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Decay estimates for spherically symmetric Yang-Mills fields in Minkowski space-time

by

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ABSTRACT. — We consider Yang-Mills fields in Minkowski space-time R^{3+1}. We prove global existence and establish decay estimates for any initial data which is sufficiently small in a specified energy norm and which is spherically symmetric in the sense of principal bundles. We consider initial data which lead to bounded solutions near the central line. The proof of the global result is achieved using wighted Sobolev norms of the Klainerman type. They not only provide global existence, but also sharp asymptotic behaviour of the solutions in time. The initial data considered here does not include Coulomb charges but includes dipole radiation, a situation which cannot be acomodated in the framework of the conformal method. A new class of solutions, which have not been considered before, is covered then by the theorem.

Key words : Yang-Mills Equations, Minkowski space-time.

RÉSUMÉ. — On démontre un théorème global d'existence d'une solution du problème de Cauchy pour les équations de Yang-Mills sur l'espace de Minkowski, pour des données petites et avec symétrie radiale, dans le sens d'une fibration principale.

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1. INTRODUCTION

We consider the Yang-Mills equations in Minkowski space-time R^{3+1}:

$$F^{;\beta}_{\alpha\beta}=0 \tag{1.1}$$

$$*F^{;\beta}_{\alpha\beta}=0 \tag{1.2}$$

F is the curvature $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu, A_\nu]$ of a potential 1-form A taking values in the Lie algebra $\mathcal{G}$ of a compact semi-simple Lie group G.

There has been a lot of research on the associated euclidean version of the system. In that case the Yang-Mills equations become a non-linear elliptic system. Here the underlying metric has Lorentz signature and the equations are hyperbolic. We look for the dynamical developments of initial data defined on a space-like submanifold. Global existence was proved by Eardley and Moncrief [6]. Their results are valid for initial data of any size, but no information about the asymptotic behaviour of the solutions is given. Another major contribution was given by Demetrius Christodoulou [2] and is based on Penrose's conformal compactification method. It works only for small initial data, but sharp decay estimates are given. The major drawback, though, is that is requires a strong fall-off rate on the initial data ($O(r^{-4})$), excluding not only charge-like solution (fall-off $O(r^{-2})$), but also dipole-type radiation (fall-off $O(r^{-5/2})$). The aim of this paper is to get decay estimates for Yang-Mills fields that correspond to dipole radiation and that cannot be accomodated within the conformal method framework. We establish global existence and characterize the asymptotic behaviour of small-amplitude solutions which are spherically symmetric, in a sense to be made precise later. The decay estimates are optimal, namely equal to their linearized counterparts. The result holds true for any semi-simple compact gauge group that admits an SU(2) subgroup and relates to work done previously by Glassey and Strauss ([10], [11]). Their results, although true for initial data of any size, are valid only for a restricted class of spherically symmetric solutions with $SU(2)$-gauge symmetry and rely heavily on the form of the spherically symmetric Ansatz.

The problem of trying to simplify the equations by looking to solutions with higher degrees of symmetry is that there could be no such solutions. A result by Coleman and Deser ([4] and [5]) prohibits the existence of finite-energy static solutions to the Yang-Mills equations. Here we consider spherical symmetry. By this we mean invariance under the combined effect of a rotation and a compensating gauge transformation. It is a remarkable fact that the Yang-Mills equations admit such solutions. This is not the case of electrodynamics where the electromagnetic field is gauge invariant and there can be no further mechanism to counterbalance the effect of a rotation. In contrast, we can prove for the Yang-Mills equations:

MAIN THEOREM. — *Let* G *a compact semi-simple gauge group admitting an* $SU(2)$ *Lie subgroup and* $(\mathrm{A}(0), \mathrm{E}(0))$, $\mathrm{A}(0):\mathrm{R}^{3+1}\to\Lambda^1\mathscr{G}$, $\mathrm{E}(0):\mathrm{R}^{3+1}\to\Lambda^1\mathscr{G}$ *an initial data set which is spherically symmetric* (1). *Define the norm:*

$$\mathrm{I}(\mathrm{A}(0), \mathrm{E}(0)) = \int_{\mathrm{R}^3} (1+r^2)\,|\,\mathrm{F}\,|^2 + (1+r^2)^2\,|\,\mathrm{DF}\,|^2 + (1+r^2)^4\,|\,\mathrm{D}^2\,\mathrm{F}\,|^2\,dx$$

where $r=|x|$ *and* F *denotes* E *and* H.

We claim that if I (A, E) *is sufficiently small, then there exists a globally defined solution of the Yang-Mills equations having* (A, E) *as initial values and which stays spherically symmetric in the gauge where the solution is regular* (*canonical gauge*). *Furthermore, the solution decays in time and satisfies the global estimate:*

$$|\,\mathrm{F}(t, .)\,|_\infty \leqq c\,(1+t)^{-1}\,\mathrm{I}(\mathrm{A}(0), \mathrm{E}(0))$$

In the interior of the domain of influence we obtain sharper estimates. We also prove a peeling theorem using light-cone coordinates.

The general spherically symmetric field is presented in the literature in two different gauges: the abelian and the canonical gauges. The abelian gauge is the most convenient gauge for exhibiting structural features of the solution. It suffers from the disadvantage that it is a singular gauge and the solution will have string-type singularities. Under some conditions one can perform a singular gauge transformation and bring the solution to a completely regular gauge, called the canonical gauge, where regularity is transparent. For the local existence theory this is unnecessary but in the course of proof of the decay estimates a global gauge, which is free of strings, is needed. Global existence is then achieved in this gauge. In the $SU(2)$ case it assumes a very simple form (The Ansatz for higher groups is similar and requires only a more cumbersome notation):

$$\mathrm{A}_0^a = \varphi\,\frac{x^a}{r} \tag{1.3}$$

$$\mathrm{A}_i^a = \psi\,\frac{x_i x^a}{r^2} + \frac{f_1}{r}\left(\delta_i^a - \frac{x_i x^a}{r^2}\right) + \frac{f_2-1}{r}\,\varepsilon^{iab}\,\frac{x^b}{r} \tag{1.4}$$

where φ, ψ, f_1 and f_2 are functions of t and r.

(1) All the terms contained in this statement will be made precise during the course of the work. In particular one examines what one means by a spherically symmetric gauge field.

The Yang-Mills equations reduce in this case to a very complicated system of equations. Glassey and Strauss consider in their papers ([10], [11]) special solutions which, after using all gauge degrees of freedom, have the form:

$$A_0^a = 0$$

$$A_i^a = \alpha(t, r)\, \varepsilon^{iab} \frac{x^b}{r}$$

Equations (1.1)-(1.2) are then a single scalar wave equation for α:

$$\square_{(3)} \alpha + \frac{2}{r^2} \alpha - \frac{3}{r} \alpha^2 + \alpha^3 - 0$$

There are, though, some subtleties associated with 1.3-1.4. One observes that the solution is everywhere regular except at the origin $r=0$. The question of regularity at the central line is very delicate. We do not attempt here to analyze the exact behaviour at the origin but merely, because we prove global existence in H^2, establish boundness near the center.

