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Quasiconvex functions, $SO(n)$ and two elastic wells

by

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ABSTRACT. – We use $W^{1,\infty}$ approximations of minimizing sequences to study the growth of some quasiconvex functions near their zero sets. We show that for $SO(n)$, the quasiconvexification of the distance function $\mathrm{dist}^2(\cdot, SO(n))$ can be bounded below by the distance function itself. In certain cases of the incompatible two elastic well structure, we establish a similar result. We also prove that for small Lipschitz perturbations of $SO(n)$ and of the two well structure, the Young measure limits of gradients supported on these perturbed sets are Dirac masses.

1. INTRODUCTION

The study of Young measures limit of gradients [KP, BFJK] supported in various compact sets in $M^{N\times n}$ and quasiconvex relaxations of certain distance functions to these sets are very important subjects in the study of martensitic phase transitions and optimal design problems (*see* [CK, BJ1, BJ2, F, K, KS]). As far as I know, explicit relaxation formulas are hard to obtain and there are only a few known examples [Da1, Da2, KS, K, D2, DR]. Therefore, the study of the behaviour of quasiconvex functions is very difficult because we cannot work on them directly. It is closely related to the analysis of quasiconvex hulls of the neighbourhoods of the zero sets for given non-negative functions and the stability problems related to Young

measures under perturbations. In this paper, we give some examples where we can estimate the growth of certain quasiconvex functions of the form

$$F(P) = Q\mathrm{dist}^p(P, K)$$

for some closed subset $K \subset M^{N\times n}$, where $M^{N\times n}$ is the space of all $N \times n$ real matrices, without knowing the exact formula of $F(P)$, and where $Q\mathrm{dist}^p(P, K)$ is the quasiconvexification of $\mathrm{dist}^p(P, K)$, and $p > 1$. We establish these results for $SO(n)$ and for two incompatible elastic wells $SO(n) \cup SO(n)H$ when H satisfies the technical condition (1.3) below. The p-th power of a distance function to a given compact set is the simplest function from geometric point of view and the quasiconvex relaxations of this type of functions give information of quasiconvex hulls of the neighbourhoods of the zero sets of the original functions.

The main results are the following.

THEOREM 1.1. – *Suppose* $n \geq 2$. *Let* $F(P) = Q\mathrm{dist}^2(P, SO(n))$ *be the quasiconvexification* (cf. [Da1, Da2], *also* see *Definition* 2.1) *of* $\mathrm{dist}^2(P, SO(n))$, $P \in M^{n\times n}$. *Then there exists a constant* $c(n) > 0$, *such that*

$$c(n)\mathrm{dist}^2(P, SO(n)) \leq Q\mathrm{dist}^2(P, SO(n)) \leq \mathrm{dist}^2(P, SO(n)) \tag{1.1}$$

for all $P \in M^{n\times n}$.

The method we use for proving of Theorem 1.1 applies to two incompatible elastic wells under a further technical condition. Let $n \geq 3$ and

$$K = SO(n) \cup SO(n)H, \tag{1.2}$$

where $SO(n)H = \{RH,\ R \in SO(n)\}$, and H is a $n \times n$ positive definite diagonal matrix satisfying

$$n + n\ \det H - \mathrm{tr}\mathrm{adj}H - \mathrm{tr}H > 0. \tag{1.3}$$

It was proved in [Ma, Sv] that the quasiconvex hull of K defined by (1.2) remains K itself under assumption (1.3). We have,

COROLLARY 1.2. – *Suppose* $n \geq 3$, K *is given by* (1.2) *with* H *satisfying* (1.3). *Then, there exists a positive constant* $c(n, H) > 0$ *such that*

$$c(n, H)\mathrm{dist}^2(P, K,) \leq Q\mathrm{dist}^2(P, K) \leq \mathrm{dist}^2(P, K). \tag{1.4}$$

We also study the Young measure limit of gradients ([KP, BFJK]) supported 'near' $SO(n)$ and $SO(n) \cup SO(n)H$. We give some estimates

of the quasiconvex hull of the ϵ-neighbourhood of these sets. This problem is related to the stability of 'one-well' and two incompatible well structure when $n = 3$. For a compact set $K \subset M^{N\times n}$, let

$$K_\alpha = \{P \in M^{N\times n},\ \mathrm{dist}(P, K) \le \alpha\}$$

and let $Q(K_\alpha)$ be the quasiconvex hull of K_α for $\alpha > 0$. We have

COROLLARY 1.3. – *Assume* (1.3). *There exist constants* $C(n) > 0$, $C(n, H) > 0$, *such that*

$$Q(SO(n)_\epsilon) \subset SO(n)_{C(n)\epsilon}$$

and

$$Q([SO(n) \cup SO(n)H]_\epsilon) \subset [SO(n) \cup SO(n)H]_{C(n,H)\epsilon}$$

for all $\epsilon > 0$.

We always assume that $\Omega \subset \mathbb{R}^n$ is a bounded open and connected set with smooth boundary. Let $f : SO(n) \to M^{n\times n}$ and $g : SO(n) \cup SO(n)H \to M^{n\times n}$ be Lipschitz functions such that

$$|f(P)| \le \epsilon, \qquad |f(P) - f(Q)| \le \epsilon|P - Q|,\ P, Q \in SO(n), \tag{1.5}$$

$$|g(P)| \le \epsilon, \qquad |g(P) - g(Q)| \le \epsilon|P - Q|,\ P, Q \in SO(n) \cup SO(n)H, \tag{1.6}$$

for some $\epsilon > 0$.

We have

THEOREM 1.4. – *Suppose that f and g are defined as above, let K_f and K_g be their graphs, respectively,* i.e.

$$K_f = \{P + f(P),\ P \in SO(n)\},$$

$$K_g = \{P + g(P),\ P \in SO(n) \cup SO(n)H\}.$$

Let (u_j) be a bounded sequence in $W^{1,p}(\Omega; \mathbb{R}^n)$, $1 < p < \infty$, such that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^n)$ and $\mathrm{dist}(Du_j, K_f) \to 0$ ($\mathrm{dist}(Du_j, K_g) \to 0$, *respectively) almost everywhere as $j \to \infty$. Then, for sufficiently small $\epsilon > 0$ and up to a subsequence, $Du_j \to Du$ almost everywhere. In other words, K_f and K_g support only trivial Young measure limit of gradients (*see *Theorem* 2.4 *for the definition of Young measures).*

Similar to Theorem 1.1, we have

COROLLARY 1.5. – *Suppose that $n \ge 2$, and let f and g satisfy* (1.5) *and* (1.6), *respectively, for $\epsilon > 0$ sufficiently small. Let $F(P) = Q\mathrm{dist}^2(P, K_f)$,*

and let $G(P) = Q\mathrm{dist}^2(P, K_g)$ *be the quasiconvexifications of* $\mathrm{dist}^2(P, K_f)$ *and* $\mathrm{dist}^2(P, K_g)$, *respectively. Then there exists a constant* $c(n, \epsilon) > 0$ *such that*

$$c(n,\epsilon)\mathrm{dist}^2(P, K_f) \leq Q\mathrm{dist}^2(P, K_f) \leq \mathrm{dist}^2(P, K_f),$$

$$c(n,\epsilon)\mathrm{dist}^2(P, K_g) \leq Q\mathrm{dist}^2(P, K_g) \leq \mathrm{dist}^2(P, K_g),$$

for all $P \in M^{n\times n}$.

The methods we use to prove Theorem 1.1 are: (i) an approximation of $W^{1,2}$ minimizing sequences by $W^{1,\infty}$ sequences (Lemma 3.1), using the maximal function method through a modified version of [Z, Lemma 3.1]; (ii) V. Šverák's idea of showing that a sequence is a Cauchy sequence [Sv]. To prove Theorem 1.1 and Corollary 1.2, we have to make use of the special structure of the sets concerned, that is, both $SO(n)$ and $SO(n) \cup SO(n)H$ have the property

$$(\mathrm{adj}P - \mathrm{adj}Q) \cdot (P - Q) \geq \alpha|P - Q|^2, \qquad P, Q \in SO(n) \cup SO(n)H$$

for some $\alpha > 0$, where $\mathrm{adj}P$ is the transpose of the cofactors of $P \in M^{n\times n}$ (*see* [Sv]). In order to use this property, the sequences under study should be in $W^{1,n}$ at least. When $n \geq 3$, our sequence is bounded in $W^{1,2}$, therefore, we have to approximate the original sequence by a bounded sequence in $W^{1,\infty}$. I do not known how to bypass Lemma 3.1 and prove the result directly. The approximation lemma (Lemma 3.1) stands on its own right. It gives an estimate of the minimum energy of quasiconvex relaxations for distance functions $\mathrm{dist}^p(\cdot, K)$ for general compact sets $K \subset M^{N\times n}$ when we only minimize the energy on a set of bounded $W^{1,\infty}$ functions.

In §2, notation and preliminaries are given which will be used to prove our main results. §3 is devoted to establishing the approximation lemma which is crucial to the proof of Theorem 1.1. In §4, we prove the results stated above. To conclude this section, we justify that the function $\mathrm{dist}^2(P, SO(n))$ is not quasiconvex itself. In fact, it is not trivial to prove it.

