists  $\lambda_0(y(x)) > 0$  such that

$$Du(x) = D\varphi(y(x)) + \lambda_0(y(x))\nu(y(x)), \quad (2.38)$$

(i.e  $Du(x) - D\varphi(y(x))$  is perpendicular to the tangential hyperplane).

Note that  $\lambda_0(y(x))$  is bounded by  $2|Du|_\infty$ . Indeed, using the homogeneity of  $\rho$ , assuming that  $|\nu(y(x))| = 1$  we have

$$|\lambda_0(y(x))\nu(y(x))| \leq |Du(x)| + |D\varphi(y(x))| \leq 2|Du|_\infty. \quad (2.39)$$



As  $u$  is a solution of (1.1), i.e.  $Du(x) \in Z_F$ , we obtain that

$$D\varphi(y(x)) + \lambda_0(y(x))\nu(y(x)) \in Z_F. \quad (2.40)$$

Letting  $x$  tend to  $y_0$ , we obtain that  $y(x)$  tends to  $y_0$ . Since  $\partial\rho_\Omega(y_0) = \{\nu(y_0)\}$  we have from Theorem 25.1 in [18] that  $\rho_\Omega$  is differentiable at  $y_0$ . By Lemma 2.9 we have that  $\partial\rho_\Omega(y(x)) = \{\nu(y(x))\}$  and  $\rho_\Omega$  is differentiable at  $y(x)$ . Using Theorem 25.5 in [18], we obtain that  $\nu(y(x))$  tends to  $\nu(y_0)$ . Also, by (2.39)  $\lambda_0(y(x))$  tends, up to a subsequence, to a limit, denoted  $\lambda_0$  when  $x$  goes to  $y_0$ . Since  $Z_F$  is closed, and  $F$  is continuous, and so is  $D\varphi$ , (2.40) implies

$$D\varphi(y_0) + \lambda_0\nu(y_0) \in Z_F.$$

As  $\lambda_0(y(x)) > 0$ , we have that  $\lambda_0 \geq 0$ . Moreover  $u$  is solution of (1.7) and so  $\lambda_0$  is uniquely determined by the equation

$$\rho(D\varphi(y_0) + \lambda_0\nu(y_0)) = 1.$$

As  $\rho(D\varphi(y_0)) < 1$ , we have that  $\lambda_0 \neq 0$  and so  $\lambda_0 > 0$ . This establishes that (1)  $\Rightarrow$  (2).

**2.** (2)  $\Rightarrow$  (1) Conversely, assume that (2.5) holds.

Using (1.8) we obtain that  $u$  defined by

$$u(x) = \inf_{y \in \partial\Omega} \{\varphi(y) + \rho^o(x - y)\}$$

is the *viscosity* solution of (1.7). We have to show that  $u$  is a *viscosity* solution of (1.1).

- Since  $u$  is a *viscosity* subsolution of (1.7), then for every  $x \in \Omega$  and  $\forall p \in D^+u(x)$ , we have from Lemma 4.2,  $p \in \text{conv}(Z_F)$  (i.e.  $\rho(p) \leq 1$ ). As (H1) is satisfied (with the convention:  $F(\xi) < 0$ ,  $\forall \xi \in \text{int}(\text{conv}(Z_F))$ ) and as  $F$  is continuous, it follows that  $F(p) \leq 0$ . So  $u$  is a *viscosity* subsolution of (1.1).
- $u$  is also a *viscosity* supersolution of (1.7), and so, for every  $x \in \Omega$  and every  $p \in D^-u(x)$  we have  $\rho(p) \geq 1$  and, from Lemma 4.2, since  $p \in \text{conv}(Z_F)$  (i.e.  $\rho(p) \leq 1$ ), we obtain  $\rho(p) = 1$ . From Lemma 2.9, there exists  $y(x) \in \partial\Omega$  where the inward unit normal is well defined such that

$$p = D\varphi(y(x)) + \lambda(y(x))\nu(y(x)).$$

Since  $\rho(p) = 1$ , then  $\lambda(y(x)) > 0$  is uniquely determined by

$$\rho(D\varphi(y(x)) + \lambda(y(x))\nu(y(x))) = 1.$$

And so from (2.5), we deduce that  $p \in Z_F$ . Thus  $F(p) = 0$ ,  $\forall p \in D^-u(x)$ . We have therefore obtained that  $u$  is a viscosity supersolution of (1.1).

The two above observations complete the proof of the sufficiency part of the theorem.  $\square$

We conclude this section with the proof of Corollary 2.8.

*Proof of Corollary 2.8.*

To prove that there exists  $\varphi \in C^1(\overline{\Omega})$  such that the problem (1.1) has no viscosity solution, it is sufficient using Theorem 2.6 to find  $y \in \partial\Omega$ , where  $\nu(y)$  the unit inward normal exists, such that

$$D\varphi(y) + \lambda\nu(y) \notin Z_F, \quad \forall \lambda > 0.$$

1. Without loss of generality, we suppose that  $0 \in \text{int}(\text{conv}(Z_F))$ . Let  $\rho$  be the gauge associated with the set  $\text{conv}(Z_F)$ . We have that  $\rho$  is differentiable for  $H^{N-1}|_{\partial(\text{conv}(Z_F))}$  almost every  $\alpha \in \partial(\text{conv}(Z_F))$ . So, since  $Z_F \neq \partial(\text{conv}(Z_F))$  and  $Z_F$  is closed, we can choose  $\alpha \in \partial(\text{conv}(Z_F)) \setminus Z_F$  such that  $\alpha$  is a point of differentiability of  $\rho$ .