The approach adopted here is a geometrical one and is based on energy estimates for weighted Sobolev-Klainerman norms (*cf.* [16] and [17]). These are global norms and yield time decay, due to the built-in asymptotics provided by the Lorentz group generators. Finally, we remark that the Yang-Mills equations contain quadratic terms. In three space dimensions this kind of terms could lead to singularities, unless a certain algebraic condition, called the null condition, is present. On this case this condition is satisfied and is a consequence of the covariant form of the equations.

The plan of the paper is the following. On section 2 we establish the notation used throughout the work. In section 3 we prove the local existence theorem we need. Our version is based directly on energy estimates for weighted Sobolev norms and is simpler than the proofs, found in the literature, which rely on semi-group theory. Sections 4, 9 and 10 contain a detailed analysis of the spherically symmetric Ansatz including a discussion on the behaviour near the central line $r=0$. The remaining sections 5, 6, 7, 8 and 11 contain the estimates that lead to global existence.

2. NOTATION

We consider Minkowski space R^{3+1} with coordinates (x^0, x^1, x^2, x^3) and standard flat metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$. We also denote $x^0 = t$ for the time coordinate. We shall use Einstein's convention of raising and lowering indices. We call a vector X time-like, null or space-like iff $\eta(X, X)$ is negative, zero or positive respectively. The infinitesimal generators X of the Lorentz group play an important role here. Denote by $T=T_0=\partial_t$, $S=x^\mu\partial_\mu$ and $O_{(i)}=\frac{1}{2}\varepsilon_{ijk}\Omega_{jk}=\varepsilon_{ijk}x_j\partial_k$. We also use the standard null frame $\{e_1, e_2, e_3, e_4\}$ where $\{e_-, e_+\}$ is the standard null pair $e_-=\partial_t$, $e_+=\partial_t+\partial_r$ and $e_A=\frac{1}{r}\xi_A$, ξ_A the standard orthonormal frame tangent to the unit sphere. Call $\tau_-^2=1+(t-r)^2$, $\tau_+^2=1+(t+r)^2$ the space-time weights associated to the mull frame. We shall also use the electromagnetic decomposition of F, defined by $E_i=F_{0i}$, $H_i=\frac{1}{2}\varepsilon_{ijk}F_{jk}$.

The gauge group is a compact semi-simple Lie group G. We denote its Lie algebra by $\mathcal{G}$ and the Lie algebra commutator by $[.,.]$. The Killing form on $\mathcal{G}$ is denoted $\langle .,. \rangle$. In the sequel we will often write '.' for this bilinear form. We also $[A \overset{\times}{;} B]=\varepsilon_{ijk}[A_j, B_k]$. Finally, we fix a basis T_a, $a=1,2,\ldots,N$ of $\mathcal{G}$ and normalize it by $T_a.T_b=\delta_{ab}$.

The Yang-Mills potential is a $\mathcal{G}$-valued 1-form $A=A_\mu dx^\mu=(A_\mu^a T_a)dx^\mu$. The Yang-Mills field-strenght of A is a 2-form $F:R^{3+1}\to\Lambda^2\mathcal{G}$ defined as $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu, A_\nu]$. D will denote the covariant derivative $D_\mu=\partial_\mu+[A_\mu, .]$. We will also refer to it by the use of semicollon ";". The gauge copies of A are denoted ${}^{(g)}A=gAg^{-1}-dg\,g^{-1}$ where $g:R^{3+1}\to G$ takes values in the group G. The corresponding curvature tensor and covariant derivative will change accordingly as ${}^{(g)}F=gFg^{-1}$ and ${}^{(g)}D^{(g)}F=g\,DF\,g^{-1}$.

Because of the tensorial nature of the equations we have to consider Lie derivative operators. The ordinary Lie derivatives are denoted by $\underline{\mathcal{L}}_X F$:

$$\underline{\mathcal{L}}_X F=X^\lambda\partial_\lambda F_{\alpha\beta}+X^\lambda_{,\alpha}F_{\lambda\beta}+X^\lambda_{,\beta}F_{\alpha\lambda}$$

and the ones containing covariant derivatives by $\mathcal{L}_X$:

$$\mathcal{L}_X F=X^\lambda D_\lambda F_{\alpha\beta}+X^\lambda_{,\alpha}F_{\lambda\beta}+X^\lambda_{,\beta}F_{\alpha\lambda}$$

This notation is a bit misleading but the confusion will arise only when we discuss the spherically symmetric Ansatz. It does not come up in the estimates that lead to the proof of theorem.

3. THE LOCAL EXISTENCE THEOREM

In this section we will prove a local existence theorem in the temporal gauge $A_0=0$. The proof is standard and simpler than existing ones but there are some subtle difficulties that requires explanation. Writing 1.1 in terms of the electromagnetic decomposition we have:

PROPOSITION 3.1. – *The Yang-Mills system* 1.1 *reduces in the temporal gauge* $A_0 = 0$ *to the following (redundant) set of equations:*

NON-DYNAMICAL EQUATIONS:

$$\mathrm{Div}_A E = 0$$
$$\mathrm{Div}_A H = 0$$

DYNAMICAL EQUATIONS:

$$D_t E + D \times H = 0$$
$$D_t H - D \times E = 0$$

CONSTITUTIVE EQUATIONS:

$$E = D_t A$$
$$H = \nabla \times A - \frac{1}{2}[A \overset{\times}{,} A]$$

The main problem is that the magnetic field H(A) is not an elliptic operator in A. If one tries to solve the Cauchy problem in terms of (A, E) then one will face loss of derivatives and regularity for H will be lost. To overcome this difficulty we must derive an auxiliary set of equations and prove that they propagate the solutions of the Yang-Mills system. This consists essentially of a system of coupled wave equations (here the covariant wave operator $\square_A := D_\mu D^\mu$ is meant). Define:

$$\Phi = \mathrm{Div}_A E \tag{3.1}$$
$$\tilde{\Phi} = \mathrm{Div}_A H \tag{3.2}$$
$$\Psi = D_t E - D \times H \tag{3.3}$$
$$\tilde{\Psi} = D_t H + D + E \tag{3.4}$$
$$\Theta = H - \nabla \times A - \frac{1}{2}[A \overset{\times}{,} A] \tag{3.5}$$

It follows from these definitions that:

THEOREM 3.1. – *The quantities defined in equations* 3.1, 3.5 *satisfy the following system of equations:*

$$D_t \Phi = D . \Psi \tag{3.6}$$
$$D_t \tilde{\Phi} = D . \tilde{\Psi} \tag{3.7}$$
$$D_t \Psi - D \times \tilde{\Psi} = D\Phi + (\square_A E + 2[E \overset{\times}{,} H]) \tag{3.8}$$
$$D_t \tilde{\Psi} + D \times \Psi = D\tilde{\Phi} + (\square_A H + [E \overset{\times}{,} E] + [H \overset{\times}{,} H]) \tag{3.9}$$
$$D_t \Theta = \tilde{\Psi} \tag{3.10}$$

