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On positive Liouville theorems and asymptotic behavior of solutions of Fuchsian type elliptic operators

by

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ABSTRACT. — Using the Harnack inequality and a scaling argument we prove some positive Liouville theorems for a Fuchsian type linear elliptic operator P on some domains in $\mathbb{R}^d$ with some prescribed boundary conditions. We also study the asymptotic behavior of the quotients of two positive solutions of linear and semilinear equations.

Key words : Asymptotic behavior, Fuchsian elliptic operators, isolated singularities, Liouville theorem, maximum principle, positive solutions.

RÉSUMÉ. — Avec l'aide de l'inégalité de Harnack et d'un argument de scaling nous démontrons des théorèmes de type Liouville pour des fonctions P-harmoniques positives d'un opérateur linéaire elliptique P de type fuchsien sur certaines régions de $\mathbb{R}^d$ avec conditions limites. Nous étudions également le comportement asymptotique des quotients de deux solutions positives des équations linéaires et semi-linéaires.

Classification A.M.S. : 35 B 05, 35 J 15.

1. INTRODUCTION

We begin with four classical theorems illustrating the circle of questions to be discussed. They involve uniqueness theorems for positive harmonic functions on the one hand and the asymptotic behavior of harmonic functions on the other.

(*a*) **The positive Liouville theorem.** *The set of all positive harmonic functions in* $\mathbb{R}^d$ *is a one-dimensional cone.*

(*b*) **The Picard principle.** *The cone of all positive harmonic functions in the punctured unit ball in* $\mathbb{R}^d$, $d \geqq 2$, *which vanish on the unit sphere* S^{d-1} *is of one dimension.*

(*c*) **The Poisson principle.** *Let* $\zeta \in S^{d-1}$. *The cone* $\mathscr{C}_\zeta$ *of all positive harmonic functions in the unit ball which vanish on* $S^{d-1} \setminus \{\zeta\}$ *is a one-dimensional cone.*

(*d*) **The Riemann removable singularity theorem.** *If* u *is a bounded harmonic function in a punctured neighborhood of the origin in* $\mathbb{R}^d$, $d \geqq 2$, *then* $\lim_{x \to 0} u(x)$ *exists.*

The first three theorems are *uniqueness theorems* while the Riemann theorem is a theorem on the *asymptotic behavior near an isolated singular point*. We shall show that these two types of theorems are closely related, apply to more general linear (and also semilinear) elliptic equations and can be treated using a unified approach.

Throughout the paper we deal with a triple (P, X, ζ). Here P is a real linear elliptic operator of second order which is defined on a domain $X \subseteqq \mathbb{R}^d$, $d \geqq 2$ and ζ is a point on the (ideal) boundary of X. More precisely, we consider the following two cases:

(i) The intersection of X with some exterior domain is an open connected truncated cone with a Lipschitz boundary and $\zeta = \infty$.

(ii) X is a domain (maybe nonsmooth and unbounded) such that the origin 0 is either an isolated component of the boundary $\Gamma = \partial X$ or 0 belongs to a Lipschitz portion of Γ. Here $\zeta = 0$.

We assume that the operator P is of the form

$$P(x, \partial_x) = -\sum_{i,j=1}^{d} a_{ij}(x)\, \partial_i \partial_j + \sum_{i=1}^{d} b_i(x)\, \partial_i + c(x), \qquad x \in X. \quad (1.1)$$

From now on, unless otherwise stated, we shall assume that P is elliptic on X and the *coefficients* a_{ij}, b_i *and* c *are real and Hölder continuous.* Furthermore, we assume that in some relative neighborhood $X' \subseteqq X$ of ζ

$$\lambda \sum_{i=1}^{d} \xi_i^2 \leqq \sum_{i,j=1}^{d} a_{ij}(x)\, \xi_i \xi_j \leqq \Lambda \sum_{i=1}^{d} \xi_i^2, \qquad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \quad (1.2)$$

for any $x \in X'$, $\xi \in \mathbb{R}^d$ and

$$|x| \sum_{i=1}^{d} |b_i(x)| + |x|^2 |c(x)| \leqq M, \qquad x \in X', \tag{1.3}$$

where λ, Λ and M are some positive constants. We shall call such an operator P *a Fuchsian elliptic operator at* ζ *which is defined on* X.

We consider the domain X as a subset of the one point compactification of $\mathbb{R}^d$ and denote by $\hat{X}$ and δX its relative closure and boundary. We are concerned with the set $\mathscr{C}_\zeta = \mathscr{C}_\zeta(P, X)$ of all positive (classical) solutions of the equation $Pu=0$ in X which has *minimal growth* at $\delta X \setminus \{\zeta\}$. So, we are concerned with *classical positive solutions* of the operator P in X which satisfy a generalized Dirichlet boundary condition except at the point ζ. (But the same methods apply to the case of *weak* or *strong* positive solutions as well, in this case one can impose weaker regularity assumptions on the coefficients of P.) We shall also discuss briefly positive solutions on X which satisfy the regular oblique derivative boundary condition.

Assume that P satisfies also the following hypothesis (H):

(H) **the equation $Pu=0$ in X has at least one positive solution** $u \in C^{2,\alpha}(X)$.

Under this assumption, it follows (*see* Lemma 5.3) that $\mathscr{C}_\zeta$ is *a nonempty convex cone* in the real vector space $C^{2,\alpha}(X)$. We denote the dimension of $\mathscr{C}_\zeta$ by $\dim \mathscr{C}_\zeta$. We prove that

$$\dim \mathscr{C}_\zeta(P, X) = \dim \mathscr{C}_\zeta(P^*, X) = 1,$$

where P^* is the formal adjoint of the operator P. Moreover, if u and v are two positive solutions of the equation $Pu=0$ in a relative neighborhood X′ of the singular point ζ which vanish on $(\delta X' \cap \delta X) \setminus \{\zeta\}$, then

$$\lim_{x \to \zeta} u(x)/v(x) \text{ exists}$$

(but maybe infinite). The above asymptotic behavior holds true also in the case where u is a positive solution of some *semilinear* equation of the form $Pu + f(x, u) = 0$ and v is a given positive solution of the linear equation. These asymptotic results generalize of course, the Riemann theorem and also the Fatou lemma.

Uniqueness properties and asymptotic behavior of positive solutions of elliptic operators were studied extensively by many mathematicians. Let us first mention some of the known results for the case $\zeta = \infty$. D. Gilbarg and J. Serrin ([8], Section 4) proved a Liouville type theorem and the existence of a limit at infinity for the case $X = \mathbb{R}^d$ under the additional assumption $c(x) \equiv 0$ (*see* [20] for a probabilistic proof). The proofs in [8] and [20] rely in a very crucial way on the facts that in this case the function $u(x) \equiv 1$ is a positive solution of the equation $Pu=0$ and that P satisfies the maximum principle (*see* also [7], p. 103, Theorem 1.11).

In [15], M. Murata proved that $\dim \mathscr{C}_\zeta = 1$ for a general Fuchsian operator in a cone X with $\zeta = \infty$ (*see* the Appendix in [15], *see* also [12]). Our more general approach is quite similar to the proof of the uniqueness result in [15]. Murata also discussed in [15] the imbedding of a boundary part $\omega \subseteqq \partial X$ into the Martin boundary. This question is closely related to some parts of our study of positive solutions near a singular boundary point.

In a recent work, H. Berestycki and L. Nirenberg ([3], Theorem 2′) proved that if u and v are two positive solutions of a Fuchsian operator at $\zeta = \infty$ in an exterior domain $D \subseteqq \mathbb{R}^d$, $d \geqq 2$, and if

$$0 < A := \liminf_{|x| \to \infty} u(x)/v(x) < \infty, \tag{1.4}$$

then $\lim_{|x| \to \infty} u(x)/v(x) = A$. The proof of the stronger and more general limit theorem given here (Theorem 7.1, part (i)) is a modification of the proof of the above result in [3]. Berestycki and Nirenberg [3], used the above result in the study of the asymptotic behavior of a solution of semilinear equations. Under some assumptions, it was shown that solutions of the semilinear equation behave at infinity like a solution of the linearized problem.

Our results extends also some of the results in [14, 19, 29]. In [19] it was assumed that $X = \mathbb{R}^d$, $d \geqq 3$ and P is a small perturbation of an operator P_0 whose Green function is equivalent to the Green function of the Laplacian in $\mathbb{R}^d$. Roughly speaking, it was proven that if P is a uniformly elliptic operator in divergence form such that

$$(1+|x|)^{1+\varepsilon} \sum_{i=1}^{d} |b_i(x)| + (1+|x|)^{2+\varepsilon} |c(x)| \leqq M, \tag{1.5}$$

with some $\varepsilon > 0$, and P admits a (positive) Green function then

(i) the Green function of P is equivalent to the Green function of $-\Delta$;

(ii) the cone $\mathscr{C}_\infty(P, \mathbb{R}^d)$ is of one dimension;

(iii) any positive solution in $\mathscr{C}_\infty(P, \mathbb{R}^d)$ is bounded and admits a positive limit at infinity.

We would like to stress here that properties (i) and (iii) do not hold for a *general* Fuchsian operator at infinity (*see* Section 9).

The Picard principle and Riemann theorem concerning the behavior of positive solutions of linear and semilinear equations in a neighborhood of an isolated singular point $\zeta = 0$ was studied extensively in the last three decades. The linear case was studied in [8, 10, 13, 16, 17, 24, 26] while the semilinear case was treated in [4, 5, 22, 23, 27, 28] (*see* also the references therein). Note that since positive solutions of Fuchsian type linear elliptic operators may admit *strong* singularities near isolated points our results include also *strong* singular behaviors of solutions of semilinear (actually sublinear) equations.

The case when ζ belongs to a Lipschitz portion of the boundary of X is closely related and actually extends the well known result that the Martin boundary of a bounded Lipschitz domain Ω is homeomorphic to its Euclidean boundary $\partial\Omega$ (*see* for example [6, 15, 25] and the references therein).

The outline of this paper is as follows. In Section 2 we introduce some basic notations, while in Section 3 we discuss positive solutions of minimal growth at some parts of δX. The proofs of our main results rely on the generalized maximum principle and some versions of the **Harnack inequality** which will be described in Section 4.

We prove in Section 5 that the cone $\mathscr{C}_\zeta \cup \{0\}$ is a nontrivial closed cone. This property holds true also for a general elliptic operator on X. In Section 6 we prove some *a priori* estimates for positive solutions of a *weakly* Fuchsian elliptic operator near the singular point ζ. The main point is that the estimates do not depend on the distance from the singular point.