2. We first prove that there exists  $y_0 \in \partial\Omega$ , where  $\nu(y_0)$  exists, with

$$\alpha + \lambda\nu(y_0) \in \text{int}(\text{conv}(Z_F)), \quad (2.41)$$

for  $\lambda < 0$  small enough. Let  $a = D\rho(\alpha)$ . By Lemma 4.3 there exists  $y_0 \in \partial\Omega$  such that the normal  $\nu(y_0)$  to  $\partial\Omega$  at  $y_0$  exists and

$$\langle a, \nu(y_0) \rangle > 0. \quad (2.42)$$

Using (2.42) and the fact that  $\rho$  is differentiable at  $\alpha$  we obtain (keeping in mind that  $\rho(\alpha) = 1$ )

$$\rho(\alpha + \lambda\nu(y_0)) = \rho(\alpha) + \lambda[\langle a, \nu(y_0) \rangle + O(\lambda)/\lambda] < 1$$

for  $\lambda < 0$  small enough. This concludes the proof of (2.41).

3. Choose  $y \in \partial\Omega$ , where  $\nu(y)$  exists, and  $\bar{\lambda} < 0$ , such that  $\beta = \alpha + \bar{\lambda}\nu(y) \in \text{int}(\text{conv}(Z_F))$  (such  $\bar{\lambda}$  exists by the previous step). Observe that by convexity of  $\rho$  we have since  $\rho(\alpha) = 1$  and  $\rho(\alpha + \bar{\lambda}\nu(y)) < 1$  that  $\rho(\alpha + \lambda\nu(y)) > 1$  for every  $\lambda > 0$ . Let  $\varphi(x) = \langle \beta, x \rangle$ . We therefore have

$$D\varphi(x) + \lambda\nu(y) = \beta + \lambda\nu(y) \notin Z_F$$

for every  $\lambda > 0$ . That is the claimed result.  $\square$

### 3. THE VIABILITY APPROACH

In the previous section, we have assumed that  $Z_F \subset \partial(\text{conv}(Z_F))$  and  $\Omega$  is convex. We have proved that a necessary and sufficient conditions for the Hamilton-Jacobi equation

$$\begin{cases} F(Du) = 0 & \text{a.e. in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \quad (3.1)$$

to admit a  $W^{1,\infty}(\Omega)$  viscosity solution is that, for any  $y \in \partial\Omega$  where there is an inward unit normal,  $\nu(y)$ , there exists  $\lambda(y) > 0$  such that

$$D\varphi(y) + \lambda(y)\nu(y) \in Z_F.$$

In this section, we no longer assume that  $Z_F \subset \partial(\text{conv}(Z_F))$  and  $\Omega$  is convex. We investigate the existence of a  $W^{1,\infty}(\Omega)$  viscosity solution for Hamilton-Jacobi equation (3.1) for any  $\varphi$  satisfying the compatibility condition  $D\varphi(y) \in \text{int}(\text{conv}(Z_F))$ .

The main result of this section is that, if

$$\partial(\text{conv}(Z_F)) \setminus Z_F \neq \emptyset,$$

then there is some affine map  $\varphi$  satisfying the compatibility condition, and for which there is no  $W^{1,\infty}(\Omega)$  viscosity solution to (3.1) (c.f. Corollary 2.8).

**THEOREM 3.1.** – *Let  $F : \mathbb{R}^N \rightarrow \mathbb{R}$  be continuous such that the set  $Z_F = \{\xi \in \mathbb{R}^N \mid F(\xi) = 0\}$  is compact and  $\partial(\text{conv}(Z_F)) \setminus Z_F \neq \emptyset$ .*

*Then for any bounded domain  $\Omega \subset \mathbb{R}^N$ , there is some affine function  $\varphi$  with  $D\varphi \in \text{int}(\text{conv}(Z_F))$  such that the problem*

$$\begin{cases} F(Du) = 0 & \text{a.e. in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

*has no  $W^{1,\infty}(\Omega)$  viscosity solution.*

The proof of Theorem 3.1 is a consequence of Theorem 3.4 below. For stating this result, we need the definition of generalized normals (see also [1]).

**DEFINITION 3.2.** – *Let  $K$  be a locally compact subset of  $\mathbb{R}^P$ ,  $x \in K$ . A vector  $v \in \mathbb{R}^P$  is tangent to  $K$  at  $x$  if there are  $h_n \rightarrow 0^+$ ,  $v_n \rightarrow v$  such that  $x + h_n v_n$  belongs to  $K$  for any  $n \in \mathbb{N}$ .*

A vector  $\nu \in \mathbb{R}^P$  is a generalized normal to  $K$  at  $x$  if for every tangent  $v$  to  $K$  at  $x$

$$\langle v, \nu \rangle \leq 0.$$

We denote by  $N_K(x)$  the set of generalized normals to  $K$  at  $x$ .

REMARK 3.3. – i) If the boundary of  $K$  is piecewise  $C^1$ , then the generalized normals coincide with the usual outward normals at any point where these normals exist.

ii) If  $\Omega$  is an open subset of  $\mathbb{R}^P$  and  $x$  belongs to  $\partial\Omega$ , then a generalized normal  $\nu \in N_{\mathbb{R}^P \setminus \Omega}(x)$  can be regarded as an interior normal to  $\Omega$  at  $x$ .