PROPOSITION 1.6. – $\mathrm{dist}^2(\cdot, SO(n)) : M^{n\times n} \to \mathbb{R}$ *is not quasiconvex.*

Proof. – Let $f(P) = \min\{|P - A|^2, |P - B|^2\}$, where $A, B \in M^{N\times n}$ are fixed matrices. It was established in [K] that the quasiconvexification Qf of f has the explicit form

$$Qf(P) = \min_{0\leq\theta\leq 1} \left\{|P - \theta A - (1-\theta)B|^2 + \theta(1-\theta)[|A - B|^2 - \lambda_{\max}]\right\},$$

where $\lambda_{\max}$ is the greatest eigenvalue of the matrix $(A-B)^T(A-B)$. Now, set $A = I$, $B = J$, where I is the $n \times n$ identity matrix,

$$J = \begin{pmatrix} -I_2 & 0 \\ 0 & I_{n-2} \end{pmatrix}$$

where I_2 and I_{n-2} are 2×2 and $(n-2) \times (n-2)$ identity matrices respectively. Consider $F(P) = \min\{|P-I|^2, |P-J|^2\}$, $P \in M^{n\times n}$, we have $F(P) \geq \operatorname{dist}^2(P, SO(n))$, since $I, J \in SO(n)$. If $\operatorname{dist}^2(P, SO(n))$ was quasiconvex, by Definition 2.2 below, $QF(P) \geq \operatorname{dist}^2(P, SO(n))$, for every $P \in M^{n\times n}$. In particular, $QF(0) \geq \operatorname{dist}^2(0, SO(n))$, where 0 is the $n \times n$ zero matrix. We have

$$\operatorname{dist}^2(0, SO(n)) = n$$

and if we notice that $\lambda_{\max} = 4$ when $A = I$, $B = J$, we have

$$\begin{aligned} QF(0) &= \min_{0\leq\theta\leq 1} \left\{|\theta I + (1-\theta)J|^2 + \theta(1-\theta)[|I-J|^2 - \lambda_{\max}]\right\} \\ &= n - 2 + \min_{0\leq\theta\leq 1} \left\{|\theta I_2 - (1-\theta)I_2|^2 + \theta(1-\theta)[|2I_2|^2 - \lambda_{\max}]\right\} \\ &= n - 2 + \min_{0\leq\theta\leq 1} \{2(2\theta-1)^2 + 4\theta(1-\theta)\} = n-1. \end{aligned}$$

Contradiction. The proof is complete. □

2. NOTATION AND PRELIMINARIES

Throughout the rest of this paper Ω is a bounded open subset of $\mathbb{R}^n$. We denote by $M^{N\times n}$ the space of real $N \times n$ matrices with the $\mathbb{R}^{Nn}$ metric; hence the norm of $P \in M^{N\times n}$ is defined by $|P| = (\operatorname{tr} P^T P)^{1/2}$, where tr is the trace operator and P^T is the transpose of P. The inner product of two matrices in $M^{N\times n}$ is $P \cdot Q = \operatorname{tr} P^T Q$. For an $n \times n$ matrix P, denote by $\operatorname{adj} P$ the transpose of the cofactors of P. $SO(n)$ is the set of all rotations with determinant 1. For a compact subset $K \subset M^{N\times n}$, let $\operatorname{conv} K$, $\operatorname{diam} K$ and $\|K\|$ be the convex hull, diameter and the norm of K, respectively, where

$$\|K\| = \sup\{|P|,\ P \in K\}.$$

We write $C_0(\Omega)$ for the space of continuous functions $\phi : \Omega \to \mathbb{R}$ having compact support in Ω, and define $C_0^1(\Omega) = C^1(\Omega) \cap C_0(\Omega)$. If

$1 \leq p \leq \infty$ we denote by $L^p(\Omega;\mathbb{R}^N)$ the Banach space of mappings $u : \Omega \to R^N$, $u = (u_1, \cdots, u_N)$, such that $u_i \in L^p(\Omega)$ for each i, with norm $\|u\|_{L^p(\Omega;\mathbb{R}^N)} = \sum_{i=1}^N \|u_i\|_{L^p(\Omega)}$. Similarly, we denote by $W^{1,p}(\Omega,\mathbb{R}^N)$ the usual Sobolev space of mappings $u \in L^p(\Omega;\mathbb{R}^N)$ all of whose distributional derivatives $\frac{\partial u_i}{\partial x_j} = D_j u_i$, $1 \leq i \leq N$, $1 \leq j \leq n$, belong to $L^p(\Omega)$. $W^{1,p}(\Omega,\mathbb{R}^N)$ is a Banach space under the norm

$$\|u\|_{W^{1,p}(\Omega,\mathbb{R}^N)} = \|u\|_{L^p(\Omega;\mathbb{R}^N)} + \|Du\|_{L^p(\Omega;M^{N\times n})},$$

where $Du = (D_j u_i)$, and we define, as usual, $W_0^{1,p}(\Omega;\mathbb{R}^N)$ to be the closure of $C_0^\infty(\Omega;\mathbb{R}^N)$ in the topology of $W^{1,p}(\Omega;\mathbb{R}^N)$.

Weak and weak $*$ convergence of sequences are written as $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$, respectively. If $H \subset M^{N\times n}$, $P \in M^{N\times n}$, then we write $H + P$ to denote the set $\{P + Q : Q \in H\}$, $jH = \{jx, x \in H\}$ for an integer $j > 0$. We define the distant function for a set $K \subset M^{N\times n}$ by

$$f(P) = \operatorname{dist}(P, K) := \inf_{Q\in K} \mid P - Q \mid .$$

DEFINITION 2.1. *(see Morrey [Mo], Ball [Bl1,Bl2], Ball, Currie and Olver [BCO]). – A continuous function $f : M^{N\times n} \to \mathbb{R}$ is* **quasiconvex** *if*

$$\int_U f(P + D\phi(x))\, dx \geq f(P)\operatorname{meas}(U)$$

for every $P \in M^{N\times n}$, $\phi \in C_0^1(U;\mathbb{R}^N)$, and every open bounded subset $U \subset \mathbb{R}^n$.

For a given function, we can consider its quasiconvexification (quasiconvex relaxation):

DEFINITION 2.2. *(see Dacorogna [Da]). – Suppose $f : M^{N\times n} \to \mathbb{R}$ is a continuous function. The* **quasiconvexification** *of f is defined by*

$$\sup\{g \leq f;\ g \text{ quasiconvex }\}$$

and will be denoted denoted by Qf.

PROPOSITION 2.3. *(see Dacorogna [Da]). – Suppose $f : M^{N\times n} \to \mathbb{R}$ is continuous, then*

$$Qf(P) = \inf_{\phi\in C_0^\infty(\Omega;\mathbb{R}^N)} \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega f(P + D\phi(x))\, dx, \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. In particular the infimum in (2.1) is independent of the choice of Ω.

We use the following theorem concerning the existence and properties of Young measures from Tartar [T]. For results in a more general context and their proofs, the reader is referred to Berliocchi and Lasry [BL], Balder [Bd] and Ball [Bl3].

THEOREM 2.4. – *Let $(z^{(j)})$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^s)$. Then there exist a subsequence $(z^{(\nu)})$ of $(z^{(j)})$ and a family $\{\nu_x\}_{x\in\Omega}$ of probability measures on $\mathbb{R}^s$, depending measurably on $x \in \Omega$, such that*

$$f(z^{(\nu)}) \overset{*}{\rightharpoonup} \langle \nu_x, f(\cdot)\rangle \qquad \text{in } L^\infty(\Omega)$$

for every continuous function $f : \mathbb{R}^s \to \mathbb{R}$.

We say that ν_x is a trivial Young measure at $x \in \Omega$ if $\nu_x = \delta_A$ for some $A \in \mathbb{R}^s$, where δ_A is the Dirac mass at A.

Suppose that $\Omega \subset \mathbb{R}^n$. A family of parametrized measures $\{\nu_x\}_{x\in\Omega}$ is called a **Young measure limit of gradients** [BFJK, KP], if it is generated by a sequence of gradients Du_j with (u_j) bounded in $W^{1,p}(\Omega,\mathbb{R}^N)$.

Let $r > 0$ and $x \in R^n$, set $B(x,r) = \{y \in \mathbb{R}^n :\mid y - x \mid< r\}$ and $\text{meas}(B(x,r)) = \omega_{n-1}r^n$, where ω_{n-1} is the area of the $n-1$ dimensional sphere.

DEFINITION 2.5. *(The Maximal Function). – Let $u \in C_0^\infty(R^n)$. We define*

$$(M^*u)(x) = (Mu)(x) + \sum_{\alpha=1}^{n}(Mu_{,\alpha})(x),$$

where we set

$$(Mf)(x) = \sup_{r>0} \frac{1}{\omega_n r^n} \int_{B(x,r)} |f(y)|\, dy$$

for every locally summable f.

