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A notion of total variation depending on a metric with discontinuous coefficients

by

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ABSTRACT. — Given a function $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, we introduce a notion of total variation of u depending on a possibly discontinuous Finsler metric. We prove some integral representation results for this total variation, and we study the connections with the theory of relaxation.

Key words : BV functions, semicontinuity, relaxation theory.

RÉSUMÉ. — Étant donnée une fonction $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, on définit une notion de variation totale de u, dépendant d'une métrique Finslérienne discontinue. On démontre quelques résultats de représentation intégrale pour cette variation totale et ses relations avec la théorie de la relaxation.

Classification A.M.S. : 49 J 45, 49 Q 25.

1. INTRODUCTION

In this paper, given a function $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, we introduce a notion of total variation of u depending on a Finsler metric $g(x, \xi)$, convex in the tangent vector ξ and possibly discontinuous with respect to the position $x \in \Omega$.

It is known that Finsler metrics arise in the context of geometry of Lipschitz manifolds (*see*, for instance, [9], [10], [40], [42], [44]). More recently, a notion of quasi-Finsler metric space has been proposed in [23], [24], [25]. In this context, problems involving geodesics and derivatives of distance functions depending on such metrics have been studied, among others, in [18], [19], [20], [21], [45]. Furthermore, an important area where metrics which depend on the position play an important role is the theory of phase transitions, in particular in the case of anisotropic and non-homogeneous media. This kind of problems is related also to the asymptotic behaviour of some singular perturbations of minimum problems in the Calculus of Variations (*see*, for instance, [4], [6], [38], [39]).

We concentrate mainly on the study of the relations between our definition and the theories of integral representation and relaxation, which constitute a proper variational setting for problems involving total variation. In order to do that, we search for a definition satisfying the following basic properties: (i) two Finsler metrics which coincide almost everywhere with respect to the Lebesgue measure give rise to the same total variation; (ii) the total variation with respect to the Finsler metric g must be $L^1(\Omega)$-lower semicontinuous on the space $BV(\Omega)$ of the functions of bounded variation in Ω. We shall start from a distributional definition, since this seems to be convenient to obtain properties (i)-(ii).

More precisely, let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary, and let $g : \Omega \times \mathbb{R}^n \to [0, +\infty[$ be a Finsler metric. Let $\phi = g^0$, where g^0 denotes the dual function of g [*see* (2.13)]. In the sequel, for simplicity of notation we shall refer our definitions and results to the function ϕ instead of the function g. If ϕ is continuous, the functional $\mathscr{J}[\phi] : BV(\Omega) \to [0, +\infty]$ defined by

$$(1.1) \qquad \mathscr{J}[\phi](u) = \int_\Omega \phi(x, \nu^u(x)) \, |Du| \qquad \forall u \in BV(\Omega),$$

where $\nu^u(x) = \dfrac{Du}{|Du|}(x)$, satisfies all previous requirements, as we shall see in the sequel, and it provides a natural definition of total variation of u in Ω with respect to ϕ (*see* theorem 5.1). However, if ϕ is not continuous, (1.1) is not the appropriate notion, since properties (i)-(ii) above are not satisfied. For instance, it is easy to realize that $\mathscr{J}[\phi]$ depends on the choice of the representative of ϕ in its equivalence class with respect to the

Lebesgue measure. The leck of properties (i)-(ii) for $\mathscr{J}[\phi]$ is basically due to the fact that the function ϕ has linear growth [*see* (2.19)] and is discontinuous. Indeed, because of the linear growth of ϕ, any lower semicontinuous functional related to ϕ must be defined in the space $\mathrm{BV}(\Omega)$. We are led then to integrate ϕ with respect to the measure $|\mathrm{D}u|$, for $u \in \mathrm{BV}(\Omega)$. But, as ϕ is discontinuous, its values on sets with zero Lebesgue measure (such as the boundaries of smooth sets) are not uniquely determined. These difficulties do not occur if, instead of the total variation, one considers the Dirichlet energy. Indeed, if $\{a_{ij}\}_{i,j}$ is a discontinuous elliptic matrix, then the integrand $a_{ij}(x)\nabla_i u \nabla_j u$ for $u \in \mathrm{W}^{1,2}(\Omega)$ gives rise to a lower semicontinuous functional (*see* [29]) which remains unchanged whenever $\{a_{ij}\}_{i,j}$ is replaced by any other matrix which coincides with $\{a_{ij}\}_{i,j}$ almost eveywhere. The lack of continuity of ϕ in the variable $x \in \Omega$ is the crucial point and the main originality of the present paper.

Our starting point is the following distributional definition. For any $u \in \mathrm{BV}(\Omega)$ we define the generalized total variation of $u \in \mathrm{BV}(\Omega)$ (with respect to ϕ) in Ω as

$$(1.2) \qquad \int_\Omega |\mathrm{D}u|_\phi = \sup \int_\Omega u \operatorname{div} \sigma \, dx,$$

where the supremum is taken over all vector fields $\sigma \in \mathrm{L}^\infty(\Omega; \mathbb{R}^n)$ with compact support in Ω such that $\operatorname{div} \sigma \in \mathrm{L}^n(\Omega)$ and $\phi^0(x, \sigma(x)) \leqq 1$ for almost every $x \in \Omega$.

Note that, as a straightforward consequence of the definition, $\int_\Omega |\mathrm{D}u|_\phi$ satisfies the basic property (i) and is $\mathrm{L}^{n/(n-1)}(\Omega)$-lower semicontinuous [actually property (ii) holds by theorem 5.1]. The choice of the class of test vector fields σ [which is obviously larger than the space $\mathscr{C}_0^1(\Omega; \mathbb{R}^n)$ of the functions σ belonging to $\mathscr{C}^1(\Omega; \mathbb{R}^n)$ which have compact support in Ω] relies on some results about the pairing between measures and functions of bounded variation (*see* [2], [3]). In remark 8.5 we show that smooth σ can be insensible to the discontinuities of ϕ, and so the space $\mathscr{C}_0^1(\Omega; \mathbb{R}^n)$ is an inadequate class of test functions for our purposes. The choice of the constraint $\phi^0 \leqq 1$ is motivated by arguments of convex analysis.

It is not difficult to prove that (1.2) coincides with the classical notion of total variation $\int_\Omega |\mathrm{D}u|$ when $\phi(x, \xi) = \|\xi\|$ [*see* (3.4)].

Our first result is an integral representation of $\int_\Omega |\mathrm{D}u|_\phi$ in terms of the measure $|\mathrm{D}u|$ (theorem 4.3), which provides a more manageable characterization of the generalized total variation. As an immediate consequence of this representation theorem, a coarea formula for $\int_\Omega |Du|_\phi$ is given (remark 4.4).

In the classical setting of relaxation theory, it is customary to present $\int_\Omega |Du|$ as a lower semicontinuous envelope, *i.e.*,

$$\int_\Omega |Du| = \inf\left\{\liminf_{h\to+\infty}\int_\Omega \|\nabla u_h\|\,dx : \{u_h\}_h \subseteq W^{1,1}(\Omega),\ u_h \xrightarrow{L^1(\Omega)} u\right\}.$$

The problem of regarding $\int_\Omega |Du|_\phi$ as a lower semicontinuous envelope of some functional defined on $BV(\Omega)$ is quite delicate. To this purpose a crucial role is played by some recent results about the integral representation of local convex functionals on $BV(\Omega)$ proven in [7]. Let us consider the functional $\mathscr{F}[\phi] : BV(\Omega) \to [0, +\infty]$ defined by

$$\mathscr{F}[\phi](u) = \begin{cases} \int_\Omega \phi(x, \nabla u(x))\,dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.3}$$

and denote by $\overline{\mathscr{F}[\phi]} : BV(\Omega) \to [0, +\infty]$ the $L^1(\Omega)$-lower semicontinuous envelope of $\mathscr{F}[\phi]$. In theorem 5.1 we prove that

$$\int_\Omega |Du|_\phi = \overline{\mathscr{F}[\phi]}(u) \qquad \forall u \in BV(\Omega).$$

Consider now the functional $\mathscr{J}[\phi]$ defined in (1.1). Since ϕ is only a Borel function, the modifications of the values of ϕ on zero Lebesgue sets must be taken into account. Precisely, let $N \subset \Omega$ be a set of zero Lebesgue measure and let ϕ_N be a representative of ϕ obtained by modifying ϕ on N as in (6.4). In theorem 6.4 we prove that $\int_\Omega |Du|_\phi$ equals the supremum, over all such sets N, of the functionals $\overline{\mathscr{J}[\phi_N]}$.

This operation of modifying ϕ on sets of zero Lebesgue measure can be dropped if ϕ is upper semicontinuous. In fact, in theorem 6.5 we prove that the $L^1(\Omega)$-lower semicontinuous envelope $\overline{\mathscr{J}[\phi]}$ of $\mathscr{J}[\phi]$ on the space $BV(\Omega)$ coincides with $\overline{\mathscr{F}[\phi]}$ (and hence with $\int_\Omega |Du|_\phi$), provided that ϕ is upper semicontinuous.

It is clear that (1.2) introduces a notion of generalized perimeter $P_\phi(E, \Omega)$ of a set E in Ω (with respect to ϕ), simply by taking $\int_\Omega |D\chi_E|_\phi$, provided $\chi_E \in BV(\Omega)$, where χ_E is the characteristic function of E. The results of section 6 can be regarded then as a one codimensional counterpart of the one dimensional results about curves proven in [18], [19], [20], [21].

If $E \subseteq \mathbb{R}^n$ is a measurable set of finite perimeter in Ω, one can consider also the quantity

$$\mathcal{J}_\phi(E, \Omega) = \inf\{\liminf_{h \to +\infty} \mathcal{J}[\phi](\chi_{E_h}) : \{\chi_{E_h}\}_h \subseteq BV(\Omega), \chi_{E_h} \xrightarrow{L^1(\Omega)} \chi_E\},$$

which stands for a lower semicontinuous envelope of $\mathcal{J}[\phi]$ by means only of sequences of characteristic functions. In theorem 6.9 we show that $\mathcal{J}_\phi(E, \Omega) = \overline{\mathcal{J}[\phi]}(\chi_E)$ for any measurable set $E \subseteq \mathbb{R}^n$ of finite perimeter in Ω and, if ϕ is upper semicontinuous, then

$$P_\phi(E, \Omega) = \mathcal{J}_\phi(E, \Omega).$$

In the special case in which $\phi(x, \xi)^2 = \sum_{i, j=1}^{n} a_{ij}(x) \xi_i \xi_j$ and $\{a_{ij}\}_{i, j}$ is a continuous coercive symmetric matrix, we prove in proposition 7.1 that

$$\int_\Omega |Du|_\phi = \int_\Omega \left(\sum_{i, j=1}^{n} a_{ij}(x) \nu_i^u \nu_j^u \right)^{1/2} |Du|.$$

However we show that, if the matrix $\{a_{ij}\}_{i, j}$ is not continuous, in general $\int_\Omega |Du|_\phi$ cannot be represented as the integral of the square root of a quadratic form, and to do that we exhibit a counterexample. The construction of the counterexample was suggested by E. De Giorgi (*see* [25], p, 117) in the context of geometry of Lipschitz manifolds, and in that setting it has been studied in [18], [19]. The same metric provides a counterexample also for our problem in codimension one, but it requires a completely different proof.

Finally, we stress that our results are still valid if we drop the hypothesis that ϕ is the dual function of the metric g. More precisely, we only suppose that ϕ is positively homogeneous of degree one in the tangent vector ξ [condition (2.12)] with linear growth [condition (2.19)], and no convexity assumption is considered.

The outline of the paper is as follows.

In section 2 we give some definitions and we recall some results on BV functions and sets of finite perimeter.

In section 3 we introduce the definition of the generalized total variation $\int_\Omega |Du|_\phi$ for $u \in BV(\Omega)$ with respect to ϕ, pointing out the connections with the classical theory.

In section 4 we prove an abstract integral representation theorem for $\int_\Omega |Du|_\phi$.

In section 5 we prove that $\int_\Omega |Du|_\phi$ coincides with the lower semicontinuous envelope on $BV(\Omega)$ of the functional $\mathcal{F}[\phi]$ defined in (1.3), and also with the lower semicontinuous envelope of a functional involving the slope of the function u with respect to ϕ.

In section 6 we prove that $\int_\Omega |Du|_\phi$ can be written as the supremum of a suitable family of functionals which are lower semicontinuous envelopes of functionals of the form (1.1). In this context the sets of zero Lebesgue measure play a central role. The final part of this section is devoted to proving that the functional $\overline{\mathcal{J}[\phi]}$, when restricted to sets of finite perimeter, can be found using only sequences of characteristic functions.

In section 7 we evaluate the generalized total variation when ϕ is the square root of a coercive quadratic form with continuous coefficients.

Finally, in section 8 we prove in detail the counterexample.

2. PRELIMINARIES

For any $x, y \in \mathbb{R}^n$, we denote by (x, y) the canonical scalar product between x and y, and by $\|x\| = (x, x)^{1/2}$ the euclidean norm of x. The absolute value of a real number r is denoted by $|r|$. If $r > 0$ and $x \in \mathbb{R}^n$, we set $B_r(x) = \{y \in \mathbb{R}^n : \|y - x\| < r\}$, and $S^{n-1} = \{y \in \mathbb{R}^n : \|y\| = 1\}$. For any set $F \subseteq \mathbb{R}^n$, we indicate by ∂F the topological boundary of F, and by $\mathrm{co}(F)$ the convex hull of F. Given two functions f, g, we denote by $f \wedge g$ (respectively $f \vee g$) the function $\min\{f, g\}$ (respectively $\max\{f, g\}$).

In what follows, Ω will be a bounded open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary. If λ is a (possibly vector-valued) Radon measure, its total variation will be denoted by $|\lambda|$. If μ is a scalar Radon measure on Ω such that λ is absolutely continuous with respect to μ, the symbols $\frac{d\lambda}{d\mu}$, or $\frac{\lambda}{\mu}$, stand for the Radon-Nikodym derivative of λ with respect to μ.

Let $B \subseteq \Omega$ be a Borel set and let λ be a scalar or vector-valued Radon measure on Ω. If $f : B \to \mathbb{R}$ is a Borel function, then the integral of f on B with respect to $|\lambda|$ will be indicated by $\int_B f |\lambda|$. We indicate by dx and by $\mathcal{H}^k$ the Lebesgue measure and the k-dimensional Hausdorff measure in $\mathbb{R}^n$ for $1 \leqq k \leqq n$, respectively. We denote by $\mathcal{N}(\Omega)$ the family of all subsets N of Ω having zero Lebesgue measure.

2.1. The space BV(Ω)

The space BV(Ω) is defined as the space of the functions $u \in L^1(\Omega)$ whose distributional gradient Du is an $\mathbb{R}^n$-valued Radon measure with bounded total variation in Ω. We indicate by ν^u the Radon-Nikodym derivative of Du with respect to $|Du|$, *i.e.*, $\nu^u(x) = \frac{Du}{|Du|}(x)$ for $|Du|$-almost every $x \in \Omega$.

We recall that, as Ω has a Lipschitz continuous boundary, the space BV(Ω) is contained in $L^{n/(n-1)}(\Omega)$ (*see* [34], § 6.1.7).

If $u \in$ BV(Ω), the total variation of Du in Ω is given by

$$(2.1) \quad \int_\Omega |Du| = \sup \left\{ \int_\Omega u \operatorname{div} \sigma \, dx : \sigma \in \mathscr{C}_0^1(\Omega; \mathbb{R}^n), \|\sigma(x)\| \leqq 1, \forall x \in \Omega \right\},$$

or, equivalently, by

$$(2.2) \quad \int_\Omega |Du| = \sup \{ \sum_{i \in I} \|Du(B_i)\| : \{B_i\}_{i \in I} \text{ is a finite Borel partition of } \Omega \}.$$

If $n = k + m$, for any $(y, z) \in \Omega \subseteqq \mathbb{R}^k \times \mathbb{R}^m$, we define

$$(2.3) \quad \int_\Omega |D_y u| = \sup \left\{ \int_\Omega u(y, z) \sum_{i=1}^k D_i \sigma_i(y, z) \, dy \, dz : \sigma \in \mathscr{C}_0^1(\Omega; \mathbb{R}^{k+m}), \sum_{i=1}^k \|\sigma_i(y, z)\|^2 \leqq 1, \quad \forall (y, z) \in \Omega \right\}.$$

Then, if $B \subseteqq \Omega$ is a Borel set, the following Fubini's type theorem holds (*see* [35, appendix]): the function $z \to \int_{B^z} |Du^z|$ is measurable for $\mathscr{H}^m$-almost every $z \in \mathbb{R}^m$, and

$$(2.4) \quad \int_B |D_y u| = \int_{\mathbb{R}^m} \left[\int_{B^z} |Du^z| \right] dz,$$

where $B^z = \{ y \in \mathbb{R}^k : (y, z) \in B \}$, and $u^z(y) = u(y, z)$.

Let E be a subset of $\mathbb{R}^n$; we denote by χ_E the characteristic function of E, *i. e.*, $\chi_E(x)=1$ if $x\in E$, and $\chi_E(x)=0$ if $x\in\mathbb{R}^n\setminus E$. Let $E\subseteq\mathbb{R}^n$ be measurable; if $\int_\Omega |D\chi_E|<+\infty$, then we say that E has finite perimeter in Ω, and we denote by $P(E,\Omega)$ its perimeter. It is well known (*see* [22]) that

$$P(E,\Omega)=\mathscr{H}^{n-1}(\Omega\cap\partial^* E),$$

where $\partial^* E$ denotes the reduced boundary of E. We recall that $\partial^* E$ is defined as the set of the points x such that there exists the Radon-Nikodym derivative $\dfrac{D\chi_E}{|D\chi_E|}(x)=\nu^E(x)=(\nu_1^E(x),\ldots,\nu_n^E(x))$ of the measure $D\chi_E$ with respect to the measure $|D\chi_E|$ at the point x, and such that $\|\nu^E(x)\|=1$. We recall also that

$$\int_{\Omega\cap E}\operatorname{div}\sigma\,dx=\int_{\Omega\cap\partial^* E}(\sigma,\nu^E)\,d\mathscr{H}^{n-1}(x)\qquad\forall\sigma\in\mathscr{C}_0^1(\Omega;\mathbb{R}^n).$$

For the definitions and the main properties of the functions of bounded variation and of sets of finite perimeter we refer to [26], [28], [30], [33], [46].