THEOREM 3.4. – Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain and let  $F : \mathbb{R}^N \rightarrow \mathbb{R}$  be continuous such that the set  $Z_F = \{\xi \in \mathbb{R}^N \mid F(\xi) = 0\}$  is compact. Let  $\varphi(y) = \langle b, y \rangle$  with  $b \in \text{int}(\text{conv}(Z_F))$ .

If  $F(\xi) < 0$  (resp.  $F(\xi) > 0$ ) for every  $|\xi|$  sufficiently large and if equation (3.1) has a  $W^{1,\infty}(\Omega)$  viscosity supersolution (resp. subsolution), then for any  $y \in \partial\Omega$ , for any non zero generalized normal  $\nu_y \in N_{\mathbb{R}^N \setminus \Omega}(y)$  to  $\Omega$  at  $y$ , there is some  $\lambda \geq 0$  such that

$$b + \lambda \nu_y \in Z_F.$$

REMARK 3.5. – In some sense, Theorem 3.4 improves the necessary part of Theorem 2.6 since we do not assume any more that  $Z_F \subset \partial(\text{conv}(Z_F))$  and that  $\Omega$  is convex. Moreover, this result gives a necessary condition of existence for sub or supersolution.

For proving Theorem 3.4 and 3.1, we assume for a moment that the following lemma holds.

LEMMA 3.6. – Let  $\Omega \subset \mathbb{R}^N$  and  $F$  be as in Theorem 3.4. If there is some  $a \in \mathbb{R}^N \setminus \{0\}$  such that

1.  $\forall \lambda \geq 0, F(\lambda a) < 0$ ,
2.  $\exists x \in \partial\Omega$  such that  $a \in N_{\mathbb{R}^N \setminus \Omega}(x)$ ,

then there is no  $W^{1,\infty}(\Omega)$  viscosity supersolution to

$$\begin{cases} F(Du) = 0 & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

*Proof of Theorem 3.4.*

Assume for instance that  $F(\xi) < 0$  for  $|\xi|$  sufficiently large. Fix  $b \in \text{int}(\text{conv}(Z_F))$  and  $a \neq 0$  for which there is some  $x \in \partial\Omega$  such that  $a \in N_{\mathbb{R}^N \setminus \Omega}(x)$ .

If  $F(b) \geq 0$ , then the result is clear because  $F$  is continuous and  $F(b + \lambda a)$  is negative for  $\lambda$  sufficiently large.

Let us now assume that  $F(b) < 0$ . Let  $u$  be a  $W^{1,\infty}(\Omega)$  supersolution to

$$\begin{cases} F(Du) = 0 & \text{a.e. in } \Omega \\ u(y) = \langle b, y \rangle & \text{on } \partial\Omega. \end{cases}$$

Set  $\tilde{F}(\xi) := F(\xi + b)$  and  $\tilde{u}(y) := u(y) - \langle b, y \rangle$ . It is easy to check that  $\tilde{u}$  is a supersolution to

$$\begin{cases} \tilde{F}(D\tilde{u}) = 0 & \text{a.e. in } \Omega \\ \tilde{u}(y) = 0 & \text{on } \partial\Omega. \end{cases}$$

So, from Lemma 3.6 there is some  $\lambda_0 \geq 0$  such that  $\tilde{F}(\lambda_0 a) \geq 0$ , i.e.,  $F(b + \lambda_0 a) \geq 0$ . Since  $F(b + \lambda a)$  is negative for  $\lambda$  sufficiently large, there is  $\lambda \geq \lambda_0$  such that  $F(b + \lambda a) = 0$ .

We have therefore proved that there is  $\lambda \geq 0$  such that  $b + \lambda a \in Z_F$ .  $\square$

*Proof of Theorem 3.1.*

Since  $F$  is continuous and  $Z_F$  is bounded,  $F(\xi)$  has a constant sign for  $|\xi|$  sufficiently large. Say it is negative.

Let  $b \in \partial(\text{conv}(Z_F)) \setminus Z_F$  and  $r > 0$  be such that  $B_r(b) \cap Z_F = \emptyset$ . From the Separation Theorem, there is some  $a \in \mathbb{R}^N$ ,  $|a| = 1$ , such that

$$\langle b, a \rangle = \sup_{\xi \in Z_F} \langle \xi, a \rangle.$$

Note that  $F(b) < 0$ . Indeed,  $F$  is continuous and  $F(b + \lambda a) < 0$  for large  $\lambda$ . Moreover,  $b + \lambda a$  never belongs to  $Z_F$  for positive  $\lambda$  because

$$\langle (b + \lambda a), a \rangle > \sup_{\xi \in Z_F} \langle \xi, a \rangle.$$

From Lemma 5.3 in Appendix 2, there is some  $x \in \partial\Omega$  and a generalized normal  $\nu_x \in N_{\mathbb{R}^N \setminus \Omega}(x)$  such that

$$\langle \nu_x, a \rangle > 0$$

Set  $0 < \epsilon = \langle \nu_x, a \rangle$ ,  $\sigma = r\epsilon/(|\nu_x| + \epsilon)$ ,  $b_\sigma = b - \sigma a$ . Let  $\lambda \geq 0$ . We are going to prove that  $b_\sigma + \lambda \nu_x \notin Z_F$ . If  $\lambda \leq \sigma/\epsilon$ , then

$$|b_\sigma + \lambda \nu_x - b| = |\lambda \nu_x - \sigma a| \leq \lambda |\nu_x| + \sigma \leq r,$$

so that  $F(b_\sigma + \lambda\nu_x) < 0$  because  $B_r(b) \cap Z_F = \emptyset$  and  $F(b) < 0$ .

