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A kinetic formulation for multi-branch entropy solutions of scalar conservation laws

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ABSTRACT. – Multivalued solutions with a limited number of branches of the inviscid Burgers equation can be obtained by solving closed systems of moment equations. For this purpose, a suitable concept of *entropy* multivalued solutions with K branches is introduced. © Elsevier, Paris

RÉSUMÉ. – On reconstruit les solutions multivoques de l'équation de Burgers sans viscosité, quand leur nombre de branches est connu *a priori*, en résolvant un système fermé d'équations de moments. A cet effet, on introduit un concept de solutions multivoques entropiques à K branches. © Elsevier, Paris

1. INTRODUCTION

It is a classical idea to solve, at least approximately, kinetic equations, set in a phase space (t, x, v), with the help of finite systems of moment equations set on the reduced space (t, x). A well known example is Grad's closure of the Boltzmann equation [G]. There has been a new interest for this approach in the recent years. Let us quote in particular Levermore's work on the Grad approximation [L]. In the present paper, we consider an academic problem, that can be seen as a model for realistic applications such as multiple arrival times in ray tracing for geophysical problems [TS], [EFO], or multiple beams in optics or plasma physics [FF], [C]. We are in particular interested in the multivalued solutions of the inviscid Burgers equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0, \tag{1.1}$$

with initial condition $u_0(x)$ valued in a bounded interval, say $[0, L]$, or equivalently, in the solution $f(t, x, v)$ of the free transport equation

$$\partial_t f + v \partial_x f = 0, \tag{1.2}$$

with initial condition

$$f(0, x, v) = f_0(x, v) = H(u_0(x) - v) H(v), \tag{1.3}$$

where H denotes the Heaviside function. A more general initial condition can be also considered, such as

$$f_0(x, v) := \begin{cases} 1 & \text{if } (x, v) \in D_0 \\ 0 & \text{otherwise} \end{cases}$$

where D_0 is a domain of $\mathbb{R}^2$ included in the slab $0 \leq v \leq L$. The exact solution is very simple

$$f(t, x, v) = f_0(x - tv, v) \tag{1.4}$$

and is a piecewise constant function with support a domain $D(t)$ of the plane (x, v) included in the slab $0 \leq v \leq L$. In particular, when f_0 is given by (1.3) with u_0 given in $C_c^1(\mathbb{R})$, then there exists a finite time $T^* = T^*(u_0) < +\infty$ such that, for $0 < t < T^*$,

$$f(t, x, v) = H(u(t, x) - v) H(v),$$

where $u(t, x)$ is the unique smooth singlevalued solution to (1.1). For larger values of t, the boundary of $D(t)$ is a curve with many projections onto the real axis and can be seen as the "graph" of the multivalued solution to the Burgers equation corresponding to the initial condition u_0. The number of branches of this multivalued solution can grow in time and depends on the number of extremal points of u_0. We are interested in finding these branches without working in the phase space (t, x, v). An elementary but key observation is that, if we *a priori* know, on a given time interval $[0, T]$, an upper bound $K \geq 1$ for the number of branches, then it is theoretically possible to recover the entire solution by solving a closed system of K moment equations. More precisely, the moments

$$m_k(t, x) = \int v^k f(t, x, v) dv, \quad k = 0, 1, 2, \ldots \tag{1.5}$$

associated to the solution of (1.2)-(1.3), satisfy the system

$$\partial_t m_k + \partial_x m_{k+1} = 0, \quad k = 0, 1, 2, \ldots . \tag{1.6}$$

This system can be closed at order $k = K - 1$ since, for every (t, x), the knowledge of the K first moments is sufficient to determine the K branches of the solution and then, express $m_K(t, x)$ as a function of $m_0(t, x), \ldots, m_{K-1}(t, x)$. The goal of this paper is to build up a closure formalism for the K moment system allowing us to recover all multivalued solutions having at most K branches by solving a non linear hyperbolic system of conservation laws.

Let us consider a simple example, when the solution has two branches

$$f(t, x, v) = H(b(t, x) - v) - H(a(t, x) - v)$$

where $0 \leq a(t, x) \leq b(t, x) \leq L$ are smooth functions. Then, for all (t, x),

$$m_0 = b - a, \quad m_1 = \frac{1}{2}(b^2 - a^2)$$

which immediately leads to

$$a = \frac{m_1}{m_0} - \frac{m_0}{2}, \quad b = \frac{m_1}{m_0} + \frac{m_0}{2},$$

thus

$$m_2 = \frac{1}{3}(b^3 - a^3) = \frac{m_1^2}{m_0} + \frac{1}{12} m_0^3.$$

Actually, if we set $\rho = m_0$ and $q = m_1$, the resulting closed system is nothing but

$$\begin{aligned} &\partial_t \rho + \partial_x q = 0 \\ &\partial_t q + \partial_x \left(\frac{q^2}{\rho} + p(\rho) \right) = 0 \end{aligned} \tag{1.7}$$

namely the isentropic gas dynamic equations with $p(\rho) = \frac{1}{12}\rho^\gamma$ and $\gamma = 3$. This system is hyperbolic with a degeneracy at $\rho = 0$, which corresponds to the case $a = b$ when the solution is singlevalued.

Our closure formalism is based on an appropriate concept of "maxwellian" functions obtained through an entropy maximization principle, as usual in Kinetic Theory (see [L], in particular). It leads in a natural way to the following kinetic formulation for multivalued solutions with *at most* K branches, in the spirit of [LPT1], [B1], [GM],

$$\partial_t f + v\partial_x f = (-1)^{K-1}(\partial_v^K)\mu \qquad \text{in } \mathcal{D}'((0,\infty) \times \mathbb{R}_x \times \mathbb{R}_v^+) \tag{1.8}$$

subject to

$$f(t,x,v) = \mathcal{M}_K(t,x,v) \qquad \text{a.e. in } (0,\infty) \times \mathbb{R}_x \times \mathbb{R}_v^+ \tag{1.9}$$

for some nonnegative measure $\mu(t,x,v)$ and where $\mathcal{M}_K$ is the above mentioned maxwellian. When $K = 2$ the formulation does not differ from Lions Perthame Tadmor kinetic formulation of the isentropic gas dynamics with $\gamma = 3$. We call these solutions *entropy K-multivalued* solutions: they will differ from the regular multivalued solutions as soon as the number of branches becomes larger than K, exactly as in the well known case $K = 1$, when shock waves form.

Finally, we observe that a simple translation of a general L^∞ initial data u_0 leads to the case of a non negative bounded initial data. Therefore, there is not lost of generality in considering non negative bounded u_0 and the exposition is simpler. Moreover, the results we show can be readily extended to the case of a general multidimensional scalar conservation law

$$\partial_t u + \sum_{i=1}^n \partial_{x_i} F_i(u) = 0. \tag{1.10}$$

The transport equation corresponding to the multivalued solution of (1.10) is

$$\partial_t f + a(v) \cdot \nabla_x f = (-1)^{K-1}(\partial_v^K)\mu$$

where $a(v) := (F_i'(v))_{i=1,\dots,n}$. Again, we consider here the case of the inviscid Burgers equation for the sake of clearness.