Following [2], [3] we set

$$X=\{\sigma\in L^\infty(\Omega;\mathbb{R}^n):\operatorname{div}\sigma\in L^n(\Omega)\}.\tag{2.5}$$

As proven in [3], theorem 1.2, if ν^Ω denotes the outer unit normal vector to $\partial\Omega$, then for every $\sigma\in X$ there exists a unique function $[\sigma.\nu^\Omega]$ belonging to $L^\infty_{\mathscr{H}^{n-1}}(\partial\Omega)$ such that

$$\left\{\begin{array}{l}\displaystyle\int_{\partial\Omega}[\sigma.\nu^\Omega]\,u\,d\mathscr{H}^{n-1}=\int_\Omega u\operatorname{div}\sigma\,dx+\int_\Omega(\sigma,\nabla u)\,dx\\ \qquad\qquad\qquad\forall u\in\mathscr{C}^1(\bar\Omega).\end{array}\right.\tag{2.6}$$

Equality (2.6) can be extended to the space $BV(\Omega)$ as follows. For every $u\in BV(\Omega)$ and every $\sigma\in X$, define the following linear functional $(\sigma.Du)$ on $\mathscr{C}_0^1(\Omega)$ by

$$\int_\Omega\psi(\sigma.Du)=-\int_\Omega u\psi\operatorname{div}\sigma\,dx-\int_\Omega u(\sigma,\nabla\psi)\,dx\qquad\forall\psi\in\mathscr{C}_0^1(\Omega).$$

The following results are proven in [2], [3].

THEOREM 2.1. – *For every* $u\in BV(\Omega)$ *and every* $\sigma\in X$, *the linear functional* $(\sigma.Du)$ *gives rise to a Radon measure on* Ω, *and*

$$\int_B|(\sigma.Du)|\le\|\sigma\|_{L^\infty(\Omega)}\int_B|Du|\quad\textit{for every Borel set } B\subseteq\Omega.\tag{2.7}$$

Moreover

$$(2.8)\qquad \int_{\partial\Omega} [\sigma . \nu^{\Omega}]\, u\, d\mathscr{H}^{n-1} = \int_{\Omega} u \operatorname{div} \sigma\, dx + \int_{\Omega} (\sigma . \mathrm{D}u).$$

Finally, there exists a Borel function $q_\sigma : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *such that*

$$(2.9)\qquad \frac{(\sigma . \mathrm{D}u)}{|\mathrm{D}u|}(x) = q_\sigma(x, \nu^u) \qquad \textit{for} \quad |\mathrm{D}u| - \text{a.e. } x \in \Omega.$$

To conclude, we recall the coarea formula, which holds for any $u \in \mathrm{BV}(\Omega)$ (*see*, for instance, [30], theorem 1.23):

$$(2.10)\qquad \int_{\Omega} |\mathrm{D}u| = \int_{\mathbb{R}} \mathrm{P}(\{u > s\}, \Omega)\, ds,$$

where $\{u > s\} = \{x \in \Omega : u(x) > s\}$ for any $s \in \mathbb{R}$.

2.2. The functions ϕ, ϕ^*, ϕ^0

Let $\phi : \Omega \times \mathbb{R}^n \to [0, +\infty]$ be a Borel function not identically $+\infty$. The function ϕ will be called convex if for any $x \in \Omega$ the function $\phi(x, .)$ is convex on $\mathbb{R}^n$. If $\phi(x, .)$ is lower semicontinuous for any $x \in \Omega$, the conjugate function $\phi^* : \Omega \times \mathbb{R}^n \to [0, +\infty]$ of ϕ is defined by

$$(2.11)\qquad \phi^*(x, \xi^*) = \sup\{(\xi^*, \xi) - \phi(x, \xi) : \xi \in \mathbb{R}^n\}.$$

As a consequence of (2.11), ϕ^* is convex and $\phi^*(x, .)$ is lower semicontinuous for any $x \in \Omega$, and, if $\phi_1 \leqq \phi_2$, then $\phi_1^* \geqq \phi_2^*$. One can prove that the biconjugate function ϕ^{**} of ϕ coincides with the convex envelope of ϕ with respect to the variable ξ, denoted by $\mathrm{co}(\phi)$ (*see*, for instance, [27], proposition 4.1).

For any Borel function $\phi : \Omega \times \mathbb{R}^n \to [0, +\infty]$ satisfying the property

$$(2.12)\qquad \phi(x, t\xi) = |t|\, \phi(x, \xi) \qquad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n, \quad \forall t \in \mathbb{R},$$

the dual function $\phi^0 : \Omega \times \mathbb{R}^n \to [0, +\infty]$ of ϕ is defined by

$$(2.13)\qquad \phi^0(x, \xi^*) = \sup\{(\xi^*, \xi) : \xi \in \mathbb{R}^n, \ \phi(x, \xi) \leqq 1\}.$$

It is immediate to verify that ϕ^0 is convex, $\phi^0(x, .)$ is lower semicontinuous, it satisfies (2.12) and, if $\phi_1 \leqq \phi_2$, then $\phi_1^0 \geqq \phi_2^0$.

For any $x \in \Omega$ let $\mathscr{Z}_x = \{\xi \in \mathbb{R}^n : \phi(x, \xi) = 0\}$. By (2.13) and (2.12), it follows

$$(2.14)\qquad \phi^0(x, \xi^*) = \begin{cases} 0 & \text{if } \xi^* = 0 \\ +\infty & \text{if } \xi^* \notin \mathscr{Z}_x^\perp, \\ \sup\{(\xi^*, \xi) : \xi \in \mathbb{R}^n, \ \phi(x, \xi) \leqq 1\} & \text{if } \xi^* \in \mathscr{Z}_x^\perp \setminus \{0\}, \end{cases}$$

where $\mathscr{Z}_x^\perp = \{\xi^* \in \mathbb{R}^n : (\xi^*, \xi) = 0, \ \forall \xi \in \mathscr{Z}_x\}$.

For any $x \in \Omega$, set

$$\{\phi_x < 1\} = \{\xi \in \mathbb{R}^n : \phi(x, \xi) < 1\}, \quad \{\mathrm{co}(\phi)_x < 1\} = \{\xi \in \mathbb{R}^n : (\mathrm{co}(\phi))(x, \xi) < 1\}.$$

Using the positive 1-homogeneity of ϕ and the linearity of the scalar product we claim

$$\phi^0(x, \xi^*) = \sup\{(\xi^*, \xi) : \xi \in \{\phi_x < 1\}\} = \sup\{(\xi^*, \xi) : \xi \in \mathrm{co}(\{\phi_x < 1\})\}$$

for any $(x, \xi^*) \in \Omega \times \mathbb{R}^n$. Indeed, the first equality is immediate. Moreover, we have that $\xi \in \mathrm{co}(\{\phi_x < 1\})$ if and only if $\xi = \sum_{i=0}^{n} \alpha_i \xi_i$, where $\alpha_i \geqq 0$, $\phi(x, \xi_i) < 1$ for any $i = 0, \ldots, n$, and $\sum_{i=0}^{n} \alpha_i = 1$. Let $j \in \{1, \ldots, n\}$ be such that $|(\xi^*, \xi_j)| = \max_{i=0, \ldots, n} |(\xi^*, \xi_i)|$. In particular, $\xi_j \in \{\phi_x < 1\}$, and

$$(\xi^*, \xi) = \sum_{i=0}^{n} \alpha_i (\xi^*, \xi_i) \leqq |(\xi^*, \xi_j)| \leqq \sup\{(\xi^*, \xi) : \xi \in \{\phi_x < 1\}\}.$$

Therefore

$$\sup\{(\xi^*, \xi) : \xi \in \{\phi_x < 1\}\} \geqq \sup\{(\xi^*, \xi) : \xi \in \mathrm{co}(\{\phi_x < 1\})\}.$$

As the opposite inequality is trivial, the claim is proven.

Moreover using [43], corollary 17.1.5, it is not difficult to prove that $\mathrm{co}(\{\phi_x < 1\}) = \{\mathrm{co}(\phi)_x < 1\}$. We deduce

$$(2.15) \quad \phi^0(x, \xi^*) = \sup\{(\xi^*, \xi) : \xi \in \{\mathrm{co}(\phi)_x < 1\}\} = (\mathrm{co}(\phi))^0(x, \xi^*)$$

for any $(x, \xi^*) \in \Omega \times \mathbb{R}^n$. In addition, by [43], theorem 15.1, it follows that $(\mathrm{co}(\phi))^{00} = \mathrm{co}(\phi)$, which implies, by (2.15), $\phi^{00} = \mathrm{co}(\phi)$. We conclude that, if $\phi(x, .)$ is lower semicontinuous for any $x \in \Omega$, then

$$(2.16) \qquad \phi^{00} = \mathrm{co}(\phi) = \phi^{**}.$$

We shall adopt the following conventions: for any $a \in [0, +\infty[$ we set $\frac{a}{+\infty} = 0$; $\frac{a}{0} = +\infty$ if $a \neq 0$ and $\frac{a}{0} = 0$ if $a = 0$. With these conventions we have

$$(2.17) \quad \phi^0(x, \xi^*) = \sup\left\{\frac{(\xi^*, \xi)}{\phi(x, \xi)} : \xi \in \mathbb{R}^n\right\} \qquad \forall x \in \Omega, \quad \forall \xi^* \in \mathbb{R}^n.$$

For later use, let us verify

$$(2.18) \quad \phi^*(x, \xi^*) = \begin{cases} 0 & \text{if } \phi^0(x, \xi^*) \leqq 1 \\ +\infty & \text{if } \phi^0(x, \xi^*) > 1 \end{cases}, \qquad \forall x \in \Omega, \quad \forall \xi^* \in \mathbb{R}^n.$$

Let $x \in \Omega$; if $\xi^* = 0$ then (2.18) is immediate. If $\xi^* \notin \mathscr{Z}_x^{\perp}$ there exists $\xi \in \mathbb{R}^n$ such that $\phi(x, \xi) = 0$ and $(\xi^*, \xi) \neq 0$. By (2.11) and (2.12) we deduce that

$\phi^*(x, \xi^*) = +\infty$. Hence (2.18) is fulfilled, since $\phi^0(x, \xi^*) = +\infty$ by (2.14). The last case, *i.e.*, $\xi^* \in \mathscr{Z}_x^\perp \setminus \{0\}$, can be proven reasoning as in [27], proposition 4.2.

Unless otherwise specified, from now on $\phi : \Omega \times \mathbb{R}^n \to [0, +\infty[$ will be a nonnegative finite-valued Borel function satisfying (2.12) and the following further property: there exists a positive constant $0 < \Lambda < +\infty$ such that

$$(2.19) \qquad 0 \leqq \phi(x, \xi) \leqq \sqrt{\Lambda} \|\xi\| \qquad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n.$$

Hence if ϕ is convex, then $\phi(x, .)$ is continuous for any $x \in \Omega$. By (2.19) and (2.17), one can verify that

$$(2.20) \qquad \sqrt{\Lambda^{-1}} \|\xi^*\| \leqq \phi^0(x, \xi^*) \qquad \forall x \in \Omega, \quad \forall \xi^* \in \mathbb{R}^n.$$

3. THE GENERALIZED TOTAL VARIATION $\int_\Omega |Du|_\phi$ OF A FUNCTION $u \in \mathrm{BV}(\Omega)$

We set

$$(3.1) \qquad \begin{cases} X_c = \{ \sigma \in X : \operatorname{spt}(\sigma) \text{ is compact in } \Omega \}, \\ \mathscr{K}_\phi = \{ \sigma \in X_c : \phi^0(x, \sigma(x)) \leqq 1 \text{ for a.e. } x \in \Omega \}, \\ \mathscr{C}_\phi = \{ \sigma \in \mathscr{C}_0^1(\Omega; \mathbb{R}^n) : \phi^0(x, \sigma(x)) \leqq 1, \ \forall x \in \Omega \}, \end{cases}$$

where the space X has been introduced in (2.5), and $\mathscr{C}_0^1(\Omega; \mathbb{R}^n) = \{ \sigma \in \mathscr{C}^1(\Omega; \mathbb{R}^n) : \operatorname{spt}(\sigma) \text{ is compact in } \Omega \}$. Observe that $\mathscr{K}_\phi$ (respectively $\mathscr{C}_\phi$) is a convex symmetric subset of X_c [respectively of $\mathscr{C}_0^1(\Omega; \mathbb{R}^n)$]; in addition $\mathscr{K}_{\phi_1} = \mathscr{K}_{\phi_2}$ if $\phi_1 = \phi_2$ almost everywhere.

Our definition of generalized total variation reads as follows.

DEFINITION 3.1. – *Let $\phi : \Omega \times \mathbb{R}^n \to [0, +\infty[$ be a Borel function satisfying conditions* (2.12) *and* (2.19). *Let $u \in \mathrm{BV}(\Omega)$; we define the generalized total variation of u with respect to ϕ in Ω as*

$$(3.2) \qquad \int_\Omega |Du|_\phi = \sup \left\{ \int_\Omega u \operatorname{div} \sigma \, dx : \sigma \in \mathscr{K}_\phi \right\}.$$

If $E \subseteq \mathbb{R}^n$ has finite perimeter in Ω, we set

$$\int_\Omega |D\chi_E|_\phi = P_\phi(E, \Omega) = \sup \left\{ \int_E \operatorname{div} \sigma \, dx : \sigma \in \mathscr{K}_\phi \right\}.$$

From the definition and the Hölder inequality, $\int_\Omega |Du|_\phi$ is the supremum of a family of functions which are continuous on $\mathrm{BV}(\Omega)$ with respect to the $L^{n/(n-1)}(\Omega)$-topology. Consequently the map $u \to \int_\Omega |Du|_\phi$ is $L^{n/(n-1)}(\Omega)$-lower semicontinuous on $BV(\Omega)$.

Note that if condition (2.19) is replaced by the stronger condition

$$(3.3) \qquad \sqrt{\lambda}\|\xi\| \leqq \phi(x, \xi) \leqq \sqrt{\Lambda}\|\xi\| \qquad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n$$

for some positive constants $0<\lambda \leqq \Lambda < +\infty$, then, as $\mathscr{K}_\phi \supseteq \mathscr{C}_\phi$, from the fact $\phi^0(x, \xi^*) \leqq \sqrt{\lambda^{-1}}\|\xi^*\|$, (2.1) and (3.2), we get

$$\int_\Omega |Du|_\phi \geqq \sqrt{\lambda} \int_\Omega |Du| \qquad \forall u \in BV(\Omega).$$

We point out that definition (3.1) and all results of sections 3, 4 and 5 do not depend on the behaviour of ϕ on sets of zero Lebesgue measure, *i.e.*, they are invariant when ϕ is replaced by any other function belonging to the same equivalence class with respect to the Lebesgue measure.

Note that, as $\phi^{000}(x, \xi^*) = \phi^0(x, \xi^*)$ for any $x \in \Omega$ and $\xi^* \in \mathbb{R}^n$ [*see* (2.15) and (2.16)], we have

$$\int_\Omega |Du|_{\phi^{00}} = \int_\Omega |Du|_\phi \qquad \forall u \in BV(\Omega).$$

Observe also that definition 3.1 generalizes the classical definition of total variation given in (2.1). Precisely, if $\phi(x, \xi) = \|\xi\|$, then

$$(3.4) \qquad \int_\Omega |Du|_\phi = \sup\left\{\int_\Omega u \operatorname{div}\sigma \, dx : \sigma \in \mathscr{C}_\phi\right\} = \int_\Omega |Du|.$$

Indeed (2.8) and (2.7) yield

$$(3.5) \quad \int_\Omega u \operatorname{div}\sigma \, dx = -\int_\Omega (\sigma . Du) \leqq \int_\Omega |(\sigma . Du)|$$
$$\leqq \|\sigma\|_{L^\infty(\Omega)} \int_\Omega |Du| \qquad \forall \sigma \in \mathscr{K}_\phi.$$

As $\phi^0(x, \xi^*) = \|\xi^*\|$, taking the supremum as $\sigma \in \mathscr{K}_\phi$ in (3.5), we get

$$\int_\Omega |Du|_\phi \leqq \int_\Omega |Du|.$$

The opposite inequality follows from the inclusion $\mathscr{K}_\phi \supseteq \mathscr{C}_\phi$.

Observe that, in general, from (3.5) and (2.20), we get

$$\int_\Omega |Du|_\phi \leqq \sqrt{\Lambda} \int_\Omega |Du| \qquad \forall u \in BV(\Omega).$$

The first equality in (3.4) is still true when ϕ is continuous and $\phi(x, \xi) > 0$ for any $(x, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$, according to the following result.

PROPOSITION 3.2. — *Let* $u \in \mathrm{BV}(\Omega)$ *and* $\phi : \Omega \times \mathbb{R}^n \to [0, +\infty[$ *be a Borel function satisfying conditions* (2.12), *and* (3.3). *Assume that the function* ϕ *is continuous. Then*

$$\int_\Omega |Du|_\phi = \sup\left\{\int_\Omega u \operatorname{div}\sigma\, dx : \sigma \in \mathscr{C}_\phi.\right\}.$$

Proof. — For any $\eta \geqq 0$ we introduce the following notations:

$$s_1(\eta) = \sup\left\{\int_\Omega u \operatorname{div}\sigma\, dx : \sigma \in X_c, \phi^0(x, \sigma(x)) \leqq 1 + \eta \textit{ for a.e. } x \in \Omega\right\}$$

$$s_2(\eta) = \sup\left\{\int_\Omega u \operatorname{div}\sigma\, dx : \sigma \in \mathscr{C}_0^1(\Omega; \mathbb{R}^n), \phi^0(x, \sigma(x)) \leqq 1 + \eta, \forall x \in \Omega\right\}.$$

Let us prove

$$s_1(\eta) \geqq s_2(\eta) \geqq s_1(0) - \eta \qquad \forall \eta > 0. \tag{3.6}$$

Inequality $s_1(\eta) \geqq s_2(\eta)$ is obvious, and it holds for any $\eta \geqq 0$.