THEOREM 3.2. — *Assume that* (A, E, H) *solve the system:*

$$\square_A E + 2[E \overset{\times}{,} H] = 0 \tag{3.11}$$

$$\square_A H + [E \overset{\times}{,} E] + [H \overset{\times}{,} H]) = 0 \tag{3.12}$$

$$D_t A = E \tag{3.13}$$

with initial data (A (0), E (0), $\dot{E}$ (0), H (0), $\dot{H}$ (0)) *satisfying the compatibility conditions* (2)*:*

$$H(0) = H(A(0)) \tag{3.14}$$

$$\dot{E}(0) = -D \times H(0) \tag{3.15}$$

$$\tilde{H}(0) = D \times E(0) \tag{3.16}$$

It follows that (A, E, H) will be a solution of the Yang-Mills system 1.1 *with the same Cauchy data.*

Proof of theorem 3.2. — Our claim consists of proving a uniqueness result, namely, that the only solution to 3.6, 3.8, 3.10 with vanishing initial data is the trivial solution. This is a simple energy argument and follows by considering the function

$$R = \frac{1}{2}|\Phi|^2 + \frac{1}{2}|\tilde{\Phi}|^2 + \frac{1}{2}|\Psi|^2 + \frac{1}{2}|\tilde{\Psi}|^2$$

We can prove now:

THEOREM 3.3 (The Local Existence Theorem). — *Consider initial data* (A, E) *satisfying the following conditions:*

(i) (A, E) *satisfies the constraint equation* $\mathrm{Div}_A E = 0$.

(ii) *The electric field* E *satisfies* $E \in H^{2,1}$.

(iii) *The magnetic field* H (A) *of the potential* A *satisfies* $H \in H^{2,1}$.

Here $H^{s,\delta}$ *denotes the weighted Sobolev space of tensors with finite norm:*

$$\| F \|_{s,\delta} = \left(\sum_{i=0}^{s} \int \sigma^{2\delta + 2i} |D^i F(x)|^2 \, dx \right)^{1/2}$$

where $\sigma^2 = 1 + |x|^2$.

It follows that there exists a unique local development (A (*t*), E (*t*)) *of the Yang-Mills system in the temporal gauge* $A_0 = 0$, *defined for some time-interval* $[0, t_*]$ *and satisfying the following conditions:*

(i') (A (*t*), E (*t*)) *satisfies the constraint equation* $\mathrm{Div}_A E = 0$ *for all* $t \in [0, t_*]$.

(ii') $E(t) \in C^0([0, t_*], H^{2,1})$.

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(2) Observe that the compatibility conditions are equivalent to prescribing Cauchy data (A (0), E (0)).

(iii') $H(t) \in C^0([0, t_*], H^{2,1})$.

(iv') $A(t) - A(0) \in C^1([0, t_*], H^{2,1})$.

Proof of the Local Existence Theorem. – The proof is a usual Picard iteration. Assume that (A_n, E_n, H_n) has been constructed. Define $(A_{n+1}, E_{n+1}, H_{n+1})$ as the solution to the following system of linear equations:

$$\square_{A_n} E_{n+1} + 2[E_n \overset{\times}{,} H_n] = 0 \tag{3.17}$$

$$\square_{A_n} H_{n+1} + [E_n \overset{\times}{,} H_n] + [H_n \overset{\times}{,} H_n] = 0 \tag{3.18}$$

$$\partial_t A_{n+1} = E_n \tag{3.19}$$

subject to the initial conditions:

$$E_{n+1}(0) = E(0) \tag{3.20}$$

$$\dot{E}_{n+1}(0) = -(D \times H)(0) \tag{3.21}$$

$$H_{n+1}(0) = H(A(0)) \tag{3.22}$$

$$\dot{H}_{n+1}(0) = (D \times E)(0) \tag{3.23}$$

$$A_{n+1}(0) = A(0) \tag{3.24}$$

Consider initial data of size R and define the closed set of multiplets $(A_n, E_n, \dot{E}_n, H_n, \dot{H}_n)$ satisfying:

$$\| A_n - A(0) \|_{H^{2,1}} \leqq 2R \tag{3.25}$$

$$\| E_n \|_{H^{2,1}} \leqq 2R \tag{3.26}$$

$$\| \dot{E}_n \|_{H^{1,1}} \leqq 2R \tag{3.27}$$

$$\| H_n \|_{H^{2,1}} \leqq 2R \tag{3.28}$$

$$\| \dot{H}_n \|_{H^{1,1}} \leqq 2R \tag{3.29}$$

The Sobolev norms refer to the connection A_n. We show that if the time-interval $[0, t_*]$ is sufficiently small, then the same estimates hold for the multiplet $(A_{n+1}, E_{n+1}, \dot{E}_{n+1}, H_{n+1}, \dot{H}_{n+1})$. The estimate for A_{n+1} follows immediatly from the estimates for E_n. The others follow easily using wave equation-type energy estimates with the multiplier $\sigma^2 \partial_t$. The rest of the argument consists of a usual contracting lemma. We will highlight the main points. In the iteration proof we will need some lemmas for the covariant wave operator $\square_A$:

LEMMA 3.1. – *Let* $A : R^{3+1} \to \mathcal{G}$ *a background potential in the temporal gauge* $A_0 = 0$ *and* $u : R^{3+1} \to \mathcal{G}$ *a Lie-algebra valued function satisfying the covariant wave equation:*

$$\square_A u = f$$

where $f : \mathrm{R}^{3+1} \to \mathscr{G}$ is a source function. Define the covariant energy of u as:

$$\mathscr{E}(u, t) = \frac{1}{2} \int |\mathrm{D}_t u|^2 + |\mathrm{D}u|^2 \, dx$$

It follows that:

$$\frac{d}{dt} \mathscr{E}(u, t) = \int f . \mathrm{D}_t u \, dx + \int \mathrm{D}_i u . [\mathrm{E}_i(\mathrm{A}), u] \, dx$$

where E (A) *denotes the electric part of the connection* A.

Proof of lemma 3.1. – The proof follows the classical one, except for the term generated by the commutation of two covariant derivatives when one integrates by parts.

