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DAO-MIN CAO

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Multiple solutions of a semilinear elliptic equation in $\mathbb{R}^N$

by

Dao-Min CAO

Wuhan Institute of Mathematical Sciences,
Academia Sinica,
P.O. Box 71007, Wuhan 430071,
P. R. China

ABSTRACT. – In this paper, we are concerned with the existence of multiple solutions of

$$-\Delta u+u=\lambda b(x)|u|^{p-1}u+c(x)|u|^{q-1}u$$

where $1<p,\ q<\dfrac{N+2}{N-2}$ if $N\geqq 3$, $1<p,\ q<+\infty$ if $N=2$, $\lambda>0$.

We obtain the existence of multiple solutions by using concentrations-compactness method and dual variational principle to establish the corresponding existence of critical points.

Key words : Semilinear elliptic equations, variation, critical point, concentration-compactness.

RÉSUMÉ. – Nous obtenons dans cet article un résultat d'existence et de multiplicité de solutions de

$$-\Delta u+u=\lambda b(x)|u|^{p-1}u+c(x)|u|^{q-1}u$$

où $1<p,\ q<\dfrac{N+2}{N-2}$, $N\geqq 3$, $1<p,\ q<+\infty$ si $N=2$, $\lambda>0$.

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Ces résultats sont prouvés à l'aide de la méthode de concentration-compacité et de principes variationnels duaux pour obtenir l'existence des points critiques correspondants.

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1. INTRODUCTION

We consider the existence of multiple solutions of the following semi-linear elliptic equation

$$(1.1)\qquad \begin{cases} -\Delta u+u=\lambda b(x)|u|^{p-1}u+c(x)|u|^{q-1}u & \text{in } \mathbb{R}^N \\ u\in H^1(\mathbb{R}^N) \end{cases}$$

where $1<p,\ q<\dfrac{N+2}{N-2}$ if $N\geqq 3$, $1<p,\ q<+\infty$ if $N=2$, $\lambda>0$ is a real number, $b(x)$ and $c(x)$ satisfy

$$(1.2)\qquad \begin{cases} b(x)\in C(\mathbb{R}^N), \quad b(x)\geqq 0 \quad \text{in } \mathbb{R}^N, \\ b(x) \xrightarrow[|x|\to\infty]{} b_\infty>0, \end{cases}$$

$$(1.3)\qquad \begin{cases} c(x)\in C(\mathbb{R}^N), \quad c(x)\geqq 0 \quad \text{in } \mathbb{R}^N, \\ c(x) \xrightarrow[|x|\to\infty]{} 0. \end{cases}$$

Existence of nontrivial solutions (positive solutions, for example) concerning (1.1) has been extensively studied even for more general nonlinearity—*see*, for instance, W. Strauss [12], H. Berestycki and P. L. Lions [4], W. Y. Ding and W. M. Ni [5], P. L. Lions [9], [10], A. Bahri and P. L. Lions [2] and the references therein. For the multiplicity of solutions we refer to H. Berestycki and P. L. Lions [4], X. P. Zhu [13] and Y. Y. Li [8].

It is known to some extent that the equation

$$(1.4)\qquad -\Delta u+u=c(x)|u|^{q-1}u \quad \text{in } \mathbb{R}^N$$

may have infinitely many solutions because (1.3) ensures that the corresponding variational functional

$$(1.5)\qquad I^*(u)=\frac{1}{2}\int|\nabla u|^2+u^2-\frac{1}{q+1}\int c(x)|u|^{q+1}$$

satisfies the (PS) (Palais-Smale) condition and the dual variational principle of A. Ambrosetti and P. Rabinowitz [1] may be applied. When λ is small, (1.1) can be taken as a small perturbation of (1.4) and thus it seems reasonable to hope that (1.1) has more and more solutions as λ tends to 0.

As mentioned in P. L. Lions ([9], [10]) that the variational functional corresponding to (1.1) defined by

$$(1.6) \qquad I_\lambda(u)=\frac{1}{2}\int |\nabla u|^2+u^2-\frac{\lambda}{p+1}\int b(x)\,|u|^{p+1}-\frac{1}{q+1}\int c(x)\,|u|^{q+1}$$

fails to satisfy the (PS) condition because of the lack of compactness of the Sobolev embedding $H^1(\mathbb{R}^N) \subsetneq L^2(\mathbb{R}^N)$.