If  $\lambda > \sigma/\epsilon$ , then

$$\langle b_\sigma + \lambda\nu_x, a \rangle \geq \langle b, a \rangle - \sigma + \lambda\epsilon > \langle b, a \rangle = \sup_{\xi \in Z_F} \langle \xi, a \rangle,$$

so that  $b_\sigma + \lambda\nu_x \notin Z_F$ .

Since  $\nu_x$  is a generalized normal to  $\mathbb{R}^N \setminus \Omega$  at  $x$  and since  $b_\sigma + \lambda\nu_x \notin Z_F$  for any  $\lambda \geq 0$ , Theorem 3.4 states that there is no *viscosity* supersolution  $W^{1,\infty}(\Omega)$  to the problem (3.1) with  $\varphi(y) = \langle b_\sigma, y \rangle$ .  $\square$

*Proof of Lemma 3.6.*

The main tool for proving Lemma 3.6 is the viability theorem. The viability theorem (c.f. Theorem 3.3.2 and 3.2.4 in [2]) states that, if  $G$  is a compact convex subset of  $\mathbb{R}^P$  and  $K$  is a locally compact subset of  $\mathbb{R}^P$ , then there is an equivalence between

i)  $\forall x \in K$ , there exists  $\tau > 0$  and a solution to

$$\begin{cases} x'(t) \in G & \text{a.e. } t \in [0, \tau), \\ x(t) \in K & \forall t \in [0, \tau), \\ x(0) = x \end{cases} \quad (3.2)$$

ii)  $\forall x \in K$ ,  $\forall \nu \in N_K(x)$ ,  $\inf_{g \in G} \langle g, \nu \rangle \leq 0$ .

As usual, the solution of the constrained differential inclusion (3.2) can be extended on a maximal interval of the form  $[0, \tau)$  such that either  $\tau = +\infty$ , or  $x(\tau)$  belongs to  $\partial K \setminus K$ .

Assume now that, contrary to our claim, there is some  $W^{1,\infty}(\Omega)$  *viscosity* supersolution  $u$  to the problem. We will proceed by contradiction.

*First step:* We claim that

$$\forall x \in \Omega, u(x) > 0. \quad (3.3)$$

Indeed, otherwise, there is some  $x \in \Omega$  minimum of  $u$ . Note that  $0 \in D^-u(x)$ , so that  $F(0) \geq 0$  because  $u$  is a *viscosity* supersolution. This is in contradiction with  $F(\lambda a) < 0$  for all  $\lambda \geq 0$ .

The proof of the lemma consists in showing that inequality (3.3) does not hold.

*Second step:* Without loss of generality we set  $|a| = 1$ . Since  $Z_F$  is compact and  $F(\lambda a) < 0$  for  $\lambda \geq 0$ , there is some positive  $\epsilon$  such that

$$\forall \lambda \geq 0, \forall \xi \in \mathbb{R}^N, \text{ if } |\xi - \lambda a| \leq \lambda \epsilon, \text{ then } F(\xi) < 0. \quad (3.4)$$

Since  $u$  is a  $W^{1,\infty}(\Omega)$  supersolution, we know, from a result due to H. Frankowska [15] (see also Lemma 5.1 in Appendix 2), that

$$\forall x \in \Omega, \forall (\nu_x, \nu_\rho) \in N_{Epi(u)}(x, u(x)), \nu_\rho < 0 \text{ and } F\left(\frac{\nu_x}{|\nu_\rho|}\right) \geq 0.$$

Let  $x \in \Omega$  and  $(\nu_x, \nu_\rho) \in N_{Epi(u)}(x, u(x))$ . Since  $F(\frac{\nu_x}{|\nu_\rho|}) \geq 0$ , we have thanks to (3.4),

$$\forall \lambda \geq 0, \left| \frac{\nu_x}{|\nu_\rho|} - \lambda a \right| > \lambda \epsilon.$$

An easy computation shows that this inequality implies

$$\langle a, \nu_x \rangle - (1 - \epsilon^2)^{1/2} |\nu_x| \leq 0.$$

Let  $G = \{a + (1 - \epsilon^2)^{1/2} B\} \times \{0\}$  where  $B$  is the closed unit ball of  $\mathbb{R}^N$ . Then the previous inequality is equivalent with the following

$$\inf_{g \in G} \langle g, (\nu_x, \nu_\rho) \rangle \leq 0,$$

so that  $K = Epi(u) \cap (\Omega \times \mathbb{R})$  is a locally compact subset such that

$$\forall x \in \Omega, \forall (\nu_x, \nu_\rho) \in N_K(x, u(x)), \inf_{g \in G} \langle g, (\nu_x, \nu_\rho) \rangle \leq 0.$$

In particular, it satisfies the condition (ii) of the viability theorem.

Thus, from the viability theorem,  $\forall (x, u(x)) \in K$ , there is a maximal solution to

$$\begin{cases} (x'(t), \rho'(t)) \in G, & \text{a.e. } t \in [0, \tau) \\ (x(t), \rho(t)) \in K, & \forall t \in [0, \tau) \\ x(0) = x, \rho(0) = u(x) \end{cases} \quad (3.5)$$

where either  $\tau = \infty$  or  $x(\tau) \in \partial\Omega$ .

Let us point out that  $\rho'(t) = 0$ , so that  $\rho(t) = u(x)$  on  $[0, \tau)$ .

