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A uniqueness result in the theory of stereo vision: Coupling Shape from Shading and Binocular Information allows Unambiguous Depth Reconstruction

by

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ABSTRACT. — We study here the mathematical consistency of coupling two classical methods in the theory of vision and surface reconstruction, namely the *shape from shading* theory and the theory of stereo vision. It is known that each of these approaches by itself is incomplete and leads to ill posed problems and multiple solutions, even under drastic simplifying assumptions. We show in this paper that, from a mathematical point of vue, these ambiguities disappear when both theories are cooperatively implemented. In section 2 we state our assumptions; then part 3 is devoted to the presentation of the binocular vision theory. Section 4 eventually studies, in the one and two dimensional case, how introducing the *shape from shading* tool leads to the uniqueness of the solution. In the annex (section 6), a few mathematical results are explained, and some experiments (in 1 D) are presented.

Key words : Stereo vision, Shape from shading.

RÉSUMÉ. — Nous étudions ici la consistance mathématique du couplage de deux méthodes classiques de reconstruction visuelle, à savoir la théorie

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du *shape from shading*, qui vise à retrouver le relief d'une surface à partir des ombres et variations d'intensité lumineuse de son image, et la théorie de la vision binoculaire (la « vision stéréo » au sens usuel). Chacune de ces approches est incomplète et conduit à des problèmes mal posés à solutions multiples. Dans cet article nous montrons que d'un point de vue mathématique ces inconvénients disparaissent lorsqu'on met à contribution de manière coopérative les deux méthodes.

Dans la section 2 on établit un certain nombre d'hypothèses simplificatrices, puis la partie suivante 3 est consacrée à la présentation de notre modèle de vision binoculaire. Finalement nous montrons (section 4) dans les cas uni- et bidimensionnel, comment l'introduction du *shape from shading* supprime les ambiguités. Le détail des résultats mathématiques de trouve en annexe, ainsi que la présentation de quelques résultats de simulations numériques (en dimension un).

1. INTRODUCTION

The main problem in stereo vision appears to be the correspondence problem, *i.e.* given two different images of the same scene, how can a computer match correctly an element of the first image to one of the second when they correspond to the same part of the scene? Several methods exist to try to solve this problem, which often differ by the kind of objects they are using in input: basically one can try to match the dots of equal brightness [7] eventually taking into account the fluctuations that necessarily occur between the two images [2]. Most stereo-matchers rather try to match image features that are supposed to be more stable and reliable, like edges, corners ([3], [8], [12]). But whatever method is used the outcoming information can never be complete and is even sometimes very sparse, especially for feature-based stereo-matchers. One problem is thus: how to fill-in the gaps? Generally the output of the matcher is smoothed in a way or another, which gives most of the time a close approximation of the right solution.

Recovering shape from shading is also an ambiguous problem ([4], [8]) and it is even clear that the reconstruction of shape cannot be achieved without using other sources of information. A. Pentland has shown that a linearization of the equations [9] could help to find an approximate solution, but he also quoted that this linearization had a meaning only under assumptions that one cannot expect to be true on a whole image (roughly as long as the direction of incident light is far enough from the normal to the surface there is an approximate proportionality between the intensity and the depth). It was shown [10] that actually, the points where the surface is lit frontally are generators of non-uniqueness of the solution of the shape from shading equation. It is very easy after a few simplifying assumptions to imagine two different surfaces giving the same image; this method needs thus necessarily to be coupled with other methods to give good results.

This paper deals with integration of stereo vision and shape from shading. More than one way can be imagined of integrating both methods: for example, one may think of using the information of the shading to fill in the spaces left by a matcher. But the aim of this short study, rather than to tell what the right way is, is to address the issue of the mathematical consistency of this integration, with as many simplifying assumptions as needed. Our main result is that coupling information of shape from shading and stereo yields a complete theoretical recovery of shape. Although our approach does not lead to a particular algorithm, we delimit some fundamental aspects of the issue and in this way can help and guide researchers in the development of algorithms; a few situations where no result can be expected are also discussed.

The assumptions that are usually made before trying to solve the problem of recovering shape from shading are, as pointed out by Pentland [9], of three kinds: assumptions on the surface shape, on the distribution of illumination and on the reflectance function. We will use assumptions of these kinds, and add a few more to make the stereo vision problem simpler. To be precise we'll make assumptions on the cameras, the surface, and their position with respect to each other. And most of all, we'll restrict our study to the case where the data as well as the brightness map is assumed to be continuous.

2. ASSUMPTIONS AND MAIN RESULT

We'll consider the following assumptions in order to simplify the shape from shading model:

1. The surface of the object we want to reconstruct is smooth (we will suppose it is C^1, *i.e.* continuously differentiable, and try to see in the end if this strong assumption can not be weakened a little.)

2. There are no shadows on the surface.

3. There is only one light source, at infinity (and the surface is not illuminating itself).

4. The reflectance is lambertian, which means that the intensity reflected by a point on the surface is proportional only to the cosine of the angle made by the direction of illumination and the normal vector to the surface.

The lambertian assumption is found in many studies. As pointed out by Pentland, it is likely that the human eye is making such an assumption when it tries to extract shape from shading.