Such a failure creates difficulties for the application of standard variational techniques. In section 2, arguing as P. L. Lions [10], we show by using the concentration-compactness principle that $I_\lambda(u)$ satisfies $(PS)_c$ condition if c belongs to an interval depending on λ which becomes large as λ tends to 0. In section 3, using a variant of the dual variational principle (dealing with unbounded even functionals) of A. Ambrosetti and P. Rabinowitz [1] we obtain the existence of multiple solutions by establishing the corresponding existence of critical points of $I_\lambda(u)$ with critical values in the interval in which $I_\lambda(u)$ satisfies $(PS)_c$ condition.

We conclude this introduction by remarking that some more general nonlinearities can be considered and similar existence results can be obtained by the arguments in this paper.

2. EXISTENCE OF A POSITIVE SOLUTION

In this section, we are concerned with the existence of a positive solution of (1.1). As preparations and for the discussion of next section, we first give some notations, definitions and auxiliary results.

Define

$$(2.1) \qquad M_\lambda=\{u\in H^1(\mathbb{R}^N)\,|\,u\not\equiv 0,\ I'_\lambda(u)u=0\}$$

$$(2.2) \qquad M_\lambda^\infty=\{u\in H^1(\mathbb{R}^N)\,|\,u\not\equiv 0,\ I_\lambda^{\infty\prime}(u)u=0\}$$

where $I_\lambda(u)$ is defined by (1.6), $I_\lambda^\infty(u)$ is defined by

$$(2.3) \qquad I_\lambda^\infty(u)=\frac{1}{2}\int |\nabla u|^2+u^2-\frac{\lambda}{p+1}\int b_\infty\,|u|^{p+1}$$

Let

$$(2.4) \qquad I_\lambda=\inf\{I_\lambda(u)\,|\,u\in M_\lambda\}$$

$$(2.5) \qquad I_\lambda^\infty=\inf\{I_\lambda^\infty(u)\,|\,u\in M_\lambda^\infty\}$$

$$(2.6)\quad I^* = \begin{cases} +\infty & \text{if} \quad c(x) \equiv 0 \quad \text{in } \mathbb{R}^N \\ \inf\{I^*(u) \mid u \in H^1(\mathbb{R}^N) \setminus \{0\},\ I^{*\prime}(u)u = 0\} & \text{if} \quad c(x) \not\equiv 0 \end{cases}$$

$$(2.7)\qquad S = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int |\nabla u|^2 + u^2}{\left(\int |u|^{p+1}\right)^{2/(p+1)}}.$$

We have

PROPOSITION 2.1. – *For each* $\lambda > 0$, $I_\lambda \leqq I^*$.

Proof. – If $c(x) \equiv 0$, then $I^* = +\infty$, thus $I_\lambda \leqq I^*$. In what follows, we assume $c(x) \not\equiv 0$.

Suppose $u \in H^1(\mathbb{R}^N)$, $u \not\equiv 0$ such that

$$(2.8)\qquad \int |\nabla u|^2 + u^2 = \int c(x) |u|^{q+1}.$$

Let $v = \bar{\sigma} u$ such that $v \in M_\lambda$, *i. e.*,

$$(2.9)\qquad \int |\nabla u|^2 + u^2 = \bar{\sigma}^{p-1} \int \lambda b(x) |u|^{p+1} + \bar{\sigma}^{q-1} \int c(x) |u|^{q+1}$$

Comparing (2.8) and (2.9) we deduce that such $\bar{\sigma}$ exists and $\bar{\sigma} \in (0, 1)$. Letting $h(\sigma) = \frac{\sigma^2}{2} \int |\nabla u|^2 + u^2 - \frac{\sigma^{q+1}}{q+1} \int c(x) |u|^{q+1}$, we have

$$h'(\sigma) = \sigma\left(\int |\nabla u|^2 + u^2 - \sigma^{q-1} \int c(x) |u|^{q+1}\right) > 0 \qquad \text{for} \quad \sigma \in (0, 1).$$

$$\begin{aligned} (2.10)\quad I_\lambda(v) &= \frac{\bar{\sigma}^2}{2} \int |\nabla u|^2 + u^2 - \frac{\bar{\sigma}^{p+1}}{p+1} \int \lambda b(x) |u|^{p+1} \\ &\quad - \frac{\bar{\sigma}^{q+1}}{q+1} \int c(x) |u|^{q+1} \\ &< \frac{\bar{\sigma}^2}{2} \int |\nabla u|^2 + u^2 - \frac{\bar{\sigma}^{q+1}}{q+1} \int c(x) |u|^{q+1} \\ &< \frac{1}{2} \int |\nabla u|^2 + u^2 - \frac{1}{q+1} \int c(x) |u|^{q+1} = I^*(u). \end{aligned}$$

Thus $I_\lambda \leqq I^*$ and we have proved Proposition 2.1.