Our main theorem concerning the uniqueness and the asymptotic behavior of positive solutions of linear Fuchsian operator is proved in Section 7. Some applications for linear and semilinear equations are discussed in Section 8. We conclude our paper in Section 9 with some examples illustrating the strictness of our results.

2. BASIC NOTATION AND NOTIONS

Let $x=(x_1, \ldots, x_d)\in\mathbb{R}^d$, denote $|x|=(\sum x_i^2)^{1/2}$; $\partial_i=\partial/\partial x_i$, $\partial_i\partial_j=\partial^2/\partial x_{ij}$. $\beta=(\beta_1, \ldots, \beta_d)$, $\beta_i=\text{integer}\geqq 0$, with $|\beta|=\sum\beta_i$, is a *multi-index*; define

$$\mathrm{D}^\beta u=\frac{\partial^{|\beta|}u}{\partial^{\beta_1}x_1 \ldots \partial^{\beta_d}x_d}.$$

We denote by $\mathrm{B_R}(x)$ (respectively, $\mathrm{B_R}$) the open ball of radius R centered at x (respectively, the origin 0) and by $\mathrm{S_R}(x)$ (resp., $\mathrm{S_R}$) its boundary. Let $0<r<\mathrm{R}$, the annulus $\mathrm{B_R}(x)\setminus \mathrm{B}_r(x)$ (respect., $\mathrm{B_R}\setminus\mathrm{B}_r$) will be denoted by $\mathrm{B}_{r,\mathrm{R}}(x)$ (respect., $\mathrm{B}_{r,\mathrm{R}}$). The exterior domain $\{x\in\mathbb{R}^d \mid |x|\geqq \mathrm{R}\}$ will be denoted by $\mathrm{D_R}$.

By a *cone* in $\mathbb{R}^d$ we mean a domain K such that

$$\mathrm{K}\cap\mathrm{D}_{\mathrm{R}_0}=\{x\in\mathrm{D}_{\mathrm{R}_0} \mid x/|x|\in\mathrm{F}\}$$

for some $\mathrm{R}_0>0$ and some Lipschitz domain $\mathrm{F}\subseteqq\mathrm{S}_1$. Sometimes we shall consider cones Ω of the form

$$\Omega=\{x\in\mathbb{R}^d\setminus\{0\} \mid x/|x|\in\mathrm{F}\},$$

where F is some Lipschitz domain $F \subseteqq S_1$. In this case we shall say that Ω is a *geometrical cone*.

For a given function u on a set V such that $V \cap S_R \neq \varnothing$ we denote

$$M_u(R, V) = \sup\{u(x) \mid x \in V \cap S_R\}, \tag{2.1}$$

$$m_u(R, V) = \inf\{u(x) \mid x \in V \cap S_R\}. \tag{2.2}$$

In the case $V = S_R$ we simply denote $M_u(R) = M_u(R, S_R)$, $m_u(R) = m_u(R, S_R)$.

The closure of a set B will be denoted by cl B. If B is a convex set in some linear space, we denote by **ex** B the set of all its extreme points.

$C = C(., \ldots, .)$ denotes a constant depending on the quantities appearing in parentheses. In a given context the same letter C may be used to denote different constants depending on the same set of arguments.

Sometimes, we shall call a (positive) solution of the equation $Pu = 0$ in a domain Ω *a (positive) solution of the operator* P in Ω.

It turns out that for most of our results it is not needed that inequalities (1.2) and (1.3) are satisfied in a neighborhood of ζ but only on some *essential* subset of it.

DEFINITION 2.1. — We say that the operator P is a *Fuchsian operator in the weak sense in* X *at* ζ if there exit real numbers $0 < a < 1 < b$ and a sequence of positive numbers $\{R_n\}$ with $R_n \to \zeta$ such that inequalities (1.2) and (1.3) are satisfied for every x in the set $\mathscr{A} = \bigcup_{n>0} A_n$, where

$$A_n = \{x \in X \mid a R_n < |x| < b R_n\}. \tag{2.3}$$

Such a set $\mathscr{A}$ is said to be *an essential set in* X *with respect to the singular point* ζ. If P is a weakly Fuchsian operator as above we simply say that $\mathscr{A}$ is *an essential set for* (P, X, ζ).

3. POSITIVE SOLUTIONS OF MINIMAL GROWTH

Consider a domain $\Omega \subseteqq \mathbb{R}^d$ and let $\hat{\Omega}$ be the closure of Ω with respect to the one point compactification of $\mathbb{R}^d$ and $\delta\Omega = \hat{\Omega} \setminus \Omega$ its boundary. Let ω be a subset of $\delta\Omega$. A set $\Omega' \subseteqq \Omega$ is called an *$\hat{\Omega}$-neighborhood of* ω if Ω' is the intersection of Ω with a relative neighborhood of ω in $\hat{\Omega}$.

Let $\omega \subseteqq \delta\Omega$ be a closed set. A solution u of the operator P *has minimal growth at* ω if u is a positive solution of the operator P in some $\hat{\Omega}$-neighborhood of ω and for any positive solution v of the operator P in an $\hat{\Omega}$-neighborhood of ω there exists a positive constant C and a set Ω' which is an $\hat{\Omega}$-neighborhood of ω such that

$$u(x) \leqq C\, v(x) \quad \text{in } \Omega'. \tag{3.1}$$

Let $\zeta \in \delta\Omega$. A solution u of the operator P *has minimal growth at* $\delta\Omega \setminus \{\zeta\}$ if there exists a neighborhood β of ζ in $\delta\Omega$ such that u is a positive solution of the operator P of minimal growth at ω for every set ω of the form $\omega = \delta\Omega \setminus \beta_\zeta$, where $\beta_\zeta \subseteq \beta$ is a neighborhood of ζ in $\delta\Omega$.

The term "*a neighborhood of infinity in* Ω" will refer to $\hat{\Omega}$-neighborhood of $\delta\Omega$. Hence, a neighborhood of infinity in Ω is any open set of the form $\Omega \setminus F$, where F is a compact set in Ω. If the operator P admits a positive solution in a connected neighborhood Ω' of infinity in Ω then P admits *a positive solution u in a neighborhood of infinity in* Ω *of minimal growth at* $\delta\Omega$ [1], we shall call such a solution a *positive solution of minimal growth in* Ω.

If P admits a positive (minimal) Green function in Ω, then P is said to be *subcritical* in Ω. Note that in this case the Green function is a positive solution in a neighborhood of infinity in Ω of minimal growth in Ω. On the other hand, in the subcritical case, all the (global) positive solutions of the operator P in Ω (which do exist) are solutions which are not of minimal growth in Ω. Moreover, if P admits (only) a positive solution in a neighborhood of infinity in Ω then for some nonnegative function W with a compact support in Ω the operator $P + W(x)$ is subcritical in Ω. So, the operators $P + W(x)$ and (hence) P admit also a positive solution in some neighborhood of infinity in Ω which does not have minimal growth in Ω [18].

If P admits a positive solution in Ω but P is not subcritical then P is said to be *a critical operator in* Ω. Moreover, in the critical case P admits (up to a constant) a unique positive solution. This solution has minimal growth in Ω and is called *a ground state* [1, 18].

Recall that if the operator P admits a positive solution in Ω then the *generalized maximum principle* holds. That is, for any domain $\Omega' \subset\subset \Omega$, if $Pu \geqq 0$ in Ω' and $\liminf_{x \to \partial\Omega'} u(x) \geqq 0$ then $u(x) \geqq 0$ in Ω' and either $u = 0$ or u is strictly positive in Ω'.

Let $\{\Omega_n\}$ be an increasing sequence of smooth bounded domains such that $\mathbf{cl}\,\Omega_n \subset \Omega_{n+1} \subset \Omega$ and $\bigcup \Omega_n = \Omega$. Let u be a positive solution of minimal growth in Ω. Without loss of generality we may assume that u is a continuous positive solution in $\Omega' = \Omega \setminus \Omega_1$. Then we have (*see* also [1]).

LEMMA 3.1. — *Let u be a positive solution of minimal growth in* Ω *which is defined and continuous in* Ω'. *Then the solution u is given by* $u(x) = \lim_{n \to \infty} u_n(x)$, *where* u_n *is the solution of the following Dirichlet problem*

$$\left.\begin{aligned} Pu_n &= 0 \quad \text{in } \Omega_n \setminus \Omega_1, \\ u_n &= u \quad \text{on } \partial\Omega_1, \\ u_n &= 0 \quad \text{on } \partial\Omega_n. \end{aligned}\right\} \tag{3.2}$$

Proof. — It is clear that $\{u_n\}$ defined by (3.2) is an increasing sequence of positive solutions and by the generalized maximum principle its limit function v is less or equal u. Hence, the solution u-v has minimal growth in Ω' where P is subcritical. Hence $v=u$. ■

Suppose now that $U \subseteqq \Omega$ is an $\hat{\Omega}$-neighborhood of $\omega \subseteqq \delta\Omega$ such that $U \subseteqq \Omega'$ and $U \cap \Omega_n$ is regular for every $n \geqq 1$. Consider the solutions of the following Dirichlet problems

$$\left.\begin{aligned} Pv_n&=0 \quad \text{in } \Omega_n \cap U,\\ v_n&=u \quad \text{on } \partial U \cap \Omega_n,\\ v_n&=0 \quad \text{on } \partial\Omega_n \cap U. \end{aligned}\right\} \tag{3.3}$$

Since clearly $u_n \leqq v_n \leqq u$ it follows that in U we have $u(x) = \lim_{n \to \infty} v_n(x)$.