The problem of stereo vision will also be simpler if we assume that:

5. We get the pictures with a set of parallel "standard stereo cameras" [11] which is described below.

6. There are no occluded zones on the images.

7. The surface can be represented by a function $Z=f(X, Y)$ where the plane (X, Y) is parallel to the focal planes of the cameras and Z remains bounded.

Note that the assumptions made on the surface and its properties correspond roughly to the description of a (not too bumpy) plaster bas-relief.

A set of parallel standard stereo cameras consists of two ideal cameras, in which the image is obtained by a simple projection through a pointwise lens, whose coördinate systems are parallel to each other and whose *focal distance* f, *i.e.* the distance between the lens and the plane of the image, is the same. This model simplifies calculations without changing the results of uniqueness, as "images of any real stereo cameras can be transformed to those of the standard stereo cameras if the parameters of these cameras are known in advance." [11]

If lens #1 and #2 are respectively in $(A/2, B, C)$ and $(-A/2, B, C)$ (*baseline length* thus is A), then a point in the scene (X, Y, Z), with $Z \leqq C$, appears on the images #1 and #2 (represented by a rectangle $R \subset \mathbb{R}$) in resp. (x_1, y) and (x_2, y) where:

$$x_1 = f\frac{X-A/2}{C-Z}, \qquad x_2 = f\frac{X+A/2}{C-Z}, \qquad y = f\frac{Y-B}{C-Z}. \tag{8}$$

Here the *epipolar lines* – intersections of the images with a plane running through both lenses – are simply the lines $y=\text{Const}$. We will have to do one last assumption useful for treating both problems:

9. The brightness that is observed in one dot (x_i, y), $i=1, 2$ of the image is exactly the light intensity emited by the point (X, Y, Z) of the surface it corresponds to (this implies that we have perfect noise-free images.)

RESULT. – Under assumptions (1-9), if the direction of illumination is not too close to the direction (X, Y) of the focal planes, and if the following boundary condition holds: the disparity is already known on the boundary of a certain set of dots of image #1 that appear also on image #2; then it should be possible to reconstruct the surface corresponding to this set without ambiguity.

3. STEREO VISION

We begin with a short study of the stereo vision problem. Assumptions (4, 5, 9) allow us to write when $I_i : R \to [0, 1]$ is the intensity map describing image $\# i$, $i = 1, 2$:

$$I_1(x_1, y) = I_2(x_2, y) \tag{10}$$

each time that (x_1, y) and (x_2, y) are corresponding points, linked by relations (8). Substracting x_2 and x_1 we find that the disparity function u is directly related to Z:

$$u = f \frac{A}{C - Z} \tag{11}$$

Let's call Ω_1 the set of all the points of the first image that appear also on the second one, we'll assume that it is a connected set. The problem is the following:

We're looking for $u(x_1, y) : \Omega_1 \to \mathbb{R}$ verifying:

$$\forall (x_1, y) \in \Omega_1, \quad I_1(x_1, y) = I_2(x_1 + u(x_1, y), y). \tag{12}$$

As this issue can be seen as a family of one-dimensional problems indexed by y it is interesting first to study the following one-dimensional issue, where:

$\Omega_1 \subset R \subset \mathbb{R}$, and we are looking for $u(x_1) : \Omega_1 \to \mathbb{R}$ with:

$$\forall x_1 \in \Omega_1, \quad I_1(x_1) = I_2(x_1 + u(x_1)). \tag{13}$$

Assumption (6) ensures that $\phi(x) = x + u(x)$ is a continuous strictly increasing function, thus an increasing homeomorphism mapping Ω_1 onto $\Omega_2 = \phi(\Omega_1)$, and we can rewrite formula (13) $I_{1|\Omega_1} = I_2 \circ \phi$. Let us set:

$$A = \{ t \in \Omega_1 \mid I_1 \text{ is constant in a neighborhood of } t \}. \tag{14}$$

We have the following theorem (shown in appendix A):

THEOREM 1. — *Assume* I_1 (*or in an equivalent way* I_2) *has a left and right limit at each* $x \in \Omega_1$ $(x \in \Omega_2) - \Omega_1$ *and* Ω_2 *being open intervals of* $\mathbb{R}$ — *and* ψ *is another increasing homeomorphism mapping* Ω_1 *onto* $\Omega_2' = \psi(\Omega_1) \subset R$ *with* $I_{1|\Omega_1} = I_2 \circ \psi$, *assume also that:*

$$\exists\, t_0 \in \bar{\Omega}_1, \quad \psi(t_0) = \phi(t_0)$$

(*when* $t_0 \notin \Omega_1$, *we consider the limits of* ψ *and* ϕ *at* t_0) *then:*

$$\Omega_2 \setminus B = \Omega_2' \setminus B, \qquad \text{and} \qquad \forall x \in \Omega_1 \setminus A, \quad \psi(x) = \phi(x);$$

where $B = \phi(A)$.