The paper is organized as follows. In section 2, we define the K-branch maxwellian functions as functions of the v variable that maximizes suitable entropy functions with prescribed K first moments. The existence and the uniqueness of the maxwellian are proved. In section 3 we introduce the kinetic formulation and we prove its equivalence with an hyperbolic system of K nonlinear conservation laws. In section 4, we get an existence theorem for entropy multivalued solutions by introducing a BGK-like approximation of (1.8)-(1.9) and showing the convergence of the approximating solutions by using the averaging lemmas of [GLPS], [LPT1]. For sake of completeness, in the appendix we prove an existence result for the solution of the BGK-like approximation through a time-splitting method.

2. K-BRANCH MAXWELLIAN FUNCTIONS

In this section, the number of branches, K, is fixed. Let us introduce $E := \left\{g \in C([0, +\infty)) \ : \ \sup_{v \geq 0} \frac{|g(v)|}{1+v^K} < +\infty\right\}$. E is a Banach space with the norm $||g|| := \sup_{v \geq 0} \frac{|g(v)|}{1+v^K}$. We consider also the set Θ of all $\theta \in C(\mathbb{R})$ for which there is a constant $\alpha \in (0,1)$ such that the K-th (distributional) derivative of θ satisfies

$$\alpha \leq \theta^{(K)}(v) \leq \alpha^{-1}.$$

Notice that such a function θ belongs to the Hölder space $C_{loc}^{K-1,1}(\mathbb{R})$ and has a constant $\gamma \in (0,1)$ such that

$$\gamma v^K - \gamma^{-1} \leq \theta(v) \leq \gamma^{-1}(v^K + 1), \ \ \forall v \geq 0.$$

In particular, the restriction of $\theta(v)$ to $[0, +\infty)$ is bounded from below, goes to $+\infty$ with v and belongs to E. Let us now denote E' the topological dual space of E, which is a subspace of the space of all bounded Radon measures on $(0, +\infty)$, and introduce the convex subset $F := \{f \in E' \ : \ 0 \leq f \leq 1\}$, which can be also seen as a subset of $L^\infty([0, +\infty))$ (with some decay properties near $v = +\infty$). For any $f \in F$, let $m(f) \in \mathbb{R}^K$ be the moment vector $m_k(f) := \langle f, v^k \rangle$, $k = 0, 1, \dots, K-1$. Then, let us introduce the set of all "attainable" moments

$$M_K := \{ \ m(f); \ f \in F \ \}$$

and, for any $m \in M_K$, the class

$$X_K(m) := \{ \ f \in F \ : \ m(f) = m \ \}.$$

The key-point of the paper is the following theorem

THEOREM 2.1. – *Given* $K \in \mathbb{N}$ *and* $m \in M_K$, *there exists a unique element* $f = \mathcal{M}_{K,m}$ *in the class* $X_K(m)$, *characterized by one of the following equivalent properties:*

(i) *there is* $-\infty < a_K \leq \cdots \leq a_1 < +\infty$, *such that for a.e.* $v \in [0, +\infty)$

$$f(v) = \sum_{j=1}^{K} (-1)^{j-1} H(a_j - v); \tag{2.1.a}$$

(ii)

$$f(v) \in \{0, 1\} \text{ a.e., vanishes for large } v \text{ and } TV_{(0,+\infty)} \leq K; \tag{2.1.b}$$

(iii) *for at least one function* $\theta \in \Theta$

$$\langle f, \theta \rangle = \inf_{g \in X_K(m)} \langle g, \theta \rangle; \tag{2.1.c}$$

(iv) *for all function* $\theta \in \Theta$

$$\langle f, \theta \rangle = \inf_{g \in X_K(m)} \langle g, \theta \rangle;$$

(v) *for at least one function* $\theta \in \Theta$, *there is* $\lambda \in \mathbb{R}^K$ *such that*

$$f(v) = H\left(\sum_{k=0}^{K-1} \lambda_k v^k - \theta(v) \right), \quad \forall v \geq 0; \tag{2.1.d}$$

(vi) *for each function* $\theta \in \Theta$, *there is* $\lambda \in \mathbb{R}^K$ *such that* (2.1.d) *holds true.*

In addition,

$$J_K^\theta(m) := \inf_{g \in X_K(m)} \langle g, \theta \rangle = \langle \mathcal{M}_{K,m}, \theta \rangle \tag{2.2}$$

can be written

$$J_K^\theta(m) = \sup_{\lambda \in \mathbb{R}^K} \left\{ \sum_{k=0}^{K-1} \lambda_k m_k - \int_0^{+\infty} \left[\sum_{k=0}^{K-1} \lambda_k v^k - \theta(v) \right]_+ dv \right\}. \tag{2.3}$$

and depends continuously on m *on each compact set* $M_K^L := \{m(f) \in M_K : \operatorname{supp} f \subseteq [0, L]\}$ *and* $\operatorname{supp} \mathcal{M}_{K,m} \subseteq [0, L]$ *holds true for any* $m \in M_K^L$.

Remarks. – Refering to the classical Kinetic Theory of Maxwell and Boltzmann, we call function $\mathcal{M}_{K,m}$ the *K-branch maxwellian* function associated with the attainable moment vector m : it is a piecewise constant function valued in $\{0, 1\}$ with at most K branches and characterization (iii) can be seen as an entropy maximization principle among all functions f subject to be valued in $[0, 1]$ with prescribed moments $m_0, \cdots, m_{K-1}$. This kind of moment problems with L^∞-type constraints, rather unusual in Kinetic Theory, is more familiar in Statistics and Probability Theory and known as the Markov Moment Problem [KN] (as mentioned to us by Elisabeth Gassiat).

Notice that, in the special case $K = 1$, $\mathcal{M}_{1,m_0}$ is the "characteristic" function of [B2], [LPT1]

$$\mathcal{M}_{1,m_0} = \begin{cases} 1 & \text{if } 0 < v < m_0 \\ 0 & \text{if } v > m_0 \end{cases}$$

and Theorem 2.1 is nothing but the "entropy lemma" used in [B2] and [LPT1].

Proof of Theorem 2.1.

Step 1: *existence of a K-branch maxwellian function through duality arguments.*

In this first step of the proof, let us show that for each $\theta \in \Theta$ there is a function $f \in X_K(m)$, temporarily denoted by $\mathcal{M}^\theta_{K,m}$, satisfying (2.1.c) and (2.3). Notice first that since m is attainable, we may introduce $f_0 \in F$ such that $m = m(f_0)$.

Let us now check that the infimum in (2.1.c) is finite. Since $\theta \in \Theta$, we already know that there is a constant $\gamma \in (0, 1)$ such that

$$\gamma v^K - \gamma^{-1} \leq \theta(v) \leq \gamma^{-1}(v^K + 1), \quad \forall v \geq 0.$$

Thus, for all $f \in X_K(m)$, $-\gamma^{-1}\langle f_0, 1\rangle \leq \langle f, \theta\rangle < +\infty$.