In particular, suppose that ω is a relatively open Lipschitz portion of $\partial\Omega$ and that the coefficients of P are Hölder continuous up to the boundary ω. Let $z \in \omega$ and consider the set $U = \Omega \cap B_r(z)$, with some $r>0$ sufficiently small. Extend P in $B_r(z)$ as an elliptic operator $\hat{P}$ with Hölder continuous coefficients and let ϑ be a positive bounded solution of the operator $\hat{P}$ in $B_r(z)$ which is bounded away from zero. The existence of such a solution ϑ is guaranteed since r is small. By considering the operator $P^\vartheta = \vartheta^{-1}(\hat{P}\vartheta)$ instead of P we may assume that $c(x)$, the zero order term of P, satisfies $c(x) \geqq 0$ in $B_r(z)$. Let w be the solution of the following Dirichlet problem

$$\left.\begin{aligned} Pw&=0 \quad \text{in } U,\\ w&=0 \quad \text{on } \omega \cap \mathrm{cl}\, U,\\ w&=u \quad \text{on } \Omega \cap \partial U. \end{aligned}\right\} \tag{3.4}$$

Then by standard elliptic methods (*see* Theorem 6.13 in [9]) w vanishes on $\omega \cap \mathrm{cl}\, U$. Now, compare w with the solution v_n of the Dirichlet problem (3.3). Recall that $v_n(x) \nearrow u(x)$, therefore, by the generalized maximum principle $w(x) \geqq v_n(x)$ in $U \cap \Omega_n$. Consequently, $w \geqq u$ in cl U. In particular, we have

LEMMA 3.2. — *Let u be a positive solution of minimal growth in a domain Ω. Let ω be a Lipschitz portion of $\partial\Omega$. Assume that either $c(x) \geqq 0$ and P has bounded coefficients near ω or that the coefficients of the operator P are Hölder continuous up to ω. Then u vanishes continuously on ω.*

4. HARNACK INEQUALITIES AND COMPACTNESS PRINCIPLE

In this section we shall describe three types of the Harnack inequality:

(i) the local Krylov-Safanov Harnack inequality,

(ii) the local boundary Harnack principle for positive solutions which vanish on the boundary,

(iii) the (up to the boundary) local Harnack inequality for positive solutions which satisfy the oblique derivative boundary condition (due to Berestycki, Caffarelli and Niremberg [2]).

(i) **The local Krylov-Safanov Harnack inequality** (*see*, for instance, [9]): *Let*

$$L(x, \partial_x) = -\sum_{i,j=1}^{d} \alpha_{ij}(x)\,\partial_i\,\partial_j + \sum_{i=1}^{d} \beta_i(x)\,\partial_i + \gamma(x); \tag{4.1}$$

be defined on a domain $\Omega \subseteqq \mathbb{R}^d$ *and assume that* α_{ij}, β_i *and* γ *are real and measurable. Furthermore, assume that for any* $x \in \Omega$

$$\lambda \sum_{i=1}^{d} \xi_i^2 \leqq \sum_{i,j=1}^{d} \alpha_{ij}(x)\,\xi_i\,\xi_j \leqq \Lambda \sum_{i=1}^{d} \xi_i^2, \quad \forall\, \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \tag{4.2}$$

$$\sum_{i=1}^{d} |\beta_i(x)| + |\gamma(x)| \leqq M, \tag{4.3}$$

where λ, Λ *and* M *are some positive constants. Then for any compact set* $K \subset \Omega$ *there exists a constant* $C > 0$ *such that*

$$C\,u(x) < u(y) \tag{4.4}$$

for every positive (strong) solution u *of the equation* $L u = 0$ *in* Ω *and all points* $x, y \in K$, *where* $C = C(\Lambda/\lambda, M, K, \Omega)$.

(ii) **The local boundary Harnack principle** (*see* [15, 6] and the references therein):

Let $\Omega \subseteqq \mathbb{R}^d$ *be a Lipschitz domain and* $z \in \partial\Omega$. *Let* L *be an operator of the form* (4.1) *which satisfies* (4.2) *and* (4.3). *Then there exist positive constants* r_0 *and* C *depending only on* λ, Λ, M *and the Lipschitz continuity of* $\partial\Omega$ *near* z *such that for any* r *with* $0 < r < r_0$ *and any positive solutions* u *and* v *of* $L u = 0$ *in* $B_{8r}(z) \cap \Omega$ *which vanish continuously on* $B_{8r}(z) \cap \partial\Omega$

$$C\,u(x)/u(z_r) < v(x)/v(z_r) \quad \text{for all } x \in B_r(z) \cap \Omega. \tag{4.5}$$

Here z_r *is a point in* $B_r(z) \cap \Omega$ *whose distance from* z *is uniformly proportional to* r.

(iii) **The up to the boundary Harnack inequality** (*see* [2], Theorem 2.1):

Let $\Omega \subseteqq \mathbb{R}^d$ *be a domain and let* u *be a positive solution of the operator* L *in* Ω, *where* L *is an operator of the form* (4.1) *which satisfies* (4.2) *and* (4.3). *Assume that for an open connected and smooth subset* Σ *of* $\partial\Omega$, u *is of class* C^1 *on* $\Omega \cup \Sigma$, *and satisfies*

$$\mu(x).\nabla u + \eta(x)\,u = 0 \qquad \text{on } \Sigma, \tag{4.6}$$

here η *and the vector* μ *are smooth on* Σ, *and* μ *is nowhere tangential to* Σ. *Then for any compact subset* K *of* $\Omega \cup \Sigma$ *there is a positive constant* C *such* *that*

$$Cu(x) < u(y) \tag{4.7}$$

for all points $x, y \in K$. *Here* C *depends only on* Λ, λ, M, K, Σ, Ω, *and the* C^2 *norms on* Σ *of* μ *and* η (*but not on* u).

We endow the space $\mathscr{S}$ of the solutions of the operator P in X and all its subsets with the compact open topology.

Let $X = \mathbb{R}^d$ and $\zeta = \infty$. It follows from the local Harnack inequality and standard Schauder estimates [9], that on $\mathscr{C}_\zeta$ this topology coincides with the topology of uniform convergence (on compact subsets in X) of functions together with their partial derivatives of first and second order. We fix some point $x_0 \in X$ and consider the set

$$\mathscr{K} = \{ u \in \mathscr{C}_\zeta \mid u(x_0) = 1 \} \tag{4.8}$$

of all *normalized positive solutions.*

Under hypothesis (H), the set $\mathscr{K}$ is nonempty. As a consequence of the local Harnack inequality and the Schauder estimates, one obtains the following **compactness principle (CP):**

(CP) the set $\mathscr{K}$ is compact with respect to the compact open topology.

The compactness principle holds true also when X is either a cone in $\mathbb{R}^d$ and $\zeta = \infty$ or X is a domain and $\zeta = 0$. Observe that in order to prove the compactness principle it is clearly enough to prove that $\mathscr{C}_\zeta \cup \{0\}$ is a closed cone. This will be proved in the next section (*see* Lemma 5.3).

5. THE CONE $\mathscr{C}_\zeta$

Let P be an elliptic operator which is defined on X and let ζ be a singular point. Recall that we consider the following two cases:

(i) The intersection of X with some exterior domain is an open connected truncated geometrical cone with a Lipschitz boundary and $\zeta = \infty$.

(ii) X is a domain (maybe nonsmooth and unbounded) such that the origin 0 is either an isolated component of its boundary Γ or 0 belongs to a Lipschitz portion of Γ. Here $\zeta = 0$.

We always assume that in some $\hat{X}$-neighborhood $\omega \subseteq \delta X$ of ζ either $c(x) \geqq 0$ or the coefficients of P are Hölder continuous up to $\omega \setminus \{\zeta\}$. Moreover, throughout this section we assume that P satisfies hypothesis (H):

(H) **the equation $Pu = 0$ in X has at least one positive solution** $u \in C^{2,\alpha}(X)$.

As in Section 3 let $\{\Omega_n\}$ be a sequence of smooth bounded domains in X which exhausts X. Let $\mathscr{A} = \bigcup_{n>0} A_n$ be an essential set in X with respect to the singular point ζ with the corresponding sequence $\{R_n\}$. Let $\mathscr{B}_n \subseteq X$

be the $\hat{X}$-neighborhood of the singular point ζ such that

$$S_n = \partial \mathscr{B}_n \cap X = S_{R_n} \cap X. \tag{5.1}$$

We denote, $W_n = X \setminus \mathrm{cl}\, \mathscr{B}_n$, and $\omega_n = \delta X \cap \hat{W}_n$.

DEFINITION 5.1. – Let P be an elliptic operator at ζ which is defined on X. We denote by $\mathscr{C}_\zeta = \mathscr{C}_\zeta(P, X)$ the cone of all positive classical solutions of the equation $Pu=0$ in X which has minimal growth at $\delta X \setminus \{\zeta\}$. That is, u has minimal growth at ω_n for all $n \geqq 1$.

In this section we shall discuss some (*a priori*) properties of the cone $\mathscr{C}_\zeta$. We prove two lemmas which are true for *general* elliptic operators on X with singular point ζ.

LEMMA 5.2. – *Let* P *be an elliptic operator which is defined on a domain* X *and has an essential set* $\mathscr{A} = \bigcup_{n>0} A_n$ *with respect to the singular point* ζ. *Assume that the operator* P *admits a positive solution in* X. *Then*

(i) *Any positive solution* u *of minimal growth in* X *is also a positive solution of minimal growth at* ω_N *for all* $N \geqq 1$.

(ii) *If* v *is a positive solution in* W_{N+1} *which has a minimal growth at* ω_{N-1} *then* v *is given in* W_{N-1} *by* $v(x) = \lim_{n\to\infty} v_n(x)$, *where* v_n *is a solution of the following Dirichlet problem*

$$\left.\begin{aligned} Pv_n &= 0 \quad \text{in } W_N \cap \Omega_n, \\ v_n &= v \quad \text{on } S_N \cap \Omega_n, \\ v_n &= 0 \quad \text{on } W_N \cap \partial\Omega_n. \end{aligned}\right\} \tag{5.2}$$

Moreover, if the coefficients of P *are Hölder continuous up to some Lipschitz portion* ω' *of* ω_{N-1} *then* u *vanishes on* ω'.

Proof. – (i) Let u be a positive solution of the operator P of minimal growth in X. We can assume that u is a solution in $X \setminus B_\varepsilon(z)$, where $z \in \mathscr{B}_{N+1}$ and $\varepsilon > 0$ is sufficiently small. Since by our assumption the set $\omega_n \setminus \omega_{N-1}$, $n \geqq N$ is a Lipschitz portion of the boundary it follows from Lemma 3.2 that u vanishes on $\omega_n \setminus \omega_{N-1}$, $n \geqq N$.

Let w be a positive solution of the operator P in some $\hat{X}$-neighborhood Ω of ω_N. We may assume that $\partial\Omega \cap X$ is smooth and $\Omega \supseteqq W_{N+1}$.