LEMMA 2.6. (cf. [S, Ch.1]). – *If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, then for every $\lambda > 0$*

$$\text{meas}(\{x \in \mathbb{R}^n : (Mf)(x) > \lambda\}) \le \frac{C(n)}{\lambda^p} \int_{\mathbb{R}^n} |f|^p\, dx.$$

LEMMA 2.7. – *If $u \in C_0^\infty(R^n)$, then $M^*u \in C^0(\mathbb{R}^n)$ and*

$$\mid u(x) \mid + \sum_{\alpha=1}^{n} \mid u_{,\alpha} \mid \le (M^*u)(x)$$

for all $x \in \mathbb{R}^n$. *Moreover* (see [20]) *if* $p > 1$, *then*

$$\|M^* u\|_{L^p(\mathbb{R}^n)} \leq c(n,p)\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

and if $p \geq 1$, *then*

$$\text{meas}(\{x \in \mathbb{R}^n : (M^* u)(x) \geq \lambda\}) \leq \frac{c(n,p)}{\lambda^p}\|u\|^p_{W^{1,p}(\mathbb{R}^n)}$$

for all $\lambda > 0$.

LEMMA 2.8. (see [AF,L]). - *Let* $u \in C_0^\infty(R^n)$ *and* $\lambda > 0$, *and set*

$$H^\lambda = \{x \in R^n : (M^* u)(x) < \lambda\}.$$

Then for every x, $y \in H^\lambda$ *we have*

$$\frac{|\, u(x) - u(y)\,|}{|\, x - y\,|} \leq C(n)\lambda.$$

LEMMA 2.9. – *Let* X *be a metric space,* E *a subspace of* X, *and* k *a positive real number. Then any* k*-Lipschitz mapping from* E *into* $\mathbb{R}$ *can be extended to a* $k-$*Lipschitz mapping from* X *into* $\mathbb{R}$.

For the proof *see* [ET, page 298].

DEFINITION 2.10. (see *[ET, page 234]*). – *Let* $\Omega \subset \mathbb{R}^n$ *be open. Let* $B \subset \mathbb{R}^p$ *be a Borel subset. A mapping*

$$f : \Omega \times B \to \bar{\mathbb{R}}(:= \mathbb{R} \cup \{-\infty, \infty\})$$

is said to be a Carathéodory function if

(1) *for almost all* $x \in \Omega$, $f(x, \cdot)$ *is continuous on* B,

(2) *for all* $a \in B$, $f(\cdot, a)$ *is measurable on* Ω.

We will not introduce the more general notion of normal integrals to which the Measurable Selection Theorem applies (*see* [ET, page 234]).

THEOREM 2.11. *(The measurable selection theorem (*see *[ET, page 236]).* – *Let* B *be a compact subset of* $\mathbb{R}^p$ *and* g *a Carathéodory function of* $\Omega \times B$. *Then, there exists a measurable mapping* $\tilde{u} : \Omega \to B$ *such that for all* $x \in \Omega$:

$$g(x, \tilde{u}(x)) = \min_{a \in B}\{g(x,a)\}.$$

A direct consequence of Theorem 2.11 is the following:

PROPOSITION 2.12. – *Let $B \subset \mathbb{R}^p$ be a compact subset and let $u : \Omega \to \mathbb{R}^p$ be an integrable mapping. Then there exists a measurable mapping $\tilde{u} : \Omega \to B$ such that for all $x \in \Omega$*

$$|u(x) - \tilde{u}(x)| = \operatorname{dist}(u(x), B).$$

We conclude this section by giving the definition of quasiconvex hull QK of a compact set $K \subset M^{N\times n}$, and some simple properties of it. We use a more restricted definition than that in [S].

DEFINITION 2.13. *(see [S]). – Let $K \subset M^{N\times n}$ be non-empty and compact. The* **quasiconvex hull** *QK of K is defined by*

$$QK = \{X \in M^{N\times n},\ f(X) \leq \sup_{Y\in K} f(Y),\ f : M^{N\times n} \to \mathbb{R} \text{ quasiconvex}\}.$$

PROPOSITION 2.14. – *For any $1 \leq p < \infty$*

$$QK = \{X \in M^{N\times n},\ Q\operatorname{dist}^p(X, K) = 0\}.$$

Proof. – Let $K_1 = \{X \in M^{N\times n},\ Q\operatorname{dist}^p(X, K) = 0\}$. Obviously, $QK \subset K_1$. Let $f : M^{N\times n} \to \mathbb{R}$ be any quasiconvex function. Let

$$\alpha_f = \sup_{X\in K} f(X)$$

and

$$f_{\alpha_f}(X) = \max\{f(X) - \alpha_f,\ 0\}.$$

It is easy to see that f_{α_f} is quasiconvex, $QK \subset f_{\alpha_f}^{-1}(0)$ and $QK = \cap_f f_{\alpha_f}^{-1}(0)$. We may assume that $f_{\alpha_f}^{-1}(0)$ is compact, otherwise, take the convex function

$$g(\cdot) = \operatorname{dist}^2(\cdot, \operatorname{conv} K),$$

which is the squared distance function to a convex set. Therefore $f_{\alpha_f} + g$ is quasiconvex. We claim that $(f_{\alpha_f} + g)^{-1}(0) \subset \operatorname{conv} K$, hence it is compact. This is easy to see because $f_{\alpha_f} \geq 0$ and $g^{-1}(0) = \operatorname{conv} K$. We have, for any fixed $1 \leq p < \infty$,

$$Q\operatorname{dist}^p(X, f_{\alpha_f}^{-1}(0)) \leq \operatorname{dist}^p(X, f_{\alpha_f}^{-1}(0)) \leq \operatorname{dist}^p(X, K)$$

for all $X \in M^{N\times n}$. Since $Q\operatorname{dist}^p(X, f_{\alpha_f}^{-1}(0))$ is quasiconvex, we have

$$Q\operatorname{dist}^p(X, f_{\alpha_f}^{-1}(0)) \leq Q\operatorname{dist}^p(X, K).$$

From [Z, Theorem 1.1] and its proof, we see that for a compact zero set $f_{\alpha_f}^{-1}(0)$ corresponding to a nonnegative quasiconvex function f_{α_f}, and for any $1 \leq p < \infty$,

$$f_{\alpha_f}^{-1}(0) = \{X \in M^{N\times n},\ Q\operatorname{dist}^p(X, f_{\alpha_f}^{-1}(0)) = 0\}.$$

Hence, $K_1 \subset f_{\alpha_f}^{-1}(0))$ for every quasiconvex function f, thus $K_1 \subset QK$. The proof is complete. □

3. THE APPROXIMATING LEMMA

In this section we establish our main approximation lemma which applies to general compact sets in $M^{N\times n}$.

LEMMA 3.1. – *Suppose that $K \subset M^{N\times n}$ is a compact subset. Let $1 \le p < \infty$. Let $P \in M^{N\times n}$ be such that*

$$0 < a = Q\mathrm{dist}^p(P, K),$$

that is,

$$a = \inf_{\phi\in C_0^\infty(D,\mathbb{R}^N)} \int_D \mathrm{dist}^p(P + D\phi, K)dx,$$

where $D = (0,1)^n \subset \mathbb{R}^n$ is the unit cube. Then there exist a minimizing sequence (u_j) bounded in $W_0^{1,p}(D,\mathbb{R}^N)$, such that

$$\lim_{j\to\infty} \int_D \mathrm{dist}^p(P + Du_j, K)dx = a,$$

$u_j \rightharpoonup 0$ in $W_0^{1,p}(D,\mathbb{R}^N)$, a bounded sequence (g_j) in $W^{1,\infty}(D,\mathbb{R}^N)$, and a constant $C(n,N,p)$ such that

$$\int_D |Du_j - Dg_j|^p dx \le C(n,N,p)a + \eta_j, \tag{3.1}$$

where $\eta_j \to 0$ as $j \to \infty$,

$$\|g_j\|_{W^{1,\infty}(D,\mathbb{R}^N)} \le C(n,N,p)[a^{1/p} + \mathrm{diam}(K)], \tag{3.2}$$

and

$$\liminf_{j\to\infty} \int_D \mathrm{dist}^p(P + Dg_j, K)dx \le C(n,N,p)a. \tag{3.3}$$

Proof. – Let $\phi_j \in C_0^\infty(D,\mathbb{R}^N)$ be a minimizing sequence

$$\int_D \mathrm{dist}^p(P + D\phi_j, K)dx = a + \epsilon_j \to a,$$

as $j \to \infty$, where $\epsilon_j \to 0$ is a nonnegative sequence. Let $K_P = \{A - P, A \in K\}$, and set

$$\|K_P\| = \sup_{A\in K}\{|A - P|\}.$$

It is easy to see that

$$\|K_P\| \leq \operatorname{dist}(P, K) + \operatorname{diam} K.$$

We have

$$\int_D \operatorname{dist}^p(P + D\phi_j, K)dx = \int_D \operatorname{dist}^p(D\phi_j, K_P)dx \to a$$

as $j \to \infty$. Since K_P is compact, we see that $(D\phi_j)$ is bounded in $W_0^{1,p}(D, \mathbb{R}^N)$; in fact, since

$$\begin{aligned} a + \epsilon_j = \int_D \operatorname{dist}^p(D\phi_j, K_p)dx &\geq 3^{-(p-1)} \\ &\times \int_D |D\phi_j|^p dx - \operatorname{dist}^p(P, K) - (\operatorname{diam} K)^p, \end{aligned}$$

we have

$$\int_D |D\phi_j|^p dx \leq 3^{p-1}(a + \epsilon_j + \operatorname{dist}^p(P, K) + (\operatorname{diam} K)^p).$$