*Third step:* Let  $x \in \partial\Omega$  be such that  $a \in N_{\mathbb{R}^N \setminus \Omega}(x) \setminus \{0\}$ . We claim that there is a solution to (3.5) starting from  $(x, u(x)) = (x, 0)$  defined on  $(0, \tau)$ .

Since  $a$  belongs to  $N_{\mathbb{R}^N \setminus \Omega}(x) \setminus \{0\}$ , from Lemma 5.2 of the Appendix 2, applied to  $C = \{a + (1 - \epsilon^2)^{1/2} B\}$ , there is some  $\alpha > 0$  such that

$$\forall c \in C, \forall b \in \mathbb{R}^N \text{ with } |b| \leq 1, \forall \theta \in (0, \alpha), \quad x + \theta(c + \alpha b) \in \Omega.$$

Since  $0 \notin C$ , we can choose also  $\alpha > 0$  sufficiently small such that  $0 \notin C + \alpha B$ , where  $B$  is the closed unit ball.

We denote by  $S$  the set

$$S = \{x + \theta(c + \alpha b), c \in C, b \in \mathbb{R}^N \text{ with } |b| \leq 1, \theta \in (0, \alpha)\}.$$

It is a subset of  $\Omega$  and  $x \in \partial S$ .

Let  $x_n \in S$  converge to  $x$ ,  $(x_n(\cdot), \rho_n(\cdot))$  be maximal solutions to (3.5) with initial data  $(x_n, u(x_n))$  defined on  $[0, \tau_n]$ . Let us first prove by contradiction that the sequence  $\tau_n$  is bounded from below by some positive  $\tau$ . Assume on the contrary that  $\tau_n \rightarrow 0^+$ . Note that

$$\forall n \in N, \quad x_n(\tau_n) \in x_n + \tau_n C,$$

because  $x'(t) \in C$  which is convex compact. Thus, for any  $n$ , there is  $c_n \in C$  such that  $x_n(\tau_n) = x_n + \tau_n c_n$ .

Since  $x_n \in S$ , for any  $n \geq N$  there are  $\theta_n \in (0, \alpha)$ ,  $b_n \in B$  and  $c'_n \in C$  such that  $x_n = x + \theta_n(c'_n + \alpha b_n)$ . Since  $x_n$  converges to  $x$  and  $0 \notin C + \alpha B$ , we have  $\theta_n \rightarrow 0^+$ . Let  $N_0$  be such that  $\forall n \geq N_0$ ,  $\theta_n + \tau_n < \alpha$ .

Then

$$x_n(\tau_n) = x + (\theta_n + \tau_n) \left[ \frac{\theta_n}{\theta_n + \tau_n} c'_n + \frac{\tau_n}{\theta_n + \tau_n} c_n + \alpha \frac{\theta_n}{\theta_n + \tau_n} b_n \right].$$

Since  $C$  is convex,

$$\frac{\theta_n}{\theta_n + \tau_n} c'_n + \frac{\tau_n}{\theta_n + \tau_n} c_n \tag{3.6}$$

belongs to  $C$ . Moreover,

$$\left| \frac{\theta_n}{\theta_n + \tau_n} b_n \right| \leq 1. \tag{3.7}$$

Thus, for any  $n \geq N_0$ ,  $x_n(\tau_n)$  belongs to  $S$  which is a subset of  $\Omega$  and we have a contradiction with  $x_n(\tau_n) \in \partial\Omega$ .



So we have proved that the sequence  $\tau_n$  is bounded from below by some positive  $\tau$ .

Since  $G$  is convex compact and since the solutions  $(x_n(\cdot), \rho_n(\cdot))$  are defined on  $[0, \tau]$ , the solutions  $(x_n(\cdot), \rho_n(\cdot))$  converge up to a subsequence to some  $(x(\cdot), \rho(\cdot))$  solution to

$$\begin{cases} (x'(t), \rho'(t)) \in G, & \text{a.e. } t \in [0, \tau) \\ (x(t), \rho(t)) \in \overline{K}, & \forall t \in [0, \tau) \\ x(0) = x, \rho(0) = u(x) = 0 \end{cases}$$

(see Theorem 3.5.2 of [2] for instance).

Since,  $x'(t) \in C$ , for any  $t \in [0, \tau]$  there is some  $c(t) \in C$  such that  $x(t) = x + tc(t)$ . Thus, for  $t \in (0, \inf\{\tau, \alpha\})$ ,  $x(t)$  belongs to  $S$  and so to  $\Omega$ .

In particular,  $(x(t), \rho(t)) = (x(t), 0)$  belongs to the epigraph of  $u$  for  $t \in (0, \tau')$  (with  $\tau' = \inf\{\tau, \alpha\}$ ), i.e.,

$$\forall t \in (0, \tau'), \quad u(x(t)) \leq 0.$$

This is in contradiction with inequality (3.3). □

#### 4. APPENDIX 1

We now state two lemmas which are well-known in the literature. The first one is a Mac Shane type extension lemma for Lipschitz functions. The second one can be found in F.H. Clarke [7] and H. Frankowska [14]. However for the sake of completeness we prove them again.