We have to prove that $Cu < w$ in Ω', where $\Omega' \subset \Omega \cap W_{N+1}$ is some $\hat{X}$-neighborhood of ω_N and C is some positive number. Let w_n be the solution of the problem

$$\left.\begin{aligned} Pw_n &= 0 \quad \text{in } \Omega \cap \Omega_n, \\ w_n &= w \quad \text{on } \partial\Omega \cap \Omega_n, \\ w_n &= 0 \quad \text{on } \Omega \cap \partial\Omega_n. \end{aligned}\right\} \tag{5.3}$$

Then $w_n \leqq w$, w_n is an increasing sequence which converges to a positive solution $\hat{w}$ and we have $\hat{w} \leqq w$. Since $\omega_{N+1} \setminus \omega_{N-1}$ is a Lipschitz portion of the boundary we may use the same argument as in the proof of Lemma 3.2 and we obtain that $\hat{w}$ vanishes on $\omega_{N+1}\setminus\omega_{N-1}$. Let $R=1/2(R_N+R_{N+1})$ and denote by W_R the set such that $W_N \subseteqq W_R \subseteqq W_{N+1}$ and $S_R=\partial W_R \cap X$. Fix some point y on $X \cap S_R$. By the Harnack boundary principle and Harnack inequality we have

$$C\,u(x) < \hat{w}(x)\,(u(y)/\hat{w}(y)) \quad \text{for all } x \in X \cap S_R. \tag{5.4}$$

So

$$C_1\,u(x) < \hat{w}(x) \quad \text{for all } x \in X \cap S_R. \tag{5.5}$$

Now solve the Dirichlet problems (5.2) in $W_R \cap \Omega_n$ once with u and once with $\hat{w}$ (instead of v) and denote the corresponding solutions by u_n and $\hat{w}_n$ respectively. It follows from (5.5) that $C_1\,u_n(x) < \hat{w}_n(x)$. By the generalized maximum principle and (3.3) we know that $u_n \nearrow u$. It is also clear that $w_n \leqq \hat{w}_n \leqq \hat{w}$, thus $\hat{w}_n \nearrow \hat{w}$. Therefore, we have

$$C_1\,u(x) < \hat{\mathrm{w}}(x) \quad \text{in } W_R \tag{5.6}$$

and u is of minimal growth at ω_N.

(ii) By Lemma 3.2 and (3.3) any positive solution u of minimal growth in X satisfies the properties of (ii) and by part (i) u has minimal growth in ω_{N-1}. On the other hand, any two positive solutions of minimal growth at a boundary portion ω are comparable on an appropriate $\hat{\Omega}$-neighborhood of $\omega \subseteqq \delta\Omega$. Therefore, any positive solution of the operator P of minimal growth in ω_N vanishes on Lipschitz portions of ω_N.

The set of solutions v_n of the problems (5.2) is a monotone sequence of positive solutions which are bounded by v. Let $\hat{v}(x) = \lim_{n\to\infty} v_n(x)$ and denote by $\tilde{v}=v-\hat{v}$, we have to prove that $\tilde{v}=0$ in W_{N-1}. Suppose that $\tilde{v}>0$. The operator P is subcritical in W_N and let $G(x)=G(x, y_0)$ be the Green function of the operator P in W_N, where $y_0 \in W_N \setminus W_{N-1}$. Then G is a positive solution in $W_N \setminus \{y_0\}$ (which is a neighborhood of infinity in W_N) and G has minimal growth in W_N.

By our assumptions the functions v and (hence also) $\tilde{v}$ have minimal growth at ω_{N-1}. Hence there exists a $\hat{X}$-neighborhood W' of ω_{N-1} and a positive constant such that $C\tilde{v} \leqq G$ in W'. Since $\tilde{v}$ and G vanish on $S_N \cup (\omega_N \setminus \omega_{N-1})$ it follows from the generalized maximum principle and the behavior of G near y_0 that $C\tilde{v} \leqq G$ in W_N. Thus $\tilde{v}$ is a positive solution in W_N which has minimal growth in W_N. But this contradicts the subcriticality of P in W_N. ■

LEMMA 5.3. — *Let* P *be an elliptic operator which is defined on a domain* X *and has an essential set* $\mathscr{A} = \bigcup_{n>0} A_n$ *with respect to the singular point* ζ. *Suppose that* P *admits a positive solution in* X. *Then* $\mathscr{C}_\zeta \cup \{0\}$ *is a closed nontrivial cone.*

Proof. – We shall first prove that the cone $\mathscr{C}_\zeta \cup \{0\}$ is closed. Let $\{u_k\} \subseteq \mathscr{C}_\zeta$ and assume that $u_k \to u$ uniformly in any compact subsets of X. We may assume that $u>0$. Let $y_n \in S_n$ be fixed (S_n is defined by (5.1)). Then $(\mathscr{M}_n)^{-1} \leqq u_k(y_n) \leqq \mathscr{M}_n$, where M_n are some positive numbers. So, by the Harnack boundary principle and the Harnack inequality we have

$$C_n u_k(x) < u_1(x)(u_k(y_n)/u_1(y_n)) \leqq \mathscr{M}_n^2 u_1(x) \quad \text{for all } x \in X \cap S_n, \tag{5.7}$$

and by Lemma 5.2 it follows that

$$C_n u_k(x) \leqq \mathscr{M}_n u_1(x) \quad \text{for all } x \in W_n. \tag{5.8}$$

Hence

$$C_n u(x) \leqq \mathscr{M}_n u_1(x) \quad \text{for all } x \in W_n \tag{5.9}$$

and u is a positive solution of minimal growth at ω_n for every $n \geqq 1$.

Now we show that $\mathscr{C}_\zeta \neq \varnothing$. If P is a critical operator in X then its ground state has minimal growth in X. Lemma 5.2 implies now that the ground state belongs to $\mathscr{C}_\zeta$.

Suppose now that P is subcritical and let y_n be any sequence in X which tends to ζ. Consider the Martin quotients

$$K_n(x) = K(x, y_n) = G(x, y_n)/G(x_0, y_n). \tag{5.10}$$

Let $k \in \mathbb{N}$ be fixed. Then by Lemma 5.2 the solutions $K_n(x)$ are normalized positive solutions of minimal growth at ω_k for all n sufficiently large. Without loss of generality, we may assume that $K_n \to K$. Observe that $K(x_0) = 1$, thus $K > 0$. The same argument as in the proof of the closedness of $\mathscr{C}_\zeta$ shows that K has minimal growth at ω_k. Since k is arbitrary large it follows that K belongs to $\mathscr{C}_\zeta$. ■

6. *A PRIORI* ESTIMATES

In this section we shall prove some *a priori* estimates for *weakly Fuchsian type elliptic operators*. Let P be a weakly Fuchsian elliptic operator at a singular point ζ which is defined on X and let $\mathscr{A} = \bigcup_{n>0} A_n$ be an essential set for (P, X, ζ). Here

$$A_n = \{x \in X \mid a R_n < |x| < b R_n\}, \tag{6.1}$$

and $0 < a < 1 < b$. Let $a' = (3a+1)/4$, $b' = (b+1)/2$, we shall also denote by A'_n the inner "annulus"

$$A'_n = \{x \in X \mid a' R_n < |x| < b' R_n\}. \tag{6.2}$$

DEFINITION 6.1. – We denote by $\mathscr{D}_n$ and $\mathscr{D}'_n$ the *dilated annuli*

$$\mathscr{D}_n = R_n^{-1} A_n = \{x \in \mathbb{R}^d \mid R_n x \in A_n\}, \tag{6.3}$$

$$\mathscr{D}'_n = R_n^{-1} A'_n = \{ x \in \mathbb{R}^d \mid R_n x \in A'_n \}. \tag{6.3'}$$

Remark 6.2. — Note that in the case when X is either a cone and $\zeta = \infty$ or X is a domain and $\zeta = 0$ is an isolated singular point of its boundary the set $\mathscr{D}_n$ is a fixed domain. On the other hand, if $\zeta = 0$ belongs to a Lipschitz portion of the boundary then $\{ \mathscr{D}_n \}$ is a sequence of bounded domains with boundaries such that their Lipschitz continuity constants are uniformly bounded. ■

LEMMA 6.3. — *Let* P *be a weakly Fuchsian elliptic operator at* ζ *which is defined on* X *and let* $\mathscr{A} = \bigcup_{n>0} A_n$ *be an essential set for* (P, X, ζ). *Let* Q *be a Lipschitz (geometrical) cone such that* $Q \cap X \subset \operatorname{cl}(Q \cap X) \setminus \{0\} \subset X'$ *where* X' *is some* $\hat{X}$*-neighborhood of* ζ. *Then*

(i) *There exists a constant* $C = C(\Lambda/\lambda, M, X, Q, a, b)$ *such that for any* $n \geq 1$ *and any positive solution* u *of the operator* P *in the "annulus"* A_n *the following inequality holds*

$$C u(x) < u(y) \quad \text{for all } x, y \in Q \cap A'_n. \tag{6.4}$$

in particular, $CM_u(R_n, Q) < m_u(R_n, Q)$.

(ii) *If* u *and* v *are two positive solutions of the operator* P *in the "annulus"* A_n *which vanish continuously on* $\Gamma \cap \operatorname{cl} A_n$ *then*

$$C u(x) < v(x) \frac{m_u(R_n, Q)}{M_v(R_n, Q)} \quad \text{for all } x \in A'_n, \tag{6.5}$$

and

$$C u(x)/v(x) < u(y)/v(y) \quad \text{for all } x, y \in X \cap S_{R_n}, \tag{6.6}$$

where C *is a positive constant which depends only on* Λ, λ, M, X.a, b *and* Q.