PROPOSITION 2.2. – *We have*

$$(2.11)\qquad I_\lambda^\infty = \frac{p-1}{2(p+1)} S^{(p+1)/(p-1)} (\lambda b_\infty)^{-(2/(p-1))}.$$

Proof. — We can easily find that

$$\text{(2.12)} \qquad S=\inf\left\{\int|\nabla u|^2+u^2 \,\Big|\, u\in H^1(\mathbb{R}^N), \int|u|^{p+1}=1\right\}$$

which has a positive minimum $\bar{u}\in H^1(\mathbb{R}^N)\cap C^2(\mathbb{R}^N)$ satisfying

$$\text{(2.13)} \qquad -\Delta u+u=S|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N$$

(*see* W. Strauss [12], P. L. Lions ([9], [10]) for examples). By Gidas, Ni and Nirenberg [7] we may assume $\bar{u}$ is radial.

On the other hand, there exists a positive radial function $\tilde{u}\in H^1(\mathbb{R}^N)\cap C^2(\mathbb{R}^N)$ achieving I_λ^∞ such that $\tilde{u}$ satisfying

$$\text{(2.14)} \qquad -\Delta u+u=\lambda b_\infty|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N$$

(*see* also W. Strauss [12], P. L. Lions ([9], [10]) for examples).

Let $\tilde{u}=\left(\dfrac{S}{\lambda b_\infty}\right)^{1/(p-1)}v$, then $v>0$ in $\mathbb{R}^N$ and solves (2.13). By the uniqueness of radial positive solution due to M. K. Kwong [11] we deduce $v\equiv\bar{u}$ and thus

$$I_\lambda^\infty=I_\lambda^\infty(\tilde{u})=\frac{p-1}{2(p+1)}\int|\nabla\tilde{u}|^2+\tilde{u}^2=\frac{p-1}{2(p+1)}S^{(p+1)/(p-1)}(\lambda b_\infty)^{-(2/(p-1))}$$

proving Proposition 2.2.

LEMMA 2.3. — *$I_\lambda(u)$ satisfies* $(PS)_c$ *condition if*

$$\text{(2.15)} \qquad c\in(-\infty, I_\lambda^\infty).$$

Proof. — Suppose $\{u_n\}\subset H^1(\mathbb{R}^N)$ such that

$$\text{(2.16)} \qquad I_\lambda(u_n)\underset{n}{\rightarrow} c\in(-\infty, I_\lambda^\infty)$$

$$\text{(2.17)} \qquad I'_\lambda(u_n)\underset{n}{\rightarrow} 0 \quad \text{in } H^1(\mathbb{R}^N)$$

It is easy to deduce from (2.16) and (2.17) that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. By choosing subsequence if necessary we assume

$$\text{(2.18)} \qquad u_0\underset{n}{\rightarrow} u_0 \quad \text{weakly in } H^1(\mathbb{R}^N).$$

By the method of concentration-compactness, as in A. Bahri and P. L. Lions [2], P. L. Lions [10], V. Benci and G. Cerami [3] we deduce that there exist a nonnegative integer k, $\{x_n^i\}(1\leq i\leq k)$ in $\mathbb{R}^N$, solutions $\bar{u}_i\in H^1(\mathbb{R}^N)$ $(1\leq i\leq k)$ of (2.14) such that (extracting subsequence if necessary)

$$\text{(2.19)} \qquad \left\|u_n-u_0-\sum_{i=1}^{k}\bar{u}_i(x-x_n^i)\right\|\underset{n}{\rightarrow} 0$$

$$c = I_\lambda(u_0) + \sum_{i=1}^{n} I_\lambda^\infty(\bar{u}_i). \tag{2.20}$$

Since $I_\lambda^\infty(\bar{u}_i) = \frac{p-1}{2(p+1)} \int |\nabla \bar{u}_i|^2 + \bar{u}_i^2 \geqq 0$ for $i = 1, \ldots, k$ if for some i, $\bar{u}_i \not\equiv 0$, then $I_\lambda^\infty(\bar{u}_i) \geqq I_\lambda^\infty$ which implies $c \geqq I_\lambda^\infty$ because $I_\lambda(u_0) \geqq 0$. Thus $\bar{u}_i \equiv 0$ for $1 \leqq i \leqq k$. Hence u_n converges to u_0 strongly and therefore Lemma 2.3 has been proved.