Let us now introduce two useful convex functions on E, valued in $(-\infty, +\infty]$,

$$\psi(g) := \begin{cases} -\langle f_0, g\rangle & \text{if } g \in P_K \\ +\infty & \text{otherwise} \end{cases}$$

where $P_K := \mathrm{Span}\{1, v, \ldots, v^{K-1}\}$,

$$\phi(g) := \int_0^{+\infty} [g(v) - \theta(v)]_+ dv$$

and consider the minimization problem

$$I = \inf\{\phi(g) + \psi(g) \colon g \in E\}. \tag{2.4}$$

Both ϕ and ψ are convex functions on E. Let us prove that there is $g_0 \in E$ such that ϕ is bounded and continuous about g_0 and $\psi(g_0)$ is finite. This will allow us to use the Fenchel-Rockafellar Theorem [Br] and assert

$$I = \max\{-\phi^*(g') - \psi^*(-g');\ g' \in E'\} \tag{2.5}$$

where ϕ^* and ψ^* are the Legendre-Fenchel transform of ϕ and ψ respectively. It is enough to take $g_0 = 0$. Indeed, $\psi(0) = 0$ and for any $g \in E$ in the ball centered at 0 of radius $\gamma/2$, we have

$$g(v) - \theta(v) \leq -(\gamma/2)\ v^K + 2\gamma^{-1}$$

so that $\phi(g)$ is bounded and continuous at $g = 0$. (Use Lebesgue's convergence theorem, for instance.) Let us now compute the Legendre-Fenchel transforms of ϕ and ψ involved in formula (2.5). First,

$$\begin{aligned}\psi^*(-g') &= \sup\{\langle -g', g\rangle - \psi(g);\ g \in E\}\\ &= \sup\left\{\sum_{k=0}^{K-1} \lambda_k(m_k(f_0) - \langle g', v^k\rangle);\ \lambda \in \mathbb{R}^K\right\}\end{aligned}$$

and

$$\psi^*(-g') = \begin{cases} 0 & \text{if } m(g') = m \\ +\infty & \text{otherwise}\end{cases} \tag{2.6}$$

follows. Then, we observe that ϕ precisely is the Legendre-Fenchel transform of $\tilde{\phi}$ defined by

$$\tilde{\phi}(g') := \begin{cases} \langle g', \theta\rangle & \text{if } g' \in F \\ +\infty & \text{if } g' \in E' \setminus F.\end{cases}$$

Indeed,

$$\begin{aligned}\phi(g) &= \sup\{\langle g', g - \theta\rangle;\ g' \in F\}\\ &= \sup\{\langle g', g\rangle - \tilde{\phi}(g');\ g' \in E'\}\\ &= \tilde{\phi}^*(g).\end{aligned}$$

Since F is a non empty closed convex subset of E', $\tilde{\phi}$ is a convex l.s.c. function on E' not identically equal to $+\infty$. Thus, the Fenchel-Moreau Theorem [Br] yields $\tilde{\phi}^{**} = \tilde{\phi}$ so that $\phi^* = \tilde{\phi}$. The last together with (2.5) and (2.6) leads to

$$I = \max\{-\langle g', \theta\rangle;\ g' \in X_K(m)\}, \tag{2.7}$$

that is $I = -J_K^\theta(m)$ (with notation (2.2)). So, we have shown the existence of a solution f, denoted by $\mathcal{M}_{K.m}^\theta$, to problem (2.1.c) since the maximum is achieved in (2.7). Moreover, we get from (2.4) a new expression for $I = -J_K^\theta(m)$. Indeed,

$$\begin{aligned}I &= \inf\left\{\int_0^{+\infty}[g(v)-\theta(v)]_+dv - \langle f_0, g\rangle;\ g\in P_K\right\}\\ &= -\sup_{\lambda\in\mathbb{R}^K}\left\{\sum_{k=0}^{K-1}\lambda_k m_k - \int_0^{+\infty}\left[\sum_{k=0}^{K-1}\lambda_k v^k - \theta(v)\right]_+ dv\right\}\\ &= -S_\theta^*(m)\end{aligned}$$

where

$$S_\theta(\lambda) := \int_0^{+\infty}\left[\sum_{k=0}^{K-1}\lambda_k v^k - \theta(v)\right]_+ dv.$$

So,

$$J_K^\theta(m) = S_\theta^*(m), \tag{2.8}$$

and (2.3) follows.

Step 2: the structure of a K-branch maxwellian function and the uniqueness.

Let us now show that (2.1.c) implies (2.1.d). In order to do that, we first prove that the supremum defining $S_\theta^*(m)$ as $S_\theta^*(m) = \sup\{\lambda\cdot m - S_\theta(\lambda);\ \lambda\in\mathbb{R}^K\}$ is achieved. We define

$$\sigma_\theta(\lambda) := \lambda\cdot m - S_\theta(\lambda) \qquad \lambda\in\mathbb{R}^K$$

and consider $\{\lambda_n\}$ a maximizing sequence. If m is attainable, we have shown that $\sup\limits_{\lambda\in\mathbb{R}^K}\sigma_\theta(\lambda) = S_\theta^*(m) = J_K^\theta(m)$ is finite. Two cases have to be investigated, according to the boundedness of sequence $\{\lambda_n\}$. If this sequence is bounded, there exists an accumulation point $\overline{\lambda}$ and by continuity $\sigma_\theta(\overline{\lambda}) = \sup\limits_{\lambda\in\mathbb{R}^K}\sigma_\theta(\lambda)$, *i.e.* the supremum is achieved. If $\{\lambda_n\}$ is unbounded, let us introduce $\overline{\lambda}_n := \frac{\lambda_n}{|\lambda_n|}$, so that $\{\overline{\lambda}_n\}$ is bounded and has an accumulation point $\overline{\lambda}$. Then,

$$\frac{\sigma_\theta(\lambda_n)}{|\lambda_n|} = \overline{\lambda}_n\cdot m - \int_0^{+\infty}\left[\sum_{k=0}^{K-1}\overline{\lambda}_{n,k}v^k - \frac{\theta(v)}{|\lambda_n|}\right]_+ dv$$

and, when $n\to+\infty$, by Fatou's lemma,

$$\int_0^{+\infty}\left[\sum_{k=0}^{K-1}\overline{\lambda}_k v^k\right]_+ dv \le \overline{\lambda}\cdot m$$

since $\frac{\sigma_\theta(\lambda_n)}{|\lambda_n|} \to 0$. Thus,

$$\int_0^{+\infty} \left[\sum_{k=0}^{K-1} \overline{\lambda}_k v^k\right]_+ dv \leq \left\langle f, \sum_{k=0}^{K-1} \overline{\lambda}_k v^k \right\rangle$$

for all $f \in X_K(m)$ and necessarily

$$f(v) = H\left(\sum_{k=0}^{K-1} \overline{\lambda}_k v^k\right), \quad \forall v \geq 0.$$