Remark 6.4. — Note that if X is an exterior domain and $\zeta = \infty$ or $\zeta = 0$ is an isolated boundary point of a domain X the cone Q can be chosen as the whole space $\mathbb{R}^d$ and Inequality (6.4) holds true in A'_n. That is,

$$C < u(x)/u(y) < C^{-1} \quad \text{for all } x, y \in A'_n. \tag{6.7}$$

Proof of Lemma 6.3. — (i) For any $n \geq 1$ define the dilated operator P_n on the dilated annulus $\mathscr{D}_n = R_n^{-1} A_n$

$$\begin{aligned} P_n(x, \partial_x) &= -\sum_{i,j=1}^{d} \alpha_{ij}^n(x)\, \partial_i \partial_j + \sum_{i=1}^{d} \beta_i^n(x)\, \partial_i + \gamma^n(x) \\ &= -\sum_{i,j=1}^{d} a_{ij}(R_n x)\, \partial_i \partial_j + R_n \sum_{i=1}^{d} b_i(R_n x)\, \partial_i + R_n^2\, c(R_n x). \end{aligned} \tag{6.8}$$

Then $\mathscr{P} = \{ P_n \mid n \geq 1 \}$ is a family of uniformly elliptic operators with bounded coefficients on the Lipschitz domains $\mathscr{D}_n$ and the bounds of the coefficients and the Lipschitz continuity constant of the boundaries do not depend on n. That is, for any $x \in \mathcal{D}_n$ and any $n > 1$

$$\lambda \sum_{i=1}^{d} \xi_i^2 \leqq \sum_{i,j=1}^{d} \alpha_{ij}^n(x) \xi_i \xi_j \leqq \Lambda \sum_{i=1}^{d} \xi_i^2, \qquad \forall \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \quad (6.9)$$

$$\sum_{i=1}^{d} |\beta_i^n(x)| + |\gamma^n(x)| \leqq \mathrm{M}. \quad (6.10)$$

Let u be some positive solution u of the operator P in the annulus A_n and define on $\mathcal{D}_n$ the function $w(x) = u(\mathrm{R}_n x)$. Then $w(x)$ is a positive solution of the operator P_n on $\mathcal{D}_n$. Therefore, by the local Harnack inequality on the dilated set $\mathrm{Q}_n = \mathrm{R}_n^{-1}(\mathrm{Q} \cap \mathrm{A}_n')$ there exists a constant $\mathrm{C} = \mathrm{C}(\Lambda/\lambda, \mathrm{M}, \mathrm{X}, \mathrm{Q}, a, b) > 0$ such that

$$\mathrm{C}\, w(\hat{x}) < w(\hat{y}) \quad (6.11)$$

for all points $\hat{x}, \hat{y} \in \mathrm{Q}_n$. Let $x, y \in \mathrm{Q} \cap \mathrm{A}_n'$ and denote $\hat{x} = x/\mathrm{R}_n$, $\hat{y} = y/\mathrm{R}_n$, then $\hat{x}, \hat{y} \in \mathrm{Q}_n$ and $x = \mathrm{R}_n \hat{x}$, $y = \mathrm{R}_n \hat{y}$. It follows from the definition of w and (6.11) that

$$\mathrm{C}\, u(x) = \mathrm{C}\, u(\mathrm{R}_n \hat{x}) = \mathrm{C}\, w(\hat{x}) < w(\hat{y}) = u(\mathrm{R}_n \hat{y}) = u(y)$$

and the first part of the lemma is proved.

(ii) Let u and v be two positive solutions as is assumed in (ii). On the set $\mathcal{D}_n$ consider the dilated operator P_n (*see* (6.8)) and the positive solutions $u_n(x) = u(\mathrm{R}_n x)$, $v_n(x) = v(\mathrm{R}_n x)$. By Inequality (6.11) we have

$$\mathrm{C}\mathrm{M}_{u_n}(1, \mathrm{Q}_n) < u_n(x) \leqq \mathrm{C}^{-1} m_{u_n}(1, \mathrm{Q}_n) \quad \text{for all } x \in \mathrm{Q}_n, \quad (6.12)$$

$$\mathrm{C}\mathrm{M}_{v_n}(1, \mathrm{Q}_n) < v_n(x) \leqq \mathrm{C}^{-1} m_{v_n}(1, \mathrm{Q}_n) \quad \text{for all } x \in \mathrm{Q}_n. \quad (6.13)$$

Let $x \in \mathcal{D}_n' \setminus \mathrm{Q}_n$ and let $y = y(x)$ be a point such that $y \in \partial \mathrm{Q}_n \cap \mathcal{D}_n'$ and $|x - y| = \mathrm{dist}(x, \mathrm{Q}_n)$. Applying the boundary Harnack principle with x and y and using (6.12) and (6.13) we obtain

$$\mathrm{C}_1\, u_n(x) < \mathrm{v}_n(x) \frac{u_n(y)}{v_n(y)} < \mathrm{C}^{-2} v_n(x) \frac{m_{u_n}(1, \mathrm{Q}_n)}{\mathrm{M}_{v_b}(1, \mathrm{Q}_n)}. \quad (6.14)$$

Now, Inequalities (6.12)-(6.14) imply the second part of the Lemma. ■

The following growth lemma for positive solutions of Fuchsian type operators is a direct consequence of Lemma 6.3.

LEMMA 6.5. — *Consider one of the following two cases:*

(i) *The operator* P *is a Fuchsian elliptic operator at* $\zeta = \infty$ *and* $\mathrm{X} = \mathbb{R}^d$, $d \geqq 2$.

(ii) P *is a Fuchsian type elliptic operator at* $\zeta = 0$ *and* X *is a punctured ball.*

There exist α, $\beta \in \mathbb{R}$ *such that any positive solution in a* $\hat{X}$-*neighborhood* $D \subseteq X$ *of* ζ *satisfies the following growth property*

$$C^{-1} |x|^{\beta} \leqq u(x) \leqq C |x|^{\alpha} \quad \text{for all } x \in D', \tag{6.15}$$

where $C = C(u)$ *is a positive constant and* $D' \subseteq D$ *is a* $\hat{X}$-*neighborhood of* ζ.

Proof. — We consider first the case $\zeta = \infty$ and $X = \mathbb{R}^d$. We can assume that u is a positive solution of the equation $Pu = 0$ in D_R for some $R > 1$. Let $R_n = 2^n R$, $n \geqq 1$, $a = 1/3$ and $b = 3$ and consider the corresponding annuli A_n and A'_n (*see* (6.1) and (6.2)). It follows from (6.7) that

$$C M_u(R_{n-1}) \leqq M_u(R_n) \leqq C^{-1} M_u(R_{n-1}). \tag{6.16}$$

Now set $C_0 = M_u(R)$. Using (6.7) n-times we obtain

$$C_0\, C^n \leqq M_u(R_n) \leqq C_0\, C^{-n}. \tag{6.17}$$

Let $x \in A'_n$ then

$$R\, 2^{n-1} < |x| < R\, 2^{n+1}.$$

So,

$$\frac{\log|x| - \log 2R}{\log 2} \leqq n \leqq \frac{\log|x| - \log R/2}{\log 2} \tag{6.18}$$

and (6.7) (6.17) and (6.18) imply that for $x \in A'_n$

$$C_1^{-1} |x|^{\beta} \leqq C_0\, C^{n+1} \leqq u(x) \leqq C_0\, C^{-n-1} \leqq C_1 |x|^{\alpha}, \tag{6.19}$$

where C_1 is a positive constant and $\beta = -\alpha = \dfrac{\log C}{\log 2}$.

The proof for the second case is obtained by obvious modifications of the above proof. ■

Remark 6.6. — (i) Suppose that P is either a Fuchsian operator on a Lipschitz cone X and $\zeta = \infty$ or a Fuchsian operator on a domain X and $\zeta = 0$ belongs to a Lipschitz portion of ∂X. Let Q be a cone as in Lemma 6.3. Using the same technique as in Lemma 6.5 we obtain that any positive solution u in a $\hat{X}$-neighborhood D of ζ satisfies Inequality (6.15) in $Q \cap D'$, where $D' \subseteq D$ is a $\hat{X}$-neighborhood of ζ. The exponents α and β depend also on Q.

(ii) Since the annulus is a connected set in $\mathbb{R}^d$ if and only if $d \geqq 2$ the proof of Lemma 6.3 does not apply in the one-dimensional case. In fact, Lemma 6.3 and the positive Liouville theorem are false for $d = 1$ since in this case the dimension of the cone $\mathscr{C}_{\infty}(P, \mathbb{R})$ equals two if and only if P is subcritical in $\mathbb{R}$ (*see* for example [14]).

(iii) Suppose that P is a Fuchsian at infinity elliptic operator on $\mathbb{R}^d$. One cannot obtain a uniform estimate of the form (6.7) for balls instead of annuli since the coefficients of the dilated operators are not uniformly bounded in a neighborhood of the origin, and as was noted above

Lemma 6.3 and the positive Liouville theorem do not hold true for $d=1$ (*see* the incorrect proof of Theorem 1 in [11]). ■

7. THE MAIN THEOREM

In this section we prove our main result concerning the uniqueness theorem for the cone $\mathscr{C}_\zeta$ and the limit theorem.

THEOREM 7.1. — *Let* P *be a weakly Fuchsian operator at* ζ *which is defined on* X. *Suppose that* P *admits a positive solution in* X. *Consider the cone* $\mathscr{C}_\zeta=\mathscr{C}_\zeta(\mathrm{P}, \mathrm{X})$ *of all positive solutions of the operator* P *in* Ω *which has minimal growth at* $\delta\mathrm{X}\setminus\{\zeta\}$.

(i) *Any quotient of two positive solutions* u *and* v *of the equation* $\mathrm{P}u=0$ *in some* $\hat{\mathrm{X}}$*-neighborhood* $\mathscr{B}$ *of the singular point* ζ *which vanish continuously on* $(\partial\mathrm{X}\cap \mathrm{cl}\,\mathscr{B})\setminus\{\zeta\}$ *admits a nontangential limit* $\lim\limits_{\substack{x\to\zeta\\ x\in\mathrm{X}}} u(x)/v(x)$. *This limit maybe infinite.*

(ii) *The cone* $\mathscr{C}_\zeta\cup\{0\}$ *is closed and* $\dim\mathscr{C}_\zeta=1$. *Moreover, in the subcritical case* $\mathscr{C}_\zeta$ *contains of exactly one Martin function while in the critical case it consists of scalar multiples of the ground state.*

(iii) *Suppose that* P *is subcritical and the coefficients of the formal adjoint operator* P* *satisfy the same local regularity assumptions as the coefficients of* P. *Then the set of all Martin functions for the operator* P* *corresponding to Martin fundamental sequences which converge in* cl X *to* ζ *is a one point set.*

Proof. — (i) Let $\mathscr{A}$ be an essential set for (P, X, ζ) and let R_n, $\mathrm{S}_n=\mathrm{X}\cap\mathrm{S}_{\mathrm{R}_n}$ and $\mathrm{C}_n=\mathscr{B}_{n-1}\setminus\mathscr{B}_{n+1}$ be the corresponding radii "spheres" and "annuli" respectively (*see* Section 5). Now set

$$a_n=m_{u/v}(\mathrm{R}_n, \mathrm{X}),\qquad b_n=\mathrm{M}_{u/v}(\mathrm{R}_n, \mathrm{X}). \tag{7.1}$$