This means that equation (13) associated with the knowlege of Ω_1 and the value of u at one point $t_0 \in \bar{\Omega}_1$ is theoretically sufficient for recovering the disparity *everywhere where* I_1 *is not locally constant.*

We could not expect a better general result as it is easy to imagine situations leading to a wrong solution and to mismatches if we don't know which points can be matched and the disparity at one of those. Think for example of periodic surfaces as corrugated iron or rows of pearls. Note also that relation (13) gives no information on the value of u in A.

The problem can be written as follows:

$$\left.\begin{array}{l} \text{let } E=\{v:\Omega_1 \to \mathbb{R} \mid x+v(x) \text{ is continuous,} \\ \qquad\qquad\qquad \text{strictly increasing from } \Omega_1 \text{ to } R\} \\ \qquad \text{and } \mathcal{F}(v)=\displaystyle\int_{\Omega_1} |I_1(x)-I_2(x+v(x))|^2\,dx, \qquad \forall v \in E \end{array}\right\} \quad (15)$$

then if v minimizes $\mathcal{F}$ on $\{v \in E,\ \exists\, t_0 \in \Omega_1\ v(t_0)=u(t_0)\}$ we have $v=u$ on $\Omega_1 \setminus A$. This formulation is for instance the one adopted by R. March [7].

4. SHAPE FROM SHADING

Let us now study the shape from shading problem, and first return to the bi-dimensional case. Once Ω_1 and the disparity u are known, the part of the surface corresponding to the set Ω_1 can be recovered simply by inverting formulas (8). We get a parametrized surface:

$$\left.\begin{aligned} X(x_1, y) &= A/2 + \frac{A}{u(x_1, y)}\, x_1 \\ Y(x_1, y) &= B + \frac{A}{u(x_1, y)}\, y \\ Z(x_1, y) &= C - \frac{A}{u(x_1, y)}\, f \end{aligned}\right\} \quad (16)$$

and with assumption (1) one can easily compute a normal vector to the surface:

$$\frac{\partial}{\partial x_1}(X, Y, Z) \times \frac{\partial}{\partial y}(X, Y, Z) = \left(\frac{A}{u}\right)^2 \begin{pmatrix} -f\dfrac{1}{u}\dfrac{\partial u}{\partial x_1} \\ -f\dfrac{1}{u}\dfrac{\partial u}{\partial y} \\ 1-\dfrac{1}{u}\left(x_1\dfrac{\partial u}{\partial x_1}+y\dfrac{\partial u}{\partial y}\right) \end{pmatrix} \quad (17)$$

We'll write $v = \log|u|$ and $\vec{x} = (x, y) \in \Omega_1$ and take as a normal vector to the surface $\vec{n} = (f\,\partial v/\partial x, f\,\partial v/\partial y, \vec{x}.\vec{\nabla} v - 1)$ where the dot denotes the scalar product in $\mathbb{R}^N$, here $\mathbb{R}^2$, and $\vec{\nabla} v = (\partial v/\partial x, \partial v/\partial y) \in \mathbb{R}^2$. We actually need a normal vector pointing towards the inside of the illuminated object (*i.e.* from top to bottom) and we can show that it is the case of this one [the simplest way of seeing this is to notice that it is continuous and points towards the bottom at $(x, y) = (0, 0)$].

Now if $\vec{\mathrm{I}}_0 = (\alpha, \beta, \gamma)$, where $\alpha^2 + \beta^2 + \gamma^2 = 1$ and $\gamma < 0$, is the direction of the incident light (we use here assumption (3)) the equation for the shape from shading problem turns to be, using assumptions (1, 2, 3, 4):

$$\mathrm{I}_1(\vec{x}) = \vec{\mathrm{I}}_0 . \frac{\vec{n}(\vec{x})}{|\vec{n}(\vec{x})|} \tag{18}$$

where $|\vec{a}| = \sqrt{\vec{a}.\vec{a}}$. This equation can also be written:

$$\mathrm{I}_1(\vec{x})\sqrt{f^2|\vec{\nabla} v|^2 + (\vec{x}.\vec{\nabla} v - 1)^2} - f(\alpha, \beta).\vec{\nabla} v - \gamma(\vec{x}.\vec{\nabla} v - 1) = 0. \tag{19}$$

This is an Hamilton-Jacobi-Belman equation and, as it is convex with respect to $\vec{\nabla} v$, it can be seen as the equation of an optimal control problem which can be studied using dynamical programming techniques ([1], [5], [6], [10]).

Now again we'll begin by restricting ourselves to the one-dimensional case even though it has less signification than in the previous section:

In this case the normal vector to the surface, which is now a line, is $(fv', xv' - 1)$ where $v = \log|u(x)|$ and $v'(x) = dv/dx(x)$. As above Ω_1 and R are now segments of $\mathbb{R}$ and we consider the formulation (18):

$$\mathrm{I}_1(x) = \vec{\mathrm{I}}_0 . \frac{(fv', xv' - 1)}{|(fv', xv' - 1)|} = \vec{\mathrm{I}}_0 . \vec{\nu} \tag{20}$$

where now $\vec{\mathrm{I}}_0 = (\alpha, \gamma)$, $\alpha^2 + \gamma^2 = 1$, $\gamma < 0$ and $\vec{\nu} = \vec{n}/|\vec{n}| = (\nu_1, \nu_2)$.