We are now going to use the preceding result to obtain the existence of a positive solution.

THEOREM 2.4. – *Suppose* $I_\lambda < I_\lambda^\infty$. *Then* (1.1) *has a positive solution.*

Proof. – By Ekeland's variational principle [6] and the definition of I_λ, there exists a minimizing sequence $\{u_n\}$ such that $\{u_n\} \subset M_\lambda$

$$I_\lambda(u_n) \underset{n}{\rightarrow} I_\lambda \tag{2.21}$$

$$I'_{\lambda \mid M_\lambda}(u_n) \underset{n}{\rightarrow} 0 \quad \text{in } H^{-1}(\mathbb{R}^N). \tag{2.22}$$

$$I'_\lambda(u_n) \underset{n}{\rightarrow} 0 \quad \text{in } H^{-1}(\mathbb{R}^N). \tag{2.23}$$

Indeed, from (2.21), $u_n \in M_\lambda$, using Sobolev inequality we can find C_1, $C_2 > 0$ such that

$$C_1 < \int |\nabla u_n|^2 + u_n^2 < C_2 \quad \text{for all } n = 1, 2, \ldots \tag{2.24}$$

Letting $J_\lambda(u) = \int |\nabla u|^2 + u^2 - \int \lambda b(x) |u|^{p+1} - \int c(x) |u|^{q+1}$, we have

$$M_\lambda = \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid J_\lambda(u) = 0 \}. \tag{2.25}$$

Thus

$$I'_\lambda(u_n) = I'_{\lambda \mid M_\lambda}(u_n) - \theta_n J'_\lambda(u_n) \tag{2.26}$$

for some $\theta_n \in \mathbb{R}$.

Since $u_n \in M_\lambda$, we have from (2.26)

$$I'_{\lambda \mid M_\lambda}(u_n) u_n - \theta_n J'_\lambda(u_n) u_n = I'_\lambda(u_n) u_n = 0 \tag{2.27}$$

$$\begin{aligned} J'_\lambda(u_n) u_n &= 2 \int |\nabla u_n|^2 + u_n^2 - (p+1) \int \lambda b(x) |u_n|^{p+1} \\ &\quad - (q+1) \int c(x) |u|^{q+1} \\ &= -(p-1) \int \lambda b(x) |u_n|^{p+1} - (q-1) \int c(x) |u_n|^{q+1}. \end{aligned} \tag{2.28}$$

Thus from (2.24), (2.28) and $u_n \in M_\lambda$ we have

$$-C_3 < J'_\lambda(u_n)\, u_n < -C_4 \tag{2.29}$$

for some constants C_3, $C_4 > 0$ independent of n.

From $I'_{\lambda \mid M_\lambda}(u_n) \underset{n}{\to} 0$, we obtain by (2.27) and (2.29) that $\theta_n \underset{n}{\to} 0$ which combined with (2.26) deduces $I'_\lambda(u_n) \underset{n}{\to}$ in $H^{-1}(\mathbb{R}^N)$. Thus (2.23) holds.

Following Lemma 2.3, we can assume (by choosing subsequence if necessary)