**LEMMA 4.1.** – *Let  $\Omega$  be a convex set of  $\mathbb{R}^N$  and  $u \in W^{1,\infty}(\Omega)$  with  $\rho(Du(x)) \leq 1$  a.e. in  $\Omega$ , then there exists an extension  $\tilde{u} \in W^{1,\infty}(\mathbb{R}^N)$  of  $u$  with  $\rho(D\tilde{u}(x)) \leq 1$  a.e. in  $\mathbb{R}^N$ .*

*Proof.*

The task here is to check that  $\tilde{u}$  given by

$$\tilde{u}(x) = \sup_{y \in \Omega} \{u(y) - \rho^o(y - x)\}, \quad \forall x \in \mathbb{R}^N.$$

satisfies the requirements of Lemma 4.1. (Note the similarity with the *viscosity* solution (1.8).)

**1.** We first show that  $\tilde{u}$  is an extension of  $u$ .

For this, it will be sufficient to show

$$\rho(Du(x)) \leq 1 \text{ a.e.} \implies u(y) - u(x) \leq \rho^o(y - x). \quad (4.1)$$

To prove (4.1) we proceed by regularization. We introduce the mollifier function

$$f(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

and the sequence  $f_n(x) = n^N f(nx)$  where  $C$  is chosen so that  $\int f = 1$ . First, we extend  $u$ , as a Lipschitz function, to the whole of  $\mathbb{R}^N$  and we still denote this extension by  $u$  (this can be done by Mac Shane lemma). We then set

$$u_n(x) = \int_{\mathbb{R}^N} f_n(x - y)u(y) \, dy.$$

It is well known that  $u_n \rightarrow u$  uniformly on every compact set. Let  $\Omega_\delta$  be the compact subset of  $\Omega$  defined by

$$\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \delta\}.$$

for  $\delta > 0$  and  $n > \frac{1}{\delta}$ . As  $\rho$  is convex and homogeneous of degree one, using Jensen inequality, we obtain that

$$\rho(D(u_n(x))) \leq \int_{\mathbb{R}^N} f_n(x - y)\rho(D(u(y))) \, dy \leq 1, \quad \forall x \in \Omega_\delta. \quad (4.2)$$

Since  $u_n$  is of class  $C^1$ , (4.2) implies that for  $x, y \in \Omega_\delta$ , there exists  $\tilde{x} \in \mathbb{R}^N$  such that

$$\begin{aligned} u_n(y) - u_n(x) &= \langle Du_n(\tilde{x}), y - x \rangle \\ &\leq \rho(Du_n(\tilde{x}))\rho^o(y - x) \\ &\leq \rho^o(y - x), \end{aligned}$$

and so, letting  $n$  tend to infinity, we obtain

$$u(y) - u(x) \leq \rho^o(y - x).$$

Letting then  $\delta$  tend to 0, we have deduced (4.1) and so,  $\tilde{u}$  is an extension of  $u$ .

2. We next show that

$$\tilde{u}(z) - \tilde{u}(x) \leq \rho^o(z - x), \quad x, z \in \mathbb{R}^N. \quad (4.3)$$

Indeed we have

$$\begin{aligned}\tilde{u}(z) - \tilde{u}(x) &= \sup_{y \in \Omega} \{u(y) - \rho^o(y - z)\} - \sup_{y \in \Omega} \{u(y) - \rho^o(y - x)\} \\ &\leq \sup_{y \in \Omega} \{-\rho^o(y - z) + \rho^o(y - x)\} \\ &\leq \rho^o(z - x).\end{aligned}$$

3. We then show that (4.3) implies that  $\rho(D\tilde{u}(x)) \leq 1$  a.e.

As  $\tilde{u}$  is a Lipschitz function we can use Rademacher theorem and obtain that for almost every  $x \in \mathbb{R}^N$

$$\lim_{h \rightarrow 0} \frac{\tilde{u}(x+h) - \tilde{u}(x) - \langle D\tilde{u}(x), h \rangle}{|h|} = 0.$$

This means that for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\frac{\tilde{u}(x+h) - \tilde{u}(x) - \langle D\tilde{u}(x), h \rangle}{|h|} \leq \epsilon.$$

for every  $|h| \leq \delta$ , and so,

$$\frac{\tilde{u}(x+h) - \tilde{u}(x) - \langle D\tilde{u}(x), h \rangle}{\rho^o(-h)} \leq \epsilon \frac{|h|}{\rho^o(-h)}.$$

From (4.3), we get that

$$-1 - \frac{\langle D\tilde{u}(x), h \rangle}{\rho^o(-h)} \leq \epsilon \frac{|h|}{\rho^o(-h)}. \quad (4.4)$$

As  $\rho$  is convex and homogeneous of degree one, we have

$$\rho(D\tilde{u}(x)) = \rho^{oo}(D\tilde{u}(x)) = \sup_{|\lambda| \leq \delta} \frac{\langle D\tilde{u}(x), \lambda \rangle}{\rho^o(\lambda)}. \quad (4.5)$$

Taking the supremum over every  $|h| \leq \delta$  in (4.4) we obtain

$$-1 + \sup_{|h| \leq \delta} \frac{\langle D\tilde{u}(x), -h \rangle}{\rho^o(-h)} \leq \sup_{|h| \leq \delta} \epsilon \frac{|h|}{\rho^o(-h)} = \epsilon D$$

where,

$$0 < \sup_{|h| \leq \delta} \frac{|h|}{\rho^o(-h)} = D < \infty.$$

Letting now  $\epsilon$  tend to 0, and using (4.5) we obtain

$$\rho(D\tilde{u}(x)) \leq 1. \quad \square$$

LEMMA 4.2. – Let  $u \in W^{1,\infty}(\Omega)$  with  $Du(y) \in \text{conv}(Z_F)$  a.e. (i.e.  $\rho(Du) \leq 1$  a.e.), then

$$D^+u(x) \cup D^-u(x) \subset \text{conv}(Z_F),$$

for every  $x \in \Omega$ .