With assumption (7) we need to have $\nu_2 < 0$ and thus $\nu_2 = -\sqrt{1 - \nu_1^2}$, and (20) implies:

$$\vec{\nu} = (\nu_1, \nu_2) = (\alpha \mathrm{I}_1 \pm \gamma\sqrt{1 - \mathrm{I}_1^2}, -\sqrt{1 - \nu_1^2}) \tag{21}$$

so that there is a choice between two possible values of $\vec{\nu}$; expect when $\mathrm{I}_1 = 1$. The C^1 solutions of (20) form therefore a set of different curves, with bifurcations from one to another at each point x where $\mathrm{I}_1(x) = 1$. We have to find the right solution among all those curves.

The answer is given by equation (13): as shown in the last section this equation under the assumptions of theorem 1 makes u uniquely determined except on the subset $\mathrm{A} \subset \Omega_1$. Equation (21) allows then to find u on A (or almost). Actually A is the union of open segments $\mathrm{K} =]x^0, x^1[$ (its connected components), on which I_1 is constant and (13) guarantees the uniqueness of $u(x^0)$ and $u(x^1)$, while (21) gives two possible values of

$u'(x)/u(x)=v'(x)$ on K:

$$v'^{+}(x)=\frac{\nu_1^+}{f\sqrt{1-(\nu_1^+)^2}+x\,\nu_1^+}$$

and

$$v'^{-}(x)=\frac{\nu_1^-}{f\sqrt{1-(\nu_1^-)^2}+x\,\nu_1^-}$$

where

$$\begin{cases}\nu_1^+=\alpha\,\mathrm{I}_1+\gamma\sqrt{1-\mathrm{I}_1^2}\\ \nu_1^-=\alpha\,\mathrm{I}_1-\gamma\sqrt{1-\mathrm{I}_1^2}\end{cases}$$

One can easily notice that

$$\begin{aligned}\text{either: }&(\forall x\in\mathrm{K},\ v'^{+}(x)\neq v'^{-}(x))\\ \text{or: }&(\forall x\in\mathrm{K},\ v'^{+}(x)=v'^{-}(x))\end{aligned}$$

and in the first case only one of these two values of v' can be compatible with the already known values $u(x^0)$ and $u(x^1)$.

In fact we have to assume a stronger boundary condition than in theorem 1 to really guaranty the uniqueness of the whole set Ω_1 of the solution of (13) and (20), as, in a few situations, there is a slight error in the above proof: for instance if the lower boundary of Ω_1 coincides with the one of A, there is a segment $]x^0, x^1[\subset \mathrm{A}$ with $x^0=\inf(\mathrm{A})=\inf(\Omega_1)$ in which $u(x^1)$ is uniquely determined by (13) but not $u(x^0)$. Therefore to ensure that u can be reconstructed *in any case* without ambiguity on the whole set Ω_1 we have to know, as a boundary condition, its value on $\partial\Omega_1=\{\inf(\Omega_1), \sup(\Omega_1)\}$.

The bi-dimensional problem is the following: $\Omega=\operatorname{int}(\Omega_1)$ is a bounded connected open set in $\mathbb{R}^2$ and we'll use as a boundary condition the (assumed known) value of u on $\partial\Omega$; now, u representing here the actual – unknown – disparity function, we would like $\tilde{u}=u$ to be the unique solution of: (with $\tilde{v}=\log|\tilde{u}|$)

$$\left.\begin{aligned}&\tilde{u}_{|\partial\Omega}=u_{|\partial\Omega};\ \text{and}\ \forall\vec{x}=(x, y)\in\Omega:\\ &\qquad \mathrm{I}_1(\vec{x})=\mathrm{I}_2(x+\tilde{u}(\vec{x}), y)\\ &\mathrm{I}_1(\vec{x})\sqrt{f^2|\vec{\nabla}\tilde{v}|^2+(\vec{x}.\vec{\nabla}\tilde{v}-1)^2}-f(\alpha, \beta).\vec{\nabla}\tilde{v}-\gamma(\vec{x}.\vec{\nabla}\tilde{v}-1)=0\end{aligned}\right\}\quad(22)$$

where $\tilde{u}\in\mathrm{C}^0(\bar{\Omega}, \mathbb{R}\setminus\{0\})\cap\mathrm{C}^1(\Omega)$ and $x+\tilde{u}(x, y)$ is for all y a strictly increasing function of x.

It is also natural to assume that:

$$\forall y,\quad \{x\in\mathrm{R}\,|\,(x, y)\in\Omega\}\ \text{is a segment of}\ \mathbb{R},$$

and this time we define two sets A and B:

$$\mathrm{A}=\{(x, y)\in\Omega\,|\,\mathrm{I}_1(t, y)=\mathrm{I}_1(x, y)\ \text{for all}\ t\ \text{in a neighborhood of}\ x\}\quad(23)$$

$$\mathrm{B}=\{(x, y)\in\mathrm{A}\,|\,\mathrm{I}_1(x, y)<1\}\quad(24)$$

We now state the following theorem:

THEOREM 2. — *If* $\vec{I}_0$, *direction of the light illuminating the scene, does not have a too low angle of incidence*, i. e. *satisfies the condition*

$$\forall (x, y) \in \Omega, \quad \alpha x + \beta y - \gamma f > 0 \tag{25}$$

then problem (22) *has a unique solution* $\tilde{u} = u$ *on* $\bar{\Omega}$.