Let $\eta > 0$, $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathscr{K}_\phi$; let Ω' be an open set such that $\operatorname{spt}(\sigma) \subset\subset \Omega' \subset\subset \Omega$, and let $\{\psi_\varepsilon\}_{\varepsilon > 0}$ be a sequence of mollifiers. Define

$$\sigma_\varepsilon = \sigma * \psi_\varepsilon = (\sigma_1 * \psi_\varepsilon, \ldots, \sigma_n * \psi_\varepsilon) \in \mathscr{C}_0^\infty(\Omega; \mathbb{R}^n)$$

for any $0 < \varepsilon < \frac{1}{2} \operatorname{dist}(\operatorname{spt}(\sigma), \partial\Omega')$. Since $\int_{\mathbb{R}^n} \psi_\varepsilon\, dx = 1$ and ϕ^0 is convex, using Jensen's Inequality (*see*, for instance, [36], lemma 1.8.2) and the uniform continuity of $\phi^0(., \xi^*)$ on $\bar{\Omega}'$ (which is a consequence of (3.3) and the continuity of ϕ), it follows that, for any $x \in \operatorname{spt}(\sigma_\varepsilon)$,

$$\begin{aligned}
\phi^0(x, \sigma_\varepsilon(x)) &= \phi^0\left(x, \int_{\mathbb{R}^n} \psi_\varepsilon(y)\, \sigma(x-y)\, dy\right) \\
&\leqq \int_{\mathbb{R}^n} \phi^0(x, \sigma(x-y))\, \psi_\varepsilon(y)\, dy \\
&= \int_{\mathbb{R}^n} \phi^0(x-y, \sigma(x-y))\, \psi_\varepsilon(y)\, dy + \int_{\mathbb{R}^n} o(\|y\|)\, \psi_\varepsilon(y)\, dy,
\end{aligned}$$

where $o(\|y\|) \to 0$ as $\|y\| \to 0$, independently of $x \in \operatorname{spt}(\sigma_\varepsilon)$. Since $\sigma \in \mathscr{K}_\phi$, we have $\phi^0(x-y, \sigma(x-y)) \leqq 1$. Using the previous inequality, if $\varepsilon > 0$ is sufficiently small, we get

$$\phi^0(x, \sigma_\varepsilon(x)) \leqq 1 + \eta \qquad \forall x \in \Omega. \tag{3.7}$$

By [3], lemma 2.2, we have

$$\int_{\Omega'} u \operatorname{div}\sigma\, dx \leqq \int_{\Omega'} u \operatorname{div}\sigma_\varepsilon\, dx + \eta = \int_\Omega u \operatorname{div}\sigma_\varepsilon\, \mathrm{dx} + \eta \leqq s_2(\eta) + \eta. \tag{3.8}$$

Then (3.7) and (3.8) yield

$$\int_{\Omega'} u \operatorname{div} \sigma\, dx \leqq s_2(\eta)+\eta; \tag{3.9}$$

taking the supremum first with respect to Ω' and then with respect to $\sigma \in \mathscr{K}_\phi$ in (3.9), we have $s_1(0) \leqq s_2(\eta)+\eta$, and this concludes the proof of (3.6).

Observe now that, using the positive 1-homogeneity of $\phi^0(x, .)$, for any $\eta>0$

$$\begin{aligned} s_2(\eta) &= \sup\left\{\int_\Omega u \operatorname{div}\sigma\, dx : \sigma \in \mathscr{C}_0^1(\Omega; \mathbb{R}^n), \phi^0\left(x, \frac{\sigma(x)}{1+\eta}\right) \leqq 1\ \forall x \in \Omega\right\} \\ &= \sup\left\{\int_\Omega u \operatorname{div}[(1+\eta)\sigma]\, dx : \sigma \in \mathscr{C}_\phi\right\} = (1+\eta)\, s_2(0). \end{aligned}$$

Hence, $s_2(\eta) \to s_2(0)$ as $\eta \to 0$, and, in a similar way, $s_1(\eta) \to s_1(0)$ as $\eta \to 0$. Taking into account definition 3.1 and passing to the limit in (3.6) as $\eta \to 0$, we obtain

$$\int_\Omega |Du|_\phi = s_1(0) = s_2(0) = \sup\left\{\int_\Omega u \operatorname{div}\sigma\, dx : \sigma \in \mathscr{C}_\phi\right\},$$

and this proves the assertion.

We remark that, as proved in remark 8.5, proposition 3.2 is false if ϕ is not continuous.

4. AN INTEGRAL REPRESENTATION THEOREM FOR $\int_\Omega |Du|_\phi$

In this section we prove an integral representation result (theorem 4.3) for the generalized total variation defined in (3.2). To this end, we recall the notion of $\mathscr{C}^1$-inf-stability (*see* [5], § 2 and [7], definition 4.2).

DEFINITION 4.1. – *Let μ be a positive Radon measure on Ω, and let* H *be a set of μ-measurable functions from Ω into $\mathbb{R}$. We say that* H *is $\mathscr{C}^1$-inf-stable if for every finite family $\{v_i\}_{i \in I}$ of elements of* H *and for every family $\{\alpha_i\}_{i\in I}$ of non-negative functions of $\mathscr{C}^1(\bar{\Omega})$ such that $\sum_{i \in I} \alpha_i(x) = 1$ for any $x \in \Omega$, there exists $v \in$ H such that $v(x) \leqq \sum_{i\in I} \alpha_i(x)\, v_i(x)$ for μ-almost every $x \in \Omega$.*

The following theorem holds (*see* [5], theorem 1, and [7], lemma 4.3).

THEOREM 4.2. — *Let* μ *be a positive Radon measure on* Ω, *and let* H *be a* $\mathscr{C}^1$*-inf-stable subset of* $L^1_\mu(\Omega)$ *Then*

$$\inf_{v \in H} \int_\Omega v \, d\mu = \int_\Omega h \, d\mu,$$

where $h = \mu - \operatorname*{ess\,inf}_{v \in H} v$.

Our representation result reads as follows.

THEOREM 4.3. — *Let* $\phi : \Omega \times \mathbb{R}^n \to [0, +\infty[$ *be a Borel function satisfying conditions* (2.12) *and* (2.19). *Then*

$$\int_\Omega |Du|_\phi = \int_\Omega h(x, \nu^u) \, |Du| \qquad \forall u \in BV(\Omega), \tag{4.1}$$

where

$$h(x, \nu^u) = (|Du| - \operatorname*{ess\,sup}_{\sigma \in \mathscr{K}_\phi} q_\sigma)(x) \qquad \textit{for} \quad |Du| - \text{a.e. } x \in \Omega. \tag{4.2}$$

Proof. — It is enough to show (4.1), since it has been proven in [7], proposition 1.8 that, if q_0 is as in (2.9), then the function $|Du| - \operatorname*{ess\,sup}_{\sigma \in \mathscr{K}_\phi} q_\sigma$ depends on u only by means of the vector ν^u, *i.e.*, the function h in formula (4.2) is well defined.

Let $u \in BV(\Omega)$; using definition 3.1, (2.8) and (2.9) we have

$$\int_\Omega |Du|_\phi = \sup_{\sigma \in \mathscr{K}_\phi} \int_\Omega (\sigma . Du) = \sup_{\sigma \in \mathscr{K}_\phi} \int_\Omega q_\sigma(x, \nu^u) \, |Du|. \tag{4.3}$$

Let $T_u : \mathscr{K}_\phi \to L^1_{|Du|}(\Omega)$ be the operator defined by $T_u(\sigma)(x) = -q_\sigma(x, \nu^u)$ for $|Du|$-almost every $x \in \Omega$, and let

$$H = \{ T_u(\sigma) : \sigma \in \mathscr{K}_\phi \}.$$

We observe that the set H is $\mathscr{C}^1$-inf-stable. Indeed, let $\{\sigma_i\}_{i \in I}$ be a finite family of elements of $\mathscr{K}_\phi$ and let $\{\alpha_i\}_{i \in I}$ be a family of non-negative functions of $\mathscr{C}^1(\bar{\Omega})$ such that $\sum_{i \in I} \alpha_i = 1$ in Ω. By the convexity of ϕ^0, it follows that $\sigma = \sum_{i \in I} \alpha_i \sigma_i$ belongs to $\mathscr{K}_\phi$; moreover, by [7], remark 1.5, we get

$$\sum_{i \in I} \alpha_i T_u(\sigma_i) = T_u(\sigma) \qquad |Du| - \text{a.e. in } \Omega.$$

Hence $\sum_{i \in I} \alpha_i T_u(\sigma_i) \in H$, and this proves that H is $\mathscr{C}^1$-inf-stable.

As $-h(x, \nu^u) = (|Du| - \operatorname{ess\,inf}_{\sigma \in \mathcal{K}_\phi} T_u(\sigma))(x)$, theorem 4.2 and formula (4.2) give

$$(4.4) \quad \inf_{\sigma \in \mathcal{K}_\phi} \int_\Omega T_u(\sigma)\,|Du| = \inf_{\sigma \in \mathcal{K}_\phi} \int_\Omega -q_\sigma(x, \nu^u)\,|Du| = -\int_\Omega h(x, \nu^u)\,|Du|.$$

Then (4.1) is a consequence of (4.3) and (4.4).

Remark 4.4. – From (4.1) and the coarea formula for BV functions [*see* (2.10)] we deduce the following coarea formula for the generalized total variation:

$$\int_\Omega |Du|_\phi = \int_{\mathbb{R}} \int_{\Omega \cap \partial^* \{u>s\}} h(x, \nu^s)\, d\mathcal{H}^{n-1}(x)\, ds$$

$$= \int_{\mathbb{R}} P_\phi(\{u>s\}, \Omega)\, ds \qquad \forall u \in BV(\Omega),$$

where ν^s denotes the outer unit normal vector to the set $\Omega \cap \partial^* \{u>s\}$.

The following lemma shows that we can replace X_c by X in the set $\mathcal{K}_\phi$ appearing in the expression of h given in (4.2), and it will be useful in the proof of theorem 5.1.

LEMMA 4.5. – *For every* $u \in BV(\Omega)$ *we have*

$$h(x, \nu^u) = (|Du| - \operatorname{ess\,sup}_{\rho \in \mathcal{M}_\phi} q_\rho)(x) \qquad \textit{for} \quad |Du| - \text{a.e. } x \in \Omega,$$

where

$$(4.5) \qquad \mathcal{M}_\phi = \{\rho \in X : \phi^0(x, \rho(x)) \leq 1 \text{ for a.e. } x \in \Omega\}.$$

Proof. – Let $A \subset\subset \Omega$ be an open set which is relatively compact in Ω, and let $\rho \in \mathcal{M}_\phi$. Choose $\sigma \in \mathcal{K}_\phi$ in such a way that $\sigma = \rho$ almost everywhere in A. Then (*see* [7], formula (1.7)) for every $u \in BV(\Omega)$ we have $q_\sigma(x, \nu^u) = q_\rho(x, \nu^u)$ for $|Du|$-almost every $x \in A$, so that

$$(4.6) \qquad \int_A q_\rho(x, \nu^u)\,|Du| = \int_A q_\sigma(x, \nu^u)\,|Du| \leq \int_A h(x, \nu^u)\,|Du|$$

for any $u \in BV(\Omega)$.

Since (4.6) holds for every $A \subset\subset \Omega$ and for any $\rho \in \mathcal{M}_\phi$, it follows

$$(|Du| - \operatorname{ess\,sup}_{\rho \in \mathcal{M}_\phi} q_\rho)(x) \leq h(x, \nu^u) \qquad \text{for} \quad |Du| - \text{a.e. } x \in \Omega.$$

As the opposite inequality is a trivial consequence of the inclusion $\mathcal{M}_\phi \supseteq \mathcal{K}_\phi$, the lemma is proven.

5. RELATIONS BETWEEN RELAXATION THEORY AND THE GENERALIZED TOTAL VARIATION

In this section we prove that the generalized total variation coincides with the lower semicontinuous envelope of certain integral functionals which are finite on $W^{1,1}(\Omega)$ (*see* theorem 5.1 and proposition 5.5).

Let $\mathscr{L} : BV(\Omega) \to [0, +\infty]$ be a functional. We denote by

$$\overline{\mathscr{L}} : BV(\Omega) \to [0, +\infty]$$

the lower semicontinuous envelope (or relaxed functional) of $\mathscr{L}$ with respect to the $L^1(\Omega)$-topology, which is defined as the greatest $L^1(\Omega)$-lower semicontinuous functional less or equal to $\mathscr{L}$. The functional $\overline{\mathscr{L}}$ can be characterized as follows:

$$(5.1) \qquad \overline{\mathscr{L}}(u) = \inf\{\liminf_{h \to +\infty} \mathscr{L}(u_h) : \{u_h\}_h \subseteq BV(\Omega),\ u_h \xrightarrow{L^1(\Omega)} u\}.$$

For the main properties of the relaxed functionals, we refer to [11], [14].

For any Borel function ϕ which satisfies conditions (2.12) and (2.19) we define the functional $\mathscr{F}[\phi] : BV(\Omega) \to [0, +\infty]$ by

$$(5.2) \qquad \mathscr{F}[\phi](u) = \begin{cases} \displaystyle\int_\Omega \phi(x, \nabla u(x))\, dx & \text{if } \ u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise} \end{cases}.$$

Clearly $\mathscr{F}[\phi^{00}] \leqq \mathscr{F}[\phi]$, and, if ϕ is convex, then $\mathscr{F}[\phi^{00}] = \mathscr{F}[\phi]$. It has been proven in [7], theorem 4.1, that $\overline{\mathscr{F}[\phi^{00}]}$ has an integral representation, and precisely there exists a Borel function $\mathscr{R}(\phi) : \Omega \times \mathbb{R}^n \to [0, +\infty[$ which satisfies conditions (2.12) and (2.19), and

$$(5.3) \qquad \overline{\mathscr{F}[\phi^{00}]}(u) = \int_\Omega [\mathbb{R}(\phi)](x, \nu^u)\,|Du| \qquad \forall u \in BV(\Omega).$$

Here

$$(5.4) \quad [\mathscr{R}(\phi)](x, \nu^u) = (|Du| - \operatorname*{ess\,sup}_{\sigma \in \mathscr{K}_\phi^*} q_\sigma)(x) \qquad \text{for } \ |Du|-\text{a.e. } x \in \Omega,$$

where

$$(5.5) \qquad \begin{aligned} \mathscr{K}_\phi^* &= \left\{\sigma \in X : \int_\Omega (\phi^{00})^*(x, \sigma(x))\, dx < +\infty\right\} \\ &= \left\{\sigma \in X : \int_\Omega \phi^*(x, \sigma(x))\, dx < +\infty\right\} \end{aligned}$$

(recall (2.16)), and X is defined in (2.5).

It is well known (*see* [30], theorem 1.17) that, if $\phi(x, \xi) = \|\xi\|$, then

$$(5.6) \qquad \int_\Omega |Du| = \overline{\mathscr{F}[\phi^{00}]}(u) = \overline{\mathscr{F}[\phi]}(u) \qquad \forall u \in \mathrm{BV}(\Omega).$$

This formula can be generalized, according to the following result.

THEOREM 5.1. – *Let* $\phi : \Omega \times \mathbb{R}^n \to [0, +\infty[$ *be a Borel function satisfying conditions* (2.12) *and* (2.19). *Then*

$$(5.7) \qquad \int_\Omega |Du|_\phi = \overline{\mathscr{F}[\phi^{00}]}(u) = \overline{\mathscr{F}[\phi]}(u) \qquad \forall u \in \mathrm{BV}(\Omega).$$

In particular, $\int_\Omega |Du|_\phi$ *is* $\mathrm{L}^1(\Omega)$*-lower semicontinuous on* $\mathrm{BV}(\Omega)$.

If in addition ϕ *is continuous and satisfies* (3.3), *then*

$$(5.8) \qquad \int_\Omega |Du|_\phi = \int_\Omega \phi^{00}(x, \nu^u)\,|Du| \qquad \forall u \in \mathrm{BV}(\Omega).$$

Proof. – Let us prove that

$$(5.9) \qquad \int_\Omega |Du|_\phi = \overline{\mathscr{F}[\phi^{00}]}(u) \qquad \forall u \in \mathrm{BV}(\Omega).$$

We observe that

$$(5.10) \qquad \mathscr{K}_\phi^* = \mathscr{M}_\phi,$$

where $\mathscr{K}_\phi^*$ and $\mathscr{M}_\phi$ are defined in (5.5) and (4.5), respectively. Indeed, for any $\sigma \in \mathrm{X}$, using (2.18) we have

$$\int_\Omega \phi^*(x, \sigma(x))\,dx < +\infty \quad \Leftrightarrow \quad \phi^0(x, \sigma(x)) \leq 1 \qquad \text{for a.e. } x \in \Omega.$$

By (5.4), (5.10) and lemma 4.5 we deduce that for any $u \in \mathrm{BV}(\Omega)$

$$(5.11) \qquad [\mathscr{R}(\phi)](x, \nu^u) = h(x, \nu^u) \qquad \text{for } |Du|-\text{a.e. } x \in \Omega.$$

Hence (5.9) follows from (4.1), (5.11) and (5.3).

We point out that, in view of lemma 4.5 and (5.10), the previous result could be obtained as a consequence of [7], formula (4.19).

Let us show that

$$(5.12) \qquad \overline{\mathscr{F}[\phi^{00}]}(u) = \overline{\mathscr{F}[\phi]}(u) \qquad \forall u \in \mathrm{BV}(\Omega).$$

Denote by $\mathscr{W} : \mathrm{BV}(\Omega) \to [0, +\infty]$ the greatest sequentially $\mathrm{W}^{1,1}(\Omega)$-weakly lower semicontinuous functional which is less or equal to $\mathscr{F}[\phi]$. By (5.1) it follows that $\overline{\mathscr{F}[\phi]}(u) \leq \mathscr{W}(u)$ for any $u \in \mathrm{BV}(\Omega)$. By [12], theorem 2.1, the functional $\mathscr{W}$ has an integral representation, *i.e.*, there exists a non-negative Borel function $g(x, \xi)$, convex in ξ (*see* [12], remark 2.1), such that

$$\mathscr{W}(u, \mathrm{A})=\int_{\mathrm{A}} g(x, \nabla u(x))\, dx$$
$$\forall u \in \mathrm{W}^{1,1}(\Omega), \quad \text{for any open set } \mathrm{A} \subseteq \Omega.$$

Here, for convenience, the functional $\mathscr{W}$ is also considered as a set function in the second variable. We set $\mathscr{W}(u)=\mathscr{W}(u, \Omega)$ for any $u \in \mathrm{BV}(\Omega)$.