Extend ϕ_j to be defined in $\mathbb{R}^n$ as a periodic function, and then let

$$u_j(x) = \frac{1}{j}\phi_j(jx)$$

for $j = 1, 2, \ldots,$ and $x \in D$. It is easy to see that (u_j) is bounded in $W_0^{1,p}(D, \mathbb{R}^N)$ and up to a subsequence, $u_j \rightharpoonup 0$ in $W_0^{1,p}(D, \mathbb{R}^N)$. Let $B(0, \Lambda)$ be a ball in $M^{N\times n}$ such that $K_P \subset B(0, \Lambda)$ and $2^{-(p-1)}\Lambda^p > \|K_P\|^p$. Extend u_j by zero outside D, we see that $u_j \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$ and $\|u_j\|_{W_0^{1,p}(\mathbb{R}^n,\mathbb{R}^N)} = \|u_j\|_{W_0^{1,p}(D,\mathbb{R}^N)}$, $u_j = (u_j^{(1)}, \ldots, u_j^{(N)})$. For each fixed j, i, define

$$H_{i,j}^\lambda = \{x \in \mathbb{R}^n : (M^* u_j^{(i)})(x) < \lambda\}, \qquad H_j^\lambda = \bigcap_{i=1}^N H_{i,j}^\lambda, \qquad \lambda \geq 3n\Lambda.$$

Lemma 2.8 ensures that for all x, $y \in H_j^\lambda$,

$$\frac{|\, u_j^{(i)}(x) - u_j^{(i)}(y)\,|}{|\, y - x\,|} \leq C(n)\lambda.$$

Let $g_j^{(i)}$ be a Lipschitz function extending $u_j^{(i)}$ outside H_j^λ with Lipschitz constant not greater than $C(n)\lambda$ (Lemma 2.9). Since H_j^λ is an open set, we have

$$g_j^{(i)}(x) = u_j^{(i)}(x), \qquad Dg_j^{(i)}(x) = Du_j^{(i)}(x)$$

for all $x \in H_j^\lambda$ and

$$\|Dg_j^{(i)}\|_{L^\infty(\mathbb{R}^n)} \leq C(n)\lambda.$$

If $H_j^\lambda = \emptyset$, set $g_j(x) \equiv 0$ for $x \in D$. If $H_j^\lambda \cap D \neq \emptyset$, then we may also assume that

$$\|g_j^{(i)}\|_{L^\infty(D)} \leq 3C(n)\lambda.$$

Indeed, fixing $y \in H_j^\lambda \cap D$, for every $x \in D$,

$$|g_j^{(i)}(x)| \leq |g_j^{(i)}(x) - g_j^{(i)}(y)| + |g_j^{(i)}(y)| \leq \sqrt{2}C(n)\lambda + \lambda,$$

where we have used Lemma 2.7 to assert that $|u_j^{(i)}(y)| \leq (M^* u_j^{(i)})(y) < \lambda$ when $y \in H_j^\lambda$. Now set $g_j = (g_j^{(1)}, \ldots, g_j^{(N)})$.

In order to estimate $\int_D \operatorname{dist}^p(Dg_j, K_P)dx$, we start from the inequality

$$\int_D |Du_j - Dg_j|^p \, dx \leq 2^{p-1} \int_{D \setminus H_j^\lambda} (|Du_j|^p + |Dg_j|^p) \, dx, \tag{3.4}$$

and find a bound for $\operatorname{meas}(D \setminus H_j^\lambda)$.

Similar to the proof of Lemma 3.1 in [Z], we have, from the definition of $H_{i,j}^\lambda$,

$$\begin{aligned} D \setminus H_{i,j}^\lambda &\subset \{x \in D : (Mu_j^{(i)})(x) \geq \lambda/2\} \\ &\cup \left\{x \in D : \sum_{\alpha=1}^n (M \frac{\partial u_j^{(i)}}{\partial x_\alpha})(x) \geq \lambda/2\right\}, \end{aligned}$$

and

$$\begin{aligned} &\left\{x \in \mathbb{R}^n : \sum_{\alpha=1}^n (MD_\alpha u_j^{(i)})(x) \geq \lambda/2\right\} \\ &\subset \bigcup_{\alpha=1}^n \left\{x \in \mathbb{R}^n : (MD_\alpha u_j^{(i)})(x) \geq \frac{\lambda}{2n}\right\}. \end{aligned}$$

Define $h : \mathbb{R}^n \to R$ by

$$h(s) = \begin{cases} 0 & \text{if } |s| < \Lambda, \\ |s| - \Lambda & \text{if } |s| \geq \Lambda, \end{cases}$$

so that we can prove that

$$\left\{x \in \mathbb{R}^n : \left(M(D_\alpha u_j^{(i)})\right)(x) \geq \frac{\lambda}{2n}\right\}$$
$$\subset \left\{x \in \mathbb{R}^n : \left(Mh(Du_j^{(i)})\right)(x) \geq \frac{\lambda}{2n} - \Lambda\right\}. \tag{3.5}$$

In fact, when $\left(M(D_\alpha u_j^{(i)})\right)(x) \geq \frac{\lambda}{2n}$, we have a sequence of $\alpha_k > 0$, $\alpha_k \to 0$ and a sequence of balls $B_k = B(x, R_k)$ such that

$$\frac{1}{\text{meas}(B_k)} \int_{B_k} \mid D_\alpha u_j^{(i)} \mid dy \geq \frac{\lambda}{2n} - \alpha_k,$$

which implies

$$\left(Mh(Du_j^{(i)})\right)(x) \geq \frac{1}{\text{meas}(B_k)} \int_{B_k \cap \{y : |Du_j^{(i)}(y)| \geq \Lambda\}} (\mid Du_j^{(i)} \mid - \Lambda)\, dy$$
$$\geq \frac{\lambda}{2n} - \frac{1}{\text{meas}(B_k)} \int_{B_k \cap \{y : |Du_j^{(i)}(y)| \leq \Lambda\}} \mid Du_j^{(i)} \mid dy$$
$$- \frac{1}{\text{meas}(B_k)} \int_{B_k \cap \{y : |Du_j^{(i)}(y)| \geq \Lambda\}} \Lambda\, dy - \alpha_k \geq \frac{\lambda}{2n} - \Lambda - \alpha_k. \tag{3.6}$$

Passing to the limit $k \to \infty$ in (3.6), we obtain (3.5) (we have chosen $\frac{\lambda}{2n} > \Lambda$).

From Lemma 2.6, we have

$$\text{meas}\left(\left\{x \in \mathbb{R}^n : (Mh(Du_j^{(i)}))(x) \geq \frac{\lambda}{2n} - \Lambda\right\}\right)$$
$$\leq \frac{C(n)}{(\frac{\lambda}{2n} - \Lambda)^p} \int_{\mathbb{R}^n} |h(Du_j^{(i)})|^p\, dx$$
$$\leq \frac{C(n)}{(\frac{\lambda}{2n} - \Lambda)^p} \int_{\{x \in D : |Du_j^{(i)}| \geq \Lambda\}} |Du_j^{(i)}|^p\, dx$$
$$\leq \frac{C(n)}{(\frac{\lambda}{2n} - \Lambda)^p} \int_{\{x \in D : |Du_j| \geq \Lambda\}} |Du_j|^p\, dx.$$

Also, from Lemma 2.6, together with the embedding theorem, we have

$$\text{meas}(\{x \in \mathbb{R}^n : (Mu_j^{(i)})(x) \geq \lambda/2\}) \leq \frac{C(n)}{(\lambda/2)^p} \int_D |u_j^{(i)}|^p\, dx := \delta_j \to 0,$$

as $j \to \infty$. Therefore

$$\begin{aligned}\operatorname{meas}(D \setminus H_j^\lambda) &\le \sum_{i=1}^{N} \operatorname{meas}(D \setminus H_{i,j}^\lambda) \\ &\le N\left(\delta_j + \frac{C(n)}{(\frac{\lambda}{2n} - \Lambda)^p} \int_{\{x\in D: |Du_j| \ge \Lambda\}} |Du_j|^p \, dx\right). \end{aligned} \tag{3.7}$$

Since ϕ_j is periodic, we have

$$\begin{aligned}\int_D \operatorname{dist}^p(Du_j, K_P)dx &= \int_D \operatorname{dist}^p(D\phi_j(jx), K_P)dx \\ &= \int_{jD} \operatorname{dist}^p(D\phi(y), K_P)\frac{dy}{j^n} \\ &= \int_D \operatorname{dist}^p(D\phi_j(x), K_P)dx = a + \epsilon_j.\end{aligned}$$

Therefore,

$$\begin{aligned}a + \epsilon_j &= \int_D \operatorname{dist}^p(Du_j, K_P)dx \\ &\ge \int_{\{x\in D, |Du_j| \ge \Lambda\}} [2^{-(p-1)}|Du_j|^p - \|K_P\|^p]dx \\ &\ge [2^{-(p-1)}\Lambda^p - \|K_P\|^p]\operatorname{meas}(\{x \in D,\ |Du_j| \ge \Lambda\}),\end{aligned}$$

which implies that

$$\operatorname{meas}(\{x \in D,\ |Du_j| \ge \Lambda\}) \le \frac{a + \epsilon_j}{2^{-(p-1)}\Lambda^p - \|K_P\|^p}.$$