Condition (25) means that the angle between $\vec{I}_0$ and $(x, y, -f)$, which is the direction of the ray connecting the point (x, y) on the camera screen Ω_1 to the corresponding real point (X, Y, Z), is always smaller than $\pi/2$.

From the first section – using stereo vision – we know that $\tilde{u}$ is known on $\Omega \setminus A$ and equals the true disparity u. By continuity it is also true on $\partial A = (\partial (\Omega \setminus A) \cap \Omega) \cup (\partial A \cap \partial \Omega)$. $A \setminus B$ is made of segments $]x^0, x^1[\times \{y\}$ with $\tilde{u}(x^0, y) = u(x^0, y)$, $\tilde{u}(x^1, y) = u(x^1, y)$ and $I_1(x, y) = 1$, $\forall x \in [x^0, x^1]$. As $I_1 = 1$ the equation (19) has a unique solution

$$\frac{\vec{\nabla} \tilde{u}}{\tilde{u}} = \vec{\nabla} \tilde{v} = \vec{\nabla} v = \frac{(\alpha, \beta)}{\alpha x + \beta y - \gamma f},$$

thus $\tilde{u} = u$ on $]x^0, x^1[\times \{y\}$. Then $\tilde{u} = u$ on $\Omega \setminus A$ and on $A \setminus B$, *i.e.* on $\Omega \setminus B$ and thus on $\overline{\Omega \setminus B}$, and we just need to prove that $\tilde{u} = u$ inside int (B) which leads to the following problem:

$$\left.\begin{array}{r} H(\vec{x}, \vec{\nabla} \tilde{v}) = 0 \text{ in int(B) bounded open set of } \mathbb{R}^2 \\ \tilde{v} = v = \log |u| \text{ on } \partial B \end{array}\right\} \tag{26}$$

where

$$H(\vec{x}, \vec{p}) = I_1(\vec{x}) \sqrt{f^2 |\vec{p}|^2 + (\vec{x}.\vec{p} - 1)^2} - f(\alpha, \beta).\vec{p} - \gamma(\vec{x}.\vec{p} - 1) u \tag{27}$$

and where:

$$\forall \vec{x} \in \text{int (B)}, \ I_1(\vec{x}) < 1 \tag{28}$$

and this problem has been shown to have a unique solution, provided (25) is true (*see* appendix B).

5. CONCLUSION

We have shown the theoretical possibility of recovering a surface such as a plaster bas-relief without ambiguity from two stereo images of the scene. The numerical resolution of this problem remains difficult, but some lessons can be drawn:

• At least in the zones where one wants to use shading information it is necessary to make an assumption of smoothness of the disparity, and thus of the surface. The above uniqueness results remain true if one just assumes $u \in C^0(\bar{\Omega}, \mathbb{R} \setminus \{0\}) \cap C^1(\text{int}(A))$ and consider equation (19) only where u is smooth. If $u \notin C^1(\mathrm{int}(A))$ it may not be possible to find the right solution: we can find C^1 solutions that won't match the boundary conditions, and there may be an infinity of non-C^1 solutions matching these conditions.

• We always used important boundary conditions to get our results, and these conditions were often necessary. (In the two-dimensional problem one can still obtain good results by using as an initial condition instead of the value of the disparity on $\partial\Omega$ its value at just one point in each epipolar line, it is even likely that with the strong continuity assumption we made knowing its value at one point should be enough in most cases). Anyway most stereo matchers won't need these conditions to work, except on very particular periodic images, so they are not a real problem for the implementation.

Two kinds of algorithms based on these ideas can be imagined:

• We can try to solve the shape from shading problem using stereo vision to select among all the solutions the right one: this has been implemented in 1 D on synthetic images and happens to give good results, that are presented in appendix C; however its 2 D implementation is a lot more difficult.

• In a completely different way we can, as it was suggested in the introduction, try to build a stereo matcher using the shape from shading information to fill in the gaps left between the matches, in a more or less integrated way. And probably, as it is seen in the last section, the ideal matcher here would match among other features the maxima of the brightness, which are the points generating "trouble" in the shape from shading problem.

APPENDIX A

PROOF OF THEOREM 1. — Assume first that $\Omega_2' = \Omega_2$. We have $I_2 \circ \psi = I_2 \circ \phi = I_1$ on Ω_1, *i.e.* $I_2 \circ \psi \circ \phi^{-1} = I_2$ on Ω_2. Let $w = \psi \circ \phi^{-1}$, w is an increasing homeomorphism mapping Ω_2 on itself with, if $x_0 = \phi(t_0) = \psi(t_0)$, $w(x_0) = x_0$. We need to show that $\forall x \in \Omega_2 \setminus B$, $w(x) = x$ where:

$$B = \phi(A) = \psi(A) = \{ t \in \Omega_2,\ I_2 \text{ is constant in a neighborhood of } t \}.$$

Consider any $x \in \Omega_2$ with $w(x) \neq x$. We must prove that $x \in B$.