*Proof.*

We first show that  $D^+u(x) \subset \text{conv}(Z_F)$ . Observe that from (4.1) we have:

$$\frac{u(x+h) - u(x)}{\rho^o(-h)} \geq -1.$$

Using the definition of  $D^+u$  we have for every  $x \in \Omega$  and  $p \in D^+u(x)$

$$\limsup_{h \rightarrow 0} \frac{u(x+h) - u(x) - \langle p, h \rangle}{|h|} \leq 0.$$

Proceeding as in Lemma 4.1, we observe that for every  $p \in D^+u(x)$ , and every  $\epsilon > 0$ , there exists  $\delta > 0$

$$\frac{u(x+h) - u(x) - \langle p, h \rangle}{|h|} \leq \epsilon,$$

for every  $|h| \leq \delta$ . We therefore get

$$-1 + \frac{\langle p, -h \rangle}{\rho^o(-h)} \leq \epsilon \frac{|h|}{\rho^o(-h)}$$

since  $\rho$  is convex and homogeneous of degree one. Taking the supremum over every  $|h| \leq \delta$ , we obtain

$$-1 + \sup_{|h| \leq \delta} \frac{\langle p, -h \rangle}{\rho^o(-h)} \leq \epsilon \sup_{|h| \leq \delta} \frac{|h|}{\rho^o(-h)}. \quad (4.6)$$

Defining

$$0 < D = \sup_{|h| \leq \delta} \frac{|h|}{\rho^o(-h)} < \infty,$$

and using (4.6), we get

$$-1 + \rho(p) \leq \epsilon D.$$

Letting  $\epsilon$  tend to 0, we obtain  $\rho(p) \leq 1$ . Using the same argument for  $D^-u(x)$  we conclude that

$$D^+u(x) \cup D^-u(x) \subset \text{conv}(Z_F). \quad \square$$

In the proof of Corollary 2.8, we used the following result (see also Lemma 5.3).

LEMMA 4.3. – *Let  $\Omega$  be a bounded, open and convex set. For every  $a \in \mathbb{R}^N \setminus \{0\}$  there exists  $y \in \partial\Omega$ , where  $\nu(y)$  the unit inward normal exists, such that*

$$\langle a, \nu(y) \rangle > 0.$$

*Proof.*

1. By the divergence theorem, we have

$$\int_{\partial\Omega} \langle a, \nu(y) \rangle \, d\sigma(y) = 0.$$

It is then clear from the above identity that the claim of this lemma will follow if we can prove that  $\langle a, \nu(y) \rangle \neq 0$  on a set of positive (relative to  $\partial\Omega$ ) measure. This will be achieved in the next step.

2. Suppose on the contrary that there exists  $a \neq 0$  for which the conclusion of the Lemma fails. We may assume without loss of generality that  $0 \in \Omega$ . Let  $\rho_\Omega$  be the gauge of  $\Omega$ . Then for each  $\mu \in \mathbb{R}$  and each  $y \in \partial\Omega$  such that  $\nu(y)$  exists we have (keeping in mind that  $\rho_\Omega(y) = 1$ )

$$\rho_\Omega(y + \mu a) \geq \rho_\Omega(y) + \mu \langle a, \nu(y) \rangle = 1,$$

and so, using that the set where  $\nu(y)$  exists is dense in  $\partial\Omega$  we deduce that

$$\rho_\Omega(y + \mu a) \geq 1, \quad (4.7)$$

for every  $y \in \partial\Omega$ . Next, take  $x \in \Omega$  and  $\bar{\mu} \in \mathbb{R}$  such that  $x + \bar{\mu}a \in \partial\Omega$ . We have, for  $y = x + \bar{\mu}a$ ,

$$1 > \rho_\Omega(x) = \rho_\Omega(y - \bar{\mu}a),$$

which is at variance with (4.7).  $\square$

## 5. APPENDIX 2

We collect here some lemmas needed throughout the proofs of Theorem 3.1 and 3.4 and Lemma 3.6. Lemma 5.1 appeared in [15], but we will give a proof for sake of completeness. Lemma 5.2 and 5.3 are well known results of non smooth analysis, although it is not easy to find a proof in the literature. We think that the proof of Lemma 5.3 is new and interesting.

LEMMA 5.1. – *If  $\Omega$  is an open subset of  $\mathbb{R}^N$  and  $u$  is a  $W^{1,\infty}(\Omega)$  supersolution of*

$$F(Du) = 0 \text{ a.e. in } \Omega,$$

*then*

$$\forall x \in \Omega, \forall (\nu_x, \nu_\rho) \in N_{Epi(u)}(x, u(x)) \setminus \{(0, 0)\}, \nu_\rho < 0 \text{ and } F\left(\frac{\nu_x}{|\nu_\rho|}\right) \geq 0.$$

Let us point out that the converse of this result holds also true (see [15]).