Consider the "annulus" C_n and let

$$\alpha_n=\min(a_{n-1}, a_{n+1}),\qquad \beta_n=\max(b_{n-1}, b_{n+1}). \tag{7.2}$$

Since the solutions $w_n=u-\alpha_n v$ and $z_n=\beta_n v-u$ are nonnegative on $\partial\mathrm{C}_n$, it follows from the generalized maximum principle that $a_n\geqq\alpha_n$, $b_n\leqq\beta_n$. Hence, there exists $\mathrm{N}\in\mathbb{N}$ such that the sequences $\{a_n\}_{n>\mathrm{N}}$ and $\{b_n\}_{n>\mathrm{N}}$ are monotone. Let $\mathfrak{a}=\lim\limits_{n\to\infty} a_n$, $\mathfrak{b}=\lim\limits_{n\to\infty} b_n$. Since in C_n

$$\alpha_n\leqq u(x)/v(x)\leqq\beta_n \tag{7.3}$$

it is enough to prove that $\mathfrak{a}=\mathfrak{b}$. We may assume that $\mathfrak{a}<\infty$. Assume that $a_n\searrow\mathfrak{a}$ (the proof for the other case is similar). So, $w_n=u-a_{n+1}v\geqq 0$ in

C_n. By lemma 6.3 (Inequality (6.6)) we have

$$C\frac{u(x)-a_{n+1}v(x)}{v(x)}=C\frac{w_n(x)}{v(x)}<\frac{w_n(y)}{v(y)}=\frac{u(y)-a_{n+1}v(y)}{v(y)} \tag{7.4}$$

for all $x, y\in X\cap S_n$. By the definition of a_n and b_n there exist sequences of nonnegative numbers ε_n and δ_n tending to zero and sequences of points $x_n, y_n \in X\cap S_n$ such that

$$u(x_n)-a_{n+1}v(x_n)=(b_n-\varepsilon_n-a_{n+1})v(x_n) \tag{7.5}$$

and

$$u(y_n)-a_{n+1}v(y_n)=(a_n+\delta_n-a_{n+1})v(y_n). \tag{7.6}$$

Combining (7.4), (7.5) and (7.6) we obtain

$$C(b_n-\varepsilon_n-a_n)\leqq C(b_n-\varepsilon_n-a_{n+1})<(a_n+\delta_n-a_{n+1}). \tag{7.7}$$

Now, let $n\to\infty$ in (7.7), we obtain $\mathfrak{a}=\mathfrak{b}$.

(ii) By lemma 5.3 the cone $\mathscr{C}_\zeta\cup\{0\}$ is a nontrivial closed cone which in the subcritical case contains at least one Martin function. By the Harnack inequality and Schauder estimates it follows that

$$\mathscr{K}=\{u\in\mathscr{C}_\zeta \mid u(x_0)=1\}$$

is a nonempty compact convex set (*see* also (4.8) and (CP)).

Suppose that the dimension of $\mathscr{C}_\zeta$ is greater than one. The set $\mathscr{K}$ is a base of the cone $\mathscr{C}_\zeta$, therefore, by our assumption and the Krein-Milman theorem, $\operatorname{ex}\mathscr{K}$ contains at least two different extreme points u and v.

It follows from part (i) that $\lim\limits_{\substack{x\to\zeta\\ x\in X}} u(x)/v(x)$ exists and without loss of generality we may assume that

$$0\leqq \lim_{\substack{x\to\zeta\\ x\in X}} u(x)/v(x)=\mathfrak{a}<\infty. \tag{7.8}$$

Let $\mathscr{B}_n$ be the $\hat{X}$-neighborhood of the singular point ζ such that $S_n=\partial\mathscr{B}_n\cap X$. There exists $N\geqq 0$ such that

$$u(x)<(\mathfrak{a}+1)v(x)\quad \text{for every } x\in\mathscr{B}_N. \tag{7.9}$$

Denote by $\mathfrak{b}=(\mathfrak{a}+1)^{-1}$. Recall that u and v has minimal growth at ω_N, thus, part (ii) of Lemma 5.2 and the generalized maximum principle imply that

$$\mathfrak{b}\,u(x)<v(x)\quad \text{for all } x\in X. \tag{7.10}$$

Consider the solution $w(x)=v(x)-\mathfrak{b}\,u(x)$, then $w(x)>0$ in X. Define $\tau=w(0)=1-\mathfrak{b}$. So, $0<\tau<1$ and the function $w_1=\tau^{-1}w\in\mathscr{K}$. Consequently, we obtain

$$v(x)=\tau\, w_1(x)+(1-\tau)\,u(x). \tag{7.11}$$

Since $v \in \operatorname{ex} \mathscr{K}$, Equation (7.11) implies that $u=v$ which contradicts our assumption. So, part (ii) is proved.

(iii) Let $G(x, y)$ be the Green function of P in X. It is well known that the Green function $G^*(x, y)$ of P^* is given by $G^*(x, y)=G(y, x)$.

For every $x \in X$ the functions $u(y)=G(y, x)$ and $v(y)=G(y, x_0)$ are positive solutions (of the operator P) of minimal growth in X. Thus, by Lemma 5.2 the solutions u and v have minimal growth at ω_n and vanish on $(\partial \mathscr{B}_n \cap \partial X) \setminus \{\zeta\}$ for every n large enough. It follows now from the first part of our theorem that

$$w(x) := \lim_{y \to \zeta} u(y)/v(y) = \lim_{y \to \zeta} G(y, x)/G(y, x_0) \tag{7.12}$$

is a well defined (maybe extended value) function.

Consider the family of functions

$$\mathscr{M} = \{ G(y, x)/G(y, x_0) \mid y \in X \}. \tag{7.13}$$

The family $\mathscr{M}$ consists of the Martin quotients of the operator P^* and we have to look for limits of sequences of functions

$$w_k(x) = G(y_k, x)/G(y_k, x_0) \quad \text{in } \mathscr{M} \quad \text{with } y_k \to \zeta.$$

Note that $\mathscr{M}$ is a family of positive normalized solutions of the operator P^* and therefore, the limit function $w(x)$ is a positive (finite) normalized solution of the operator P^*. Moreover, by (7.12) any sequence $\{ w_k(x) \}$ tend to $w(x)$ as $y_k \to \zeta$. Consequently, there is exactly one limit function as $y_k \to \zeta$. Hence, there is only one fundamental sequence in the Martin boundary of X with respect to P^* corresponding to all sequences in X which converge to ζ in cl X. ■

Remark 7.2. — (i) The operator P^* is critical (subcritical) in X if and only if P is critical (respectively, subcritical) in X. Recall also, that in the critical case the cone of positive solutions of a critical operator P in a domain Ω (and therefore, also of the operator P^*) is always one dimensional.

(ii) The relationships between the Martin boundary at infinity of P^* and the limits at infinity of quotients of two positive solutions of minimal growth at infinity of the operator P are stressed in [21]. ■

The following corollary is the extension of the Liouville positive theorem, the Picard principle and the Poisson principle for the case of smooth boundary.

COROLLARY 7.3. — (i) *Assume that* P *is a weakly Fuchsian operator at infinity in a Lipschitz cone* X *and assume that* P *admits a positive solution in* X. *Suppose that the coefficients of* P *and* P^* *are Lipschitz continuous up to the boundary* ∂X. *Then the cones of positive solutions of the operators* P *and* P^* *that vanish on* ∂X *are of one dimension.*

(ii) *Assume that* P *is a weakly Fuchsian operator at the isolated singular point* $\zeta=0$ *in a Lipschitz bounded domain* X *and assume that* P *admits a positive solution in* $X\setminus\{0\}$. *Suppose that the coefficients of* P *and* P* *are Lipschitz continuous in* $\mathrm{cl}\,X\setminus\{0\}$. *Then the cones of positive solutions of the operators* P *and* P* *in* $X\setminus\{0\}$ *that vanish on* ∂X *are of one dimension.*

(iii) *Assume that* P *is a weakly Fuchsian operator at the boundary point* $\zeta=0$ *in a Lipschitz bounded domain X. Suppose that the coefficients of* P *and* P* *are Lipschitz continuous up to* $\partial X\setminus\{0\}$. *Then the cones of positive solutions of the operators* P *and* P* *that vanish on* $\partial X\setminus\{0\}$ *are of one dimension.*

8. APPLICATIONS

In this section we present some applications of Theorem 7.1 to the theory of positive solutions of linear and also semilinear equations. We first deal with the behavior of solutions which do not have a constant sign.

COROLLARY 8.1. – *Let* P *be a Fuchsian type operator at infinity which is defined on* $X=\mathbb{R}^d$, $d\geqq 2$. *Assume that* P *admits a positive solution* u *in* X. *Let* v *be a solution of the equation* $Pu=0$ *in* $\mathbb{R}^d$ *which changes its sign in* R^d. *Then*

$$\lim_{R\to\infty} M_u(R)/M_{|v|}(R)=0. \tag{8.1}$$

Proof. – Assume that u and v satisfy the assumptions above and assume that (8.1) does not hold true. Therefore, there exists $\varepsilon>0$ and a sequence $\{R_n\}$, $R_n\to\infty$, such that $M_u(R_n)/M_{|v|}(R_n)>\varepsilon$. Without loss of generality, we may assume that

$$M_u(R_n)>\varepsilon M_v(R_n) \quad \text{for all } n\geqq 1. \tag{8.2}$$

By Lemma 6.3 (*i*) and (8.2) we have

$$Cu(x)>M_u(R_n)>\varepsilon M_v(R_n)\geqq \varepsilon v(x) \quad \text{for all } x\in S_{R_n} \text{ and } n\geqq 1. \tag{8.3}$$

Hence, the generalized maximum principle implies that $\varepsilon^{-1}Cu(x)>v(x)$ for all $x\in B_{R_n}$ and all $n\geqq 1$. So, the function $w(x):=\varepsilon^{-1}Cu(x)-v(x)$ is a global positive solution of the operator P. It follows from Corollary 7.3 (i) that $w=\alpha u$ for some positive number α. So, $v=(\varepsilon^{-1}C-\alpha)u$ and v does not change its sign in $\mathbb{R}^d$ which contradicts our assumption on v. ■

Remark 8.2. – (i) Suppose that u in Corollary 8.1 is a positive solution of the operator P only in a neighborhood of infinity.