We will estimate only the plain L^2-norms without weights. The weighted norms are estimated in a similar way. We start with the first derivatives. Assume that 3.25 hold true for all iterates up to step n. Using lemma 3.1:

$$\begin{aligned} \frac{d}{dt} \mathscr{E}_{n+1} = \int & -2 [\mathrm{E}_n \overset{\times}{;} \mathrm{H}_n] . \mathrm{D}_t \mathrm{E}_{n+1} \\ & - ([\mathrm{E}_n \overset{\times}{;} \mathrm{H}_n] + [\mathrm{H}_n \overset{\times}{;} \mathrm{H}_n]) . \mathrm{D}_t \mathrm{H}_{n+1} \, dx \\ & + \int \mathrm{DE}_{n+1} . [\mathrm{E}(\mathrm{A}_n), \mathrm{E}_{n+1}] \, dx + \int \mathrm{DH}_{n+1} . [\mathrm{E}(\mathrm{A}_n), \mathrm{H}_{n+1}] \, dx \end{aligned}$$

$\mathrm{E}(\mathrm{A}_n)$ is the electric part of connection A_n. Remarking that $\mathrm{E}(\mathrm{A}_n) = \partial_t \mathrm{A}_n = \mathrm{E}_{n-1}$:

$$\begin{aligned} \frac{d}{dt} \mathscr{E}_{n+1} \leq \mathscr{E}_{n+1}^{1/2} . & \left(\int |\mathrm{E}_n|^2 . |\mathrm{H}_n|^2 + |\mathrm{E}_n|^2 . |\mathrm{H}_n|^2 + |\mathrm{H}_n|^2 . |\mathrm{H}_n|^2 \, dx \right)^{1/2} \\ & + \mathscr{E}_{n+1}^{1/2} . \left(\int |\mathrm{E}_{n-1}|^2 . |\mathrm{E}_{n+1}|^2 \, dx \right)^{1/2} \\ & + \mathscr{E}_{n+1}^{1/2} . \left(\int |\mathrm{E}_{n-1}|^2 . |\mathrm{H}_{n+1}|^2 \, dx \right)^{1/2} \end{aligned}$$

We apply the Sobolev embedding:

$$\left(\int |\mathrm{E}|^4 \, dx \right)^{1/4} \leq c \left(\int |\mathrm{DE}|^2 \, dx \right)^{1/2}$$

and find:

$$\frac{d}{dt} \mathscr{E}_{n+1} \leq c \, \mathrm{R} \, \mathscr{E}_{n+1} + c \, \mathrm{R}^2 \, \mathscr{E}_{n+1}^{1/2}$$

It will follow that:

$$\mathscr{E}_{n+1}(t) \leq \mathscr{E}_{n+1}(0) \, e^{c\mathrm{R}t} + \mathrm{R} \, (e^{c\mathrm{R}t} - 1)$$

now $\mathscr{E}_{n+1}(0) \leqq R$ and if we take t in the interval $[0, t_*]$ with t_* so small that $t_* \leqq (R c)^{-1} \log(3/2)$, then we get $\mathscr{E}_{n+1}(t) \leqq 2R$ and the bounds are recovered. We estimate now the higher derivatives. Commuting 3.17 with D we will be able to estimate all second derivatives, with the exception of $D_t^2 E_{n+1}$, $D_t^2 H_{n+1}$. To complete the estimates we must also commute the equations with D_t. We appeal to the following lemma:

LEMMA 3.2. – *Let* A, *u and f* Lie *algebra valued functions as in* 3.1. *It follows that* $D_\alpha u$ *will satisfy the equation:*

$$\Box_A (D_\alpha u) = D_\alpha f + 2[F_{\alpha\mu}, D^\mu u] + [F_{\alpha\mu}^{;\mu}, u]$$

and obtain:

$$\begin{aligned}
\Box_{A_n}(DE_{n+1}) = & -2[DE_n \overset{\times}{,} H_n] - 2[E_n \overset{\times}{,} DH_n] \\
& + 2[E_n, DE_{n+1}] + 2[H_n, DE_{n+1}] \\
& + 2[E_n, D_t E_{n+1}] + 2[F_{\alpha\mu}(A_n)^{;\mu}, E_{n+1}] \\
\Box_{A_n}(D_t E_{n+1}) = & -2[D_t E_n \overset{\times}{,} H_n] - 2[E_n \overset{\times}{,} D_t H_n] \\
& + 2[E_n, D_t E_{n+1}] + 2[E_n, D_t E_{n+1}] \\
& + 2[F_{\alpha\mu}(A_n)^{;\mu}, E_{n+1}]
\end{aligned}$$

(Similarly for the magnetic part.)

Remark that for the exact solution F of the Yang-Mills equations, the term $F_{\alpha\mu}^{;\mu} = 0$. For an approximating background connection A_n $F(A_n)_{\alpha\mu}^{;\mu}$ will be a linear function of DE_n, DH_n, $D_t E_n$, $D_t H_n$, which are lower order terms that have been estimated before. We apply one again the energy inequalities. Calling $\mathscr{E}_{n+1}$ the energy associated with the first derivatives and proceeding as before:

$$\frac{d}{dt}\mathscr{E}_{n+1} \leqq 2 c R \mathscr{E}_{n+1} + 4 c R^2 \mathscr{E}_{n+1}^{1/2}$$

Taking t_* smaller, if necessary, we obtain the bounds we desired. The only task left is the estimate of the undifferentiated fields E_{n+1}, H_{n+1}. This follows from the estimates for the first derivatives and a simple integration in time.

The rest of the construction is standard and consists of showing that the sequence contracts in a low norm. We just remark though that in order to apply a fixed-point argument we must work in a fixed Sobolev space of iterates, meaning that we must refer all bounds to the original connection $A(0)$ and use $H^{2,1}(R^{3+1}, A(0))$. A simple comparison argument implies:

$$\|A_n - A(0)\|_{H^{2,1}} \leqq 3R \tag{3.30}$$

$$\|E_n\|_{H^{2,1}} \leqq 3R \tag{3.31}$$

$$\|\dot{E}_n\|_{H^{1,1}} \leqq 3R \tag{3.32}$$

$$\|H_n\|_{H^{2,1}} \leqq 3R \tag{3.33}$$

$$\|\dot{H}_n\|_{H^{1,1}} \leqq 3R \tag{3.34}$$

The Sobolev norms refer now to the connection $A(0)$. The comparison argument consists of writing $D_{A(0)} = D_{A_{n-1}} + [A(0) - A_{n-1}, .]$. The last term is treated via an integration in time. To estimate the differences between two iterates we need:

LEMMA 3.3. – *Let* $\tilde{A}$, $A : R^{3+1} \to \Lambda^1 \mathscr{G}$ *two connections,* $\tilde{D}$, D *the corresponding derivatives and* $\tilde{\Box}$, $\Box$ *the corresponding wave operators. For every function* $f : R^{3+1} \to \mathscr{G}$ *we have:*

$$\begin{aligned}\tilde{\Box} f = \Box f - [D(\tilde{A} - A), f] - 2[\tilde{A} - A, Df] \\ - [\tilde{A} - A, [\tilde{A} - A, f]]\end{aligned}$$

The proof is a straightforward computation.