By definition of $\mathscr{W}$, we then have

$$(5.13) \qquad \int_{\mathrm{B}} g(x, \nabla u(x))\, dx \leqq \int_{\mathrm{B}} \phi(x, \nabla u(x))\, dx \qquad \forall u \in \mathrm{W}^{1,1}(\Omega),$$

for any Borel set $\mathrm{B} \subseteq \Omega$.

We claim that there exists $\mathrm{N} \in \mathscr{N}(\Omega)$ such that $g(x, \xi) \leqq \phi(x, \xi)$ for any $(x, \xi) \in (\Omega \setminus \mathrm{N}) \times \mathbb{R}^n$. Assume by contradiction that there exists a measurable set $\mathrm{B} \subseteq \Omega$ of positive Lebesgue measure such that we can find a function $\xi : \mathrm{B} \rightarrow \mathbb{R}^n$ with

$$(5.14) \qquad g(x, \xi(x)) > \phi(x, \xi(x)) \qquad \forall x \in \mathrm{B}.$$

Without loss of generality, we can suppose that B is a Borel set. By the Aumann-von Neumann Selection Theorem (*see* [13], theorem III.22) we can assume that the function $x \mapsto \xi(x)$ is Borel. Moreover, as $\mathrm{B}=\bigcup_{k \in \mathbb{N}}\{x \in \mathrm{B} : \|\xi(x)\| \leqq k\}$, we can also suppose that the function $x \mapsto \xi(x)$ is bounded on B. Let us define $\xi(x)=0$ for every $x \in \Omega \setminus \mathrm{B}$. By [1], theorem 1, for any $\varepsilon>0$ there exist a function $\bar{u} \in \mathrm{W}^{1,1}(\Omega)$ and a Borel set M with $\mathscr{H}^n(\mathrm{M})<\varepsilon$ such that $\nabla \bar{u}(x)=\xi(x)$ for almost every $x \in \Omega \setminus \mathrm{M}$. Taking ε in such a way that $\mathrm{B} \setminus \mathrm{M}$ has positive Lebesgue measure, by (5.14) we obtain

$$\int_{\mathrm{B} \setminus \mathrm{M}} g(x, \nabla \bar{u}(x))\, dx > \int_{\mathrm{B} \setminus \mathrm{M}} \phi(x, \nabla \bar{u}(x))\, dx,$$

which contradicts (5.13), and proves the claim.

Therefore, recalling that ϕ^{00} coincides with the convex envelope of ϕ and that g is convex, we have that $g(x, \xi) \leqq \phi^{00}(x, \xi)$ for any $(x, \xi) \in (\Omega \setminus \mathrm{N}) \times \mathbb{R}^n$. The opposite inequality follows by recalling that the convexity of ϕ^{00} yields the $\mathrm{W}^{1,1}(\Omega)$-weak lower semicontinuity of $\mathscr{F}[\phi^{00}]$ (*see*, for instance, [11], theorem 4.1.1). Hence

$$(5.15) \qquad \begin{cases} \overline{\mathscr{F}[\phi]}(u) \leqq \mathscr{W}(u)=\displaystyle\int_{\Omega} \phi^{00}(x, \nabla u(x))\, dx=\mathscr{F}[\phi^{00}](u) \\ \qquad \forall u \in \mathrm{W}^{1,1}(\Omega). \end{cases}$$

Then, by (5.15) and the definition of $\overline{\mathscr{F}[\phi^{00}]}$, we get $\overline{\mathscr{F}[\phi]} \leqq \mathscr{F}[\phi^{00}]$ on $\mathrm{BV}(\Omega)$, which implies $\overline{\mathscr{F}[\phi]} \leqq \overline{\mathscr{F}[\phi^{00}]}$ on $\mathrm{BV}(\Omega)$. As the opposite inequality is trivial, the proof of (5.12) [and hence of (5.7)] is complete.

Assume now that ϕ is continuous and satisfies (3.3); then ϕ^{00} is also continuous. By (5.15) and [15], theorem 3.1, we have

$$(5.16)\qquad \overline{\mathscr{F}[\phi^{00}]}(u)=\bar{\mathscr{W}}(u)=\int_{\Omega}\phi^{00}(x, \nu^u)\,|Du| \qquad \forall u\in BV(\Omega).$$

Hence (5.8) follows from (5.7) and (5.16).

Remark 5.2. – Note that theorem 5.1 provides an integral representation on $BV(\Omega)$ of the $L^1(\Omega)$-lower semicontinuous envelope of the functional $\mathscr{F}[\phi]$ when ϕ is not convex.

Remark 5.3. – We recall that, if ϕ satisfies (2.12), it is continuous, convex, and verifies (2.19) instead of (3.3), then the functional $\mathscr{F}[\phi]$ is not necessarily $L^1(\Omega)$-lower semicontinuous on $W^{1,1}(\Omega)$, and hence formula (5.8) does not hold. This fact was observed by Aronszajn (*see* [40], p. 54) and exploited afterwards in [15], example 4.1.

Observe that (5.7) gives

$$P_\phi(E, \Omega)=\overline{\mathscr{F}[\phi^{00}]}(\chi_E)=\overline{\mathscr{F}[\phi]}(\chi_E)$$

for any measurable set $E\subseteq\mathbb{R}^n$ of finite perimeter in Ω. Moreover, if ϕ is convex, continuous, and satisfies (3.3), by (5.8) we get

$$P_\phi(E, \Omega)=\int_{\Omega\cap\partial^* E}\phi(x, \nu^E)\,d\mathscr{H}^{n-1}(x),$$

where $\nu^E(x)$ denotes the generalized outer unit normal vector to $\partial^* E$ at the point x.

Assume now that ϕ satisfies condition (3.3); for any $u\in\mathscr{C}^1(\Omega)$ define the slope of u with respect to ϕ by

$$(5.17)\qquad |\nabla_\phi|\,u(\xi)=\limsup_{\substack{\eta\neq\xi\\ \eta\to\xi}}\frac{|u(\eta)-u(\xi)|}{\phi^0(\xi, \eta-\xi)} \qquad \forall\xi\in\Omega.$$

LEMMA 5.4. – *For any $u\in\mathscr{C}^1(\Omega)$ we have*

$$(5.18)\qquad |\nabla_\phi|\,u(\xi)=\phi^{00}(\xi, \nabla u(\xi)) \qquad \forall\xi\in\Omega.$$

Proof. – Let $u\in\mathscr{C}^1(\Omega)$ and $\xi\in\Omega$; let B be a ball centered at ξ and contained in Ω. For any $\eta\in B$, there exists a point $\tau_{\xi,\eta}\in B$ between ξ and η such that $u(\eta)-u(\xi)=(\nabla u(\tau_{\xi,\eta}), \eta-\xi)$. Then, by the definition of upper limit, the positive 1-homogeneity of ϕ^0 and the continuity of

$\phi^{00}(\xi, .)$ (which is a consequence of the convexity), we have

$$|\nabla_\phi| u(\xi) = \limsup_{\substack{\eta \neq \xi \\ \eta \to \xi}} \frac{|(\nabla u(\tau_{\xi,\eta}), \eta - \xi)|}{\phi^0(\xi, \eta - \xi)}$$

$$= \lim_{\varepsilon \to 0} \sup_{\substack{\|\eta - \xi\| \leq \varepsilon \\ \eta \neq \xi}} \frac{|(\nabla u(\tau_{\xi,\eta}), \varepsilon^{-1}(\eta - \xi))|}{\phi^0(\xi, \varepsilon^{-1}(\eta - \xi))}$$

$$= \lim_{\varepsilon \to 0} \phi^{00}(\xi, \nabla u(\tau_{\xi,\eta})) = \phi^{00}(\xi, \nabla u(\xi)),$$

that is (5.18).

PROPOSITION 5.5. – *Let* $\phi : \Omega \times \mathbb{R}^n \to [0, +\infty[$ *be a Borel function satisfying conditions* (2.12) *and* (3.3). *Let* $\mathscr{G} : \mathrm{BV}(\Omega) \to [0, +\infty]$ *be the functional defined by*

$$\mathscr{G}(u) = \begin{cases} \displaystyle\int_\Omega |\nabla_\phi| u \, d\xi & \text{if } u \in \mathscr{C}^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $|\nabla_\phi| u$ *is defined in* (5.17). *Then*

$$\mathscr{G}(u) = \mathscr{F}[\phi^{00}](u) \qquad \forall u \in \mathscr{C}^1(\Omega), \tag{5.19}$$

which yields

$$\overline{\mathscr{G}}(u) = \int_\Omega |Du|_\phi \qquad \forall u \in \mathrm{BV}(\Omega). \tag{5.20}$$

Proof. – Formula (5.19) follows from (5.18) and the definition of $\mathscr{F}[\phi^{00}]$. Formula (5.20) follows from (5.19) and (5.7).

Remark 5.6. – Assume that the function ϕ of the statement of theorem 5.1 is convex and independent of x. Then for any $u \in \mathrm{BV}(\Omega)$ we have

$$\int_\Omega |Du|_\phi = \sup \left\{ \sum_{i \in \mathrm{I}} \phi(Du(\mathrm{B}_i)) : \{\mathrm{B}_i\}_{i \in \mathrm{I}} \text{ is a finite Borel partition of } \Omega \right\} \tag{5.21}$$

[compare with (2.2)]. Indeed the right hand side of (5.21) equals $\int_\Omega \phi(\nu^u) |Du|$ (*see* [31]), which coincides also with $\int_\Omega |Du|_\phi$ by formula (5.8).

6. THE ROLE PLAYED BY SETS OF ZERO LEBESGUE MEASURE

Let $u \in BV(\Omega)$. We recall that the value of $\int_\Omega |Du|_\phi$ is independent of the choice of the representative of ϕ in its equivalence class, while, as $|Du|$ can be concentrated on sets of $\mathcal{N}(\Omega)$, any integral with respect to $|Du|$ takes into account the behaviour of the integrand on sets of zero Lebesgue measure. This difficulty is overcomed by considering special representatives ϕ_N of ϕ for $N \in \mathcal{N}(\Omega)$ [*see* (6.4)], by relaxing the functional $\mathcal{J}[\phi_N]$ and finally considering the $\sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N]}$ (theorem 6.4).

We recall the following definition (*see* [7], § 1.3).

Let $h_1, h_2 : \Omega \times \mathbb{R}^n \to [0, +\infty]$ be two functions. We define the relations $h_1 \preceq h_2$ and $h_1 \simeq h_2$ by

$$(6.1)\quad \begin{cases} h_1 \preceq h_2 \Leftrightarrow \forall u \in BV(\Omega) \quad h_1(x, \nu^u) \leqq h_2(x, \nu^u) \quad \text{for } |Du|-\text{a.e. } x \in \Omega, \\ h_1 \simeq h_2 \Leftrightarrow \forall u \in BV(\Omega) \quad h_1(x, \nu^u) = h_2(x, \nu^u) \quad \text{for } |Du|-\text{a.e. } x \in \Omega. \end{cases}$$

Let $\phi : \Omega \times \mathbb{R}^n \to [0, +\infty[$ be an arbitrary Borel function satisfying conditions (2.12) and (2.19). We recall (*see* (1.1) and (1.3)) that the functional $\mathcal{J}[\phi] : BV(\Omega) \to [0, +\infty]$ is defined by

$$(6.2)\qquad \mathcal{J}[\phi](u) = \int_\Omega \phi(x, \nu^u)\,|Du| \qquad \forall u \in BV(\Omega),$$

and that $\mathcal{F}[\phi] : BV(\Omega) \to [0, +\infty]$ is defined by

$$(6.3)\qquad \mathcal{F}[\phi](u) = \begin{cases} \int_\Omega \phi(x, \nabla u(x))\,dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

If $N \in \mathcal{N}(\Omega)$ we get

$$(6.4)\qquad \phi_N(x, \xi) = \begin{cases} \phi(x, \xi) & \text{if } x \in \Omega \setminus N \text{ and } \xi \in \mathbb{R}^n, \\ \sqrt{\Lambda}\,\|\xi\| & \text{if } x \in N \text{ and } \xi \in \mathbb{R}^n, \end{cases}$$

and by ϕ_N^{00} we denote the bidual function of ϕ_N (*see* § 2). Then

$$\phi_N^{00}(x, \xi) = \begin{cases} \phi^{00}(x, \xi) & \text{if } x \in \Omega \setminus N \text{ and } \xi \in \mathbb{R}^n, \\ \sqrt{\Lambda}\,\|\xi\| & \text{if } x \in N \text{ and } \xi \in \mathbb{R}^n, \end{cases}$$

and ϕ_N and ϕ_N^{00} satisfy condition (2.12) and (2.19); moreover, ϕ_N^{00} is convex. Obviously $\mathcal{J}[\phi_N^{00}] \leqq \mathcal{J}[\phi_N]$,

$$(6.5)\quad N_1, N_2 \in \mathcal{N}(\Omega),\ N_1 \subseteq N_2 \Rightarrow \forall u \in BV(\Omega) \quad \begin{cases} \mathcal{J}[\phi_{N_1}](u) \leqq \mathcal{J}[\phi_{N_2}](u) \\ \mathcal{J}[\phi_{N_1}^{00}](u) \leqq \mathcal{J}[\phi_{N_2}^{00}](u). \end{cases}$$

Furthermore, since $\phi_N(x, \xi)=\phi(x, \xi)$ and $\phi_N^{00}(x, \xi)=\phi^{00}(x, \xi)$ for almost every $x\in\Omega$ and every $\xi\in\mathbb{R}^n$, we have

$$(6.6)\quad \begin{cases} \mathcal{J}[\phi_N](u)=\mathcal{F}[\phi](u), \\ \mathcal{J}[\phi_N^{00}](u)=\mathcal{F}[\phi^{00}](u) \end{cases} \qquad \forall N\in\mathcal{N}(\Omega), \qquad \forall u\in W^{1,1}(\Omega).$$

We point out that, in general, since we do not require the continuity of ϕ^{00}, the functional $\mathcal{J}[\phi_N^{00}]$ is not lower semicontinuous on $BV(\Omega)$, even if ϕ^{00} is convex.

We recall that the functionals $\overline{\mathcal{J}[\phi^{00}]}$ and $\overline{\mathcal{J}[\phi_N^{00}]}$ have an integral representation. Indeed, consider for example the functional $\mathcal{J}[\phi^{00}]$. As ϕ^{00} is convex and satisfies conditions (2.12) and (2.19), one can prove that $\mathcal{J}[\phi^{00}]$ satisfies all hypotheses of theorem 6.4 of [17], which implies that $\mathcal{J}[\phi^{00}]$ satisfies the J-property (*see* [17], definition 2.2). Hence, by [17], theorem 2.5 and [16], proposition 18.6, one infers that $\overline{\mathcal{J}[\phi^{00}]}$ is a measure. The same arguments hold for $\overline{\mathcal{J}[\phi_N^{00}]}$. It follows that, in view of the general results concerning the integral representation of convex functionals on $BV(\Omega)$ proven in [7], we can define the functions $\mathcal{S}(\phi)$, $\mathcal{S}(\phi_N)$, $\mathcal{R}(\phi):\Omega\times\mathbb{R}^n\to[0, +\infty[$ by

$$(6.7)\qquad \overline{\mathcal{J}[\phi^{00}]}(u)=\int_\Omega [\mathcal{S}(\phi)](x, \nu^u)\,|Du| \qquad \forall u\in BV(\Omega),$$

$$(6.8)\qquad \overline{\mathcal{J}[\phi_N^{00}]}(u)=\int_\Omega [\mathcal{S}(\phi_N)](x, \nu^u)\,|Du| \qquad \forall u\in BV(\Omega),$$

$$(6.9)\qquad \overline{\mathcal{F}[\phi^{00}]}(u)=\int_\Omega [\mathcal{R}(\phi)](x, \nu^u)\,|Du| \qquad \forall u\in BV(\Omega)$$

(*see* also (5.3)). Moreover, given $u\in BV(\Omega)$, we denote by $\mathcal{E}(\phi, u):\Omega\to[0, +\infty[$ the function

$$(6.10)\qquad \begin{cases} |\mathcal{E}(\phi, u)](x)=(|Du|-\operatorname*{ess\,sup}_{N\in\mathcal{N}(\Omega)}[\mathcal{S}(\phi_N)])(x) \\ \qquad \text{for } |Du|-\text{a.e. } x\in\Omega. \end{cases}$$

As already observed, the function $\mathcal{R}(\phi)$ satisfies conditions (2.12), (2.19); moreover, the same holds for $\mathcal{S}(\phi)$ and $\mathcal{S}[\phi_N]$ (*see* [7], theorem 5.1).

For later use, let us verify that

$$(6.11)\qquad \mathcal{R}(\phi)\simeq\mathcal{S}(\mathcal{R}(\phi)).$$

By (6.2) and (6.9) it follows

$$\mathcal{J}[\mathcal{R}(\phi)](u)=\int_\Omega \mathcal{R}(\phi)(x, \nu^u)\,|Du|=\overline{\mathcal{F}[\phi^{00}]}(u) \qquad \forall u\in BV(\Omega).$$

Then, from the previous equality and the lower semicontinuity of $\overline{\mathscr{F}[\phi^{00}]}$, we have

$$\overline{\mathscr{J}[\mathscr{R}(\phi)]}(u)=\overline{\overline{\mathscr{F}[\phi^{00}]}}(u)=\overline{\mathscr{F}[\phi^{00}]}(u)=\int_\Omega \mathscr{R}(\phi)(x, \nu^u)\,|Du|$$

for any $u\in BV(\Omega)$. As previously, $\overline{\mathscr{J}[\mathscr{R}(\phi)]}$ satisfies all hypotheses of [7], theorem 5.1, hence it has an integral representation of the type

$$\overline{\mathscr{J}[\mathscr{R}(\phi)]}(u)=\int_\Omega [\mathscr{S}(\mathscr{R}(\phi))](x, \nu^u)\,|Du|.$$

This, together with the previous chain of equality, gives (6.11).