If the operator P is a subcritical Fuchsian (at infinity) elliptic operator then assertion (8.1) holds even for such a solution u. We do not know if in the critical case assertion (8.1) is true for a positive solution near infinity which does not have minimal growth at infinity.

(ii) If the operator P admits a positive solution in a neighborhood of infinity (but not a global positive solution) it may happen that (8.1) does not hold true for a positive solution near infinity u and a global solution v (which changes its sign) as the following example shows:

Consider the function $u(x)=\log|x|$ in the exterior domain $D_{1/2}\subseteqq\mathbb{R}^2$ and extend it as a smooth function $v(x)$ in the whole space $\mathbb{R}^2$ such that v is negative in B_1. Let $P=-\Delta+W(x)$, where $W(x)=\Delta v(x)/v(x)$. So, v is a smooth global solution of the operator P which changes its sign in $\mathbb{R}^d$ and $u(x)$ is a positive solution of the operator P in a neighborhood of infinity. On the other hand, $\lim_{R\to\infty} M_u(R)/M_{|v|}(R)=1$. Note, that in this example P does not admit positive solutions in $\mathbb{R}^2$, since v does not satisfy the generalized maximum principle in B_1. ■

COROLLARY 8.3. — *Let* P *be a Fuchsian type operator at infinity which is defined on* $X=\mathbb{R}^d$, $d\geqq 2$ *and satisfies hypothesis* (H) *in* $\mathbb{R}^d$. *There exists a real number* α *which depends on* P *such that if* v *is a solution of the operator* P *in* $\mathbb{R}^d$ *such that* $v(x_0)>0$ *at some point* $x_0\in\mathbb{R}^d$ *and*

$$v(x)>-C|x|^\alpha \tag{8.4}$$

for some positive number C *and* $|x|$ *large enough, then* v *is the unique* (*up to a constant*) *positive solution of the operator* P *in* $\mathbb{R}^d$.

In particular, if the number α *which satisfies the property above is nonnegative then* P *satisfies the following Liouville property: If* v *is a solution as above and* v *is bounded* (*below*) *then* v *is the unique* (*up to a constant*) *positive solution of the operator* P *in* X. *Moreover, if* $c(x)\geqq 0$ *then* $\alpha\geqq 0$.

Proof. — Let u be a positive solution of the operator P in $\mathbb{R}^d$. It follows from Lemma 6.5 (i) that there exists $\alpha\in\mathbb{R}$ such that $u(x)>C|x|^\alpha$. So, if v is a global solution which satisfies Inequality (8.4) with the above number α then

$$v(x)>-C|x|^\alpha>-u(x)\quad\text{for all } |x| \text{ large enough.} \tag{8.5}$$

Therefore, by the generalized maximum principle the function $w=v+u$ is a positive global solution of the operator P. Corollary 7.3 implies that there exists a positive number μ such that $w=\mu u$. Consequently, $v=(\mu-1)u$ and v is of a constant sign. By our assumption $v(x_0)>0$, hence, v is the unique (up to a constant) positive solution of the operator P in $\mathbb{R}^d$.

Suppose now that $c(x)\geqq 0$ and let u be the unique positive solution in $\mathscr{K}$. If $\liminf_{|x|\to\infty} u(x)=0$, then it follows from Lemma 6.3 that $\lim_{|x|\to\infty} u(x)=0$ which contradicts the maximum principle. Therefore, $u\geqq C$

for some $C>0$ and hence $\alpha \geqq 0$. It is clear now that the assertion of the lemma with $\alpha \geqq 0$ implies that P satisfies the above Liouville property. ∎

The following lemma is a representation theorem for positive solutions of a Fuchsian operator near an isolated point.

LEMMA 8.4. – *Assume that* P *is a weakly Fuchsian operator at the isolated singular point* $\zeta=0$ *in a Lipschitz bounded domain* X *with coefficients which are Lipschitz continuous in* $\operatorname{cl} X \setminus \{0\}$. *Furthermore, assume that* P *admits a positive solution in* $X \setminus \{0\}$. *Let* u *be a positive solution of the operator* P *in* $X \setminus \{0\}$ *which is continuous in* $\operatorname{cl} X \setminus \{0\}$. *Then*

$$u=u_0+u_1 \tag{8.6}$$

where $u_0 \in \mathscr{C}_0(P, X) \cup \{0\}$ *and* u_1 *is a nonnegative solution which equals* u *on* ∂X. *Moreover, the representation* (8.6) *is unique.*

If in addition, $c(x) \geqq 0$ *then* u_1 *is bounded. Furthermore, if* $u_0 \neq 0$, P *is Fuchsian at the origin and*

$$|x|^2 c(x)>m>0, \tag{8.7}$$

then there exist negative numbers α *and* β *such that* $|x|^\alpha \leqq u_0(x) \leqq |x|^\beta$ *for small* $|x|$.

Proof. – Let w_n be the solution of the problem

$$\left.\begin{aligned} P w_n&=0 \quad \text{in } X \setminus B_{1/n}, \\ w_n&=u \quad \text{on } \partial X, \\ w_n&=0 \quad \text{on } \partial B_{1/n}. \end{aligned}\right\} \tag{8.8}$$

Then $w_n \leqq u$, $\{w_n\}$ is an increasing sequence which converges to a nonnegative solution u_1 and we have $u_1 \leqq u$. Note that $u_1=0$ if and only if $u=0$ on ∂X. Let $u_0=u-u_1$. Suppose that $u_0 \neq 0$ then u_0 is a positive solution of the operator P in $X \setminus \{0\}$. Moreover, since the boundary ∂X is Lipschitz u_0 vanishes on ∂X. Therefore, Corollary 7.3 (ii) implies that $u_0 \in \mathscr{C}_0$ and that u_0 is uniquely determined by its value at some point $x_0 \in X \setminus \{0\}$.

If $c(x) \geqq 0$ then $v \equiv 1$ is a positive supersolution. Using the maximum principle we obtain that $w_n \leqq M_u(\partial X)$. Thus, u_1 is bounded.

Suppose that P is Fuchsian at the origin and (8.7) is satisfied. We find on computation that for some negative number α, the function $|x|^\alpha$ is a positive supersolution of the operator P in X. Suppose that $\liminf_{|x| \to 0} u_0(x)/|x|^\alpha=0$ then Lemma 6.3 implies that $\lim_{|x| \to 0} u_0(x)/|x|^\alpha=0$. Consider the supersolution $v_M(x)=|x|^\alpha - M u_0(x)$, the maximum principle implies that v_M is positive in $X \setminus \{0\}$ for every $M>0$ which contradicts our assumption that u_0 is positive. Therefore, for some $\varepsilon>0$, $\varepsilon |x|^\alpha \leqq u_0(x)$ near the origin. The inequality $u_0(x) \leqq |x|^\beta$ follows from Lemma 6.5. ∎

We formulate now an analog representation theorem for a Fuchsian elliptic operator at infinity on a Lipschitz cone. The proof is left to the reader.

LEMMA 8.5. — *Assume that* P *is a weakly Fuchsian operator at* $\zeta=\infty$ *on a Lipschitz cone* X *with coefficients which are Lipschitz continuous up to* ∂X. *Furthermore, assume that* P *satisfies hypothesis* (H). *Let* u *be a positive solution of the operator* P *in* X *which is continuous in* cl X. *Then*

$$u=u_0+u_1 \tag{8.9}$$

where $u_0 \in \mathscr{C}_\infty(\mathrm{P}, \mathrm{X}) \cup \{0\}$ *and* u_1 *is a nonnegative solution which equals* u *on* ∂X. *Moreover, the representation* (8.9) *is unique.*

As in [3], we are now ready to use our results on the asymptotic behavior of positive solutions of the linear operator P in the study of the asymptotic behavior of a positive solution of a *semilinear* equation. We show that the solution behaves like a solution of the linearized problem. First we deal with a semilinear equation on a cone (*see* [3], for the analog results and proofs for the case where $\mathrm{X}=\mathbb{R}^d$ or a strip).

THEOREM 8.6. — *Let* P *be a Fuchsian elliptic operator at infinity which is defined on a cone* X *and assume in addition that*

$$\liminf_{|x|\to\infty} |x|^2 c(x)>0. \tag{8.10}$$

Suppose that $f(x, u)$ *is a given function which satisfies, for some* $\delta>0$,

$$|x|^2 |f(x, u)| \leqq \mathrm{C} u^{1+\delta} \quad \text{for } u>0 \text{ small and } |x| \text{ large.} \tag{8.11}$$

Let u *and* v *be positive solutions of the respective equations*

$$\mathrm{P}u+f(x, u)=0, \qquad \mathrm{P}v=0, \tag{8.12}$$

in a neighborhood E *of infinity in* X *which vanish continuously on* $\Gamma \cap \partial\mathrm{E}$. *Assume also that* u *tends to zero at infinity. Then*

$$\lim_{|x|\to\infty} u(x)/v(x)=\mathrm{A}\geqq 0, \tag{8.13}$$

where $\mathrm{A}>0$ *if and only if* v *is a positive solution of minimal growth at infinity.*

Proof. — If v and w are positive solutions of the operator P in E which vanish on $\Gamma \cap \partial\mathrm{E}$ and v has minimal growth at infinity while w does not then by Theorem 6.3 (ii) $\lim_{|x|\to\infty} v(x)/w(x)=0$. Therefore, it is enough to prove the theorem for a positive solution v of minimal growth at infinity.

Using our assumption (8.10), we find on computation that for some negative number α, the function $|x|^\alpha$ is a positive supersolution of the operator P in some exterior domain in $\mathbb{R}^d$. Thus, $\lim_{|x|\to\infty} v(x)=0$.

Now, the proof of the theorem follows from Theorem 7.1 (i) in exactly the same way that Theorem 3 in [3] follows from Theorem 2 therein (*see* also the proof of Theorem 8.7). ■

The next theorem concerns the asymptotic behavior of positive solutions of semilinear equations near an isolated singular point.