From the above two sets of inequalities, we have

$$\begin{aligned}&2^{-(p-1)} \int_{\{x\in D, |Du_j| \ge \Lambda\}} |Du_j|^p dx \\ &\le (a + \epsilon_j) + \|K_P\|^p \operatorname{meas}(\{x \in D,\ |Du_j| \ge \Lambda\})] \\ &\le (a + \epsilon_j)\left\{1 + \frac{\|K_P\|^p}{2^{-(p-1)}\Lambda^p - \|K_P\|^p}\right\} \\ &= (a + \epsilon_j)\frac{2^{-(p-1)}\Lambda^p}{2^{-(p-1)}\Lambda^p - \|K_P\|^p}.\end{aligned}$$

Hence

$$\int_{\{x\in D, |Du_j| \ge \Lambda\}} |Du_j|^p dx \le (a + \epsilon_j)\frac{2^{(p-1)}\Lambda^p}{\Lambda^p - 2^{(p-1)}\|K_P\|^p}.$$

Consequently,

$$\operatorname{meas}(D \setminus H_j^\lambda) \leq N\left(\delta_j + (a+\epsilon_j)\frac{2^{p-1}\Lambda^p}{(\Lambda^p - 2^{p-1}\|K_P\|^p)}\frac{C(n)}{(\frac{\lambda}{2n}-\Lambda)^p}\right). \tag{3.8}$$

Hence,

$$\begin{aligned} &2^{p-1}\int_{D\setminus H_j^\lambda} |Dg_j|^p dx \\ &\leq 2^{p-1}C^p(n)\lambda^p\left[N\delta_j + N(a+\epsilon_j)\frac{2^{(p-1)}\Lambda^p}{\Lambda^p - 2^{p-1}\|K_P\|^p}\frac{C(n)}{(\frac{\lambda}{2n}-\Lambda)^p}\right]. \end{aligned} \tag{3.9}$$

Also, from

$$a+\epsilon_j = \int_D \operatorname{dist}^p(Du_j, K_P)dx$$

we deduce that

$$a+\epsilon_j \geq \int_{D\setminus H_j^\lambda} [2^{-(p-1)}|Du_j|^p - \|K_P\|^p]dx,$$

thus,

$$\begin{aligned} 2^{p-1}\int_{D\setminus H_j^\lambda} |Du_j|^p &\leq 2^{2(p-1)}(a+\epsilon_j) + 2^{2(p-1)}\|K_P\|^p \operatorname{meas}(D\setminus H_j^\lambda) \\ &\leq 2^{2(p-1)}(a+\epsilon_j) + 2^{2(p-1)}\|K_P\|^p N \\ &\quad\times\left(\delta_j + (a+\epsilon_j)\frac{2^{p-1}\Lambda^p}{(\Lambda^p - 2^{p-1}\|K_P\|^p)}\frac{C(n)}{(\frac{\lambda}{2n}-\Lambda)^p}\right). \end{aligned} \tag{3.10}$$

Therefore

$$\begin{aligned} &\int_D |Du_j - Dg_j|^p\, dx \leq 2^{p-1}\int_{D\setminus H_j^\lambda} (|Du_j|^p + |Dg_j|^p)\, dx \\ &\leq 2^{p-1}C^p(n)\lambda^p\left[N\delta_j + N(a+\epsilon_j)\frac{2^{(p-1)}\Lambda^p}{\Lambda^p - 2^{p-1}\|K_P\|^p}\frac{C(n)}{(\frac{\lambda}{2n}-\Lambda)^p}\right] \\ &+ 2^{2(p-1)}(a+\epsilon_j) + 2^{2(p-1)}\|K_P\|^p \operatorname{meas}(D\setminus H_j^\lambda) \\ &\leq 2^{2(p-1)}(a+\epsilon_j) + [2^{p-1}C^p(n)\lambda^p + 2^{2(p-1)}\|K_P\|^p]N \\ &\times\left(\delta_j + (a+\epsilon_j)\frac{2^{p-1}\Lambda^p}{(\Lambda^p - 2^{p-1}(\operatorname{diam}K)^p)}\frac{C(n)}{(\frac{\lambda}{2n}-\Lambda)^p}\right). \end{aligned} \tag{3.11}$$

Given $A, B \in M^{N\times n}$, if we take $Q \in K_P$ such that $|A-Q| = \mathrm{dist}(A, K_P)$, we have

$$\begin{aligned}\mathrm{dist}^p(A, K_P) = |A-Q|^p &\geq 2^{-(p-1)}|B-Q|^p - |A-B|^p \\ &\geq 2^{-(p-1)}\mathrm{dist}^p(B, K_P) - |A-B|^p,\end{aligned}$$

and so

$$\begin{aligned}a + \epsilon_j &= \int_D \mathrm{dist}^p(Du_j, K_P)dx \\ &\geq 2^{-(p-1)}\int_D \mathrm{dist}^p(Dg_j, K_P)dx - \int_D |Du_j - Dg_j|^p dx.\end{aligned}$$

Therefore, from (3.11) we obtain

$$\begin{aligned}&\int_D \mathrm{dist}^p(Dg_j, K_P)dx \\ &\leq 2^{p-1}(a+\epsilon_j) + 2^{p-1}\int_D |Du_j - Dg_j|^p dx \leq 2^{p-1}(a+\epsilon_j) + 2^{p-1} \\ &\times \Bigg\{ 2^{2(p-1)}(a+\epsilon_j) + [2^{p-1}C^p(n)\lambda^p + 2^{2(p-1)}\|K_P\|^p]N \\ &\left(\delta_j + (a+\epsilon_j)\frac{2^{p-1}\Lambda^p}{(\Lambda^p - 2^{p-1}\|K_P\|^p)}\frac{C(n)}{(\frac{\lambda}{2n} - \Lambda)^p}\right)\Bigg\}. \qquad (3.12)\end{aligned}$$

Now, taking $\Lambda = 2\|K_P\|$, $\lambda = 6n\|K_P\|$, we have

$$\int_D \mathrm{dist}^p(Dg_j, K_P)dx \leq C(N,p)\delta_j + C_1(n,N,p)(a+\epsilon_j) \qquad (3.13)$$

which yields (3.3) after letting $j \to \infty$. Also

$$|Dg_j(x)| \leq C(n)\lambda = 6nC(n)\|K_P\| \leq C_1(n)(\mathrm{diam}K + \mathrm{dist}(P,K)) \qquad (3.14)$$

almost everywhere in D and

$$|g_j(x)| \leq C(n)\lambda \leq C(n)(\mathrm{diam}K + \mathrm{dist}(P,K)).$$

Since $\mathrm{dist}^p(P, \mathrm{conv}K)$ is a convex function and $\mathrm{dist}^p(P, \mathrm{conv}K) \leq \mathrm{dist}^p(P,K)$, we have, from the definition of quasiconvexification,

$$\mathrm{dist}^p(P, \mathrm{conv}K) \leq Q\mathrm{dist}^p(P,K) = a.$$

Therefore,

$$\mathrm{dist}(P,K) \leq a^{1/p} + \mathrm{diam}(\mathrm{conv}K) = a^{1/p} + \mathrm{diam}K,$$

thus,

$$|Dg_j(x)| \leq C_2(n)(\mathrm{diam}K + a^{1/p}), \quad |g_j(x)| \leq C_3(n)(\mathrm{diam}K + a^{1/p}). \quad \square \tag{3.15}$$

From the numerical analysis point of view, when we minimize the energy in a bounded subset of $W^{1,\infty}$ and compare the minimum with the $W^{1,p}$ minimum, we have proved the following result

COROLLARY 3.2. – *Suppose that $K \subset M^{N\times n}$ is compact. Then there exist $C_1(n,N,p) > 0$, $C_2(n,N,p) > 0$, such that*

$$\begin{aligned}\inf\Big\{ \int_D \mathrm{dist}^p(P+Dg,K)dx,\ \|g\|_{W^{1,\infty}(D,\mathbb{R}^N)} \\ \leq C_2(n,p)(\mathrm{dist}^p(P,K) + \mathrm{diam}K)\Big\} \\ \leq C_1(n,N,p)Q\mathrm{dist}^p(P,K).\end{aligned}$$

The only fact in this Corollary to be remarked is that the $W^{1,\infty}$ bound of g depends on $\mathrm{dist}^p(P,K)$, while in (3.15) it depends on $a^{1/p}$. In fact, they are equivalent because of the following inequalities,

$$\mathrm{dist}^p(P,\mathrm{conv}K) \leq Q\mathrm{dist}^p(P,K) = a \leq \mathrm{dist}^p(P,K).$$

Remark 3.1. – From the construction of g_j in the proof of Lemma 3.1, we see that even if a minimizing sequence is not bounded in $W^{1,\infty}$, we may find a $W^{1,\infty}$-sequence such that near the zero points of $Q\mathrm{dist}^p(\cdot,K)$, it serves as an approximate minimizing sequence.