If, for instance, $x_0 < x$, then:

$$\text{either:}\quad x_0 = w(x_0) < w(x) < x,$$
$$\text{or:}\quad x_0 = w^{-1}(x_0) < w^{-1}(x) < x.$$

In the first case by induction we find that

$$\forall n \in \mathbb{N},\quad x_0 < w^{n+1}(x) < w^n(x) < x$$

and therefore:

$$\exists\, x^{\infty} = \lim_{n \to \infty} w^{n}(x) \in [x_0, x[\subset \Omega_2$$

and we have $w(x^{\infty}) = x^{\infty}$ as w is continuous. Let $I = I_2(x) = I_2(w(x))$, we have $I = I_2(w^n(x))$, $\forall n \in \mathbb{N}$, and as I_2 has a right limit at x^{∞}, $I = \lim_{n \to \infty} I_2(w^n(x)) = I_2(x^{\infty} + 0)$.

Now if $t \in]x^{\infty}, w^{-1}(x)[$ we have

$$x^{\infty} = w^n(x^{\infty}) < w^n(t) < w^{n-1}(x), \quad \forall n \in \mathbb{N}$$

thus $\lim_{n \to \infty} w^n(t) = x^{\infty}$ and therefore:

$$\lim_{n \to \infty} I_2(w^n(t)) = I_2(x^{\infty} + 0) = I$$

which leads, as $\forall n \in \mathbb{N}$, $I_2(t) = I_2(w^n(t))$, to:

$$I_2(t) = I,$$

We thus proved that $I_2 = I = \text{Const.}$ on $]x^{\infty}, w^{-1}(x)[\ni x$, therefore $x \in B$. The second case is not different from this one: we just have to replace w by w^{-1}, and the proof is the same.

Now if we drop the assumption $\Omega_2' = \Omega_2$, theorem 1 remains true. For instance of the right of $x_0 = \phi(t_0) = \psi(t_0) \in \Omega_2 \cap \Omega_2'$ we may have:

$$\Omega_2' \cap [\psi(t_0), +\infty[= [x_0, r[$$

and

$$\Omega_2 \cap [\phi(t_0), +\infty[= [x_0, s[$$

with $r < s \leqq +\infty$, $w = \psi \circ \phi^{-1}$ defines an increasing homeomorphism mapping $[x_0, s[$ on $[x_0, r[$; we have $w(r) < w(s) = r < s$ and can show (just the same way as above), if $x^{\infty} = \lim_{n \to \infty} w^n(r)$, that I_2 is constant on $]x^{\infty}, s[\ni r$ and therefore I_1 is constant on $]t^{\infty}, \sup(\Omega_1)[$ where $t^{\infty} = \psi^{-1}(x^{\infty}) = \phi^{-1}(x^{\infty})$ (because $w(x^{\infty}) = x^{\infty}$ and $\sup(\Omega_1) = \psi^{-1}(r) = \phi^{-1}(s)$). Therefore $]t^{\infty}, \sup(\Omega_1)[\subset A$ and, using the above demonstration after having replaced Ω_1 with $[t_0, t^{\infty}[$ and Ω_2, Ω_2' with $[x_0, x^{\infty}[$, we find that $\psi = \phi$ on $(\Omega_1 \setminus A) \cap [t_0, +\infty[$, and we obviously deal the same way on the left of t_0.

APPENDIX B

Elisabeth Rouy and Agnès Tourin have studied the shape from shading equation using techniques developed for optimal control, such as viscosity solutions [5]. We are using here the following theorem [1]:

THEOREM 3. — *Let* Ω *be a bounded connected open set in* $\mathbb{R}^N$ *and*

$$H(\vec{x}, \vec{p}): \quad \Omega \times \mathbb{R}^N \to \mathbb{R}$$

satisfy the following properties:

1. H *is continuous with respect to* $\vec{x}$ *and* $\vec{p}$.
2. H *is convex with respect to* $\vec{p}$.
3. $\exists\, \omega : \mathbb{R}_+ \to \mathbb{R}_+$, *continuous and equal to* 0 *in* 0, *such that:*

$$\forall \vec{x}, \vec{y} \in \Omega, \quad \forall \vec{p} \in \mathbb{R}^N, \qquad |H(\vec{x}, \vec{p}) - H(\vec{y}, \vec{p})| \leqq \omega(|\vec{x}-\vec{y}|(1+|\vec{p}|))$$

4. $\exists\, \bar{u} \in C^0(\bar{\Omega}) \cap C^1(\Omega)$,

$$\forall \vec{x} \in \Omega, \quad H(\vec{x}, \vec{\nabla} \bar{u}) < 0.$$

($\bar{u}$ *is a strict sub-solution of the following problem.*)