Similarly, using the lower semicontinuity of $\overline{\mathscr{J}[\phi^{00}]}$, one has

$$\mathscr{S}(\phi)\simeq\mathscr{S}(\mathscr{S}(\phi)).$$

Finally,

$$\mathscr{R}(\mathscr{R}(\phi))\simeq\mathscr{R}(\phi).$$

Indeed, $\mathscr{F}[\mathscr{R}(\phi)]\leqq\mathscr{F}[\phi]$, and passing to the lower semicontinuous envelopes, it follows $\mathscr{R}(\mathscr{R}(\phi))\preceq\mathscr{R}(\phi)$. Moreover, by definition, $\overline{\mathscr{F}[\phi^{00}]}\leqq\mathscr{F}[\mathscr{R}(\phi)]$; hence, taking again the lower semicontinuous envelopes, and recalling (6.11), we get $\mathscr{R}(\phi)\simeq\mathscr{S}(\mathscr{R}(\phi))\preceq\mathscr{R}(\mathscr{R}(\phi))$.

LEMMA 6.1. – *We have*

$$\mathscr{S}(\phi_N)\preceq\mathscr{R}(\phi)\qquad\forall N\in\mathscr{N}(\Omega). \tag{6.12}$$

Proof. – Let $N\in\mathscr{N}(\Omega)$; by (6.6) and (6.3) it follows that $\mathscr{J}[\phi_N^{00}](u)\leqq\mathscr{F}[\phi^{00}](u)$ for every $u\in BV(\Omega)$. Consequently

$$\overline{\mathscr{J}[\phi_N^{00}]}(u)\leqq\overline{\mathscr{F}[\phi^{00}]}(u)\qquad\forall N\in\mathscr{N}(\Omega),\quad\forall u\in BV(\Omega). \tag{6.13}$$

By (6.8), (6.9), and (6.13), we deduce $\mathscr{S}(\phi_N)\preceq\mathscr{R}(\phi)$ for every $N\in\mathscr{N}(\Omega)$, that is (6.12).

Note that, in general, the relation "$\preceq$" in (6.12) cannot be replaced by "$\simeq$" as the following example shows.

Example 6.2. – Let $n=1$, $\Omega=]-1, 1[$, $\phi(x, \xi)=a(x)|\xi|=\phi^{00}(x, \xi)$, where

$$a(x)=\begin{cases}1 & \text{if } x\in]-1, 0[\cup]0, 1[,\\ \dfrac{1}{2} & \text{if } x=0.\end{cases}$$

Then [*see* (5.6)] we have $\overline{\mathscr{F}[\phi^{00}]}(u)=\int_\Omega |Du|$ for any $u\in BV(\Omega)$, so that $\overline{\mathscr{F}[\phi^{00}]}(\chi_{]0,1[})=1$. Take $N\in\mathscr{N}(\Omega)$ with $0\notin N$. Then $\phi_N^{00}(x, \xi)=\phi^{00}(x, \xi)$

for any $x \in]-1, 1[$ and any $\xi \in \mathbb{R}$, so that

$$\overline{\mathscr{J}[\phi_N^{00}]}(u) \leq \mathscr{J}[\phi_N^{00}](u) = \mathscr{J}[\phi^{00}](u) = \int_{-1}^{0} |Du| + \int_{0}^{1} |Du| + \frac{1}{2} |Du|(\{0\}),$$

and in particular $\overline{\mathscr{J}[\phi_N^{00}]}(\chi_{]0,1[}) \leq \frac{1}{2}$.

LEMMA 6.3. – *Let* $\phi : \Omega \times \mathbb{R}^n \to [0, +\infty[$ *be a Borel function satisfying conditions* (2.12) *and* (2.19). *Then*

$$\sup_{N \in \mathscr{N}(\Omega)} \overline{\mathscr{J}[\phi_N^{00}]}(u) = \int_{\Omega} [\mathscr{E}(\phi, u)](x) |Du| \qquad \forall u \in BV(\Omega), \tag{6.14}$$

where the funcion $\mathscr{E}(\phi, u)$ *is defined in* (6.10).

Proof. – Let $u \in BV(\Omega)$; in view of [37], proposition II.4.1, we can select a countable family $\{N_i\}_{i \in \mathbb{N}} \subseteq \mathscr{N}(\Omega)$ (which depends on u) such that

$$\sup_{N \in \mathscr{N}(\Omega)} \overline{\mathscr{J}[\phi_N^{00}]}(u) = \sup_{i \in \mathbb{N}} \overline{\mathscr{J}[\phi_{N_i}^{00}]}(u), \tag{6.15}$$

$$[\mathscr{E}(\phi, u)](x) = \sup_{i \in \mathbb{N}} [\mathscr{S}(\phi_{N_i})](x, \nu^u) \qquad \text{for } |Du| - \text{a.e. } x \in \Omega. \tag{6.16}$$

Given $N_1, N_2 \in \mathscr{N}(\Omega)$ with $N_1 \subseteq N_2$, according to (6.5) we have $\overline{\mathscr{J}[\phi_{N_1}^{00}]}(u) \leq \overline{\mathscr{J}[\phi_{N_2}^{00}]}(u)$ for every $u \in BV(\Omega)$, and hence

$$\mathscr{S}(\phi_{N_1}) \preceq \mathscr{S}(\phi_{N_2}). \tag{6.17}$$

Consequently, it is not restrictive to assume that the family $\{N_i\}_{i \in \mathbb{N}}$ is increasing, *i. e.*, $N_i \subseteq N_{i+1}$ for any $i \in \mathbb{N}$. Using (6.15), (6.17), the Monotone Convergence Theorem and (6.16), it follows

$$\begin{aligned}
&\sup_{N \in \mathscr{N}(\Omega)} \int_{\Omega} [\mathscr{S}(\phi_N)](x, \nu^u) |Du| = \sup_{i \in \mathbb{N}} \int_{\Omega} [\mathscr{S}(\phi_{N_i})](x, \nu^u) |Du| \\
&\quad = \lim_{i \to +\infty} \int_{\Omega} [\mathscr{S}(\phi_{N_i})](x, \nu^u) |Du| = \int_{\Omega} \lim_{i \to +\infty} [\mathscr{S}(\phi_{N_i})](x, \nu^u) |Du| \\
&\quad = \int_{\Omega} \sup_{i \in \mathbb{N}} [\mathscr{S}(\phi_{N_i})](x, \nu^u) |Du| = \int_{\Omega} [\mathscr{E}(\phi, u)](x) |Du|,
\end{aligned}$$

and this proves (6.14).

We are now in a position to prove the main result of this section.

THEOREM 6.4. – *Let* $\phi : \Omega \times \mathbb{R}^n \to [0, +\infty[$ *be a Borel function satisfying conditions* (2.12) *and* (2.19). *Then*

$$\int_{\Omega} |Du|_{\phi} = \sup_{N \in \mathscr{N}(\Omega)} \overline{\mathscr{J}[\phi_N]}(u) \qquad \forall u \in BV(\Omega). \tag{6.18}$$

Proof. – We first claim that for any $u \in BV(\Omega)$

$$(6.19)\qquad [\mathscr{E}(\phi, u)](x) = [\mathscr{R}(\phi)](x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

To this aim we shall prove

$$[\mathscr{E}(\phi, u)](x) \leqq [\mathscr{R}(\phi)](x, \nu^u) \leqq [\mathscr{E}(\mathscr{R}(\phi), u)](x) \leqq [\mathscr{E}(\phi, u)](x) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

Let $u \in BV(\Omega)$; according to (6.12), we have $\mathscr{S}(\phi_N) \preceq \mathscr{R}(\phi)$ for any $N \in \mathscr{N}(\Omega)$, and this implies

$$[\mathscr{E}(\phi, u)](x) \leqq [\mathscr{R}(\phi)](x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

Moreover, (6.11) and $\mathscr{R}(\phi) \preceq \mathscr{R}(\phi)_N$ give

$$\mathscr{R}(\phi) \simeq \mathscr{S}(\mathscr{R}(\phi)) \preceq \mathscr{S}(\mathscr{R}(\phi)_N).$$

Consequently

$$[\mathscr{R}(\phi)](x, \nu^u) \leqq [\mathscr{E}(\mathscr{R}(\phi), u)](x) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

To conclude the proof of the claim, we must show

$$(6.20)\qquad [\mathscr{E}(\mathscr{R}(\phi), u)](x) \leqq [\mathscr{E}(\phi, u)](x) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

By definition (6.9) and reasoning as in (5.13) and below, it follows that there exists $N_0 \in \mathscr{N}(\Omega)$ such that

$$(6.21)\qquad [\mathscr{R}(\phi)](x, \xi) \leqq \phi^{00}(x, \xi) \quad \forall x \in \Omega \setminus N_0, \quad \forall \xi \in \mathbb{R}^n.$$

Take $N \in \mathscr{N}(\Omega)$ such that $N \supseteqq N_0$. Then (6.21) yields

$$[\mathscr{R}(\phi)_N](x, \xi) \leqq \phi_N^{00}(x, \xi)$$

for every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. Therefore $\mathscr{S}(\mathscr{R}(\phi)_N) \preceq \mathscr{S}(\phi_N)$ for any $N \in \mathscr{N}(\Omega)$ such that $N \supseteqq N_0$. Recalling that the functional $\overline{\mathscr{J}[\mathscr{R}(\phi)_N]}$ and $\overline{\mathscr{J}[\phi_N^{00}]}$ are increasing, if considered as functions of N [*see* for example (6.5)], we deduce that $[\mathscr{E}(\mathscr{R}(\phi), u)](x) \leqq [\mathscr{E}(\phi, u)](x)$ for every $u \in BV(\Omega)$ and for $|Du|$-almost every $x \in \Omega$, *i.e.*, (6.20).

Note that, in view of lemma 6.3, relation (6.19) can be equivalently rewritten as

$$(6.22)\qquad \sup_{N \in \mathscr{N}(\Omega)} \overline{\mathscr{J}[\phi_N^{00}]}(u) = \overline{\mathscr{F}[\phi^{00}]}(u) \quad \forall u \in BV(\Omega).$$

Observe now that, by (6.22) and (5.12).

$$(6.23)\qquad \sup_{N \in \mathscr{N}(\Omega)} \overline{\mathscr{J}[\phi_N]}(u) \geqq \sup_{N \in \mathscr{N}(\Omega)} \overline{\mathscr{J}[\phi_N^{00}]}(u) = \overline{\mathscr{F}[\phi^{00}]}(u) = \overline{\mathscr{F}[\phi]}(u) \quad \forall u \in BV(\Omega).$$

On the other hand

$$\mathscr{F}[\phi](u) \geqq \mathscr{J}[\phi_N](u) \qquad \forall N \in \mathscr{N}(\Omega), \quad \forall u \in BV(\Omega),$$

so that, passing to the lower semicontinuous envelopes and taking the supremum with respect to $N \in \mathscr{N}(\Omega)$, we get

$$\overline{\mathscr{F}[\phi]}(u) \geqq \sup_{N \in \mathscr{N}(\Omega)} \overline{\mathscr{J}[\phi_N]}(u) \qquad \forall u \in BV(\Omega).$$

This inequality, together with (6.23) gives

$$\overline{\mathscr{F}[\phi]}(u) = \sup_{N \in \mathscr{N}(\Omega)} \overline{\mathscr{J}[\phi_N]}(u) \qquad \forall u \in BV(\Omega). \tag{6.24}$$

Then (6.18) is a consequence of (5.7) and (6.24).

Observe that, as a particular case of (6.18), we deduce

$$P_\phi(E, \Omega) = \sup_{N \in \mathscr{N}(\Omega)} \overline{\mathscr{J}[\phi_N]}(\chi_E) \tag{6.25}$$

for any measurable set $E \subseteq \mathbb{R}^n$ of finite perimeter in Ω.

6.1. Relaxation of $\mathscr{F}[\phi]$ and $\mathscr{J}[\phi]$ when ϕ is upper semicontinuous

In this subsection we specialize our results in the case in which ϕ is upper semicontinuous. For a counterpart of the following results in the case of curves we refer to [18], theorem 3.3.

THEOREM 6.5. – *Let $\phi : \Omega \times \mathbb{R}^n \to [0, +\infty[$ be a Borel function satisfying conditions* (2.12) *and* (2.19). *Assume that ϕ is upper semicontinuous on $\Omega \times \mathbb{R}^n$. Let $\mathscr{F}[\phi]$, $\mathscr{J}[\phi]$ be the functionals defined in* (5.2) *and* (6.2), *respectively. Then*

$$\overline{\mathscr{F}[\phi]}(u) = \overline{\mathscr{J}[\phi]}(u) \qquad \forall u \in BV(\Omega). \tag{6.26}$$

Proof. – The inequality $\overline{\mathscr{F}[\phi]} \geqq \overline{\mathscr{J}[\phi]}$ follows immediately from the definitions of $\mathscr{F}[\phi]$ and $\mathscr{J}[\phi]$. Let us prove the opposite inequality. As ϕ is upper semicontinuous, there exists a decreasing sequence $\{\phi_k\}_k$ of continuous functions defined on $\Omega \times S^{n-1}$ such that

$$0 \leqq \phi_k(x, \xi) \leqq \sqrt{\Lambda}\,\|\xi\|, \qquad \phi(x, \xi) = \inf_{k \in \mathbb{N}} \phi_k(x, \xi) \qquad \forall x \in \Omega, \quad \forall \xi \in S^{n-1}.$$

Let $u \in BV(\Omega)$, and let $\{u_h\}_h \subseteq W^{1,1}(\Omega)$ be a sequence of functions converging to u in $L^1(\Omega)$ and such that $\int_\Omega \|\nabla u_h\|\, dx \to |Du|(\Omega)$ as $h \to +\infty$ (*see* [30], theorem 1.17). For any k we have

$$\overline{\mathscr{F}[\phi]}(u) \leqq \liminf_{h \to +\infty} \mathscr{F}[\phi](u_h) \tag{6.27}$$

$$\leqq \liminf_{h \to \infty} \mathscr{F}[\phi_k](u_h) = \liminf_{h \to \infty} \mathscr{J}[\phi_k](u_h).$$

By using a result due to Reshetnyak (*see* [41] and [32], appendix) we have

$$\lim_{h\to+\infty} \mathscr{I}[\phi_k](u_h)=\int_\Omega \phi_k(x,\nu^u)\,|Du|=\mathscr{I}[\phi_k](u),$$

for any $k\in\mathbb{N}$; hence, from (6.27), we get

$$\overline{\mathscr{F}[\phi]}(u)\leqq \mathscr{I}[\phi_k](u) \qquad \forall k\in\mathbb{N},\quad \forall u\in \mathrm{BV}(\Omega). \tag{6.28}$$

Let us fix $u\in\mathrm{BV}(\Omega)$; by (6.28) and $\overline{\mathscr{F}[\phi^{00}]}=\overline{\mathscr{F}[\phi]}$, there exists a set $\mathrm{F}\subseteqq\Omega$ such that $|Du|(\mathrm{F})=0$ and

$$[\mathscr{R}(\phi)](x,\nu^u)\leqq\phi_k(x,\nu^u) \qquad \forall x\in\Omega\setminus\mathrm{F},\quad \forall k\in\mathbb{N}. \tag{6.29}$$

Take $\varepsilon>0$ and $x\in\Omega\setminus\mathrm{F}$. By definition, there exists $k\in\mathbb{N}$ such that

$$\phi_k(x,\nu^u(x))\leqq\phi(x,\nu^u(x))+\varepsilon,$$

which, together with (6.29), implies that

$$[\mathscr{R}(\phi)](x,\nu^u(x))\leqq\phi(x,\nu^u(x))+\varepsilon.$$

Since this inequality holds for any $\varepsilon>0$, for any $u\in\mathrm{BV}(\Omega)$, and for $|Du|$-almost every $x\in\Omega$, we deduce that $\mathscr{R}(\phi)\preceq\phi$. This implies

$$\overline{\mathscr{F}[\phi]}(u)\leqq\mathscr{I}[\phi](u) \qquad \forall u\in\mathrm{BV}(\Omega).$$

Passing to the lower semicontinuous envelopes, we get the assertion.

Remark 6.6. – We observe that theorem 6.5 provides an integral representation on $\mathrm{BV}(\Omega)$ of the $\mathrm{L}^1(\Omega)$-lower semicontinuous envelope of $\mathscr{I}[\phi]$ when ϕ is not convex.

COROLLARY 6.7. – *Let $\phi:\Omega\times\mathbb{R}^n\to[0,+\infty[$ be a Borel function satisfying conditions* (2.12) *and* (2.19). *Assume that ϕ is upper semicontinuous on $\Omega\times\mathbb{R}^n$. Then*

$$\overline{\mathscr{F}[\phi]}(u)=\int_\Omega |Du|_\phi=\overline{\mathscr{I}[\phi]}(u)=\overline{\mathscr{I}[\phi_{\mathrm{N}}]}(u)$$
$$\forall \mathrm{N}\in\mathscr{N}(\Omega),\qquad \forall u\in\mathrm{BV}(\Omega).$$

Proof. – By theorems 5.1, 6.4, and 6.5, for any $\mathrm{N}\in\mathscr{N}(\Omega)$ we have

$$\overline{\mathscr{F}[\phi]}(u)=\int_\Omega |Du|=\sup_{\mathrm{N}\in\mathscr{N}(\Omega)}\overline{\mathscr{I}[\phi_{\mathrm{N}}]}(u)$$
$$\geqq\overline{\mathscr{I}[\phi_{\mathrm{N}}]}(u)\geqq\overline{\mathscr{I}[\phi]}(u)=\overline{\mathscr{F}[\phi]}(u) \qquad \forall u\in\mathrm{BV}(\Omega).$$

The following example shows that the previous result fails when ϕ is not upper semi-continuous. Precisely, we prove that the inequality $\overline{\mathscr{I}[\phi]}<\overline{\mathscr{I}[\phi_{\mathrm{N}}]}$ can hold for some function ϕ and some $\mathrm{N}\in\mathscr{N}(\Omega)$.