We estimate $E_{n+1} - E_n$. The magnetic part is similar. Applying 3.3 to $A(0)$, A_{n-1} and A_n:

$$\Box_{A(0)}(E_{n+1} - E_n) = g_{n+1} \tag{3.36}$$

where $g_{n+1} = g_{n+1}^{(1)} + g_{n+1}^{(2)} + g_{n+1}^{(3)} + g_{n+1}^{(4)}$ is:

$$g_{n+1}^{(1)} = 2([E_n \overset{\times}{,} H_n] - [E_{n-1} \overset{\times}{,} H_{n-1}])$$

$$g_{n+1}^{(2)} = [D_{A_{n-1}}(A(0) - A_{n-1}), E_n] - [D_{A_n}(A(0) - A_n), E_{n+1}]$$

$$g_{n+1}^{(3)} = 2([A(0) - A_n, D_{A_n} E_{n+1}] - [A(0) - A_n, D_{A_n} E_{n+1}])$$

$$\begin{aligned} g_{n+1}^{(4)} = [A(0) - A_{n-1}, [A(0) - A_{n-1}, E_n]] \\ - [A(0) - A_n, [A(0) - A_n, E_{n+1}]] \end{aligned}$$

We will prove:

$$\|E_{n+1} - E_n\|_{1,1} \leqq c R 2^{-(n+1)} \tag{3.37}$$

$$\|\dot{E}_{n+1} - \dot{E}_n\|_{0,1} \leqq c R 2^{-(n+1)} \tag{3.38}$$

$$\|H_{n+1} - H_n\|_{1,1} \leqq c R 2^{-(n+1)} \tag{3.39}$$

$$\|\dot{H}_{n+1} - \dot{H}_n\|_{0,1} \leqq c R 2^{-(n+1)} \tag{3.40}$$

In particular, both series:

$$\sum_{n=0}^{\infty} (E_{n+1} - E_n) \quad \text{and} \quad \sum_{n=0}^{\infty} (H_{n+1} - H_n)$$

converge in $H^{1,1}$ to elements E and H. This is the difficult part of argument. We proceed by induction and assume that 3.37 hold up to step n. Call:

$$\mathcal{R}_{n+1}=\frac{1}{2}\int_{R^3}|DE_{n+1}-DE_n|^2+|D_t E_{n+1}-D_t E_n|^2\,dx$$

Applying lemma 3.1 to the equation 3.36:

$$\begin{aligned}\frac{d}{dt}\mathcal{R}_{n+1}&=\int g_{n+1}\,.\,D_t(E_{n+1}-E_n)\\&\quad+\int D(E_{n+1}-E_n)\,.\,[E_{n-1},E_{n+1}-E_n]\\&\leqq \mathcal{R}_{n+1}^{1/2}\|g_{n+1}\|+c\,R\,\mathcal{R}_{n+1}\end{aligned}$$

The inhomogeneous term g_{n+1} is bounded as follows. For $g_{n+1}^{(1)}$:

$$\begin{aligned}g_{n+1}^{(1)}&=2[E_n \overset{\times}{,} H_n]-2[E_{n-1}\overset{\times}{,} H_{n-1}]\\&=2[E_n-E_{n-1}\overset{\times}{,} H_n]+2[E_{n-1}\overset{\times}{,} H_n-H_{n-1}]\end{aligned}$$

and therefore:

$$\begin{aligned}\|g_{n+1}^{(1)}\|^2&\leqq c\int|E_n-E_{n-1}|^2|H_n|^2+|H_n-H_{n-1}|^2|E_n|^2\,dx\\&\leqq c\|E_n-E_{n-1}\|_{L^4}^2\|H_n\|_{L^4}^2+c\|H_n-H_{n-1}\|_{L^4}^2\|E_n\|_{L^4}^2\,dx\\&\leqq c\|DE_n-DE_{n-1}\|^2\|DH_n\|^2\\&\quad+c\|DH_n-DH_{n-1}\|^2\|DE_n\|^2\,dx\\&\leqq(c\,R^2\,2^{-n})^2(3\,R)^2\end{aligned}$$

If follows that:

$$\|g_{n+1}^{(1)}\|\leqq c\,R^2\,2^{-n}$$

For $g_{n+1}^{(2)}$:

$$\begin{aligned}g_{n+1}^{(2)}&=[D_{A_{n-1}}(A(0)-A_{n-1}),E_n]-[D_{A_n}(A(0)-A_n),E_{n+1}]\\&=[D_{A_{n-1}}(A(0)-A_{n-1}),E_n-E_{n+1}]\\&\quad+[D_{A(0)}(A_n-A_{n-1}),E_{n+1}]\end{aligned}$$

Now:

$$\begin{aligned}\|g_{n+1}^{(2)}\|^2&\leqq\int|D_{A_{n-1}}(A(0)-A_{n-1})|^2|E_n-E_{n+1}|^2\,dx\\&\quad+\int|D_{A(0)}(A_n-A_{n-1})|^2|E_{n+1}|^2\,dx\end{aligned}$$

$$\leqq \left(\int | \mathrm{D}_{\mathrm{A}_{n-1}} (\mathrm{A}(0) - \mathrm{A}_{n-1}) |^4 \, dx \right)^{1/2} \left(\int | \mathrm{E}_n - \mathrm{E}_{n+1} |^4 \, dx \right)^{1/2}$$

$$+ | \mathrm{E}_{n+1} |_\infty^2 \int | \mathrm{D}_{\mathrm{A}(0)} (\mathrm{A}_n - \mathrm{A}_{n-1}) |^2 \, dx$$

$$\leqq | \mathrm{D}\mathrm{D}_{\mathrm{A}_{n-1}} (\mathrm{A}(0) - \mathrm{A}_{n-1}) \|^2 \, \| \mathrm{D}\mathrm{E}_n - \mathrm{D}\mathrm{E}_{n+1} \|^2$$

$$+ | \mathrm{E}_{n+1} |_\infty^2 \, \| \mathrm{D}_{\mathrm{A}(0)} (\mathrm{A}_n - \mathrm{A}_{n-1}) \|^2$$

$$\leqq \| \mathrm{D}\mathrm{D}_{\mathrm{A}_{n-1}} (\mathrm{A}(0) - \mathrm{A}_{n-1}) \|^2 \, \mathscr{R}_{n+1}$$

$$+ | \mathrm{E}_{n+1} |_\infty^2 \, \| \mathrm{D}_{\mathrm{A}(0)} (\mathrm{A}_n - \mathrm{A}_{n-1}) \|^2$$

$$\leqq c \, \| \mathrm{D}\mathrm{D}_{\mathrm{A}(0)} (\mathrm{A}(0) - \mathrm{A}_{n-1}) \|^2 \, \mathscr{R}_{n+1}$$

$$+ | \mathrm{E}_{n+1} |_\infty^2 \, \| \mathrm{D}_{\mathrm{A}(0)} (\mathrm{A}_n - \mathrm{A}_{n-1}) \|^2$$

Using now the Sobolev embedding and 3.30 we get $| \mathrm{E}_{n+1} |_\infty^2 \leqq c \, \mathrm{R}$. The remaining terms involving the potentials A can be bounded by R^2 just by writing $\mathrm{A}_k = \mathrm{A}(0) + \int_0^t \mathrm{E}_{k-1}$ and commuting it with D. It will follow that:

$$\| g_{n+1}^{(2)} \| \leqq c \, \mathrm{R}^2 \, 2^{-n}$$

The other error terms $g_{n+1}^{(3)}$ and $g_{n+1}^{(4)}$ are treated in the same fashion. We conclude:

$$\frac{d}{dt} \mathscr{R}_{n+1} \leqq c \, \mathrm{R} \, \mathscr{R}_{n+1} + c \, \mathrm{R}^2 \, 2^{-n} \, \mathscr{R}_{n+1}^{1/2}$$

Because all iterates have the same Cauchy data then $\mathscr{R}_{n+1}(0) = 0$ and then:

$$\mathscr{R}_{n+1} \leqq c \, \mathrm{R}^2 \, 2^{-n} (e^{c\mathrm{R}t} - 1)$$

Observe that we can choose t_* independently of n such that $c \, \mathrm{R} \, t_* < \log(3/2)$ and therefore:

$$\mathscr{R}_{n+1} \leqq c \, \mathrm{R}^2 \, 2^{-n} . \frac{1}{2} = c \, \mathrm{R}^2 \, 2^{-(n+1)}$$

We recover this way the bounds for the difference of the derivatives. The bounds for the differences of the undifferentiated fields follow from the previous bounds by integration in time and the corresponding bounds for $\mathrm{D}_t \mathrm{E}_{n+1} - \mathrm{D}_t \mathrm{E}_n$. The estimates for the magnetic part are similar and we can conclude finally that the sequence contracts.