This means, in particular, that $X_K(m)$ has a single element $f = f_0$ which necessarily satisfies

$$\int_0^{+\infty} \theta(v) f(v) dv = J_K^\theta(m).$$

Next we observe that, since the K-th derivative of θ is positive, the function $\sum_{k=0}^{K-1} \lambda_k v^k - \theta(v)$ has at most K zeros on $\mathbb{R}$ for any $\lambda \in \mathbb{R}^K$. Hence, we can find $\mu \in \mathbb{R}^K$ such that the function $\sum_{k=0}^{K-1} \mu_k v^k - \theta(v)$ has exactly the same zeros of $\sum_{k=0}^{K-1} \overline{\lambda}_k v^k$ on $\mathbb{R}^+$ with the same multiplicity, so that

$$f(v) = H\left(\sum_{k=0}^{K-1} \mu_k v^k - \theta(v)\right), \quad \forall v \geq 0.$$

Thus,

$$S_\theta(\mu) = \int_0^{+\infty} \left(\sum_{k=0}^{K-1} \mu_k v^k - \theta(v)\right) f(v) dv = \mu \cdot m - J_K^\theta(m),$$

which yields

$$\sigma_\theta(\mu) = \mu \cdot m - S_\theta(\mu) = J_K^\theta(m) = \sup_{\lambda \in \mathbb{R}^K} \sigma_\theta(\lambda)$$

and shows that μ is a maximizer for σ_θ.

So, we have found in all cases, a maximizer $\lambda \in \mathbb{R}^K$ for $\lambda \cdot m - S_\theta(\lambda)$. As a consequence, we get that the solution $f = \mathcal{M}_{K,m}^\theta$ to the minimization problem (2.1.c), found at the previous step, satisfies (iv), namely (2.1.d), precisely for the $\lambda \in \mathbb{R}^K$ that we have just found. Indeed, equality (2.3) reads

$$\langle f, \theta \rangle = \sum_{k=0}^{K-1} \lambda_k \langle f, v^k \rangle - \int_0^{+\infty} \left[\sum_{k=0}^{K-1} \lambda_k v^k - \theta(v)\right]_+ dv.$$

Since f is valued in $[0, 1]$, f has necessarily the form (2.1.d) and satisfies (iv).

The proof of uniqueness for the solution $f = \mathcal{M}^{\theta}_{K,m}$ is straightforward. Indeed, the set of all solutions of (2.1.c) is convex, and by (2.1.d), all these solutions must be valued a.e. in the discrete set $\{0, 1\}$. This is possible only if there is a unique solution.

Step 3: *Equivalence and independence of* $\mathcal{M}^{\theta}_{K,m}$ *on* θ.

Let us show that properties (i)$\cdots$(vi) are equivalent. Obviously, (i) and (ii) are equivalent. (Notice that the total variation is computed on the open set $(0, +\infty)$ although some of the a_j may lie in $(-\infty, 0]$.) Next, we know from the previous step that (iii) implies (v). Now, assume (v) and write (2.1.d). Then (i) and (2.1.a) immediately follows since

$$v \rightarrow \sum_{k=0}^{K-1} \lambda_k v^k - \theta(v)$$

has a negative K-the derivative and, therefore, has at most K zeros on the real line. Moreover, $(\sum_{k=0}^{K-1} \lambda_k v^k - \theta(v) \rightarrow -\infty$ as $v \rightarrow +\infty$. Next, assume (i) and consider any function $\theta \in \Theta$. Then we can solve the Vandermonde problem and find $\lambda \in \mathbb{R}^K$ (depending on θ) such that

$$\sum_{k=0}^{K-1} \lambda_k v^k = \theta(v)$$

vanishes on the real line if and only if v is one of the a_j for $j = 1, \dots, K$. Therefore (vi) and (2.1.d) immediately follows. Now, assume (vi) and (2.1.d). Then, for each $g \in X_K(m)$, we have

$$\begin{aligned}
\langle f, \theta \rangle &= \int_0^{+\infty} H\Big(\sum_{k=0}^{K-1} \lambda_k v^k - \theta(v)\Big)\theta(v)dv \\
&= \sum_{k=0}^{K-1} \lambda_k m_k - \int_0^{+\infty} \Big[\sum_{k=0}^{K-1} \lambda_k v^k - \theta(v)\Big]_+ dv \\
&\leq \sum_{k=0}^{K-1} \lambda_k m_k - \int_0^{+\infty} \Big(\sum_{k=0}^{K-1} \lambda_k v^k - \theta(v)\Big) g(v)dv = \langle g, \theta \rangle.
\end{aligned}$$

which implies (iv). The proof of equivalence is complete since (iv) implies necessarily (iii). The independence of the K-branch maxwellian function is a straightforward consequence of this equivalence and its uniqueness.

Step 4: supp$\mathcal{M}_{K,m} \subseteq [0, L]$ *for any* $m \in M_K^L$.

Observe first that the maxwellian $\mathcal{M}_{K,m}$ has compact support in $[0, +\infty)$, as follows from (2.1.d), since $(\sum_{k=0}^{K-1} \lambda_k v^k - \theta(v)) \to -\infty$ as $v \to +\infty$. Let us assume that $m \in M_K^L$ where $M_K^L := \{m(f) \in M_K : \mathrm{supp} f \subseteq [0, L]\}$ and show that the corresponding maxwellian is necessarily supported by $[0, L]$. For any $R \in \mathbb{R}^+$, we consider $\theta_R \in \Theta$ defined by

$$\theta_R(v) := v^K + (v - L)_+^K R.$$

If $\mathcal{M}_{K,m}(v) = 1$ on $(a, b) \subseteq [L, +\infty)$, then,

(2.9)
$$\frac{L^{K+1}}{K+1} \geq \langle f, v^K \rangle = \langle f, \theta_R \rangle \geq \langle \mathcal{M}_{K,m}, \theta_R \rangle \geq \int_a^b \theta_R(v) dv \to +\infty,$$

when $R \to +\infty$ and we get a contradiction.

Step 5: *continuity of* $J_K^\theta(m)$.