Exemple 6.8. – Let $\Omega=\mathrm{B}_2(0)$, $0<\lambda<1$, and set

$$\phi(x,\xi)=\begin{cases}\|\xi\| & \text{if } x\in\Omega\setminus\partial\mathrm{B}_1(0)\\ \sqrt{\lambda}\,\|\xi\| & \text{if } x\in\partial\mathrm{B}_1(0).\end{cases}$$

Clearly ϕ is convex and $\phi(., \xi)$ is lower semicontinuous. Take $N=\partial B_1(0)\in\mathcal{N}(\Omega)$. Then

$$\overline{\mathcal{J}[\phi]}(u)\leqq\mathcal{J}[\phi](u),\qquad \overline{\mathcal{J}[\phi_N]}(u)=\int_\Omega |Du|\quad \forall u\in BV(\Omega).$$

It follows

$$\overline{\mathcal{J}[\phi]}(\chi_{B_1(0)})\leqq\sqrt{\lambda}\,\mathcal{H}^{n-1}(\partial B_1(0))<\mathcal{H}^{n-1}(\partial B_1(0))=\overline{\mathcal{J}[\phi_N]}(\chi_{B_1(0)}).$$

6.2. Relaxation of $\mathcal{J}[\phi]$ by means of sequences of characteristic functions

We conclude this section with a theorem showing that, if $E\subseteqq\mathbb{R}^n$ is a measurable set of finite perimeter in Ω, then, to calculate $\overline{\mathcal{J}[\phi]}(\chi_E)$ we can restrict ourselves to the class of all approximating sequences which consist of characteristic functions. This result will be useful in lemma 8.3.

Precisely, define

$$(6.30)\qquad \mathcal{J}_\phi(E,\Omega)=\inf\{\liminf_{h\to+\infty}\mathcal{J}[\phi](\chi_{E_h}):\{\chi_{E_h}\}_h\subseteqq BV(\Omega),\ \chi_{E_h}\xrightarrow{L^1(\Omega)}\chi_E\}.$$

Then the following result holds [compare with (6.25)].

THEOREM 6.9. — *Let $\phi:\Omega\times\mathbb{R}^n\to[0,+\infty[$ be a Borel function satisfying conditions* (2.12) *and* (2.19), *and let* E *be a measurable set of finite perimeter in Ω. Then*

$$(6.31)\qquad \mathcal{J}_\phi(E,\Omega)=\overline{\mathcal{J}[\phi]}(\chi_E).$$

In particular, if ϕ is upper semicontinuous on $\Omega\times\mathbb{R}^n$, then

$$(6.32)\qquad \mathcal{J}_\phi(E,\Omega)=P_\phi(E,\Omega).$$

Proof. — Let $E\subseteqq\mathbb{R}^n$ be a measurable set of finite perimeter in Ω. To prove (6.31) it is enough to show that $\mathcal{J}_\phi(E,\Omega)\leqq\overline{\mathcal{J}[\phi]}(\chi_E)$, since the opposite inequality follows immediately from the definitions. To do that, it will be sufficient to find a sequence $\{E_h\}_h\in\mathbb{R}^n$ of measurable sets of finite perimeter in Ω such that

$$(6.33)\qquad \begin{cases}\chi_{E_h}\to\chi_E\quad\text{in } L^1(\Omega)\quad\text{as } h\to+\infty,\\ \text{and}\\ \lim\limits_{h\to+\infty}\mathcal{J}[\phi](\chi_{E_h})=\overline{\mathcal{J}[\phi]}(\chi_E).\end{cases}$$

Indeed, using (6.33) and formula (6.30) one realizes

$$\overline{\mathcal{J}[\phi]}(\chi_E)=\lim_{h\to+\infty}\mathcal{J}[\phi](\chi_{E_h})\geqq\mathcal{J}_\phi(E,\Omega),$$

which is the assertion.

Let $\{u_h\}_h \subseteq \mathrm{BV}(\Omega)$ be a sequence of functions converging to χ_E in $L^1(\Omega)$ and such that $\mathscr{J}[\phi](\chi_E) = \lim_{h \to +\infty} \mathscr{J}[\phi](u_h)$. Let us show that we can assume

$$(6.34) \qquad 0 \leqq u_h \leqq 1 \qquad \forall h \in \mathbb{N}.$$

Since $u_h \in \mathrm{BV}(\Omega)$ for any h, from the coarea formula (2.10) the set $\{u_h > s\}$ has finite perimeter in Ω for almost every $s \in \mathbb{R}$. Hence there exists a sequence of positive real numbers $\{\varepsilon_h\}_h$ converging to zero as $h \to +\infty$ such that $-\varepsilon_h \vee u_h \wedge (1-\varepsilon_h) \in \mathrm{BV}(\Omega)$ for any h. Define $v_h = u_h + \varepsilon_h$. Then, for any h, we have $v_h \in \mathrm{BV}(\Omega)$, $0 \vee v_h \wedge 1 \in \mathrm{BV}(\Omega)$, and $\mathrm{D}v_h = \mathrm{D}u_h$ (as measures), which implies $\mathscr{J}[\phi](u_h) = \mathscr{J}[\phi](v_h)$. Define $w_h = 0 \vee v_h \wedge 1$. It is easy to verify that $\| w_h - \chi_E \|_{L^1(\Omega)} \leqq \| u_h - \chi_E \|_{L^1(\Omega)}$ for any h, which gives $\lim_{h \to +\infty} w_h = \chi_E$ in $L^1(\Omega)$, and to verify that $\mathscr{J}[\phi](w_h) \leqq \mathscr{J}[\phi](u_h)$. This proves that we can assume condition (6.34).

Using Cavalieri's formula and (6.34) we have

$$\int_\Omega |u_h - \chi_E| \, dx = \int_0^1 \int_\Omega \chi_{\{|u_h - \chi_E| > t\}} \, dx \, dt \qquad \forall h \in \mathbb{N}.$$

Hence there exists a subsequence (still denoted by $\{u_h\}_h$) such that

$$(6.35) \qquad \begin{aligned} \mathscr{H}^n(\{|u_h - \chi_E| > t\}) &= \mathscr{H}^n(E \cap \{u_h < 1 - t\}) \\ &+ \mathscr{H}^n((\Omega \setminus E) \cap \{u_h > t\}) \to 0 \quad \text{for a.e. } t \in [0, 1] \quad \text{as } h \to +\infty. \end{aligned}$$

Let $s \in]0, 1[$, and choose t whith $0 < t < s < 1 - t < 1$ and in such a way that (6.35) is fulfilled. Then $\{u_h \leqq s\} \subseteq \{u_h < 1 - t\}$ and $\{u_h > s\} \subseteq \{u_h > t\}$ for any h. Hence, from (6.35), we have

$$\mathscr{H}^n(E \Delta \{u_h > s\}) \leqq \mathscr{H}^n(E \cap \{u_h < 1 - t\}) + \mathscr{H}^n((\Omega \setminus E) \cap \{u_h > t\}) \to 0$$

as $h \to +\infty$ (here Δ denotes the symmetric difference of sets). Consequently

$$(6.36) \qquad \lim_{h \to +\infty} \chi_{\{u_h > s\}} = \chi_E \quad \text{in } L^1(\Omega) \quad \forall s \in]0, 1[.$$

Let now $\varepsilon > 0$ be a small number. We shall show that there exists a sequence $\{s_h\}_h \subseteq [\varepsilon, 1 - \varepsilon]$ such that

$$(6.37) \qquad \begin{cases} \{u_h > s_h\} \text{ has finite perimeter in } \Omega \qquad \forall h \in \mathbb{N}, \\ \lim_{h \to +\infty} \chi_{\{u_h > s_h\}} = \chi_E \quad \text{in } L^1(\Omega), \\ \mathscr{J}[\phi](\chi_{\{u_h > s_h\}}) \leqq \dfrac{1}{1 - 2\varepsilon} \mathscr{J}[\phi](u_h) \qquad \forall h \in \mathbb{N}. \end{cases}$$

Using the coarea formula and (6.34) we have

$$\begin{aligned}\mathcal{J}[\phi](u_h)&=\int_\Omega \phi(x, \nu^{u_h})\,|Du_h|\\ &=\int_0^1\int_{\partial^*\{u_h>s\}}\phi(x, \nu^{\{u_h>s\}})\,d\mathcal{H}^{n-1}(x)\,ds\\ &=\int_0^1 \mathcal{J}[\phi](\chi_{\{u_h>s\}})\,ds \geqq \int_\varepsilon^{1-\varepsilon}\mathcal{J}[\phi](\chi_{\{u_h>s\}})\,ds \qquad \forall h\in\mathbb{N}.\end{aligned}$$

Therefore for any $h\in\mathbb{N}$ there exists a measurable set $S(h)\subseteqq[\varepsilon, 1-\varepsilon]$ such that $\mathcal{H}^1(S(h))>0$ and

$$\mathcal{J}[\phi](u_h)\geqq(1-2\varepsilon)\,\mathcal{J}[\phi](\chi_{\{u_h>s\}}) \qquad \forall s\in S(h).$$

For any $h\in\mathbb{N}$ choose $s_h\in S(h)$ such that $\{u_h>s_h\}$ has finite perimeter in Ω. For a subsequence (still denoted by $\{s_h\}_h$) we have $s_h\to s_0$ as $h\to+\infty$, and $s_0\in[\varepsilon, 1-\varepsilon]$. Let us prove

$$\lim_{h\to+\infty}\chi_{\{u_h>s_h\}}=\chi_E \quad \text{in } L^1(\Omega). \tag{6.38}$$

As $\lim_{h\to+\infty} s_h=s_0$, we can assume that there exists $\delta>0$ such that $s_h\in[s_0-\delta, s_0+\delta]\subseteqq]0, 1[$ for any h. Then $\{u_h>s_0+\delta\}\subseteqq\{u_h>s_h\}\subseteqq\{u_h>s_0-\delta\}$ for any h. But (6.36) yields $\lim_{h\to+\infty}\chi_{\{u_k>s_0-\delta\}}=\lim_{h\to+\infty}\chi_{\{u_h>s_0+\delta\}}=\chi_E$ in $L^1(\Omega)$. Consequently $\lim_{h\to+\infty}\chi_{\{u_h>s_h\}}=\chi_E$ in $L^1(\Omega)$, that is (6.38). Hence all properties required in (6.37) are fulfilled. Take now $\varepsilon=\dfrac{1}{n}$, for $n\in\mathbb{N}$, and let $n\to+\infty$. Using a diagonal argument and (6.37) we have that $\{u_{h(n)}>s_{h(n)}\}$ has finite perimeter in Ω for any n, $\lim_{n\to+\infty}\chi_{\{u_{h(n)}>s_{h(n)}\}}=\chi_E$ in $L^1(\Omega)$, and

$$\begin{aligned}\lim_{n\to+\infty}\mathcal{J}[\phi](\chi_{\{u_{h(n)}>s_{h(n)}\}})&=\lim_{n\to+\infty}\mathcal{J}[\phi](u_{h(n)})\\ &=\lim_{h\to+\infty}\mathcal{J}[\phi](u_h)=\overline{\mathcal{J}[\phi]}(\chi_E).\end{aligned}$$

This concludes the proof of (6.31).

If ϕ is upper semicontinuous, then (6.32) follows from (6.31) and (6.26).

7. SQUARE ROOTS OF QUADRATIC FORMS

In this section we evaluate $\int_\Omega |Du|_\phi$ when ϕ^2 is a uniformly elliptic quadratic form with regular coefficients. Let $A=\{a_{ij}\}_{i,j}\in\mathscr{C}^0(\Omega;\mathbb{R}^{n\times n})$ be a symmetric matrix such that

$$(7.1)\qquad \lambda\|\xi\|^2\leqq\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\leqq\Lambda\|\xi\|^2\qquad\forall x\in\Omega,\quad\forall\xi\in\mathbb{R}^n,$$

for some $0<\lambda\leqq\Lambda<+\infty$.

Setting

$$(7.2)\qquad \phi(x,\xi)=\left(\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\right)^{1/2}\qquad\forall x\in\Omega,\quad\forall\xi\in\mathbb{R}^n,$$

we have that ϕ is convex and satisfies conditions (2.12) and (3.3). Then (5.8) yields

$$(7.3)\qquad \int_\Omega|Du|_\phi=\int_\Omega\left(\sum_{i,j=1}^{n}a_{ij}(x)\nu_i^u\nu_j^u\right)^{1/2}|Du|\qquad\forall u\in BV(\Omega).$$

In particular, for any measurable set $E\subseteq\mathbb{R}^n$ of finite perimeter in Ω, we have

$$P_\phi(E,\Omega)=\int_{\Omega\cap\partial^*E}\left(\sum_{i,j=1}^{n}a_{ij}(x)\nu_i^E\nu_j^E\right)^{1/2}d\mathscr{H}^{n-1}(x).$$

For the sake of completeness, we shall prove formula (7.3) in a more direct way.

PROPOSITION 7.1. – *Let* $A=\{a_{ij}\}_{i,j}\in\mathscr{C}^0(\Omega;\mathbb{R}^{n\times n})$ *be a symmetric matrix which satisfies condition* (7.1), *and let* ϕ *be defined as in* (7.2). *Then relation* (7.3) *holds.*

Proof. – Let $u\in BV(\Omega)$; using proposition 3.2, formula (2.8), and [7], proposition 1.3 (v), it follows

$$\int_\Omega|Du|_\phi=\sup\left\{\int_\Omega u\,\mathrm{div}\,\sigma\,dx:\sigma\in\mathscr{C}_\phi\right\}=\sup\left\{\int_\Omega(\sigma,Du):\sigma\in\mathscr{C}_\phi\right\},$$

where $\mathscr{C}_\phi$ is defined in (3.1), and $\int_\Omega(\sigma,Du)=\sum_{i=1}^{n}\int_\Omega\sigma_i D_i u$. Using (4.1), (4.2), and (2.9) we get

$$(7.4)\qquad \int_\Omega|Du|_\phi=\int_\Omega h(x,\nu^u)|Du|,$$

where $h=|Du|-\operatorname*{ess\,sup}_{\sigma\in\mathscr{C}_\phi}(\sigma,\nu^u)$.

In view of (7.4), to prove (7.3) it will be enough to show

$$(7.5)\qquad h(x, \nu^u)=\left(\sum_{i, j=1}^{n} a_{ij}(x)\nu_i^u \nu_j^u\right)^{1/2} \qquad \text{for} \quad |Du|-\text{a.e. } x\in\Omega.$$

For any $x\in\Omega$, let $A^{-1}(x)=\{a^{ij}(x)\}_{i,j}$ be the inverse matrix of $A(x)$. It is not difficult to prove that $\phi^0(x, \xi^*)=\sqrt{(A^{-1}(x)\xi^*, \xi^*)}$ for any $(x, \xi^*)\in\Omega\times\mathbb{R}^n$. Moreover, one can prove that $\sqrt{A^{-1}}\in\mathscr{C}^0(\Omega;\ \mathbb{R}^{n\times n})$. We then have

$$\begin{aligned}\mathscr{C}_\phi=\{\sigma\in\mathscr{C}_0^1(\Omega;\ \mathbb{R}^n):\|\sqrt{A^{-1}(x)}\,\sigma(x)\|\leqq 1\ \forall x\in\Omega\}\\ \subseteqq\{\sqrt{A}\,\sigma:\sigma\in\mathscr{C}_0^0(\Omega;\ \mathbb{R}^n),\ \|\sigma(x)\|\leqq 1\ \forall x\in\Omega\}.\end{aligned}$$

Hence, for any $u\in BV(\Omega)$, it follows

$$\begin{aligned}h\leqq |Du|-\text{ess sup}\{(\sigma, \sqrt{A}\,\nu^u):\sigma\in\mathscr{C}_0^0(\Omega;\ \mathbb{R}^n), \|\sigma(y)\|\leqq 1\ \forall y\in\Omega\}\\ \leqq |Du|-\text{ess sup}\{(\sigma, \sqrt{A}\,\nu^u):\sigma\in L^\infty_{|Du|}(\Omega;\ \mathbb{R}^n),\\ \|\sigma(y)\|\leqq 1 \text{ for } |Du|-\text{a.e. } y\in\Omega\},\end{aligned}$$

which implies that, for $|Du|$-almost every $x\in\Omega$,

$$h(x, \nu^u)\leqq\|\sqrt{A(x)}\,\nu^u\|=\left(\sum_{i, j=1}^{n} a_{ij}(x)\nu_i^u \nu_j^u\right)^{1/2}.$$

The opposite inequality is a consequence of (5.7) and the fact that the functional

$$\int_\Omega\left(\sum_{i, j=1}^{n} a_{ij}(x)\nu_i^u \nu_j^u\right)^{1/2}|Du|$$

is $L^1(\Omega)$-lower semicontinuous on $BV(\Omega)$ (*see* [16], theorem 3.1). This proves (7.5), and concludes the proof of (7.3).

8. A COUNTEREXAMPLE

Let $\{a_{ij}\}_{i,j}$ be a symmetric matrix satisfying (7.1) and let ϕ be defined as in (7.2). In this section we show that, if $\{a_{ij}\}_{i,j}$ is highly discontinuous, then $\int_\Omega |Du|_\phi$ has not, in general, an integral representation with an integrand of the same type of ϕ, *i.e.*, which is the square root of a quadratic form.