This completes the proof of the local existence theorem.

4. SPHERICALLY SYMMETRIC GAUGE FIELDS

We define here the spherically symmetric Yang-Mills fields. This subject has been studied extensely by many authors and we shall only outline the main points (*cf.* references [20], [19], [15], [14], [8] and [21]).

DEFINITION 4.1. — *Consider the principal bundle* $E = R^{3+1} \times G$ *and an action* $SO(3) \times E \to E$ *of the rotation group. If* ω *is a connection* 1*-form on* E *then we say that* ω *is spherically symmetric iff* $s^* \omega = \omega$ *for every element* s *in* $SO(3)$, *where* s^* *is the pull-back induced by the bundle automorphism* $s. : x \mapsto s.x$.

Consider now the canonical action of SO(3) on the base space R^{3+1}. The problem one encounters is that there is no canonical procedure to uniquely lift the $SO(3)$-action on Minkowski space to the whole of E. It can proved that all possible lifts of the action will be in correspondence with homomorphisms $\lambda : SU(2) \to G$ (*cf.* [23]). One says that this mapping determines the type of spherical symmetry of the gauge field F_A. Degenerate cases will correspond to configurations which are either reducible to abelian U(1)-gauge fields or to classical configurations for which no compensating gauge transformation required. This is the case when λ is the trivial homomorphism. The non-abelian configurations described here correspond to the case when λ is an embedding of the rotation group into the gauge group. This can only happen of course when G admits an SU(2) subgroup.

The symmetric potentials are obtained by setting $\Omega = \lambda_* (\gamma_3)$ where γ_3 denotes the generator of the isotropy group of the points in the z-axis. Introducing standard spherical coordinates (θ, φ) the potential will assume the following form:

$$A_0 = a_0 \tag{4.1}$$

$$A_r = a_r \tag{4.2}$$

$$A_\theta = a_\theta \tag{4.3}$$

$$A_\varphi = a_\varphi \sin\theta + \cos\theta\, \Omega \tag{4.4}$$

where the functions a_0, a_r, a_θ and a_φ are functions of (t, r) only satisfying the constraint relations:

$$[a_0, \Omega] = [a_r, \Omega] = 0 \tag{4.5}$$

$$[a_\theta, \Omega] = -a_\varphi \tag{4.6}$$

$$[a_\varphi, \Omega] = a_\theta \tag{4.7}$$

A complete derivation of this Ansatz can be found in [19] and [1] (*See* also [14]). The invariant connections are unique up to gauge transformations that are independent of the angles and take values in the subalgebra generated by Ω. For this reason the gauges 4.1-4.4 are called abelian.

Despite its structural advantages, the abelian gauge is unfortunately a singular gauge. Besides the obvious problem at the origin, the potential has on this gauge string singularities at the lines $\theta=0$ and $\theta=\pi$. This is a gauge artifact and has to do with the fact that (θ, φ) is a coordinate system on S^2-{north pole} only. Introducing a second coordinate system on S^2-{south pole} one obtains a second gauge where the potential is now singular at $\theta=\pi$. If the element Ω satisfies the conditions:

$$e^{4\pi\Omega}=\mathrm{I} \tag{4.8}$$

then we can use the gauge transformations $g=e^{\varphi\Omega}$ and $g=e^{-\varphi\Omega}$ to remove the strings. Observe that the strings at $\theta=0$ and $\theta=\pi$ can be only individually, but not simultaneously, removed.

For the sake of the global existence argument one needs a gauge in which the potentials have good space-regularity. In particular, one must be assured that there exists a gauge in which the string-singularities disappear. This gauge is called in the monopole literature the canonical or the no-string gauge. If the element Ω satisfy 4.8 then we can gauge the string away by means of a singular map. The existence of the canonical gauge is tied to the existence of *su* (2)-subalgebras and one can prove (*cf.* [15]):

PROPOSITION 4.1 (Canonical Gauge). — *There exists a singular gauge transformation bringing the class of abelian gauges* 4.1-4.4 *to a class where the gauge potential* A *assumes the following form:*

$$\mathrm{A}_0=\sum_{l=1}^{\mathrm{N}} \varphi_l(t, r)\mathscr{P}_l \tag{4.9}$$

$$\mathrm{A}_i=\sum_{l=1}^{\mathrm{N}} \psi_l(t, r)\mathscr{P}_l$$

$$+\frac{1}{r}\left[\sum_{l=1}^{\mathrm{N}} (a_{1l}[\mathrm{T}_m, \mathscr{P}_l]\,\varepsilon_{mjn}\frac{x_n}{r}+a_{2l}[\mathrm{T}_j, \mathscr{P}_l])-\mathrm{T}_j\right]\varepsilon_{ijk}\frac{x_k}{r} \tag{4.10}$$

where a_{1l} and a_{2l} are functions of t, r alone, $\mathrm{T}_i=\lambda_*(O_{(i)})$ and $\mathscr{P}_l$ is defined in terms of su(2) representations as follows:

$$\mathscr{P}_j=\sum_{m=-l_j}^{m=l_j} \mathrm{Y}^{l_j}{}_{-m}\left(\frac{x}{r}\right)\mathscr{Y}^{l_j}_m$$

The functions $Y^{l}{}_{\underline{j}\,m}$ *are the standard spherical harmonic functions on the sphere and* $\mathcal{Y}^{l}_{m}$ *are a basis for an* $su(2)$ *representation of dimension* $2l+1$ *labeled by the third eigenvalue* ([3]).

This class is called the class of canonical gauges. Its elements are completely regular except at the central line $r=0$. The proof of proposition 4.1 is a straight-forward computation which consists of composing maps of the '*t* Hooft-type. To avoid unnecessary complications due to cumbersome notation we will assume in the sequel that $G=SU(2)$. In this case the canonical Ansatz reduces to 1.3-1.4.