Given $\overline{m} \in M_K^L$, let m_ε be any sequence in M_K^L converging to $\overline{m}$. Let $f_\varepsilon = \mathcal{M}_{K,m_\varepsilon}$. Since f_ε is bounded in $L^\infty(\mathbb{R}^+)$ uniformly in ε, there exists a subsequence f_{ε_j} and $f \in L^\infty(\mathbb{R}^+)$ such that $f_{\varepsilon_j} \rightharpoonup f$ in $L^\infty(\mathbb{R}^+)$ weak $*$. Moreover, $0 \leq f(v) \leq 1$ a.e. $v \in \mathbb{R}^+$, supp$f \subseteq [0, L]$ and $\int_0^{+\infty} v^k f(v) dv = \overline{m}_k$, $k = 0, \ldots, K-1$. Then $f \in X_K(\overline{m})$ and $\overline{m} \in M_K^L$, *i.e.* M_K^L is compact. Next, we observe that f_ε has total variation $TV_{(0,+\infty)}(f_\varepsilon)$ bounded by K uniformly in ε. Hence, by Helly Theorem and passing to a subsequence if necessary, $f_{\varepsilon_j}(v) \to f(v)$ a.e. $v \in \mathbb{R}^+$. The last implies that f is valued in $\{0, 1\}$ as f_ε and $TV_{(0,+\infty)}(f) \leq K$. Therefore, f is the K-branch maxwellian corresponding to $\overline{m}$ and

$$(2.10) \quad J_K^\theta(m_{\varepsilon_j}) = \int_0^{+\infty} \theta(v) f_{\varepsilon_j}(v) dv \to \int_0^{+\infty} \theta(v) f(v) dv = J_K^\theta(\overline{m}).$$

Since the limit in (2.10) holds true for the sequence m_ε, the continuity of $J_K^\theta(m)$ on M_K^L is proved. By the same argument, we also have

COROLLARY 2.3. – *For any* $1 \leq p < +\infty$ *and* $L > 0$, $\{\mathcal{M}_{K,m(f)}, f \in F : \mathrm{supp} f \subseteq [0, L]\}$ *is a compact subset of* $L^p(\mathbb{R}^+)$.

COROLLARY 2.4. – *For any sequence* $f_\varepsilon \in \{f \in F : \mathrm{supp} f \subseteq [0, L]\}$ *weakly convergent to* f, $\mathcal{M}_{K,m(f_\varepsilon)}$ *converges for a.e.* v *to* $\mathcal{M}_{K,m(f)}$.

3. ENTROPY K-MULTIVALUED SOLUTIONS

We define an *entropy K-multivalued* solution to be any measurable function $f(t,x,v)$ on $(0,\infty)\times\mathbb{R}_x\times\mathbb{R}_v^+$ valued in $[0,1]$ such that

$$\partial_t f + v\partial_x f + (-\partial_v)^K \mu = 0 \qquad \text{in } \mathcal{D}'((0,\infty)\times\mathbb{R}_x\times\mathbb{R}_v^+) \tag{3.1}$$

for some nonnegative Radon measure $\mu(t,x,v)$ on $(0,\infty)\times\mathbb{R}_x\times\mathbb{R}_v^+$, subject to

$$f(t,x,v) = \mathcal{M}_{K,m(f(t,x,\cdot))}(v) \qquad \text{a.e. } (t,x,v)\in(0,\infty)\times\mathbb{R}_x\times\mathbb{R}_v^+. \tag{3.2}$$

Equation (3.1) is a nonlinear evolution equation with a non local nonlinearity $(\partial_v^K)\mu$ arising from the nonlinear constraint (3.2) of which $\mu\geq 0$ can be seen as the Lagrange multiplier. This formulation can already be found in [LPT1], [LPT2], when $K=1$ and $K=2$ respectively, with a clear connection with the Burgers equation in the case $K=1$. In the same way as Lions, Perthame and Tadmor, we get the equivalence between the above kinetic formulation and a closed K moments system derived from (1.6).

THEOREM 3.1. – *A measurable function f with* $\mathrm{supp}_v f(t,x,\cdot)\subseteq[0,L]$ *satisfying* (3.2) *is an* entropy K-multivalued *solution if and only if, for every smooth function θ, the distribution*

$$\partial_t \int_0^{+\infty} \theta(v) f(t,x,v)dv + \partial_x \int_0^{+\infty} v\theta(v) f(t,x,v)dv \tag{3.3}$$

is non positive if the K-th derivative of θ is everywhere positive and null if this derivative is identically zero. Moreover, the moment vector $m=m(f)$ is a distributional solution to the nonlinear hyperbolic system of conservation laws

$$\partial_t m_k + \partial_x m_{k+1} = 0, \qquad k=0,1,\ldots,K-1 \tag{3.4}$$

obtained by closing $m_K = J_K^\theta(m_1,\ldots,m_{K-1})$ with $\theta(v)=v^K$, and to the entropy inequalities

$$\partial_t J_K^\theta(m) + \partial_x Z_K^\theta(m) \leq 0, \tag{3.5}$$

where $Z_K^\theta(m) := \int_0^{+\infty} v\theta(v)\mathcal{M}_{K,m}(v)dv$, for all smooth function θ with positive K-th derivative.

Since the proof is a straightforward generalization of [LPT1], [LPT2], we skip it. (See, for example, the proof of Theorem 3 in [LPT2].) Just notice that μ is supported in the v variable in the same interval $[0, L]$ as f. Also notice that the total mass of μ is finite as soon as, at time 0, $f(t = 0, x, v)$ is supported in v in $[0, L]$ and has a finite integral. Indeed, after multiplying (3.1) by v^K and integrating in x and v, we find

$$K! \int_{\mathbb{R}_+ \times \mathbb{R}^2} \mu(dt, dx, dv) \le \int v^K f(0, x, v) dx dv \tag{3.6}$$

and the right hand side can be bounded by $L^K \int f(0, x, v) dx dv$.

Remark 3.2. – For all smooth function θ with positive K-th derivative, the function $J_K^\theta(m)$ defined in Theorem 2.1 is a convex entropy for system (3.4), the convexity in $m \in M_K$ following from formula (2.3). The associated entropy fluxes are of course $\int_0^{+\infty} v\theta(v)\mathcal{M}_{K,m}(v)dv$. As a matter of fact, (3.4) can be diagonalized as K uncoupled inviscid Burgers equations. Indeed, according to (3.2) and (2.1.a), we can write the solution f of (3.1) (3.2) as

$$f(t, x, v) = \sum_{j=1}^{K} (-1)^{j-1} H(a_j(t, x) - v),$$

where $a_1 \ge a_2 \ge \cdots \ge a_K$ (with some ambiguity when these inequalities are not all strict). When they are distinct and smooth, the a_i are solutions to the inviscid Burgers equation

$$\partial_t a + \partial_x \left(\frac{1}{2} a^2 \right) = 0$$

and provide Riemann-invariants for system (3.4).

Remark 3.3. – Kruzhkov entropy solutions for the inviscid Burgers equation correspond to the case $K = 1$, as shown in [LPT1], and isentropic gas dynamics with $\gamma = 3$ corresponds to $K = 2$ as in [LPT2]. In this case, we found that $J_2^\theta(m)$ is the weak kinetic entropy computed in [LPT2] as the fundamental solution to the wave equation for entropies.