LEMMA 5.2. – *Let  $\Omega$  be an open subset of  $\mathbb{R}^N$ ,  $x \in \partial\Omega$  and  $a \in N_{\mathbb{R}^N \setminus \Omega}(x)$  with  $a \neq 0$ . Let  $C$  be a compact subset of  $\mathbb{R}^N$  be such that*

$$\inf_{c \in C} \langle c, a \rangle > 0.$$

*Then there is some  $\alpha > 0$  such that*

$$\forall c \in C, \forall b \in \mathbb{R}^N \text{ with } |b| \leq 1, \forall \theta \in (0, \alpha), \quad x + \theta(c + \alpha b) \in \Omega.$$

LEMMA 5.3. – *If  $\Omega \subset \mathbb{R}^N$  is open and bounded, then, for any  $a \in \mathbb{R}^N \setminus \{0\}$ , there is some  $x \in \partial\Omega$  and a generalized normal  $\nu_x \in N_{\mathbb{R}^N \setminus \Omega}(x)$  such that*

$$\langle \nu_x, a \rangle > 0.$$

*Proof of Lemma 5.1.*

Let  $(\nu_x, \nu_\rho) \neq (0, 0)$  be a generalized normal to  $Epi(u)$  at  $(x, u(x))$ . We have to prove that  $\nu_\rho < 0$  and  $\nu_x/|\nu_\rho|$  belongs to  $D^-u(x)$ .

Since  $(x, u(x)) + t(0, 1)$  belongs to  $Epi(u)$  for  $t > 0$ ,  $(0, 1)$  is tangent to  $Epi(u)$  at  $(x, u(x))$ , and so  $\langle (0, 1), (\nu_x, \nu_\rho) \rangle \leq 0$ . In particular,  $\nu_\rho \leq 0$ .

Assume for a while that  $\nu_\rho = 0$ . Then,  $\nu_x \neq 0$ . Set  $h_n := 1/n$ . Since  $u$  is Lipschitz, the sequence

$$\frac{(x + h_n \nu_x, u(x + h_n \nu_x)) - (x, u(x))}{h_n} \quad (5.1)$$

is bounded and it converges, up to a subsequence, to some  $(\nu_x, \theta)$  which is tangent to  $Epi(u)$  at  $(x, u(x))$ .

Thus  $\langle (\nu_x, 0), (\nu_x, \theta) \rangle > \leq 0$  which is impossible since  $\nu_x \neq 0$ . So  $\nu_\rho < 0$ .

Set  $p := \nu_x / |\nu_\rho|$ . We now have to check that,  $\forall v \in \mathbb{R}^N$ ,

$$\liminf_{h \rightarrow 0^+} \frac{u(x + hv) - u(x) - h \langle p, v \rangle}{h} \geq 0.$$

Fix  $v \in \mathbb{R}^N \setminus \{0\}$  and denote by  $\theta$  the lower limit as above. Since  $u$  is Lipschitz,  $\theta$  is finite. We have to prove that  $\theta \geq 0$ .

Let  $\{h_n\}$  be a sequence converging to 0 such that

$$\frac{u(x + h_n v) - u(x) - h_n \langle p, v \rangle}{h_n} \quad (5.2)$$

converge to  $\theta$ .

Note that

$$\frac{(x + h_n v, u(x + h_n v)) - (x, u(x))}{h_n} \quad (5.3)$$

converges to  $(v, \langle p, v \rangle + \theta)$ . Thus  $(v, \langle p, v \rangle + \theta)$  is tangent to  $Epi(u)$  at  $(x, u(x))$  and

$$\langle (v, \langle p, v \rangle + \theta), (\nu_x, \nu_\rho) \rangle \leq 0.$$

This implies that

$$\langle v, \nu_x \rangle + \left\langle \left( \frac{\nu_x}{-\nu_\rho} \right), v \right\rangle \nu_\rho + \theta \nu_\rho \leq 0.$$

So  $\theta \geq 0$  because  $\nu_\rho < 0$ .

Since  $u$  is a supersolution and  $\nu_x / |\nu_\rho| \in D^-u(x)$ , we deduce from Lemma 2.5,  $F(\nu_x / |\nu_\rho|) \geq 0$ .  $\square$

*Proof of Lemma 5.2.*

Assume that, contrary to our claim, for any  $n > 0$  there are  $0 < \theta_n \leq \frac{1}{n}$ ,  $c_n \in C$ ,  $b_n \in B$  with  $x + \theta_n(c_n + \frac{1}{n}b_n) \notin \Omega$ .

Then  $c_n$  converges, up to a subsequence, to some  $c \in C$ . Clearly  $c$  is tangent to  $\mathbb{R}^N \setminus \Omega$  at  $x$ .

Since  $a \in N_{\mathbb{R}^N \setminus \Omega}(x)$ , this implies that  $\langle a, c \rangle \leq 0$ , which is in contradiction with the assumption.  $\square$

*Proof of Lemma 5.3.*

Assume that the conclusion of the lemma is false. Then

$$\forall x \in \partial\Omega, \forall \nu_x \in N_{\mathbb{R}^N \setminus \Omega}(x), \langle \nu_x, a \rangle \leq 0.$$

This means (from the viability Theorem (again !)) applied to the closed set  $K := \mathbb{R}^N \setminus \Omega$  and  $G := a$ ) that for any  $x \in \partial\Omega$ , the solution to  $x'(t) = a$ ,  $x(0) = x$  remains in  $K$  forever.

Let now  $y$  belong to  $\Omega$ . Since  $\Omega$  is bounded, there is some  $\tau$  sufficiently large such that  $x - \tau a \notin \Omega$ . The previous remark applied to  $x - \tau a$  yields that  $x(t) = x - \tau a + ta$  belongs to  $\mathbb{R}^N \setminus \Omega$  for any  $t \geq 0$ , which, for  $t = \tau$ , is in contradiction with  $x \in \Omega$ .  $\square$

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