THEOREM 8.7. — *Let* P *be a Fuchsian elliptic operator at the origin which is defined in some punctured neighborhood of the origin. Assume in addition that*

$$\liminf_{|x| \to 0} |x|^2 c(x) > m > 0. \tag{8.14}$$

Suppose that $f(x, u)$ *is a given function which satisfies, for some* $0 < \delta < 1$,

$$|x|^2 |f(x, u)| \leqq C u^\delta \qquad \text{for} \quad |x| \text{ small and } u > 0 \text{ large.} \tag{8.15}$$

Let u *and* v *be positive solutions of the respective equations*

$$Pu + f(x, u) = 0, \qquad Pv = 0, \tag{8.16}$$

in some punctured neighborhood of the origin. Assume also that u *and* v *tend to infinity as* $|x|$ *tends to zero. Then there exists positive number* A *such that*

$$\lim_{|x| \to 0} u(x)/v(x) = A > 0, \tag{8.17}$$

Proof. — Consider the elliptic operator $\hat{P} = P + \rho(x)$, where $\rho(x) = f(x, u)/u$. Then by (8.15) $\hat{P}$ is Fuchsian operator at the origin and u is a positive solution of the equation $\hat{P}u = 0$ in some punctured neighborhood of the origin. Recall that if u_0 and u_1 are positive supersolutions of an elliptic operator L then for any $0 < \gamma < 1$ the function $(u_0)^\gamma (u_1)^{1-\gamma}$ is also a supersolution of the operator L. Moreover, by (8.14) the function $1(x) \equiv 1$ is a positive supersolution of the operator P. Therefore, it is natural to consider the functions $w_\pm(x) = v(x) \pm v(x)^\gamma$ as a super- (respectively, sub-) solution of the operator $\hat{P}$, where $0 < \gamma < 1$ will be determined later on. Indeed, we find on computation that

$$\hat{P} w_+ \geqq c(1-\gamma) v^\gamma + \rho(v^\gamma + v) \geqq |x|^{-2} \{ m(1-\gamma) v^\gamma - u^{\delta-1}(v^\gamma + v) \}. \tag{8.18}$$

It follows from Lemma 8.4 that there exist negative numbers α and β such that $|x|^\alpha \leqq u(x)$ and $v(x) \leqq |x|^\beta$ near the origin. Hence

$$u(x)^{\delta-1} \leqq |x|^{\alpha(\delta-1)} \leqq v(x)^{\alpha(\delta-1)/\beta} \tag{8.19}$$

near the origin. Now, by choosing γ close enough to one we obtain that

$$\hat{P} w_+ = \hat{P}(v + v^\gamma) \geqq 0. \tag{8.20}$$

Similarly,

$$\hat{P} w_- = \hat{P}(v - v^\gamma) \leqq 0. \tag{8.21}$$

Let N be a large enough natural number. For $n \geqq N$ let w_n be the solution of the following Dirichlet problem in the annulus $B_{1/N,\,1/n}$:

$$\left.\begin{aligned} \hat{P}\,w_n = 0 \quad &\text{in } B_{1/N,\,1/n} \\ w_n = w_+ \quad &\text{on } \partial B_{1/N,\,1/n}. \end{aligned}\right\} \tag{8.22}$$

By the maximum principle $w_- \leqq w_n \leqq w_+$ and therefore, w_n is a decreasing sequence which converges to a positive solution $\hat{w}$ of the operator $\hat{P}$ in $B_{1/N} \setminus \{0\}$. Since

$$v - v^\gamma = w_- \leqq \hat{w} \leqq w_+ = v + v^\gamma,$$

we obtain

$$\lim_{|x| \to 0} \hat{w}(x)/v(x) = 1. \tag{8.23}$$

Decompose $u = u_0 + u_1$ and $\hat{w} = \hat{w}_0 + \hat{w}_1$ in $B_{1/N} \setminus \{0\}$ according to Theorem 8.4. So, u_0, $\hat{w}_0 \in \mathscr{C}_0(\hat{P}, B_{1/N})$ and u_1 and $\hat{w}_1$ are bounded. Since $\dim \mathscr{C}_0 = 1$ we have $\hat{w}_0 = C\,u_0$ for some $C > 0$. Therefore,

$$\lim_{|x| \to 0} \hat{w}(x)/u(x) = C. \tag{8.24}$$

Combining (8.23) and (8.24) we see that the theorem is proved with $A = 1/C$. ■

We conclude this section with some remarks on possible extensions of our results to another type of boundary conditions (oblique derivative boundary conditions) and to another type of unbounded domains (strips).

Remark 8.8. — (i) Consider a Fuchsian at infinity elliptic operator P on a cone X with a smooth boundary Γ and let $\mathscr{C}_\zeta(P, X, B)$ be the cone of all positive solutions of the operator P in X which are of class C^1 near the boundary Γ and satisfy the oblique derivative boundary condition

$$B\,u = \mu(x).\nabla u + \gamma(x)\,u = 0 \quad \text{on } \Gamma. \tag{8.25}$$

Here the function γ and the unit vector μ are smooth on the boundary Γ and μ is nowhere tangential to Γ. We would like to obtain a positive Liouville theorem and asymptotic behavior results also in this case.

To this end, we need to use the up to the boundary Harnack inequality for the dilated operator P_n (for the definition of P_n, see (6.8)) and then we may apply the same techniques as in Section 7. Therefore, we need to assume that the boundary operator B is of Fuchsian type. That us, we assume that

$$\gamma(x) \geqq 0 \qquad \text{and} \qquad \mu(x).\nu(x) \geqq \varepsilon, \tag{8.26}$$

and

$$(1 + |x|)\,|\gamma(x)| \leqq M \qquad \text{for all} \quad x \in \Gamma, \tag{8.27}$$

here M and ε are positive constants and $\nu(x)$ is the unit outward normal vector to Γ at the point x.

But since the constant in this Harnack inequality depends also on the C^2 norms of μ and γ, we do have to add appropriate decay assumptions on the first and the second derivatives of these functions. Namely, in addition to (8.27) we assume that for every multi-index α, $0 \leqq |\alpha| \leqq 2$

$$(1+|x|)^{|\alpha|+1} |D^\alpha \gamma(x)| + (1+|x|)^{|\alpha|} |D^\alpha \mu(x)| \leqq M, \quad \text{for all } x \in \Gamma. \tag{8.28}$$

Now, it is clear from the arguments used throughout our paper that under the assumptions (8.26)-(8.28) the Dirichlet boundary condition can be replaced by the oblique derivative boundary condition (8.25). In particular, the dimension of the cone of all positive solutions of the operator P in the cone X which satisfy (8.25) is at most one. Moreover, the quotient of two positive solutions of the operator P at a neighborhood of infinity in X satisfying (8.25) admits a limit at infinity.

(ii) The results of Section 7 are valid also in the case where X is a semi-infinite strip in $\mathbb{R}^d$. In this case it is assumed that $X = [0, \infty) \times \Omega$, where Ω is a bounded smooth domain in $\mathbb{R}^{d-1}$, P is a uniformly elliptic operator with bounded coefficients in X, and positive solutions of the operator P are assumed to satisfy the Dirichlet boundary condition on $\Gamma = \partial X$ (or even the regular oblique derivative boundary condition (8.25), provided that (8.26) is satisfied and the C^2 norms of μ and γ are bounded on Γ). The case of the Neumann condition on a semi-infinite strip was studied intensively in [3]. ■

9. EXAMPLES

In this section we shall present some examples which will demonstrate the sharpness of our result.

The first three examples show that unlike the case of a Fuchsian type elliptic operator with $c(x) \equiv 0$ or the case of a small perturbation of $-\Delta$, the operator P may admit a positive solution in $\mathbb{R}^d$ which is not bounded or with no limit at infinity. Moreover, in our case the Green function of P (if it exists) may not behave at infinity like the Green function of the Laplacian.

EXAMPLE 9.1. — Let $u(x) = 2 + \sin(\log|x|)$, and define $V(x) = \Delta u/u$. Clearly, $1 \leqq u(x) \leqq 3$, $|V(x)| \leqq M/(1+|x|)^2$ and by Corollary 7.3, u is the unique (up to a constant) positive solution of the Fuchsian at infinity equation

$$Pu = -\Delta u + V(x) u = 0, \quad \text{in } \mathbb{R}^d. \tag{9.1}$$

But $\lim_{x \to \infty} u(x)$ does not exist.

Note that in Example 9.1 positive solutions behave like a radial function near infinity. The following example shows a behavior of a different type for positive solutions at infinity.

EXAMPLE 9.2. — Let $f : S^{d-1} \to [-1, 1]$ be a real nonconstant smooth function. For $x \in \mathbb{R}^d$, $|x| \geqq 1$ let $u(x) = 2 + f(x/|x|)$, and extend the function u as a smooth positive function on $\mathbb{R}^d$. Let $V(x) = \Delta u/u$. Then $|V(x)| \leqq M/(1+|x|)^2$ and $u(x)$ is a bounded positive function which is bounded away from zero. Moreover, $u(x)$ is the unique (up to a constant) positive solution of the Fuchsian at infinity equation

$$P u = -\Delta u + V(x) u = 0 \quad \text{in } \mathbb{R}^d, \tag{9.2}$$

but $\lim_{x \to \infty} u(x)$ does not exist.

EXAMPLE 9.3. — Let $c(x) \geqq 0$ be a smooth function which equals 1 outside the unit disk and zero in a neighborhood of the origin and let t be a real number. Consider the Fuchsian at infinity operator

$$P = -\Delta + tc(x)/|x|^2,$$

If $\mathfrak{d} = ((2-d)^2 + 4t) > 0$, then P admits positive solutions in a neighborhood of infinity which behave like $|x|^{\tau_\pm}$, where $\tau_\pm = (2 - d \pm \mathfrak{d}^{1/2})/2$. The functions $|x|^{\tau_\pm}$ describe the behavior of positive solutions which do not have minimal growth at infinity and of positive solutions of minimal growth at infinity respectively.

The next example shows that the results are sharp in the sense of the exponents which appear in (1.3).

EXAMPLE 9.4. — Consider the operator $P = -\Delta + tc(x)|x|^{-\mu}$, where $c(x) \geqq 0$ is a smooth function which equals 1 outside the unit disk and zero in some neighborhood of the origin, $0 \leqq \mu < 2$ and $t > 0$. Then $\mathscr{C}_\infty$ is an infinite dimensional cone (*see* [14]).

QUESTION 9.5. — Suppose that P is a Fuchsian operator at infinity on $\mathbb{R}^d$ and $c(x) \geqq 0$. Then the maximum principle and Lemma 6.3 imply that the normalized positive solution of the operator P in $\mathbb{R}^d$ is bounded away from zero (*see* also Corollary 8.3) and a positive solution in a neighborhood of infinity of minimal growth at infinity is bounded. Is it true that any positive solution of the operator P in an exterior domain D admits a limit at infinity?

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