Coming back to the multivalued solution of the inviscid Burgers equation, it is clear that any "classical" multivalued solution with K branches is a trivial solution to (3.1)-(3.2) with measure $\mu \equiv 0$. If, after some time, new branches develop, then this multivalued solution will differ from the entropy K-multivalued solution, just as in the well known case $K = 1$, when shocks develop (see [B2], [LPT1]). Anyway, for smooth initial data, there exist an increasing sequence $T_l \in (0, +\infty]$, $T_l = T_l(u_0)$, such that the "classical" multivalued solution to the Burgers equation is an entropy l-multivalued solution in the time interval $(0, T_l)$. Notice, finally, that it is worth considering a slightly more general framework when $f_0(x, v)$ is the "signature" function of a domain D_0 contained in a slab $0 \leq v \leq R$, not necessarily limited by the x-axis and the graph of a singlevalued function, *i.e.*

$$f_0(x, v) := \begin{cases} 1 & \text{if } (x, v) \in D_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, if f is the solution of the free transport equation (1.2) with the above initial condition, it holds true that

THEOREM 3.5. – *f is a solution of* (3.1)-(3.2) *with $\mu \equiv 0$ if and only if for a.e. (t, x) the set $\{v \in \mathbb{R}^+ : f(t, x, v) = 1\}$ has at most N connected components where $N = \frac{K+1}{2}$ if K is odd and $N = \frac{K}{2}$ if K is even.*

We conclude this section showing compactness property of averages in the v variable of entropy K-multivalued solutions. This result is a straightforward consequence of the velocity averaging lemma in [LPT1] that is an appropriate adaptation of the averaging lemmas shown in a rather complete way in [DLM].

PROPOSITION 3.6. – *Let $f_\varepsilon \in L^\infty((0, \infty) \times \mathbb{R}_x \times \mathbb{R}_v^+)$ verifies* (3.1) *for some nonnegative measure μ_ε on $(0, \infty) \times \mathbb{R}_x \times \mathbb{R}_v^+$, bounded uniformly in $\varepsilon > 0$. If f_ε is bounded uniformly in ε, then $\int_0^{+\infty} f_\varepsilon \psi dv$ belongs to a compact set of $L^p_{loc}((0, \infty) \times \mathbb{R}_x)$, $1 < p < \infty$ for any $\psi \in L^{p'}(\mathbb{R}_v^+)$ with compact support.*

Proof. – We only need to show that there exists $s \in [0, 1)$, $m \geq 0$ and g_ε belonging to a compact set of $L^p_{loc}((0, \infty) \times \mathbb{R}_x \times \mathbb{R}_v^+)$ such that

$$(-1)^{K-1}(\partial_v^K)\mu_\varepsilon = (1 - \Delta_{x,t})^{s/2}(1 - \Delta_v)^{m/2} g_\varepsilon. \tag{3.8}$$

Indeed, the last is a consequence of the Sobolev imbedding theorems since $W^{s,p'}((0, \infty) \times \mathbb{R}_x \times \mathbb{R}_v^+) \hookrightarrow C^0((0, \infty) \times \mathbb{R}_x \times \mathbb{R}_v^+)$ for $1 > s > 3/p'$, so that μ_ε is bounded in $W^{-s,p}((0, \infty) \times \mathbb{R}_x \times \mathbb{R}_v^+)$ and (3.8) holds true with $m = s + K$. ■

4. EXISTENCE OF ENTROPY K-MULTIVALUED SOLUTIONS

In order to prove the existence of entropy K-multivalued solutions to (3.1)-(3.2), we consider the BGK-like approximation

$$\partial_t f_\varepsilon + v \partial_x f_\varepsilon = \frac{1}{\varepsilon}\left[\mathcal{M}_{K, m(f_\varepsilon)} - f_\varepsilon\right] \tag{4.1}$$

and we pass to the limit as ε goes to 0 in it. The existence result is then a consequence of compactness properties of the solutions of (4.1). The main tools used are the properties of the K-branch maxwellian arising by the basic Theorem 2.1 and the averaging lemmas. Since we are interested in the existence of an entropy K-multivalued solution, in this section the existence of a solution of (4.1) is assumed. It will be proved for sake of completeness in the appendix.

Let $f^0(x,v)$ be a measurable function on $\mathbb{R} \times [0,+\infty)$, valued in $\{0,1\}$, such that $f^0(x,\cdot) = \mathcal{M}_{K,m(f^0(x,\cdot))}$ for a.e. $x \in \mathbb{R}$, $\mathrm{supp}_v f^0(x,\cdot) \subseteq [0,L]$ for some $L \in \mathbb{R}^+$ and

$$\int f^0(x,v)dxdv < +\infty .$$

Next, let us observe that, by Duhamel's principle, equation (4.1) admits the following equivalent integral representation

$$f_\varepsilon(t,x,v) = e^{-t/\varepsilon} f^0(x-tv,v) + \frac{1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}(t-s)} \mathcal{M}_{K,m(f_\varepsilon(s,x-(t-s)v,v))}(v)ds$$

when we take f^0 as the initial condition. Then, the properties of the maxwellian yield immediately that $0 \le f_\varepsilon \le 1$ and $\mathrm{supp}_v f_\varepsilon \subseteq [0,L]$. Moreover, for every smooth function θ, the distribution on $(0,T) \times \mathbb{R}_x$

$$\partial_t \int_0^{+\infty} \theta(v) f_\varepsilon(t,x,v)dv + \partial_x \int_0^{+\infty} v\theta(v) f_\varepsilon(t,x,v)dv$$

is non positive if the K-th derivative of θ is everywhere positive and null if this derivative is identically zero. Thus (just as in Theorem 3.1), this is possible if and only if there exists a non negative Radon measure μ_ε on $(0,T) \times \mathbb{R}_x \times \mathbb{R}_v^+$ such that

$$-(-\partial_v)^K \mu_\varepsilon = \frac{1}{\varepsilon}\left[\mathcal{M}_{K,m(f_\varepsilon)} - f_\varepsilon\right].$$

In addition,

$$K! \int \mu_\varepsilon(dt,dx,dv) \le L^K \int f^0(x,v)dxdv < +\infty,$$

i.e. μ_ε has total mass bounded uniformly in ε. However, f_ε is not an entropy K-multivalued solution since (3.2) is not satisfied a.e., in general. Anyway, the uniform bounds of the sequence $\{f_\varepsilon\}$ and the family $\{\mu_\varepsilon\}$ allow us to pass to the limit as $\varepsilon \to 0$ in order to obtain the following result.

THEOREM 4.2. – *Under the above hypotheses on the initial condition* f^0, *there exists a subsequence* f_ε *and an entropy* K*-multivalued solution* f *such that* f_ε *converges to* f *in* L^∞ *weak-*$*$, $m(f) \in C([0,+\infty), \mathcal{D}'(\mathbb{R}_x))$ *and* $f(0,\cdot,\cdot) = f^0$.

Proof. – Since f_ε and μ_ε are bounded uniformly in ε and supported in v in $[0,L]$, there exist a bounded measurable function $f(t,x,v)$ a bounded nonnegative Radon measure $\mu(t,x,v)$, on $(0,T)\times\mathbb{R}_x\times\mathbb{R}_v^+$, such that, up to subsequences, $f_\varepsilon \rightharpoonup f$ and $\mu_\varepsilon \rightharpoonup \mu$ weakly. Moreover, $0\le f(t,x,v)\le 1$, $\mathrm{supp}_v f\subseteq[0,L]$ and $\partial_t f + v\partial_x f + (-\partial_v)^K\mu = 0$ on $\mathcal{D}'((0,T)\times\mathbb{R}_x\times\mathbb{R}_v^+)$.