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.1. – Let $F(P) = Q\mathrm{dist}^2(P,SO(n))$ for $P \in M^{n\times n}$. It is known (*see* [Z]) that $F^{-1}(0) = SO(n)$. Let $0 < a = F(P)$ for some P, *i.e.*

$$a = F(P) = \inf_{\phi_j \in C_0^\infty(D)} \int_D \mathrm{dist}^2(P + D\phi, SO(n))dx.$$

By Lemma 3.1, we find (u_j) and (g_j) satisfying (3.1)-(3.15) with $p = 2$. We have that

$$\int_D \mathrm{dist}^2(P + Dg_j, SO(n))dx \leq C(n)\delta_j + C_1(n)(a + \epsilon_j) \tag{4.1}$$

and by (3.15)

$$|Dg_j(x)| \leq C_2(n)(\mathrm{diam}SO(n) + a^{1/2}).$$

We may assume that, up to a subsequence, $g_j \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(D, \mathbb{R}^n)$. From the measurable selection theorem (*see* Proposition 2.13), for each $i = 1, 2, \ldots$, there exists a measurable mapping $P_i : D \to SO(n)$, such that

$$\mathrm{dist}(P + Dg_i(x), SO(n)) = |P_i(x) - (P + Dg_i(x))|$$

almost everywhere in D. Let

$$n_i(x) = (P + Dg_i(x)) - P_j(x).$$

Following Šverák [S], we consider the integral

$$I_{ij} = \int_D [\mathrm{adj}(P+Dg_j) - \mathrm{adj}(P+Dg_i)] \cdot [(P+Dg_j) - (P+Dg_i)]dx, \tag{4.2}$$

for $i, j = 1, 2, \ldots$. From the weak continuity property of null-Lagrangians [B1,R], if we let $i \to \infty$, then $j \to \infty$,

$$\lim_{j\to\infty} \lim_{i\to\infty} I_{ij} = 0.$$

Now, since $P_i(x), P_j(x) \in SO(n)$ almost everywhere, we have

$$\begin{aligned} \mathrm{adj}P_j &= P_j,\ \mathrm{adj}(P_j + n_j) = \mathrm{adj}P_j + C(P_j, n_j), \\ \mathrm{adj}P_i &= P_i,\ \mathrm{adj}(P_i + n_i) = \mathrm{adj}P_i + C(P_j, n_i), \end{aligned}$$

where $\mathrm{adj}P_i + C(P_j, n_i)$ is the expansion of $\mathrm{adj}(P_j + n_j)$, and

$$|C(P_j, n_j)| \leq C(n)|n_j|(1 + |n_j| + \ldots |n_j|^{n-2}).$$

We then have

$$
\begin{aligned}
I_{ij} &= \int_D [\mathrm{adj}(P_j + n_j) - \mathrm{adj}(P_i + n_i)] \cdot [(P_j + n_j) - (P_i + n_i)] dx \\
&= \int_D \left\{ \left[(\mathrm{adj}P_j - \mathrm{adj}P_i) + (C(P_j, n_j) - C(P_i, n_i)) \right] \right. \\
&\quad \left. \cdot \left[(P_j - P_i) + (n_j - n_i) \right] \right\} dx \\
&= \int_D [(P_j + n_j) - (P_i + n_i)] \cdot [(P_j + n_j) - (P_i + n_i)] dx \\
&\quad + \int_D [(C(P_j, n_j) - n_j) - (C(P_i, n_i) - n_i)] \cdot [(P_j + n_j) - (P_i + n_i)] dx \\
&\geq \int_D |Dg_j - Dg_i|^2 dx \\
&\quad - \int_D |(C(P_j, n_j) - n_j) - (C(P_i, n_i) - n_i)|^2 dx \\
&\quad - \frac{1}{4} \int_D |Dg_j - Dg_i|^2 dx. \qquad (4.3)
\end{aligned}
$$

Here we have used the facts that

$$(\mathrm{adj}P - \mathrm{adj}Q) \cdot (P - Q) = |P - Q|^2$$

if P, $Q \in SO(n)$, $ab \leq a^2 + \frac{b^2}{4}$ for real numbers and that $P_i + n_i = P + Dg_i$. From (4.3) we get

$$
\begin{aligned}
&\int_D |Dg_j - Dg_i|^2 dx \\
&\leq C(n) \int_D \left[|n_i|^2 \left(\sum_{k=0}^{2(n-2)} |n_i|^k \right) + |n_j|^2 \left(\sum_{k=0}^{2(n-2)} |n_j|^k \right) \right] dx + C(n)|I_{ij}|, \qquad (4.4)
\end{aligned}
$$

where $C(n) > 0$ is a constant depending only on n. Since $|P| \leq a^{1/2} + \mathrm{diam}SO(n)$,

$$
\begin{aligned}
|n_i(x)| &= |P + Dg_j - P_i| = \mathrm{dist}(P + Dg_i, SO(n)) \\
&\leq |Dg_j| + a^{1/2} + C_1(n) \leq C(n)(1 + a^{1/2}),
\end{aligned}
$$

and so, from (4.4) and (4.1) we conclude that

$$\begin{aligned}\int_D |Dg_j - Dg_i|^2 dx &\le C(n)(\sum_{k=0}^{2(n-2)} a^{k/2}) \int_D [\mathrm{dist}^2(P + Dg_j, SO(n)) \\ &+ \mathrm{dist}^2(P + Dg_i, SO(n))]dx + I_{ij} \\ &\le C(n)\left(\sum_{k=0}^{2(n-2)} a^{k/2}\right)[\delta_j + \delta_i + a + \epsilon_i + \epsilon_j] + CI_{ij}. \end{aligned} \tag{4.5}$$

Now we find a lower bound of

$$\int_D |Dg_j - Dg_i|^2 dx.$$

Recalling the estimate (3.11) on $\int_D |Dg_i - Du_i|^p dx$, we have, for our choice of λ, and Λ,

$$\begin{aligned}&\int_D |Dg_j - Dg_i|^2 dx \\ &= \int_D |(Dg_j - Du_j) + (Du_j - Du_i) + (Du_i - Dg_i)|^2 dx \\ &\ge \frac{1}{3}\int_D |Du_j - Du_i|^2 dx - \int_D |Dg_j - Du_j|^2 dx - \int_D |Dg_i - Du_i|^2 dx \\ &\ge \frac{1}{3}\int_D |Du_j - Du_i|^2 - C(n)a - \eta_i - \eta_j, \end{aligned} \tag{4.6}$$

where $\lim_{j\to\infty} \eta_j = 0$, $\lim_{i\to\infty} \eta_i = 0$, and $C(n, N, p) = C(n)$, $N = n$, $p = 2$ is a positive constant.

Combining (4.5) and (4.6), we see that

$$\int_D |(P+Du_j)-(P+Du_i)|^2 dx \le C(n)\left(\sum_{k=0}^{2(n-2)} a^{k/2}[a + \eta_i + \eta_j]\right) + C|I_{ij}|. \tag{4.7}$$

Since the integral on the left hand side of (4.7) is lower-semicontinuous as $i \to \infty$ and $u_i \rightharpoonup 0$ in $W_0^{1,2}(D, \mathbb{R}^n)$, letting $i \to \infty$ we have

$$\int_D |(P + Du_j) - P|^2 dx \le C(n)\left(\sum_{k=0}^{2(n-2)} a^{k/2}(a + \eta_j)\right) + \lim_{i\to\infty} C|I_{ij}|. \tag{4.8}$$

By applying the measurable selection lemma (Theorem 2.13) to the function $|P + Du_j(x) - Q|$ for $Q \in SO(n)$, we can find a measurable mapping $Q_j : \Omega \to SO(n)$, such that

$$\mathrm{dist}(P + Du_j(x), SO(n)) = |P + Du_j(x) - Q_j(x)|$$

for almost every $x \in \Omega$. Therefore the following inequalities hold almost everywhere,

$$\begin{aligned} |(P + Du_j(x)) - P|^2 &\geq \frac{1}{2}|P - Q_j(x)|^2 - |P + Du_j(x) - Q_j(x)|^2 \\ &\geq \frac{1}{2}\mathrm{dist}(P, SO(n))^2 - \mathrm{dist}(P + Du_j(x), SO(n))^2. \end{aligned}$$

and also

$$\begin{aligned} \mathrm{dist}(P, SO(n))^2 \leq 2 \int_D \mathrm{dist}(P + Du_j(x), SO(n))^2 dx \\ + C(n)\left(\sum_{k=0}^{2(n-2)} a^{k/2}(a + \eta_j)\right) + \lim_{i\to\infty} C|I_{ij}|. \quad (4.9) \end{aligned}$$

As u_j is a minimizing sequence, we pass to the limit as $j \to \infty$ and we obtain

$$\mathrm{dist}(P, SO(n))^2 \leq C(n)\left(\sum_{k=0}^{2(n-2)} a^{k/2}\right) a. \quad (4.10)$$