Remarks. — 1. The canonical gauge admits a residual group of gauge symmetries:

$$g=\exp\left(\theta(t, r)\frac{x^a}{r}T_a\right) \tag{4.11}$$

They act on 1.3-1.4 as follows:

$$^{(g)}A_0^a=(\varphi-\theta_t)\frac{x^a}{r}$$

$$^{(g)}A_i^a=(\psi-\theta_r)\frac{x_i x^a}{r^2}+\frac{^{(g)}f_1}{r}\left(\delta_i^a-\frac{x_i x^a}{r^2}\right)+\frac{^{(g)}f_2-1}{r}\varepsilon^{iab}\frac{x^b}{r}$$

The transverse components $^{(g)}f_1$, $^{(g)}f_2$ suffer a rotation:

$$^{(g)}f_1=f_1\cos\theta-f_2\sin\theta$$
$$^{(g)}f_2=f_1\sin\theta+f_2\cos\theta$$

2. For any potential A in the canonical gauge, there exists an element g of the form 4.11 that brings A to the temporal gauge $A_0=0$, *i.e.*, $^{(g)}A_0=0$. A similar fact holds also for the radial gauge $A_r=0$.

3. The Ansatz 4.9 presented here is not adequate for the point of view of partial differential equations. One needs a characterization of the invariance condition in terms of gauge-invariant differential operators. By commuting these operators with the equations of motion one can set-up energy estimates that will lead to the preservation of the Ansatz by the non-linear flow. The problem of defining gauge invariant angular momentum operators, which incidentally consists of another way to derive the most general invariant connection, has been studied in [7] (*cf.* also [13]). One can prove:

THEOREM 4.1 (Angular Momentum Operators). — *Let* $\hat{\mathcal{L}}_{O_{(i)}}$, $i=1,2,3$, *denote the gauge-invariant angular momentum operators:*

$$\hat{\mathcal{L}}_{O_{(i)}}=\underline{\mathcal{L}}_{O_{(i)}}+[T_i, .] \tag{4.12}$$

([3]) *Cf.* [15].

It follows that a connection A *is spherically symmetric in the sense of Ansatz* 4.9-4.10 *if and only if:*

$$\hat{\mathscr{L}}_{O_{(i)}} A = 0$$

Remarks. — 1. The operators $\hat{\mathscr{L}}_{O_{(i)}}$ satisfy the commutation rules:

$$[\hat{\mathscr{L}}_{O_{(i)}}, \hat{\mathscr{L}}_{O_{(j)}}] - \varepsilon_{ijk} \hat{\mathscr{L}}_{O_{(k)}} = 0$$

Physically $\underline{\mathscr{L}}_{O_{(i)}}$ measures the orbital angular momentum, while $[T_i, .]$ measures the isopin contribution to the total angular momentum.

2. We also remark that if $\hat{\mathscr{L}}_{O_{(i)}} A = 0$ then the curvature F of the potential A will also satisfy $\underline{\mathscr{L}}_{O_{(i)}} F + [T_i, F] = 0$. Proposition 4.1 is a particular case of the result proved in [13]. For a direct proof *see* [22].

In general, when applied to configurations which are non-symmetric *a priori*, one has in general:

LEMMA 4.1. — *The angular momentum operators* 4.12 *satisfy the following properties:*

(i) *The operators* 4.12 *are well defined,* i.e., *they are left invariant with respect to the transformations* 4.11 *preserving the canonical class of gauges.*

(ii) *The operators* $\hat{\mathscr{L}}_O$ *satisfy the following commutation identity:*

$$(\hat{\mathscr{L}}_{O_{(i)}} F_{\alpha\beta})^{;\beta} - \hat{\mathscr{L}}_{O_{(i)}} (F^{;\beta}_{\alpha\beta}) = [F_{\alpha\beta}, \hat{\mathscr{L}}_{O_{(i)}} A^{\beta}]$$

(iii) *If* $F = D_A A$ *expresses the field in terms of* A, *then one has the following relation for the angular derivative of* F:

$$\hat{\mathscr{L}}_O F = D \times (\hat{\mathscr{L}}_O A)$$

The proof of the lemma is a simple calculation.

5. THE INHOMOGENEOUS EQUATIONS

We consider the following system of linear equations:

$$W^{;\beta}_{\alpha\beta} = J_{\alpha} \tag{5.1}$$

$$*W^{;\beta}_{\alpha\beta} = *J_{\alpha} \tag{5.2}$$

W is an arbitrary 2-form $W : R^{3+1} \to \Lambda^2 \mathscr{G}$, *W denotes its Hodge dual and the covariant derivative refers to a fixed background connection A. The 1-forms $J, ^*J : R^{3+1} \to \Lambda^1 \mathscr{G}$ are given 1-forms called the current forms. These equations arise every time one comutes system 1.1-1.2 with a Lie derivative operator and obtain inhomogeneous equations whose source terms consist of non-linear error terms generated by the commutation with the vector fields used in the definition of the global energy norm. We will examine the field W in detail.

Relative to the slicing of Minkowski space induced by T, one defines the electric and the magnetic parts of W as $E_i = W_{io}$, $H = \frac{1}{2}\varepsilon_{ijk} W_{jk}$. Both E and H determine W at all points in space-time and are tangential to the hyperplanes $t = \text{Const}$. The energy-momentum tensor of a 2-form field W is defined as:

$$Q(X, Y) = \frac{1}{2}(\langle i_X W, i_Y W \rangle + \langle i_X * W, i_Y * W \rangle)$$

In local coordinates:

$$Q_{\mu\nu} = \frac{1}{2}(W^{\alpha}_{\mu} . W_{\alpha\nu} + {}^*W^{\alpha}_{\mu} . {}^*W_{\alpha\nu}) \tag{5.3}$$

The energy-momentum tensor Q is a symmetric traceless 2-tensor satisfying the positivity condition $Q(X, Y) \geqq 0$ for any pair of non space-like future-directed vectors X and Y. Computing the divergence:

PROPOSITION 5.1. – *Let* Q *be the energy-momentum tensor of a* 2-*form field* W *satisfying equations* 5.1-5.2. *It follows that:*

$$Q^{;\nu}_{\mu\nu} = W_{\mu\alpha} . J^{\alpha} + {}^*W_{\mu\alpha} . {}^*J^{\alpha}$$

The proof of this proposition is a simple manipulation with tensors and can be achieved by rewriting equations 5.1-5.2 in the form:

$$W_{[\alpha\beta;\nu]} = \varepsilon_{\mu\alpha\beta\nu} * J^{\mu}$$
$$*W_{[\alpha\beta;\nu]} = -\varepsilon_{\mu\alpha\beta\nu} J^{\mu}$$

The uselfulness of the tensor Q relies on the fact that it can be used to set energy norms which will satisfy, in the small-amplitude regime, an almost conservation law. Introduce the momentum vector $P_{\mu} = Q_{\mu\nu} X^{\nu}$. If X is conformally Killing then because of the traceless property of Q we have $\partial . P = (Q^{;\mu}_{\mu\nu}) X^{\nu}$. Integrating this equation on a time slab $R^3 \times [0, t_*]$ one obtains:

PROPOSITION 5.2. – *Let* W *a* 2-*form field satisfying* 5.1-5.2 *in a time interval* $[0, t_*]$. *For a conformally Killing vector field* X *one has the following identity:*

$$\int_{t=t_*} P^0(W)\, dx = \int_{t=0} P^0(W)\, dx + \int_0^{t_*} dt \int_{x_0=t} \{ W_{\mu\alpha} . J^{\alpha} + {}^*W_{\mu\alpha} . {}^*J^{\alpha} \} X^{\mu}\, dx$$

We are only interested in vectors X that lead to a positive quantity $P^0 = Q(T, X)$. This will be the case if X is a future directed timelike vector field. It happens that for R^{n+1} the only timelike conformally Killing vector fields are the time-translation $T=\frac{\partial}{\partial t}$ and the conformal accelleration $K_0=(1+t^2+r^2)\frac{\partial}{\partial t}+2\,tx^i\frac{\partial}{\partial x^i}$. Here we shall use $X=K_0$.