Let us now prove that f satisfies (3.2) which will complete the proof. From Proposition 3.6, we first notice that the moments

$$m_k^\varepsilon(t,x) := \int_0^{+\infty} v^k f_\varepsilon(t,x,v)dv$$

belong to a compact set of $L^p_{loc}((0,T)\times\mathbb{R}_x)$, $p\in(1,\infty)$, for any $k\in\mathbb{N}$. Therefore, there exists a subsequence m_k^ε converging a.e. (t,x) to $m_k(t,x) := \int_0^{+\infty} v^k f(t,x,v)dv$ solutions to (3.4)-(3.5). Next, we observe that $[\mathcal{M}_{K,m(f_\varepsilon)} - f_\varepsilon] \rightharpoonup 0$ in $\mathcal{D}'((0,T)\times\mathbb{R}_x\times\mathbb{R}_0^+)$ weak $*$. Then, we get for any $\phi\in\mathcal{D}((0,T)\times\mathbb{R}_x)$, $\phi\ge 0$, and any smooth function θ with positive K-th derivative,

$$\int J_K^\theta(m(t,x))\phi(t,x)dxdt \le \int \theta(v)f(t,x,v)\phi(t,x)dvdxdt$$

(according to (2.2), observing that $f\in X_K(m(f))$)

$$= \lim_{\varepsilon\to 0}\int \theta(v)f_\varepsilon(t,x,v)\phi(t,x)dvdxdt$$

$$= \lim_{\varepsilon\to 0}\int \theta(v)\mathcal{M}_{K,m(f_\varepsilon(t,x,v))}(v)\phi(t,x)dvdxdt$$

(as just observed above)

$$= \lim_{\varepsilon\to 0}\int J_K^\theta(m^\varepsilon(t,x))\phi(t,x)dxdt$$

(by definition (2.2) of J_K^θ)

$$= \int J_K^\theta(m(t,x))\phi(t,x)dxdt,$$

the last inequality being consequence of the continuity of J_K^θ on the compact sets M_K^L and of the Lebesgue's convergence theorem. So, f must satisfy (3.2) and the proof is now complete. ■

Remark 4.3. – (i) As long as the "classical" multivalued solution to the inviscid Burgers equation has no more than K branches, the semi-discrete scheme provides the exact solution. The same phenomenon was already pointed out in [B2] when $K = 1$. (ii) The uniqueness problem is open.

Remark 4.4. – In order to obtain an existence result for entropy K-multivalued solutions, one could consider a time splitting method with a free transport step followed by a projection on the maxwellian functions. This method generalizes the "transport-collapse" method of [B2] (see also [B1], [GM]) for the approximation of the entropy solution ($K = 1$). In the case $K > 1$, a technical problem arises : the strong compactness of the vector moment associated to the approximating solution and computed at the discrete times is not obvious. So the proof of the convergenge of the splitting method is still an interesting open problem.

5. APPENDIX

As mentioned in the introduction, this appendix is devoted to the proof of the existence of a solution for the BGK-like approximation

$$\partial_t f_\varepsilon + v\partial_x f_\varepsilon = \frac{1}{\varepsilon}\left[\mathcal{M}_{K,m(f_\varepsilon)} - f_\varepsilon\right] \tag{5.1}$$

with initial condition f^0 given as in the previous section. Since the existence result is obtained by a time splitting method, we closely follow the method introduced in [DM] and we skip some technical details. Notice that a more direct but less constructive proof can be obtained by Schauder's fixed point theorem.

Let us denote $\Delta t := TN^{-1}$ the time step with $N \in \mathbb{N}$, and $T > 0$. Let f^n and g^n be respectively the solution of

$$\begin{aligned}&\partial_t f^n + v\partial_x f^n = 0\\ &f^n(n\Delta t, x, v) = g^{n-1}(n\Delta t, x, v)\end{aligned}$$

and

$$\begin{aligned}\partial_t g^n &= \frac{1}{\varepsilon}\left[\mathcal{M}_{K,m(g^n)} - g^n\right]\\ g^n(n\Delta t, x, v) &= f^n((n+1)\Delta t, x, v)\end{aligned}$$

in the time interval $[n\Delta t, (n+1)\Delta t]$, $n = 0, \ldots, N-1$, where $g^{-1} := f^0$. It holds true that f^n and g^n belong to $C([n\Delta t, (n+1)\Delta t]; L^1(\mathbb{R}_x \times \mathbb{R}_v^+))$. Moreover, the moment vector $m(g^n)$ is constant with respect to $t \in$

$[n\Delta t, (n+1)\Delta t]$ as well as $\mathcal{M}_{K,m(g^n)}$. Therefore, by Duhamel's principle, g^n can be written as

$$g^n(t,x,v) = e^{-(t-n\Delta t)/\varepsilon} g^n(n\Delta t, x, v) + [1 - e^{-(t-n\Delta t)/\varepsilon}] \mathcal{M}_{K,m(g^n(n\Delta t,x,v))}(v).$$

Next, let us define on $[0,T] \times \mathbb{R}_x \times \mathbb{R}_v^+$

$$f_\Delta(t,x,v) := f^n(t,x,v), \quad g_\Delta(t,x,v) := g^n(t,x,v) \quad t \in [n\Delta t, (n+1)\Delta t).$$

The functions f_Δ and g_Δ verify in $\mathcal{D}'((0,T) \times \mathbb{R}_x \times \mathbb{R}_v^+)$ respectively the following equations

(5.2)
$$\partial_t f_\Delta + v\partial_x f_\Delta = \frac{1}{\varepsilon} \sum_{n=1}^{N-1} \left(\int_{(n-1)\Delta t}^{n\Delta t} [\mathcal{M}_{K,m(g_\Delta)} - g_\Delta](s) ds \right) \delta(t - n\Delta t)$$

and

(5.3)
$$\partial_t g_\Delta = \frac{1}{\varepsilon} [\mathcal{M}_{K,m(g_\Delta)} - g_\Delta] + \sum_{n=1}^{N-1} \left[\int_{n\Delta t}^{(n+1)\Delta t} -v\partial_x f_\Delta(s,x,v) ds \right] \delta(t - n\Delta t).$$

Moreover, $0 \leq f_\Delta,\ g_\Delta \leq 1$, $\mathrm{supp}_v f_\Delta \subseteq [0,L]$, $\mathrm{supp}_v g_\Delta \subseteq [0,L]$, and for Proposition 3.6 the vector moment $m(f_\Delta)$ lies in a strongly compact set of $L^p_{\mathrm{loc}}([0,T] \times \mathbb{R})$, $p \in (1,+\infty)$, while, following [DM], it can be proved that

LEMMA 5.1. – *The moment vector $m(g_\Delta)$ lies in a strongly compact set of $L^p([0,T] \times \mathbb{R})$, $p \in [2,+\infty)$.*