Since we also have

$$\mathrm{dist}^2(P, \mathrm{conv}SO(n)) \leq Q\mathrm{dist}^2(P, SO(n)) \leq \mathrm{dist}^2(P, SO(n)), \quad (4.11)$$

and when $|P| \geq 3\|\mathrm{conv}SO(n)\|$ we obtain

$$\frac{\mathrm{dist}^2(P, \mathrm{conv}SO(n))}{\mathrm{dist}^2(P, SO(n))} \geq \left(\frac{|P| - \|\mathrm{conv}SO(n)\|}{|P| + \|\mathrm{conv}SO(n)\|}\right)^2 \geq \frac{1}{4},$$

while, when $|P| \leq 3\|\mathrm{conv}SO(n)\|$, we have

$$a^{1/2} \leq \mathrm{dist}(P, SO(n)) \leq |P| + \|\mathrm{conv}SO(n)\| \leq 5\|\mathrm{conv}SO(n)\|,$$

we conclude that

$$\sum_{k=0}^{2(n-2)} a^{k/2} \leq \sum_{k=0}^{2(n-2)} (5\|\mathrm{conv}SO(n)\|)^k,$$

which, together with (4.10) yields

$$\mathrm{dist}(P, SO(n))^2 \leq C(n)Q\mathrm{dist}^2(P, SO(n)),$$

for some $C(n) > 0$. Therefore we conclude that there exists $c(n) > 0$ (in fact $c(n) = 1/C(n)$) such that

$$c(n)\mathrm{dist}(P, SO(n))^2 \leq Q\mathrm{dist}(P, SO(n))^2 \leq \mathrm{dist}(P, SO(n))^2$$

for all $P \in M^{n\times n}$. $\square$

Proof of Corollary 1.2. – In the proof of Theorem 1.1 we used the fact that if $P, Q \in SO(n)$,

$$(\mathrm{adj}P - \mathrm{adj}Q) \cdot (P - Q) = |P - Q|^2.$$

It was observed by Šverák in [Sv] (also *see* [Ma]) that under the condition (1.2), there exists an $\alpha(H) > 0$, such that

$$(\mathrm{adj}P - \mathrm{adj}Q) \cdot (P - Q) \geq \alpha(H)|P - Q|^2, \tag{4.12}$$

for all $P, Q \in K = SO(n) \cup SO(n)H$.

We denote by λ_{max} the greatest eigenvalue of H and

$$\Lambda = \frac{n + n\det H - \mathrm{tr}\mathrm{adj}H - \mathrm{tr}H}{\sum_{i=1}^{n}(1 + \lambda_i)^2},$$

we see that if $P, Q \in SO(n)$, (4.11) holds for $\alpha = 1$. If $P, Q \in SO(n)H$, (4.11) holds for $\alpha = \det H/\lambda_{\max}^2$. This is because we can write $P = P_1H$, $Q = Q_1H$ for some P_1, $Q_1 \in SO(n)$ and notice that H is a diagonal matrix, so that

$$\begin{aligned}&[\mathrm{adj}(P_1H) - \mathrm{adj}(Q_1H)] \cdot [P_1H - Q_1H]\\ &\quad = \mathrm{tr}[\mathrm{adj}H(P_1 - Q_1)^T(P_1 - Q_1)H] = \det H|P_1 - Q_1|^2,\end{aligned}$$

while

$$|P_1H - P_2H|^2 \leq \lambda_{\max}^2|P_1 - Q_1|^2.$$

Hence we reach the conclusion. Finally, if $P \in SO(n)H$ and $Q \in SO(n)$, then we write $P = RH$, where $R \in SO(n)$, and since $\mathrm{tr}RH \leq \mathrm{tr}H$, $\mathrm{tr}HR \leq \mathrm{tr}H$ $R \in SO(n)$ and H is a diagonal matrix with positive entries, we see that

$$\begin{aligned}&(\mathrm{adj}P - \mathrm{adj}Q) \cdot (P - Q)\\ &\quad = \mathrm{tr}[(\det HH^{-1}R^T - Q^T)(RH - Q)]\\ &\quad = n\det H + n - \mathrm{tr}(\det HH^{-1}R^TQ + Q^TRH)\\ &\quad \geq n\det H + n - \mathrm{tr}\mathrm{adj}H - \mathrm{tr}H.\end{aligned}$$

Since

$$|RH - Q|^2 \leq \sum_{i=1}^{n}(1+\lambda_i)^2,$$

it is clear that (4.11) is satisfied setting

$$\alpha = \frac{n + n\det H - \operatorname{tr}\operatorname{adj} H - \operatorname{tr} H}{\sum_{i=1}^{n}(1+\lambda_i)^2} = \Lambda.$$

Take

$$\alpha(H) = \min\left\{1, \frac{\det H}{\lambda_{\max}^2}, \Lambda\right\},$$

and (4.11) holds for all $P, Q \in K$. We may follow the proof of Theorem 1.1 to prove Corollary 1.2. The only place in the proof of Theorem 1.1 where $\alpha(H)$ is involved is the last inequality (4.3). Also the constants $C(n)$, $c(n)$ in Theorem 1.1 are replaced by $C(n, H)$ and $c(n, H)$, respectively. □

Remark 4.1. – In fact Matos [Ma] proved that if for some eigenvalue λ_k of H, we have

$$(1-\lambda_k)(1-\det H/\lambda_k) > 0,$$

then the quasiconvex hull of K remains itself. It turns out that, it is enough to assume this condition in Corollary 1.2. What we need is a variation of (4.11). Let $E_\epsilon = (e_{ij})$ be a matrix such that $e_{ij} = 0$ if $i \neq j$, $e_{ii} = \epsilon$ for some $\epsilon > 0$ sufficiently small if $i \neq k$ and $e_{kk} = 1$. We consider the form

$$[E(\operatorname{adj} P - \operatorname{adj} Q)] \cdot [P - Q].$$

Then this form is still a null Lagrangian. We also have, for $P \in SO(n)H$, $Q \in SO(n)$,

$$\begin{aligned}
&[E(\operatorname{adj} P - \operatorname{adj} Q)] \cdot [P - Q] \\
&\quad \geq \epsilon[\sum_{j\neq k}(1-\lambda_j)(1-\det H/\lambda_j)] + (1-\lambda_k)(1-\det H/\lambda_k) \\
&\quad \geq \alpha(H,\epsilon)|P-Q|^2,
\end{aligned}$$

when $\epsilon > 0$ is small enough. Therefore the conclusion of Corollary 1.2 is still true under this weaker condition.

Proof of Corollary 1.3. – Let $F(P) = Q\operatorname{dist}^2(P, SO(n))$, and let $P \in Q[SO(n)_\epsilon]$. Then by proposition 2.15 we have

$$Q\operatorname{dist}^2(P, SO(n)_\epsilon) = 0,$$

and since

$$\mathrm{dist}^2(P, SO(n)) \leq (\epsilon + \mathrm{dist}(P, SO(n)_\epsilon))^2 \leq 2\epsilon^2 + 2\mathrm{dist}^2(P, SO(n)_\epsilon),$$

using Theorem 1.1 we obtain

$$\begin{aligned}\mathrm{dist}^2(P, SO(n)) &\leq C(n)Q\mathrm{dist}^2(P, SO(n)) \\ &\leq C(n)(\epsilon^2 + Q\mathrm{dist}^2(P, SO(n)_\epsilon)) \\ &\leq C(n)\epsilon^2.\end{aligned}$$

The proof is finished for $SO(n)$. The other case is similar. □

Proof of Theorem 1.4. – We only give a proof for $f : SO(n) \to M^{n\times n}$. The proof for g is similar. Let $\{\nu_x\}_{x\in\Omega}$ is a family of Young measure limit of gradients supported on K_f. If we can show that ν_x is trivial for almost every x, the proof will be finished. It is known [Z, KP] that if a Young measure limit of gradients has bounded support, then the Young measure can be generated by a bounded sequence in $W^{1,\infty}$. Next we notice that if ϵ is small enough, then the mapping $Y = X + f(X)$ from $SO(n)$ to K_f is invertible and the inverse is continuous. Let $Q(\cdot) : \Omega \to M^{n\times n}$ be measurable. If we apply the measurable selection theorem, we may find a measurable mapping $P : \Omega \to K_f$, such that $|P(x) - Q(x)| = \mathrm{dist}(Q(x), K_f)$, where $P(x) = R(x) + f(R(x))$, and $R : \Omega \to SO(n)$ is measurable. Now, let (u_j) be a bounded sequence in $W^{1,\infty}(\Omega, \mathbb{R}^N)$ such that the Young measures generated by (Du_j) is supported in K_f and $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega, \mathbb{R}^N)$. For each j, we may find a measurable mapping $R_j : \Omega \to SO(n)$ such that

$$|Du_j(x) - R_j(x) - f(R_j(x))| = \mathrm{dist}(Du_j(x), K_f) \to 0$$

almost everywhere. Let $n_j = Du_j(x) - R_j(x) - f(R_j(x))$. As in the proof of Theorem 1.1, setting

$$I_{ij} = \int_\Omega [\mathrm{adj}Du_j - \mathrm{adj}Du_i] \cdot [Du_j - Du_i]dx,$$