Let $I=]0, 2[$, $\Omega=I\times I$, and let $\{q_h\}_{h=1}^{+\infty}$ be a countable dense subset of I. Define

$$C=\{t\in I:|t-q_h|\geqq 2^{-h}\ \forall h\geqq 1\},\qquad A=I\setminus C.$$

Then A is an open dense subset of I with

$$0<\mathscr{H}^1(\mathrm{A})\leqq \sum_{h\geqq 1} 2^{-h}=1<\mathscr{H}^1(\mathrm{I})=2,$$

and C is a closed set without interior. We recall that, by the Lebesgue Differentiation Theorem, almost every $t\in \mathrm{C}$ has density one for C, *i.e.*,

$$\lim_{\rho\to 0^+} \frac{\mathscr{H}^1(]t-\rho, t+\rho[\cap \mathrm{C})}{2\rho}=1 \qquad \text{for a.e. } t\in \mathrm{C}.$$

Define $\mathrm{E}=(\mathrm{A}\times \mathrm{I})\cup(\mathrm{I}\times \mathrm{A})$; then E is an open dense subset of Ω and $\Omega\setminus \mathrm{E}=\mathrm{C}\times \mathrm{C}$ is closed and without interior. Let $\Lambda\geqq 2$ be a positive real number, and let $\phi, \psi:\Omega\times \mathbb{R}^2\to[0, +\infty[$ be defined by

$$(8.1)\qquad \left\{\begin{array}{l} \phi(x, \xi)=\begin{cases}\|\xi\| & \text{if } x\in \mathrm{E},\\ \sqrt{\Lambda}\,\|\xi\| & \text{if } x\in \mathrm{C}\times \mathrm{C}\end{cases}\\ \psi(x, \xi)=\begin{cases}\|\xi\| & \text{if } x\in \mathrm{E},\\ |\xi_1|+|\xi_2| & \text{if } x\in \mathrm{C}\times \mathrm{C},\end{cases}\end{array}\right.$$

where $\xi=(\xi_1, \xi_2)$. Obviously ϕ and ψ are convex, and ψ is not the square root of a quadratic form. Observe also that $\psi(., \xi)$ is not lower semicontinuous, that $\phi(., \xi)$ is upper semicontinuous and that $\psi\leqq\phi$. Consider the functionals $\mathscr{F}[\phi]$ and $\mathscr{I}[\phi]$ defined as in (6.2), (6.3) respectively and let $\mathscr{R}(\phi)$, $\mathscr{S}(\phi)$ be the integrands which correspond to $\overline{\mathscr{F}[\phi]}$ and $\overline{\mathscr{I}[\phi]}$ as in (6.9) and (6.7). Our aim is to prove that

$$[\mathscr{R}(\phi)](x, \xi)=[\mathscr{S}(\phi)](x, \xi)=\psi(x, \xi) \qquad \forall \xi\in \mathbb{R}^2 \qquad \text{for a.e. } x\in \Omega.$$

Let $u\in \mathrm{BV}(\Omega)$; since ϕ is symmetric (*i.e.*, $\phi(x, \xi)=\phi(x, -\xi)$), we have $\mathscr{I}[\phi](u)=\mathscr{I}[\phi](-u)$, which yields $\overline{\mathscr{I}[\phi]}(u)=\overline{\mathscr{I}[\phi]}(-u)$. Then also $\mathscr{S}(\phi)$ is symmetric, in the sense that for any $u\in \mathrm{BV}(\Omega)$ we have

$$(8.2)\qquad [\mathscr{S}(\phi)](x, \nu^u)=[\mathscr{S}(\phi)](x, -\nu^u) \qquad \text{for } |Du|-\text{a.e. } x\in \Omega.$$

Let $\{e_1, e_2\}$ be the canonial basis of $\mathbb{R}^2$. For any pair $\{n, \nu\}$ of unit vectors mutually orthogonal, let $\mathrm{R}\{n, \nu\}$ be the family of all bounded open rectangles having sides parallel to n and ν and which are contained in Ω.

LEMMA 8.1. – *We have*

$$(8.3)\qquad \overline{\mathscr{I}[\phi]}(\chi_{\mathrm{R}})=\mathrm{P}(\mathrm{R}, \Omega) \qquad \forall \mathrm{R}\in \mathrm{R}\{e_1, e_2\}.$$

Proof. – Since $\phi(x, \xi)\geqq\|\xi\|$ for any $x\in\Omega$ and any $\xi\in\mathbb{R}^n$, we deduce

$$\overline{\mathscr{I}[\phi]}(u)\geqq\int_\Omega |Du| \qquad \forall u\in \mathrm{BV}(\Omega).$$

In particular, $\overline{\mathscr{I}[\phi]}(\chi_{\mathrm{R}})\geqq \mathrm{P}(\mathrm{R}, \Omega)$ for any $\mathrm{R}\in \mathrm{R}\{e_1, e_2\}$. Let us prove the opposite inequality. Let $\mathrm{R}\in \mathrm{R}\{e_1, e_2\}$; using the density of the set A in

I, it is easy to find a sequence $\{R_h\}_h \subseteq R\{e_1, e_2\}$ with the properties $\partial R_h \subseteqq E$ for any h, $\lim_{h \to +\infty} P(R_h, \Omega) = P(R, \Omega)$, and $\lim_{h \to +\infty} \chi_{R_h} = \chi_R$ in $L^1(\Omega)$. As $\partial R_h \subseteqq E$, we have $\mathscr{J}[\phi](\chi_{R_h}) = P(R_h, \Omega)$ for any h. We deduce that $P(R, \Omega) = \lim_{h \to +\infty} \mathscr{J}[\phi](\chi_{R_h}) \geqq \overline{\mathscr{J}[\phi]}(\chi_R)$, and this concludes the proof.

LEMMA 8.2. – *Let* ψ, $\mathscr{S}(\phi)$ *be defined as in* (8.1) *and* (6.7), *respectively. Then*

$$(8.4)\quad [\mathscr{S}(\phi)](x, \xi) = \|\xi\| = \psi(x, \xi) \qquad \forall \xi \in \mathbb{R}^2 \qquad \textit{for} \ \text{a.e. } x \in E,$$

and

$$\|\xi\| \leqq [\mathscr{S}(\phi)](x, \xi) \leqq \psi(x, \xi) \qquad \forall \xi \in \mathbb{R}^2 \qquad \textit{for} \ \text{a.e. } x \in C \times C.$$

Proof. – As $\phi(x, \xi) \geqq \|\xi\|$ for any $(x, \xi) \in \Omega \times \mathbb{R}^2$, we have $[\mathscr{S}(\phi)](x, \xi) \geqq \|\xi\|$ for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^2$. But $\overline{\mathscr{J}[\phi]} \leqq \mathscr{J}[\phi]$, which gives $\mathscr{S}(\phi) \preceq \phi$ [*see* (6.1)]. In particular $[\mathscr{S}(\varphi)](x, \xi) \leqq \phi(x, \xi) = \|\xi\|$ for every $\xi \in \mathbb{R}^2$ and almost every $x \in E$, and (8.4) is proven. In order to conclude the proof, it remains to verify

$$(8.5)\quad [\mathscr{S}(\phi)](x, \xi) \leqq |\xi_1| + |\xi_2| \qquad \forall \xi \in \mathbb{R}^2 \qquad \text{for a.e. } x \in C \times C.$$

For any $x = (x_1, x_2) \in \Omega$, let δ, $\varepsilon > 0$ be sufficiently small in such a way that $R_{\delta, \varepsilon}(x) =]x_1 - \delta, x_1 + \delta[\times]x_2 - \varepsilon, x_2 + \varepsilon[$ is contained in Ω. From (8.3) we have

$$(8.6)\quad \overline{\mathscr{J}[\phi]}(\chi_{R_{\delta, \varepsilon}(x)}) = \int_{x_1 - \delta}^{x_1 + \delta} [\mathscr{S}(\phi)]((s, x_2 - \varepsilon), e_2)\, ds + \int_{x_1 - \delta}^{x_1 + \delta} [\mathscr{S}(\phi)]((s, x_2 + \varepsilon), -e_2)\, ds + \int_{x_2 - \varepsilon}^{x_2 + \varepsilon} [\mathscr{S}(\phi)]((x_1 - \delta, t), e_1)\, dt + \int_{x_2 - \varepsilon}^{x_2 + \varepsilon} [\mathscr{S}(\phi)]((x_1 + \delta, t), -e_1)\, dt = 4\delta + 4\varepsilon.$$

We want now to pass to the limit in (8.6) as $\delta, \varepsilon \to 0$. Since the translation operator is a continuous map from $L^1(\Omega)$ to $L^1(\Omega)$ (*see*, for instance, [8], lemma 4.4), for any open interval I′ which is relatively compact in I, we have

$$(8.7)\quad \lim_{\varepsilon \to 0} \int_{\Omega'} |[\mathscr{S}(\phi)]((s, t - \varepsilon), e_2) - [\mathscr{S}(\phi)]((s, t), e_2)|\, ds\, dt = 0,$$

where $\Omega' = I' \times I'$. Hence there exist a sequence $\{\varepsilon_h\}_h$ of positive real number converging to zero as $h \to +\infty$ and a set $M_2^- \in \mathcal{N}(I')$ such that

$$\lim_{h \to +\infty} \int_{I'} |[\mathcal{S}(\phi)]((s, x_2 - \varepsilon_h), e_2) - [\mathcal{S}(\phi)]((s, x_2), e_2)| \, ds = 0 \qquad \forall x_2 \in I' \setminus M_2^-.$$

In particular, for any $x_1 \in I'$ and any $\delta > 0$ sufficiently small we have

$$(8.8) \quad \lim_{h \to +\infty} \int_{x_1 - \delta}^{x_1 + \delta} |[\mathcal{S}(\phi)]((s, x_2 - \varepsilon_h), e_2) - [\mathcal{S}(\phi)]((s, x_2), e_2)| \, ds = 0 \qquad \forall x_2 \in I' \setminus M_2^-.$$

Similarly we can find $M_2^+ \in \mathcal{N}(I')$ and a sequence of positive real numbers (still denoted by $\{\varepsilon_h\}_h$) converging to zero such that formula (8.8) holds with e_2 replaced by $-e_2$ and $x_2 - \varepsilon_h$ replaced by $x_2 + \varepsilon_h$, for any $x_2 \in I' \setminus M_2^+$. In the same way we can find M_1^-, $M_1^+ \in \mathcal{N}(I')$ and a sequence $\{\delta_h\}_h$ of positive real numbers converging to zero such that, for any $x_2 \in I'$,

$$(8.9) \quad \lim_{h \to +\infty} \int_{x_2 - \varepsilon}^{x_2 + \varepsilon} |[\mathcal{S}(\phi)]((x_1 \pm \delta_h, t), \mp e_1) - [\mathcal{S}(\phi)]((x_1, t), \mp e_1)| \, dt = 0 \qquad \forall x_1 \in I' \setminus M_1^\pm,$$

provided that ε is sufficiently small. Replacing ε by ε_h, letting $\varepsilon_h \to 0$ and keeping δ fixed (respectively replacing δ by δ_h, letting $\delta_h \to 0$ and keeping ε fixed) in (8.6), using (8.8) [respectively (8.9)], the symmetry [*see* property (8.2)] and the boundedness of $\mathcal{S}(\phi)$, we deduce, for δ and ε sufficiently small,

$$(8.10) \quad \begin{cases} \displaystyle\int_{x_1 - \delta}^{x_1 + \delta} [\mathcal{S}(\phi)]((s, x_2), e_2) \, ds = 2\delta \\ \displaystyle\int_{x_2 - \varepsilon}^{x_2 + \varepsilon} -\varepsilon [\mathcal{S}(\phi)]((x_1, t), e_1) \, dt = 2\varepsilon \end{cases} \qquad \forall x = (x_1, x_2) \in \Omega' \setminus M,$$

where $M = ((M_1^- \cup M_1^+) \times I') \cup (I' \times (M_2^- \cup M_2^+))$ belongs to $\mathcal{N}(\Omega')$ by Fubini-Tonelli's Theorem.

Let us prove

$$(8.11) \quad [\mathcal{S}(\phi)](x, e_i) = \|e_i\| = \psi(x, e_i) \qquad \text{for a.e. } x \in \Omega, \quad \text{for } i = 1, 2.$$

The second equality in (8.10) can be rewritten as

$$\int_{x_2 - \varepsilon}^{x_2 + \varepsilon} \{[\mathcal{S}(\phi)]((x_1, t), e_1) - \|e_1\|\} \, dt = 0 \qquad \forall x_1 \in I' \setminus (M_1^- \cup M_1^+), \qquad \forall x_2 \in I'.$$

Since the previous equality holds for any $\varepsilon>0$ sufficiently small and for every $x_2\in \mathrm{I}'$, we infer that, for any $x_1\in \mathrm{I}'\setminus(\mathrm{M}_1^-\cup \mathrm{M}_1^-)$ we have

$$[\mathcal{S}(\phi)]((x_1, x_2), e_1)=\|e_1\| \qquad \text{for a. e. } x_2\in \mathrm{I}'.$$

Hence, by Fubini-Tonelli's Theorem, we have $[\mathcal{S}(\phi)](x, e_1)=\|e_1\|$ for almost every $x\in\Omega'$. Since this is true for any open set Ω' which is relatively compact in Ω, (8.11) is proven for $i=1$. The proof of (8.11) for $i=2$ is similar.

By the convexity and the positive 1-homogeneity of $\overline{\mathcal{J}[\phi]}$ and since $\overline{\mathcal{J}[\phi]}$ is symmetric, using (8.11) it follows that $[\mathcal{S}(\phi)](x, \xi)\leqq|\xi_1|+|\xi_2|$ for every $\xi\in\mathbb{R}^2$ and almost every $x\in\Omega$, that is (8.5).

LEMMA 8.3. – *Let* ϕ *be defined as in* (8.1). *Then for any measurable set* $\mathrm{T}\subseteqq\Omega$ *of finite perimeter in* Ω *we have*

$$\overline{\mathcal{J}[\phi]}(\chi_\mathrm{T})\geqq\int_\mathrm{C}\mathcal{H}^0(\partial^*\mathrm{T}^{x_1})\,dx_1+\int_\mathrm{C}\mathcal{H}^0(\partial^*\mathrm{T}^{x_2})\,dx_2,$$

where $\mathrm{T}^{x_1}=\{x_2\in\mathrm{I}:(x_1, x_2)\in\mathrm{T}\}$, $\mathrm{T}^{x_2}=\{x_1\in\mathrm{I}:(x_1, x_2)\in\mathrm{T}\}$, *and* $\partial^*\mathrm{T}^{x_i}$ *denotes the reduced boundary of the one-dimensional section* T^{x_i} *of* T *for* $i=1, 2$.

Proof. – For any $u\in\mathrm{BV}(\Omega)$ and any Borel set $\mathrm{B}\subseteqq\Omega$, using an approximation argument, ti is easy to show

$$\int_\mathrm{B}|\mathrm{D}_1 u|+\int_\mathrm{B}|\mathrm{D}_2 u|\leqq\sqrt{2}\int_\mathrm{B}|\mathrm{D}u|\leqq\sqrt{\Lambda}\int_\mathrm{B}|\mathrm{D}u|, \tag{8.12}$$

$$\int_\mathrm{B}|\mathrm{D}_i u|\leqq\int_\mathrm{B}|\mathrm{D}u| \qquad \text{for} \quad i=1, 2, \tag{8.13}$$

where $\int_\mathrm{B}|\mathrm{D}_i u|$ is defined in (2.3).

Let $\mathrm{T}\subseteqq\Omega$ be a measurable set of finite perimeter in Ω. Since $\mathrm{E}=(\mathrm{A}\times\mathrm{I})\cup(\mathrm{I}\times\mathrm{A})\supseteqq(\mathrm{A}\times\mathrm{C})\cup(\mathrm{C}\times\mathrm{A})$, and $(\mathrm{A}\times\mathrm{C})\cap(\mathrm{C}\times\mathrm{A})=\varnothing$, recalling (8.12) and (8.13), we deduce

$$\begin{aligned}
\mathcal{J}[\phi](\chi_\mathrm{T})&\geqq\int_{\mathrm{A}\times\mathrm{C}}|\mathrm{D}\chi_\mathrm{T}|+\int_{\mathrm{C}\times\mathrm{A}}|\mathrm{D}\chi_\mathrm{T}|+\sqrt{\Lambda}\int_{\mathrm{C}\times\mathrm{C}}|\mathrm{D}\chi_\mathrm{T}|\\
&\geqq\int_{\mathrm{A}\times\mathrm{C}}|\mathrm{D}_1\chi_\mathrm{T}|+\int_{\mathrm{C}\times\mathrm{A}}|\mathrm{D}_2\chi_\mathrm{T}|+\int_{\mathrm{C}\times\mathrm{C}}|\mathrm{D}_1\chi_\mathrm{T}|+\int_{\mathrm{C}\times\mathrm{C}}|\mathrm{D}_2\chi_\mathrm{T}|\\
&=\int_{\mathrm{I}\times\mathrm{C}}|\mathrm{D}_1\chi_\mathrm{T}|+\int_{\mathrm{C}\times\mathrm{I}}|\mathrm{D}_2\chi_\mathrm{T}|\\
&=\int_\mathrm{C}\left[\int_\mathrm{I}|\mathrm{D}\chi_\mathrm{T}^{x_2}|\right]dx_2+\int_\mathrm{C}\left[\int_\mathrm{I}|\mathrm{D}\chi_\mathrm{T}^{x_1}|\right]dx_1,
\end{aligned}$$

where $\chi_T^{x_i}$ are the one-dimensional sections of χ_T, for $i=1, 2$ [*see* formula (2.4)]. Let $\{\chi_{T_h}\}_h \subseteq BV(\Omega)$ be a sequence of characteristic functions of sets of finite perimeter in Ω converging to χ_T in $L^1(\Omega)$. By Fubini-Tonelli's Theorem, there exists a subsequence (still denoted by $\{\chi_{T_h}\}_h$) such that for $\mathscr{H}^1$-almost every $x_i \in C$ the sequence $\{\chi_{T_h}^{x_i}\}_h$ converges to $\chi_T^{x_i}$ in $L^1(I)$, for $i=1, 2$. Hence the previous inequality, Fatou's Lemma, and the lower semicontinuity of the total variation applied to the one dimensional sections, imply

$$\begin{aligned}\liminf_{h\to+\infty} \mathscr{J}[\phi](\chi_{T_h}) &\geqq \int_C \liminf_{h\to+\infty}\left[\int_I |D\chi_{T_h}^{x_1}|\right] dx_1 \\ &\quad + \int_C \liminf_{h\to+\infty}\left[\int_I |D\chi_{T_h}^{x_2}|\right] dx_2 \\ &\geqq \int_C\left[\int_I |D\chi_T^{x_1}|\right] dx_1 + \int_C\left[\int_I |D\chi_T^{x_2}|\right] dx_2 \\ &= \int_C \mathscr{H}^0(\partial^* T^{x_1})\, dx_1 + \int_C \mathscr{H}^0(\partial^* T^{x_2})\, dx_2.\end{aligned}$$

Consequently, applying (6.31) of theorem 6.9, we obtain

$$\overline{\mathscr{J}[\phi]}(\chi_T) = \mathscr{J}_\phi(T, \Omega) \geqq \int_C \mathscr{H}^0(\partial^* T^{x_1})\, dx_1 + \int_C \mathscr{H}^0(\partial^* T^{x_2})\, dx_2, \tag{8.14}$$

and this concludes the proof.