Then problem

$$\begin{cases} H(\vec{x}, \vec{\nabla} u) = 0 \\ u = u^0 \quad \text{on } \partial\Omega \end{cases}$$

has at most one viscosity solution $u \in C^0(\bar{\Omega})$

There is no use going into the details and define what a viscosity solution is, it is just enough to point out that any classical C^1-solution will be a viscosity solution of the problem. We have to assume that I_1 is lipschitz-continuous to have property number 3 satisfied. This is true with all the assumptions we made in the first section. The convexity of H with respect to $\vec{p}$ is easy to check, and finally conditions (25) and (28) guarantee the existence of a simple strict sub-solution:

$$\bar{v} = \log(\bar{u}) = \log(|\alpha x + \beta y - \gamma f|).$$

APPENDIX C

EXPERIMENTS. — We show here some results of a method of solving in 1 D the shape from shading problem on a pair of signals, using the stereo information to select the right solution. We implemented three very short programs:

1. The first one builds the pair of synthetic stereo 1 D-images. The output of this program is a pair of signals, whose integer values range from 0 (zero intensity) to 255 (maximum intensity). Its input are: a signal representing the shape (the "object"), the direction of illumination, and the position of the cameras (their common height and their abscissas) — the user can also choose their focal length and the width of their screens. We must make sure that the viewed object is entirely seen by both screens, in

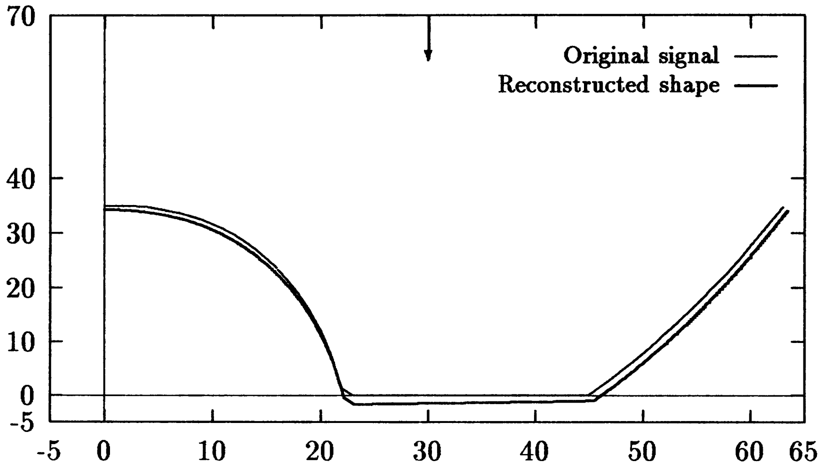

FIG. 1. – A 1 D-shape and its reconstruction.

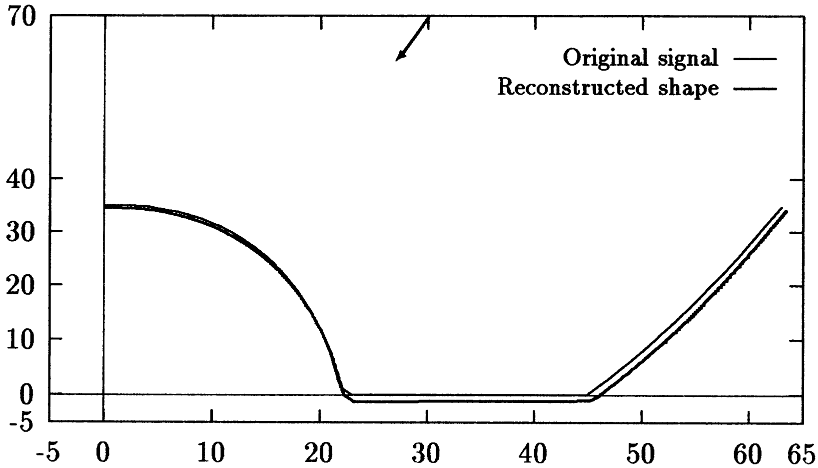

FIG. 2. – The same shape under different illumination conditions.

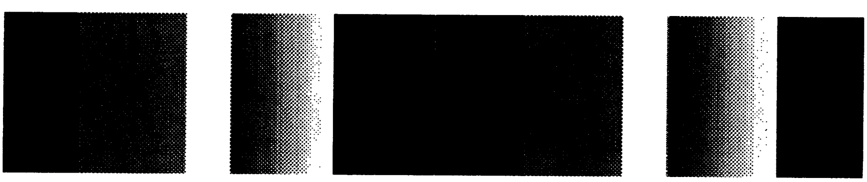

FIG. 3. – The images (vertical illumination).

FIG. 4. – The images corresponding to Figure 2.

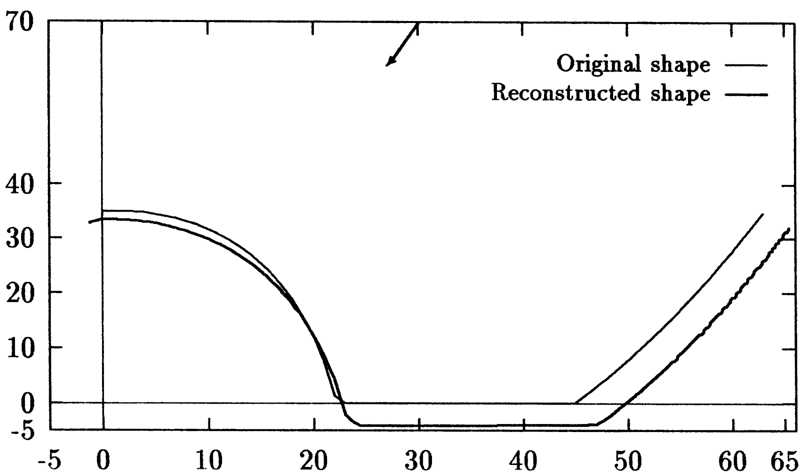

FIG. 5. – Same conditions as in Figure 2 but with a 256 pixels wide image.

such a way that it is easy to find the two sets Ω_1 and Ω_2 defined in section 3, and thus to get boundary conditions for the disparity. But the method should still work if one end of the object disappears from one or both images, as having just one boundary condition, that can be measured on any end of the object, is enough to recover the disparity and the shape.