we have $\lim_{i\to\infty}\lim_{j\to\infty} I_{ij} = 0$. We also have

$$\begin{aligned}I_{ij} = \int_\Omega &[\mathrm{adj}(R_j(x) + f(R_j(x)) + n_j(x)) \\ &- \mathrm{adj}(R_i(x) + f(R_i(x)) + n_i(x))] \cdot [(R_j(x) + f(R_j(x)) + n_j) \\ &- (R_i(x) + f(R_i(x)) + n_i(x))]dx \\ = \int_\Omega &\{[\mathrm{adj}R_j + f(R_j) + n_j] - [\mathrm{adj}R_i + f(R_i) + n_i]\}\end{aligned}$$

$$
\begin{aligned}
&\cdot \{(R_j + f(R_j) + n_j) - (R_i + f(R_i) + n_i)\} dx \\
&+ \int_\Omega \{[C(R_j, f(R_j), n_j) - C(R_i, f(R_i), n_i)] \\
&-[(f(R_j) + n_j) - (f(R_j) + n_j)]\} \\
&\cdot \{(R_j + f(R_j) + n_j) - (R_i + f(R_i) + n_i)\} dx \\
= I_{ij}^1 &+ I_{ij}^2,
\end{aligned}
$$

where $C(R_j, f(R_j), n_j)$ is defined by the expansion of the determinants

$$\mathrm{adj}(R_j + f(R_j) + n_j) = \mathrm{adj} R_j + C(R_j, f(R_j), n_j).$$

Therefore we have, due to the fact that $R_j \in SO(n)$ and f satisfies (1.5), that

$$
\begin{aligned}
&|C(R_j, f(R_j), n_j) - C(R_i, f(R_i), n_i)| \\
&\quad \le C(n) \Bigg[\sum_{k=1}^{n-2} (|n_j|^k + |f(R_j)|^k + |n_i|^k + |f(R_i)|^k) |R_j - R_i| \\
&\quad + \sum_{k=1}^{n-2} (|n_j|^k + |f(R_j)|^k + |n_i|^k + |f(R_i)|^k)(|n_i| + |n_j|) \\
&\quad + \sum_{k=1}^{n-2} (|n_j|^k + |f(R_j)|^k)| + |n_i|^k + |f(R_i)|^k) |f(R_j) - f(R_i)| \Bigg] \\
&\quad \le C(n) \left(\epsilon |R_j - R_i| + \sum_{k=1}^{n-1} (|n_j|^k + |n_i|^k) \right). \qquad (4.13)
\end{aligned}
$$

Since $Du_j = R_j + f(R_j) + n_j$, from (1.5), and if we choose $\epsilon \le 1/2$, we have

$$|R_j - R_i| \le 2|Du_j - Du_i| + 2(|n_j| + |n_i|).$$

Thus we have,

$$
\begin{aligned}
I_{ij}^2 &\ge - \int_\Omega C(n)(2\epsilon |Du_j - Du_i| + 2\epsilon(|n_j| + |n_i|) \\
&\quad + \sum_{k=1}^{n-1} (|n_j|^k + |n_i|^k) |Du_i - Du_j| dx \\
&\ge -2C(n)\epsilon \int_\Omega |Du_j - Du_i|^2 dx \\
&\quad - C(n) \int_\Omega \sum_{k=1}^{n-1} (|n_j|^k + |n_i|^k) |Du_i - Du_j| dx. \qquad (4.13)
\end{aligned}
$$

Obviously,

$$I_{ij}^1 = \int_\Omega |Du_j - Du_i|^2 dx. \tag{4.14}$$

Choosing $\epsilon = \min\{1/2, 1/(4C(n))\}$, we see that if we combine (4.13) and (4.14), we have

$$\int_\Omega |Du_j - Du_i|^2 dx \leq CI_{ij} + C(n) \int_\Omega \sum_{k=1}^n (|n_j|^k + |n_i|^k)|Du_i - Du_j| dx. \tag{4.15}$$

Since $|n_j| \to 0$, when $j \to 0$ in $L^p(\Omega)$ for any $p > 1$, and (Du_j) is bounded in L^∞, passing to the limit in $i \to \infty$, and using the fact that the functional on the left hand side of (4.15) is lower semicontinuous, then letting $j \to \infty$ in (4.15), we see that

$$\int_\Omega \left(\int_{M^{n\times n}} |\lambda - Du(x)|^2 d\nu_x(\lambda) \right) dx = 0.$$

Therefore $\nu_x = \delta_{Du(x)}$ almost everywhere.

The proof for K_g is similar. □

The proof for Corollary 1.5 is similar to that for Theorem 1.1 and it is left to the reader.

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REFERENCES

[AF] E. Acerbi and N. Fusco, Semicontinuity problems in the calculus of variations, *Arch. Rational Mech. Anal.*, Vol. **86**, 1984, pp. 125-145.

[Bd] E. J. Balder, A general approach to lower semicontinuity and lower closure in optimal control theory, *SIAM J. Control and Optimization*, Vol. **22**, 1984, pp. 570-597.

[BL] H. Berliocchi and J. M. Lasry, Intégrandes normales et mesures paramétrées en calcul des variations, *Bull. Soc. Math. France*, Vol. **101**, pp. 129-184.

[Bl1] J. M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.*, Vol. **63**, 1977, pp. 337-403.

[Bl2] J. M. Ball, Constitutive inequalities and existence theorems in nonlinear elasticity, in "*Nonlinear Analysis and Mechanics*: Heriot - Watt Symposium." Vol. **1** (edited by R. mJ. Knops), Pitman, London, 1977.

[Bl3] J. M. BALL, A version of the fundamental theorem of Young measures, in *Partial Differential Equations and Continuum Models of Phase Transitions,* (edited by M. Rascle, D. Serre and M. Slemrod), 1989, 207-215, Springer-Verlag.

[BCO] J. M. BALL, J. C. CURRIE and P. J. OLVER, Null Lagrangians, weak continuity, and variational problems of arbitrary order, *J. Functional Anal.*, Vol. **41**, 1981, pp. 135-174.

[BFJK] K. BHATTACHARYA, N. B. FIROOZY, R. D. JAMES and R. V. KOHN, Restrictions on Microstructures, *Proc. Royal Soc. Edinburgh A* Vol. **124**, 1994, pp. 843-878

[BJ1] J. M. BALL and R. D. JAMES, Fine phase mixtures as minimizers of energy, *Arch. Rational Mech. Anal.*, Vol. **100**, 1987, pp. 13-52.

[BJ2] J. M. BALL and R. D. JAMES; Proposed experimental tests of a theory of fine microstructures and the two-well problem, *Phil. Royal Soc. Lon.*, Vol. **338A**, 1992, pp. 389-450.

[CK] M. CHIPOT and D. KINDERLEHRER, Equilibrium configurations of crystals, *Arch. Rational Mech. Anal.*, Vol. **103**, 1988, pp. 237-277.

[Da1] B. DACOROGNA, Weak Continuity and Weak Lower Semicontinuity of Nonlinear Functionals, *Lectures Notes in Math.,* Springer-Verlag, Berlin, Vol. **922**, 1980.

[Da2] B. DACOROGNA, Direct Methods in the Calculus of Variations, *Springer*, 1989.

[DR] H. LE DRET and A. RAOULT, Enveloppe quasi-convexe de la densité d'énergie de Saint Venant-Kirchhoff, *C. R. Acad. Sci. Paris, Série I*, Vol. **318**, 1994, pp. 93-98.

[ET] I. EKELAND and R. TEMAM, Convex Aanlysis and Variational Problems, *North-Holland*, 976.

[F] I. FONSECA, The lower quasiconvex envelope of the stored energy function for an elastic crystal, *J. Math. Pures et Appl.*, Vol. **67**, 1988, pp. 175-195.

[K] R. V. KOHN, The relaxation of a double-well energy, *Cont. Mech. Therm.*, Vol. **3**, 1991, pp. 981-1000.

[KP] D. KINDERLEHRER and P. PEDREGAL, Characterizations of Young measures generated by gradients, *Arch. Rational Mech. Anal.*, Vol. **115**, 1991, pp. 329-365.

[KS] R. V. KOHN and D. STRANG, Optimal design and relaxation of variational problems I, II, III, *Comm. Pure Appl. Math.*, Vol. **39**, 1986, pp. 113-137, 139-182, 353-377.

[L] F. C. LIU, A Luzin type property of Sobolev functions, *Ind. Univ. Math. J.*, Vol. **26**, 1977, pp. 645-651.

[Ma] J. MATOS, Young measures and the absence of fine microstructures in the $\alpha - \beta$ quarts phase transition, *preprint.*

[Mo] C. B. Jr MORREY, Multiple integrals in the calculus of variations, *Springer*, 1966.

[Sv] V. SVERÁK, On the problem of two wells, *preprint.*

[St] E. M. STEIN, Singular Integrals and Differentiability Properties of Functions, *Princeton University Press, Princeton* , 1970.

[T] L. TARTAR, Compensated compactness and applications to partial differential equations, *Nonlinear Analysis and Mechanics*: Heriot-Watt Symposium (edited by R. J. Knops) Vol. **IV**, 1979, pp. 136-212.

[Z] K.-W. ZHANG, A construction of quasiconvex functions with linear growth at infinity, *Ann. Sc. Norm Sup. Pisa, Serie IV*, Vol. **XIX**, 1992, pp. 313-326.

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