THEOREM 8.4. – *Let* ψ, $\mathscr{R}(\phi)$ *and* $\mathscr{S}(\phi)$ *be defined as in* (8.1), (6.9) *and* (6.7), *respectively. Then*

$$[\mathscr{R}(\phi)](x, \xi) = [\mathscr{S}(\phi)](x, \xi) = \psi(x, \xi) \qquad \forall \xi \in \mathbb{R}^2 \quad \textit{for a.e. } x \in \Omega.$$

In particular, for every linear $u \in W^{1,1}(\Omega)$, *we have*

$$\overline{\mathscr{F}[\phi]}(u) = \overline{\mathscr{J}[\phi]}(u) = \mathscr{J}[\psi](u) = \int_\Omega |Du|_\phi, \tag{8.15}$$

and here $\mathscr{J}|\phi]$ *is not lower semicontinuous on* $BV(\Omega)$.

Proof. – The equality $\overline{\mathscr{F}[\phi]}(u) = \overline{\mathscr{J}[\phi]}(u)$ for any $u \in BV(\Omega)$ is a consequence of theorem 6.2, being ϕ upper semicontinuous, and it implies that $[\mathscr{R}(\phi)](x, \xi) = [\mathscr{S}(\phi)](x, \xi)$ for every $\xi \in \mathbb{R}^n$ and almost every $x \in \Omega$. In this particular case, we can give a simple proof of this fact without using any previous result. Indded, the inequality $\overline{\mathscr{F}[\phi]} \geqq \overline{\mathscr{J}[\phi]}$ is an immediate consequence of the definition $\mathscr{F}[\phi]$ and $\mathscr{J}[\phi]$. Let us prove the opposite inequality. Given $u \in BV(\Omega)$, by [30], theorem 1.17, there exists a sequence $\{u_h\}_h$ of functions of class $\mathscr{C}^\infty(\Omega)$ convering to u in $L^1(\Omega)$, and such that

$$\int_\Omega |Du| = \lim_{h\to+\infty} \int_\Omega \|\nabla u_h\|\, dx. \tag{8.16}$$

As $\int_O |Du| \leqq \liminf_{h \to +\infty} \int_O \|\nabla u_h\| \, dx$ for every open set $O \subseteqq \Omega$, using (8.16) we deduce that, for any closed set $F \subseteqq \Omega$,

$$(8.17) \qquad \int_F |Du| \geqq \liminf_{h \to +\infty} \int_F \|\nabla u_h\| \, dx.$$

Hence, by (8.16), (8.17), and the definition of ϕ, it follows

$$\begin{aligned}
\overline{\mathscr{F}[\phi]\, u)} &\leqq \liminf_{h \to +\infty} \mathscr{F}[\phi](u_h) \\
&= \liminf_{h \to +\infty} \left\{ \int_\Omega \|\nabla u_h\| \, dx + (\sqrt{\Lambda} - 1) \int_{\Omega \setminus E} \|\nabla u_h\| \, dx \right\} \\
&\leqq \int_\Omega |Du| + (\sqrt{\Lambda} - 1) \int_{\Omega \setminus E} |Du| = \mathscr{J}[\phi](u).
\end{aligned}$$

Passing to the lower semicontinuous envelopes, we obtain

$$\overline{\mathscr{F}[\phi]}(u) \leqq \overline{\mathscr{J}[\phi]}(u)$$

for every $u \in BV(\Omega)$.

The equality $\int_\Omega |Du|_\phi = \overline{\mathscr{F}[\phi]}(u)$ is proven in theorem 5.1, and actually it holds for any $u \in BV(\Omega)$.

Let us prove that $[\mathscr{S}(\phi)](x, \xi) = \psi(x, \xi)$ for every $\xi \in \mathbb{R}^2$ and almost every $x \in \Omega$. In view of (8.4) and (8.5) it will be sufficient to show

$$(8.18) \quad [\mathscr{S}(\phi)](x, \xi) \geqq |\xi_1| + |\xi_2| \qquad \forall \xi \in \mathbb{R}^2 \quad \text{for a.e. } x \in C \times C.$$

By the Lebesgue Differentiation Theorem there exists $N \in \mathscr{N}(\Omega)$ such that any point $x = (x_1, x_2) \in (C \times C) \setminus N$ has the property that x_i has density one for C, for $i = 1, 2$. Fix n, ν two unit vectors mutually orthogonal. For any $x \in \Omega \setminus N$ and any $\delta, \varepsilon > 0$ sufficiently small, let $R_{\delta, \varepsilon}(x) \in R\{n, \nu\}$ be the rectangle centered at x and contained in Ω given by $R_{\delta, \varepsilon}(x) = \{x + sn + t\nu : |s| < \delta, \ |t| < \varepsilon\}$. Let $L_\delta(x)$ be the median line of $R_{\delta, \varepsilon}(x)$ in the direction of n, *i.e.*, $L_\delta(x) = \{x + sn : |s| < \delta\}$. Using the continuity of the translation operator in $L^1(\Omega)$ and property (8.2), reasoning exactly as in lemma 8.2 (*see* formula (8.7) and below), there exists $Z \in \mathscr{N}(\Omega)$ (depending on ν) such that

$$(8.19) \qquad \lim_{\varepsilon \to 0} \overline{\mathscr{J}[\phi]}(\chi_{R_{\delta, \varepsilon}(x)}) = 2 \int_{L_\delta(x)} [\mathscr{S}(\phi)](s, \nu) \, ds \qquad \forall x \in \Omega \setminus Z.$$

Using inequality (8.14) applied with $T=R_{\delta,\varepsilon}(x)$, passing to the limit as $\varepsilon\to 0$, from (8.19) we get

$$(8.20)\quad 2\int_{L_\delta(x)}[\mathscr{S}(\phi)](s,\nu)\,ds \geq 2\,[\mathscr{H}^1(C\cap\pi_1(L_\delta(x)))+\mathscr{H}^1(C\cap\pi_2(L_\delta(x)))]$$

for any $x\in\Omega\setminus Z$, where π_1 and π_2 are the canonical projection onto the coordinate axes.

For any $h\in\mathbb{R}$, let $L^h(x)$ be the part of the line parallel to $L_\delta(x)$ shifted of the factor h in the direction of ν which is contained in Ω, *i.e.*,

$$L^h(x)=\{x+tn+h\nu : t\in\mathbb{R}\}\cap\Omega,$$

and let $L^h_\delta(x)=\{x+tn+h\nu : |t|<\delta\}\subseteq L^h(x)$. By the Lebesque Differentiation Theorem, for any $h\in\mathbb{R}$ such that $L^h(x)\neq\varnothing$, there exists $M_h\in\mathscr{N}(\mathbb{R})$ such that

$$(8.21)\quad \begin{cases} \displaystyle\lim_{\delta\to 0}\frac{1}{\mathscr{H}^1(L^h_\delta(y))}\int_{L^h_\delta(y)}[\mathscr{S}(\phi)](s,\nu)\,ds=[\mathscr{S}(\phi)](y,\nu) \\ \qquad\forall\, y\in L^h(x)\setminus M_h. \end{cases}$$

Define $M=\bigcup_{h\in\mathbb{R}} M_h$. Obviously, M depends on ν. By Fubini-Tonelli's Theorem $M\in\mathscr{N}(\Omega)$ and (8.21) holds for any $x\in\Omega\setminus M$.

Using (8.21) and (8.20) we deduce that

$$(8.22)\quad \begin{cases} \displaystyle[\mathscr{S}(\phi)](x,\nu)\geq \lim_{\delta\to 0}\frac{\mathscr{H}^1(C\cap\pi_1(L_\delta(x)))+\mathscr{H}^1(C\cap\pi_2(L_\delta(x)))}{\mathscr{H}^1(L_\delta(x))} \\ \qquad\forall\, x\in\Omega\setminus K_\nu, \end{cases}$$

where $K_\nu=N\cup Z\cup M$ is a set of zero Lebesgue measure and depends on ν. If $x\notin N$ (*i.e.*, x_i has density one for C, for $i=1, 2$), using elementary trigonometric arguments, we have

$$\lim_{\delta\to 0}\frac{\mathscr{H}^1(C\cap\pi_1(L_\delta(x)))+\mathscr{H}^1(C\cap\pi_2(L_\delta(x)))}{\mathscr{H}^1(L_\delta(x))} = \lim_{\delta\to 0}\frac{\mathscr{H}^1(\pi_1(L_\delta(x)))+\mathscr{H}^1(\pi_2(L_\delta(x)))}{\mathscr{H}^1(L_\delta(x))}=|\nu_1|+|\nu_2|,$$

which, together with (8.22), gives

$$(8.23)\qquad [\mathscr{S}(\phi)](x,\nu)\geq|\nu_1|+|\nu_2|\qquad \forall\,\nu\in\mathbf{S}^1,\quad \forall\, x\in\Omega\setminus K_\nu.$$

The theorem then follows, since (8.18) is a consequence of (8.23) and the positive 1-homogeneity of $\mathscr{S}(\phi)$.

The following remark justifies the choice of the class $\mathscr{K}_\phi$ in the definition of $\int_\Omega |Du|_\phi$.

Remark 8.5. — Observe that

$$\phi^0(x, \xi) = \begin{cases} \|\xi\| & \text{if } x \in E \\ \sqrt{\Lambda^{-1}}\,\|\xi\| & \text{if } x \in C \times C. \end{cases} \tag{8.24}$$

Take a vector field σ on Ω. If σ belongs to $\mathscr{C}_0^1(\Omega; \mathbb{R}^2)$, and if $\phi^0(x, \sigma(x)) \leqq 1$ for any $x \in \Omega$, by the density of E and (8.24) it follows that $\|\sigma(x)\| \leqq 1$ for any $x \in \Omega$. We deduce

$$\sup\left\{\int_\Omega u \operatorname{div} \sigma\, dx : \sigma \in \mathscr{C}_\phi\right\} \leqq \int_\Omega |Du|,$$

where $\mathscr{C}_\phi$ is defined in (3.1).

The opposite inequality follows from (2.1), (8.24), and the inequality $0 < \sqrt{\Lambda^{-1}} < 1$. However, as ϕ is upper semicontinuous, by corollary 6.7 we have

$$\overline{\mathscr{I}[\phi]}(u) = \int_\Omega |Du|_\phi \qquad \forall u \in BV(\Omega).$$

Using (8.15), if $u \in W^{1,1}(\Omega)$ is a linear function and $u \neq 0$, we have

$$\begin{aligned} \sup\left\{\int_\Omega u \operatorname{div} \sigma\, dx : \sigma \in \mathscr{K}_\phi\right\} &= \int_\Omega |Du|_\phi = \overline{\mathscr{I}[\phi]}(u) \\ &= \mathscr{I}[\psi](u) > \int_\Omega |Du| = \sup\left\{\int_\Omega u \operatorname{div} \sigma\, dx : \sigma \in \mathscr{C}_\phi\right\}. \end{aligned}$$

This shows that it is necessary to consider discontinuous test vector fields in definition 3.1 of the generalized total variation.

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REFERENCES

[1] G. Alberti, A lusin Type Theorem for Gradients, *J. Funct. Anal.*, Vol. **100** (1), 1991, pp. 110-118.

[2] G. Anzellotti, Traces of Bounded Vectorfields and the Divergence Theorm, preprint Univ. Trento, 1983.

[3] G. Anzellotti, Pairings Between Measures and Bounded Functions and Compensated Compactness, *Ann. Mat. Pura e Appl.*, (4), Vol. **135**, 1983, pp. 293-318.

[4] A. C. BARROSO and I. FONSECA, *Anisotropic Singular Perturbations. The Vectorial Case*, Research Report n. 92-NA-015, Carnegie Mellon University, 1992.

[5] G. BOUCHITTÉ and M. VALADIER, Integral Representation of Convex Functionals on a Space of Measures, *J. Funct. Anal.*, Vol. **80**, 1988, pp. 398-420.

[6] G. BOUCHITTÉ, Singular Perturbations of Variational Problems Arising from a Two-Phase Transition Model, *Appl. Math. and Opt.*, Vol. **21**, 1990, pp. 289-315.

[7] G. BOUCHITTÉ and G. DAL MASO, *Integral Representation and Relaxation of Convex Local Functionals on* BV(Ω), preprint SISSA, April 1991, to appear on *Ann. Scuola Norm. Sup.*, Pisa.

[8] H. BREZIS, *Analyse Fonctionnelle. Théorie et Applications*, Masson, Paris, 1983.

[9] H. BUSEMANN and W. MAYER, On the Foundations of Calculus of Variations, *Trans. Amer. Math., Soc.*, Vol. **49**, 1941, pp. 173-198.

[10] H. BUSEMANN, Metric Methods in Finsler Spaces and in the Foundations of Geometry, *Ann. of Math. Studies*, Vol. **8**, Princeton Univ. Press, Princeton, 1942.

[11] G. BUTTAZZO, *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations*, Longman, Harlow, 1989.

[12] G. BUTTAZZO and G. DAL MASO, Integral Representation and Relaxation of Local Functionals, *Nonlinear Analysis, Theory, Methods & Applications*, Vol. **9**, 1985, pp. 515-532.

[13] C. CASTAING and M. VALADIER, Convex Analysis and Measurable Multifunctions, *Lect. Not. in Math.*, Vol. **580**, Springer-Verlag, Berlin, 1977.

[14] B. DACOROGNA, *Direct Methods in the Calculus of Variations*, Springer-Verlag, Berlin, 1989.

[15] G. DAL MASO, Integral Representation on BV(Ω) of Γ-Limits of Variational Integrals, *Manuscripta Math.*, Vol. **30**, 1980, pp. 387-416.

[16] G. DAL MASO, *An Introduction to* Γ-*Convergence*, preprint SISSA, 1992, Trieste, to appear on Birkhäuser.

[17] G. DAL MASO and L. MODICA, A General Theory of Variational Functionals, *in: Topics in Functional Analysis* (1980-1981), *Quaderni Scuola Norm. Sup. Pisa*, Pisa, 1980, pp. 149-221.

[18] G. DE CECCO, *Geometria sulle Varietá di Lipschitz*, preprint Univ. di Roma "La Sapienza", 1992, pp. 1-38.

[19] G. DE CECCO and G. PALMIERI, Distanza Intriseca su una Varietà Riemanniana di Lipschitz, *Rend. Sem. Mat. Univ. Pol. Torino*, Vol. **46**, 1988, pp. 157-170.

[20] G. DE CECCO and G. PALMIERI, Lenght of Curves on Lip manifolds, *Rend. Mat. Acc. Lincei*, Vol. **1**, (9), 1990, pp. 215-221.

[21] G. DE CECCO and G. PALMIERI, Integral Distance on a Lipschitz Riemannian Manifold, *Math. Zeitschrift*, Vol. **207**, 1991, pp. 223-243.

[22] E. DE GIORGI, Nuovi Teoremi Relativi alle Misure $(r-1)$-dimensionali in uno Spazio a r Dimensioni, *Ricerche Mat. IV*, 1955, pp. 95-113.

[23] E. DE GIORGI, Conversazioni di Matematica, *Quaderni Dip. Mat. Univ. Lecce*, 1988-1990, Vol. **2**, 1990.

[24] E. DE GIORGI, Su Alcuni Problemi Comuni all'Analisi e alla Geometria, *Note di Matematica, IX-Suppl.*, 1989, pp. 59-71.

[25] E. DE GIORGI, *Alcuni Problemi Variazionali della Geometria*, Conference in onore di G. Aquaro, Bari, Italy, 9 Nov. 1990, 1991, pp. 112-125, Conferenze del Sem. Mat. Bari.

[26] E. DE GIORGI, F. COLOMBINI and L. C. PICCININI, *Frontiere Orientate di Misura Minima e Questioni Collegate*, Quaderni della Classe di Scienze della Scuola Normale Superiore di Pisa, Pisa, 1972.

[27] I. EKELAND and R. TEMAM, *Convex Analysis and Variational Problems*, North-Holland, New York, 1976.

[28] H. FEDERER, Geometric Measure Theory, Springer-Verlag, Berlin, 1968.

[29] N. FUSCO and G. MOSCARIELLO, L^2-Lower Semicontinuity of Functionals of Quadratic Type, *Ann. di Mat. Pura e Appl.*, Vol. **129**, 1981, pp. 305-326.
[30] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984.
[31] C. GOFFMAN and J. SERRIN, Sublinear Functions of Measures and Variational Integrals, *Duke Math. J.*, Vol. **31**, 1964, pp. 159-178.
[32] S. LUCKHAUS and L. MODICA, The Gibbs-Thomson Relation within the Gradient Theory of Phase Transitions, *Arch. Rat. Mech. An*, Vol. **1**, 1989, pp. 71-83.
[33] U. MASSARI and M. MIRANDA, *Minimal Surfaces of Codimension One*, North-Holland, New York, 1984.
[34] V. G. MAZ'YA, *Sobolev Spaces*, Springer-Verlag, Berlin, 1985.
[35] M. MIRANDA, Superfici Cartesiane Generalizzate ed Insiemi di Perimetro Localmente Finito sui Prodotti Cartesiani, *Ann. Scuola Norm. Sup.*, Pisa, Vol. **3**, 1964, pp. 515-542.
[36] C. B. MORREY, *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, Berlin, 1966.
[37] J. NEVEU, *Bases Mathématiques du Calcul des Probabilités*, Masson, Paris, 1980.
[38] N. OWEN, Non Convex Variational Problems with General Singular Perturbations, *Trans. Am. Math. Soc.* Vol. **310**, 1988, pp. 393-404.
[39] N. OWEN and P. STERNBERG, Non Convex Variational Problems with Anisotropic Perturbations, *Nonlin. Anal.*, Vol. **16**, 1991, pp. 705-719.
[40] C. PAUC, *La Méthode Métrique en Calcul des Variations*, Hermann, Paris, 1941.
[41] YU G. RESHETNYAK, Weak Convergence of Completely Additive Vector Functions on a Set , *Siberian Math. J.*, Vol. **9**, 1968, pp. 1039-1045.
[42] W. RINOW, *Die Innere Geometrie der Metrischen Räume*, Springer-Verlag, Berlin, 1961.
[43] R. T. ROCKAFELLAR, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1972.
[44] N. TELEMAN, The Index of Signature Operators on Lipschitz Manifolds, *Inst. Hautes Études Sci. Publ. Math.*, Vol. **58**, 1983, pp. 39-78.
[45] S. VENTURINI, *Derivations of Distance Functions on* $\mathbb{R}^n$, preprint, 1993.
[46] A. I. VOL'PERT, The Space BV and Quasilinear Equation, *Math. USSR Sbornik*, Vol. **2**, 1967, pp. 225-267.

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