$$u_n \underset{n}{\to} u_0 \quad \text{strongly in } H^1(\mathbb{R}^N).$$

By Sobolev inequality, we have $I_\lambda > 0$. Thus u_0 is a nontrivial solution of (1.1). Letting $u_0 = u_0^+ + u_0^-$, where $u_0^+ = \max\{u_0, 0\}$, $u_0^- = u_0 - u_0^+$, we have $I_\lambda(u_0) = I_\lambda(u_0^+) + I_\lambda(u_0^-)$. Since $I'_\lambda(u_0^\pm)\, u_0^\pm = 0$, *i.e.*, $u_0^\pm \in M_\lambda$ if $u_0^\pm \not\equiv 0$ we have $I_\lambda(u_0^\pm) \geqq I_\lambda$ if $u_0^\pm \not\equiv 0$. Therefore $u_0^+ \equiv 0$ or $u_0^- \equiv 0$. Without loss of generality, assume $u_0^- \equiv 0$. Thus $u_0 \geqq 0$ in $\mathbb{R}^N$. It follows from standard regularity method and maximum principle that $u_0 \in C^2(\mathbb{R}^N)$, $u_0 > 0$ in $\mathbb{R}^N$. Thus, we conclude the proof of Theorem 2.4.

COROLLARY 2.5. — *Suppose* (1.2) *holds*, $c(x)$ *satisfies*

$$\begin{cases} c(x) \in C(\mathbb{R}^N), & c(x) \geqq 0 \quad \textit{in } \mathbb{R}^N, \\ c(x) \xrightarrow[|x| \to \infty]{} 0, & c(x) \not\equiv 0 \quad \textit{in } \mathbb{R}^N. \end{cases} \tag{2.30}$$

Then (1.1) *has a positive solution provided*

$$\lambda \in \left(0, \left[\frac{p-1}{2(p+1)\, I^*}\right]^{(p-1)/2} S^{(p+1)/2}\, b_\infty^{-1}\right). \tag{2.31}$$

Proof. — From (2.31) we have

$$I^* < \frac{p-1}{2(p+1)}\, S^{(p+1)/(p-1)}\, (\lambda\, b_\infty)^{-(2/(p-1))} = I_\lambda^\infty \tag{2.32}$$

which combined with Proposition 2.1 implies

$$I_\lambda < I_\lambda^\infty. \tag{2.33}$$

Thus, by Theorem 2.4 we know (1.1) has a positive solution.

We end this section by a few remarks.

Remark 2.6. — The fact that if $I_\lambda < I_\lambda^\infty$ then I_λ has a minimum has been proved in P. L. Lions ([9], [10]). We reprove this fact for the sake of completeness.

Remark 2.7. — Consider the following equation

$$-\Delta u + u = Q(x)\, |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N \tag{2.35}$$

where $Q(x) \in C(\mathbb{R}^N)$, $Q(x) \geqq 0$ in $\mathbb{R}^N$, $Q(x) \to \bar{Q} > 0$ as $|x| \to \infty$.

(2.35) can be obtained by taking $\lambda=1$, $Q(x)\equiv b(x)$, $c(x)\equiv 0$ in (1.1). From Theorem 2.4 we can deduce the corresponding results concerning the existence of positive solution of (2.35) in section 3 of W. Y. Ding and W. M. Ni [5] [for the case $Q(x)\to\bar{Q}$ as $|x|\to\infty$]. Corollary 2.5 gives a type of precise condition under which $I_\lambda < I_\lambda^\infty$.

Suppose $Q(x)=\lambda b(x)+c(x)$, where $b(x)$ satisfies (1.2) and

$$(2.36)\qquad (b_\infty - b(x))\log(1+|x|)\to +\infty \quad \text{as } |x|\to\infty$$

$c(x)$ satisfies (2.30) with $\operatorname{supp} c(x)$ bounded.

Corollary 2.5 ensures the existence of positive solution if λ is properly small. It should be pointed out that in this case $Q(x)$ does not satisfy the condition proposed by A. Bahri and P. L. Lions in [2].

3. EXISTENCE OF MULTIPLE SOLUTIONS

First of all, let us state a variant of the dual variational principle of A. Ambrosetti and P. Rabinowitz [1] dealing with unbounded even functionals.

Let E be a Banach space, B_r be the ball in E centered at 0 with radius r, ∂B_r be the boundary of B_r. $A\subset E$ is called symmetric if $u\in A$ implies $-u\in A$. Let

$$(3.1)\qquad \Sigma=\{A \mid A\subset E\setminus\{0\},\ A \text{ is closed and symmetric}\}$$

For $A\subset\Sigma$, $\nu(A)$ denotes the genus of A. We set for $f\in C^1(E,\mathbb{R})$

$$(3.2)\qquad E_+=\{u\in E \mid f(u)\geqq 0\}$$

$$(3.3)\qquad H=\{h \mid h\in C(E,E),\ h \text{ is odd homeomorphism } h(B_1)\subset E_+\}$$

$$(3.4)\qquad \Gamma_n=\{A\subset\Sigma \mid A \text{ is compact, } \nu(A\cap h(\partial B_1))\geqq n \text{ for any } h\in H\}$$

Replacing (PS) by $(PS)_c$ condition, we have the following lemma proved exactly as in [1].