2. The second program finds the disparity, with the method described by formulas (20), (21) and following. It needs as input the two images, the direction of the light and the focal length of the cameras. The program builds the two possible solutions, and selects one each time there is a possible bifurcation, *i.e.* when the intensity reaches a maximum. To this purpose, two energies are computed [of the kind of formula (15)], one for each solution, and the solution with lower energy is kept.

3. The last program actually rebuilds the original shape from the disparity computed by program #2. It needs to know the baseline length, *i.e.* the difference of the abscissas of the two cameras, and the focal length.

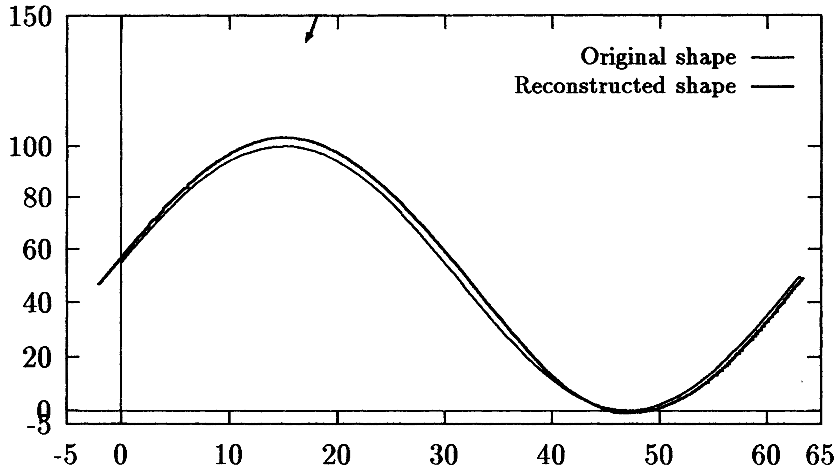

FIG. 6. – An example. Cameras are at (16, 150) and (20, 150).

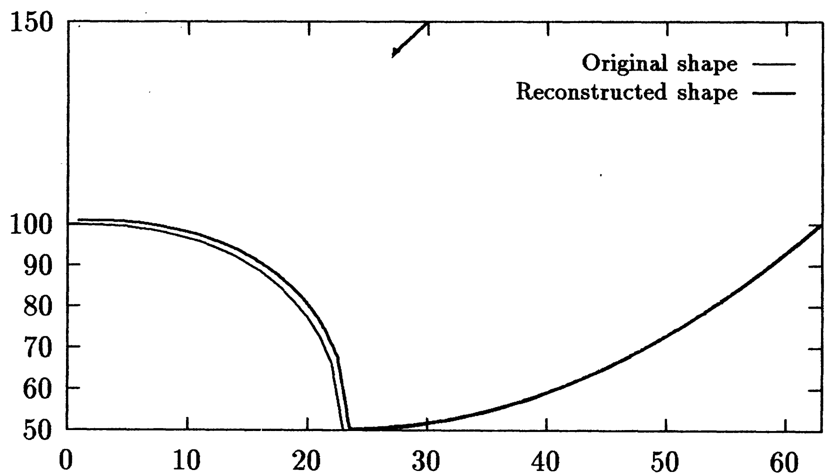

FIG. 7. – Here cameras are at (28, 150) and (32, 150).

The absolute position of the cameras is also needed if we want to translate the reconstructed shape to its original absolute position, in order to compare it to the original data.

On Figures 1, 2 we plotted a signal and the corresponding reconstructed shape in two different situations: with a vertical illumination, and with an illumination making an angle of 20 degrees with the vertical direction (the direction of illumination is indicated by a small vector at the top of each figure). In both Figures, the cameras were placed at positions (28, 70) and (32, 70), and their focal length was 10 – while the x abscissa on the screen ranges from -15 to $+15$. Figures 3 and 4 show the images viewed by the left and right cameras, with the direction of illumination shown respectively in Figure 1 and in Figure 2. (These images are reversed with respect to the original shape by the cameras). The results are good. The error is mainly due to the fact that the initial value for the disparity (measured, in our program, on the left of the images, which corresponds to the right end of the original shape), is necessarily rounded off by the fact our 1 D-"images" are discrete signals. Here they were 1,024 pixels wide, which is a lot and gives a good precision. We also made experiments with smaller images (256 pixels wide, *Fig.* 5): as expected, the initial value is less accurate and a bigger error propagates through the solution. However the right solution is still found, and this method appears to be quite robust in many cases. Other examples are given in Figures 6 and 7.

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