Proof. – Let us denote $m_k^\Delta := m_k(g_\Delta)$. Then, for any $\phi \in C_0^1(\mathbb{R})$ and $k = 0,\ldots,K-1$, we have from (5.3)

$$(5.4) \quad \frac{d}{dt} \int m_k^\Delta(t,x)\phi(x)dx = \sum_{n=1}^{N-1} \delta(t - n\Delta t) \int_{n\Delta t}^{(n+1)\Delta t} \left[\int v^{k+1} f_\Delta(s,x,v)\phi'(x) dx dv \right] ds.$$

Since the right-hand side of (5.4) is a Radon measure on $[0,T]$ bounded uniformly in Δt, the sequence $\int m_k^\Delta(t,x)\phi(x)dx$ is bounded in $BV([0,T])$ uniformly in Δt. The last implies that for any sequence of mollifiers ρ_ε, the sequence $m_k^\Delta *_x \rho_\varepsilon$ is strongly compact in $L^1([0,T] \times \mathbb{R})$ with respect to Δt. Since $m_k^\Delta *_x \rho_\varepsilon$ is also bounded in $L^\infty([0,T] \times \mathbb{R})$, uniformly in

Δt and ε, the sequence $m_k^\Delta *_x \rho_\varepsilon$ is stronlgy compact in $L^2([0,T] \times \mathbb{R})$. Next, thanks to the identity

$$m_k^\Delta = m_k^\Delta *_x \rho_\varepsilon + (m_k^\Delta - m_k^\Delta *_x \rho_\varepsilon),$$

the sequence m_k^Δ lies in a strongly compact set of $L^2([0,T] \times \mathbb{R})$ if the term $(m_k^\Delta - m_k^\Delta *_x \rho_\varepsilon)$ is small in $L^2([0,T] \times \mathbb{R})$ when ε goes to 0 uniformly in Δt. Moreover

$$\|m_k^\Delta - m_k^\Delta *_x \rho_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R})} \le \int dy\, \rho_\varepsilon(y) \int |m_k^\Delta(t,x) - m_k^\Delta(t,x-y)|^2 dx dt.$$

Therefore, it is sufficient to prove that

$$I_y := \sup_{\Delta t > 0} \int |m_k^\Delta(t,x) - m_k^\Delta(t,x-y)|^2 dx dt$$

is small when $y \in B(0,\varepsilon)$. Since

$$I_y = \sup_{\Delta t>0} \Delta t \sum_{n=0}^{N-1} \int |m_k^\Delta(n\Delta t, x) - m_k^\Delta(n\Delta t, x-y)|^2 dx$$

and $m_k^\Delta(n\Delta t, \cdot) \in (L^1 \cap L^2)(\mathbb{R})$, denoting $\widehat{h}$ the Fourier transform with respect to x and $m_k^n(x) := m_k^\Delta(n\Delta t, x)$, we have

$$\begin{aligned} I_y &= \sup_{\Delta t>0} \Delta t \sum_{n=0}^{N-1} \int |1 - e^{-iy\xi}|^2 |\widehat{m_k^n}(\xi)|^2 d\xi \\ &\le \sup_{\Delta t>0} \Delta t \sum_{n=0}^{N-1} \int_{|\xi|\le R} (yR)^2 |\widehat{m_k^n}(\xi)|^2 d\xi \\ &\quad + 4 \sup_{\Delta t>0} \Delta t \sum_{n=0}^{N-1} \int_{|\xi|>R} |\widehat{m_k^n}(\xi)|^2 d\xi \\ &\le TC\|f^0\|^2_{L^1(\mathbb{R}^2)} y^2 R^3 + 4 \sup_{\Delta t>0} \Delta t \sum_{n=0}^{N-1} \int_{|\xi|>R} |\widehat{m_k^n}(\xi)|^2 d\xi. \end{aligned}$$

It rests to prove that

$$\sup_{\Delta t>0} \Delta t \sum_{n=0}^{N-1} \int_{|\xi|>R} |\widehat{m_k^n}(\xi)|^2 d\xi$$

is small when R is sufficently large. In order to do that, we take the Fourier transform of equation (5.2), we use the Duhamel representation of the solution of the transformed equation and we take the moments of that solution. The reader can found the technical details in Lemma 4 of [DM] where the authors prove the convergence of the time splitting method for the Boltzmann equation. ■

The previous lemma allow us to obtain the existence result. Indeed, it holds true that $f_\Delta - g_\Delta \rightharpoonup 0$ since for any $\phi \in \mathcal{D}((0,T)\times\mathbb{R}_x\times\mathbb{R}_v^+)$, we have

$$\begin{aligned}
&\left|\int [f_\Delta(t,x,v) - g_\Delta(t,x,v)]\phi(t,x,v)dtdxdv\right| \\
&\leq \left|\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int [f^n(t,x,v) - f^n((n+1)\Delta t,x,v)]\phi(t,x,v)dxdvdt\right| \\
&\quad + \left|\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int [g^n(n\Delta t,x,v) - g^n(t,x,v)]\phi(t,x,v)dxdvdt\right| \\
&= \left|\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} dt\int dxdv\int_t^{(n+1)\Delta t} v\partial_x f^n(s,x,v)ds\phi(t,x,v)\right| \\
&\quad + \left|\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} dt\int dxdv\int_t^{n\Delta t} \partial_t g^n(s,x,v)ds\phi(t,x,v)\right| \\
&\leq C\|\partial_x\phi\|_{L^1}\Delta t + \frac{2}{\varepsilon}\Delta t\|\phi\|_{L^1}.
\end{aligned}$$

Next, let us denote f and $\mathcal{M}$ the L^∞ weak $*$ limit of f_Δ and $\mathcal{M}_{K,m(g_\Delta)}$ respectively. Equation (5.2) becomes when Δt goes to 0

$$\partial_t f + v\partial_x f = \frac{1}{\varepsilon}[\mathcal{M} - f].$$

Let us now prove that $\mathcal{M} = \mathcal{M}_{K,m(f)}$. For any $\phi \in \mathcal{D}((0,T)\times\mathbb{R}_x)$ and any smooth function θ with positive K-th derivative, we have

$$\begin{aligned}
&\int \theta(v)\mathcal{M}(t,x,v)\phi(t,x)dvdxdt \\
&= \lim_{\Delta t\to 0}\int \theta(v)\mathcal{M}_{K,m(g_\Delta(t,x,v))}(v)\phi(t,x)dvdxdt \\
&= \lim_{\Delta t\to 0}\int J_K^\theta(m(g_\Delta)(t,x))\phi(t,x)dxdt \\
&= \lim_{\Delta t\to 0}\int J_K^\theta(m(f)(t,x))\phi(t,x)dxdt,
\end{aligned}$$

the last inequality being a consequence of Lemma 5.1. Therefore, $J_K^\theta(m(f)(t,x)) = \int\theta(v)\mathcal{M}(t,x,v)dv$ a.e. (t,x) and the uniqueness of the maxwellian gives us the proof.

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