LEMMA 3.1. – *Suppose* $f\in C^1(E,\mathbb{R})$ *satisfies*

(C1) $f(0)=0$ *and there exist* ρ, $\alpha>0$ *such that* $f(u)>0$ *for any* $u\in B_\rho\setminus\{0\}$, $f(u)\geqq\alpha$ *for all* $u\in\partial B_\rho$;

(C2) *for any finite dimensional subspace* $E^n\subset E$, $E^n\cap E_+$ *is bounded;*

(C3) $f(u)=f(-u)$.

Set

$$(3.5)\qquad b_n=\inf_{A\in\Gamma_n}\sup\{f(u) \mid u\in A\},\qquad n=1,2,\ldots$$

Then

(i) $\Gamma_n\neq 0$ *for* $n=1, 2, \ldots$, $b_n\geqq\alpha$;

(ii) b_n *is a critical level if* f *satisfies* $(PS)_c$ *condition for* $c=b_n$.

Furthermore, if $b=b_n=\ldots=b_{n+m}$, *then* $\nu(K_b)\geqq m+1$, *where*

$$K_b=\{u\in E \mid f(u)=b, f'(u)=0\}.$$

In what follows, we always take $E=H^1(\mathbb{R}^N)$ and use the same notations Σ, B_r, ∂B_r and $\nu(A)$. Let

$$E_\lambda=\{u\in H^1(\mathbb{R}^N) \mid I_\lambda(u)\geqq 0\} \tag{3.6}$$

$$E_*=\{u\in H^1(\mathbb{R}^N) \mid I^*(u)\geqq 0\} \tag{3.7}$$

$$H_\lambda=\{h\in C(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)),\ h \text{ is odd homeomorphism}, h(B_1)\subset E_\lambda\} \tag{3.8}$$

$$H_*=\{h\in C(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)),\ h \text{ is odd homeophormism}, h(B_1)\subset E_*\} \tag{3.9}$$

Obviously $E_\lambda\subset E_*$, $H_\lambda\subset H_*$.

PROPOSITION 3.2. – *If* $b(x)$ *satisfies* (1.2), $c(x)$ *satisfies*

$$\left\{\begin{array}{l} c(x)\in C(\mathbb{R}^N), \quad c(x)\geqq 0 \quad \text{in } \mathbb{R}^N, \\ \operatorname{meas}\{x\in\mathbb{R}^N \mid c(x)=0\}=0, \\ c(x)\to 0 \quad \text{as} \quad |x|\to\infty \end{array}\right. \tag{3.10}$$

Then $I_\lambda(u)$ *and* $I^*(u)$ *satisfy* (C1), (C2) *and* (C3) *in the previous lemma.*

Proof. – The verification of (C1) and (C3) is trivial. We only show that (C2) holds for $I_\lambda(u)$ [resp. $I^*(u)$]. We argue by way of contradiction. Suppose there exists a m dimensional subspace $E^m\subset H^1(\mathbb{R}^N)$, a sequence $\{u_n\}\subset E^m\cap E_\lambda$ (resp. $\{u_n\}\subset E_*\cap E^m$) such that $\|u_n\|\underset{n}{\to}+\infty$. Let $e_1, e_2, \ldots, e_m$ be the basis of E_m. Then

$$u_n=t_1^n e_1+\ldots+t_m^n e_m \tag{3.13}$$

for some $t_n=(t_1^n, \ldots, t_m^n)\in\mathbb{R}^m$.

Set $|t_n|=\max\limits_{1\leqq i\leqq m}|t_i^n|$, we have $|t_n|\underset{n}{\to}+\infty$.

$$\int|\nabla u_n|^2+u_n^2=0(|t_n|^2) \tag{3.14}$$

$$\int b(x)|u_n|^{p+1}\geqq 0 \tag{3.15}$$

$$\int c(x)|u_n|^{q+1}\geqq C_5|t_n|^{q+1} \quad \text{for } n \text{ large enough} \tag{3.16}$$

where $C_5>0$ is some constant.

(3.14), (3.15) and (3.16) deduce $I_\lambda(u_n)<0$ for n larger enough [resp. $I^*(u_n)<0$ for n large enough], which contradicts $u_n\in E_\lambda$ (resp. $u_n\in E_*$).

Define

(3.17) $\Gamma_\lambda^n = \{ A \subset \Sigma \mid A \text{ is compact and } \nu(A \cap h(\partial B_1)) \geqq n$ for any $h \in H_\lambda \}$, $n = 1, 2, \ldots,$

(3.18) $\Gamma_*^n = \{ A \subset \Sigma \mid A \text{ is compact and } \nu(A \cap h(\partial B_1)) \geqq n$ for any $h \in H_* \}$, $n = 1, 2, \ldots,$

(3.19) $$c_\lambda^n = \inf_{A \in \Gamma_\lambda^n} \max \{ I_\lambda(u) \mid u \in A \}, \qquad n = 1, 2, \ldots,$$

(3.20) $$c_*^n = \inf_{A \in \Gamma_*^n} \max \{ I_*(u) \mid u \in A \}, \qquad n = 1, 2, \ldots,$$

By the definitions we have

(3.21) $$\Gamma_\lambda^n \supset \Gamma_*^n \qquad \text{for} \quad n = 1, 2, \ldots$$

Suppose (3.10) holds then by Proposition 3.2 and Lemma 3.1, $\Gamma_*^n \neq \varnothing$ for each $n = 1, 2, \ldots$, and consequently $c_*^n < +\infty$.

Let

$$\lambda_k = \left[\frac{p-1}{2(p+1) c_*^k} \right]^{(p-1)/2} S^{(p+1)/2} b_\infty^{-1}, \qquad k = 1, 2, \ldots$$

We have

THEOREM 3.3. – *Suppose* (1.2) *and* (3.10) *hold. Then for each* $n = 1, 2, \ldots$, (1.1) *has n pair of solutions* $\{ -u_i u_i \}$, $i = 1, \ldots, n$ *if* $\lambda \in (0, \lambda_n)$.

Proof. – By the definition of c_λ^n, c_*^n, $n = 1, 2, \ldots$ we have

$$\begin{aligned} c_\lambda^n &= \inf_{A \in \Gamma_\lambda^n} \max \{ I_\lambda(u) \mid u \in A \} \\ &\leqq \inf_{A \in \Gamma_*^n} \max \{ I_\lambda(u) \mid u \in A \} \\ &\leqq \inf_{A \in \Gamma_*^n} \max \{ I^*(u) \mid u \in A \} \\ &= c_*^n. \end{aligned}$$

Thus

(3.23) $$c_\lambda^n \leqq c_*^n \qquad \text{for} \quad n = 1, 2, \ldots$$

Next we claim that for each c_λ^k, $k = 1, \ldots, n$, $I_\lambda(u)$ satisfies $(PS)_c$ condition.

Indeed, $\lambda < \lambda_n$ implies

$$\lambda < \left[\frac{p-1}{2(p+1) c_*^n} \right]^{(p-1)/2} S^{(p+1)/2} b_\infty^{-1}.$$

Thus

$$c_*^n < \frac{p-1}{2(p+1)} S^{(p+1)/(p-1)} (\lambda b_\infty)^{-(2/(p-1))} = I_\lambda^\infty$$

which combining with (3.23) deduces

$$c_\lambda^n < I_\lambda^\infty. \tag{3.24}$$

On the other hand, obviously we have

$$c_\lambda^1 \leqq \ldots \leqq c_\lambda^n. \tag{3.25}$$

Thus, by Lemma 2.3, $I_\lambda(u)$ satisfies $(PS)_c$ condition for c_λ^k, $k=1, 2, \ldots, n$. Following Lemma 3.1, $I_\lambda(u)$ has at least n different critical points $u_i \in H^1(\mathbb{R}^N)$ $(1 \leqq i \leqq n)$ such that $I_\lambda(u_i) = c_\lambda^i$ $(1 \leqq i \leqq n)$. Since $I_\lambda(u)$ is a even functional $-u_i$ is critical point either $(1 \leqq i \leqq n)$, $\{-u_i, u_i\}$ are the solutions we are looking for. Hence we have proved Theorem 3.3.

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