The non-linearities contained in the Yang-Mills equations are quadratic. To achieve global existence we have to find an appropriate version of the null condition. In our context this is equivalent to considering the decomposition of the tensor F with respect to a light-cone coordinate system. Relative to the standard null frame of Minkowski space we define:

DEFINITION 5.1. – *Given a 2-form* W *on Minkowski space, we define the null decomposition* $\{\alpha(W), \underline{\alpha}(W), \rho(W), \sigma(W)\}$ *of* W *as:*

$$\underline{\alpha}_A=F_{A3}$$
$$\alpha_A=F_{A4}$$
$$\rho=\frac{1}{2}F_{34}$$
$$\rho=F_{12}$$

The components α and $\underline{\alpha}$ are 1-forms tangent to the spheres $S^2(r)$ and ρ and σ are scalars, totaling 6 independent components. One defines similarly $*\alpha$, $*\underline{\alpha}\,*\rho$ and $*\sigma$. W and *W are determined completely by its null components. In terms of null coordinates the energy density P_0 is:

$$P_0=Q(W)(T_0, K_0)=\tau_-^2\,|\underline{\alpha}|^2+\tau_+^2\,|\alpha|^2+1/2\,(\tau_-^2+\tau_+^2)(\rho^2+\sigma^2)$$

At the end of this section we would like to write down the expression of the inhomogeneous equations 5.1-5.2 in the null frame. For any 1-form u which is tangent to the spheres $r=$Const., define the spherical operators $\not{D}\,.\,u=\not{D}_A\,u_A$, $\not{D}\times u=\varepsilon_{AB}(\not{D}_A\,u_B-\not{D}_B\,u_A)$. We can prove:

PROPOSITION 5.3. – *Let* $W:R^{3+1}\to\Lambda^2\mathcal{G}$ *a 2-form field satisfying system* 5.1-5.2 *with current forms* $J_{,}^{*}J:R^{3+1}\to\Lambda^1\mathcal{G}$. *In terms of the null decomposition of* W *equations* 5.1 *can be written as:*

$$D_4\,\underline{\alpha}_A+\frac{1}{r}\underline{\alpha}_A+\not{D}_A\,\rho-\varepsilon_{AB}\,\not{D}_B\,\sigma=J_A \tag{5.4}$$

$$D_3\,\alpha_A-\frac{1}{r}\alpha_A-\not{D}_A\,\rho-\varepsilon_{AB}\,\not{D}_B\,\sigma={*J}_A \tag{5.5}$$

$$D_4\,\rho+\frac{2}{r}\rho+\not{D}\,.\,\alpha=-J_4 \tag{5.6}$$

$$D_3\,\rho-\frac{2}{r}\rho-\not{D}\,.\,\underline{\alpha}=J_3 \tag{5.7}$$

$$\mathrm{D}_4\,\sigma+\frac{2}{r}\sigma+\not{\mathrm{D}}\times\alpha=-{}^*\mathrm{J}_4 \tag{5.8}$$

$$\mathrm{D}_3\,\sigma-\frac{2}{r}\sigma+\not{\mathrm{D}}\times\underline{\alpha}={}^*\mathrm{J}_3 \tag{5.9}$$

The derivation is similar to that done for the Maxwell equations (*cf.* [3]).

6. THE ENERGY NORMS AND THE COMPARISON LEMMA

To obtain the crucial L^∞ control on the curvature F one has to resort to a weighted Sobolev norm that encodes information about the different asymptotic behaviour of the null components of F. We then relate this weighted norm to the basic quantities which will stay bounded under the non-linear flow. This is in principle not so obvious since the Yang-Mills equations do not supply equations along all null directions for every null component of F. First we introduce some notation.

Call $\mathcal{Q}_k=\int_{\mathbf{R}^3} \mathrm{Q}(\mathrm{T}_0,\mathrm{K}_0)(\mathcal{L}_{\mathrm{S}}^k\mathrm{F})\,dx$. In terms of the null decomposition:

$$\mathcal{Q}_k=\int_{\mathbf{R}^3}\tau_-^2\,|\underline{\alpha}(\mathcal{L}_{\mathrm{S}}^k\mathrm{F})|^2+\tau_+^2\,|\alpha(\mathcal{L}_{\mathrm{S}}^k\mathrm{F})|^2$$
$$+1/2\,(\tau_-^2+\tau_+^2)\,(\rho(\mathcal{L}_{\mathrm{S}}^k\mathrm{F})^2+\sigma(\mathcal{L}_{\mathrm{S}}^k\mathrm{F})^2\,dx$$

The weighted L^2 norms that will encode the space-time information are defined as follows:

$$\mathcal{F}_m={}^{(\mathrm{I})}\mathcal{F}_m+{}^{(\mathrm{E})}\mathcal{F}_m$$

where:

$${}^{(\mathrm{I})}\mathcal{F}_m=\sum_{i=1}^{m}\int_{|x|\leqq 1+t/2} t^{2+2i}\,|\mathrm{D}^i\mathrm{F}|^2\,dx$$

and:

$${}^{(\mathrm{E})}\mathcal{F}_0=\int_{|x|\geqq 1+t/2}\tau_-^2\,|\underline{\alpha}|^2+\tau_+^2\,|\alpha|^2+\tau_\rho^2\,\rho^2+\tau_+^2\,\sigma^2\,dx$$

$${}^{(\mathrm{E})}\mathcal{F}_m={}^{(\mathrm{E})}\mathcal{F}_{m,1}+{}^{(\mathrm{E})}\mathcal{F}_{m,2}$$

$${}^{(\mathrm{E})}\mathcal{F}_{1,1}=\int_{|x|\geqq 1+t/2}\tau_-^2\,\tau_+^2\,|\mathrm{D}_4\,\underline{\alpha}|^2+\tau_+^4\,|\mathrm{D}_3\,\alpha|^2+\tau_+^4\,|\mathrm{D}_4\,\rho|^2\,dx$$
$$+\int_{|x|\geqq 1+t/2} r_-^2\,\tau_+^2\,|\mathrm{D}_3\,\rho|^2+\tau_+^4\,|\mathrm{D}_4\,\sigma|^2+\tau_-^2\,\tau_+^2\,|\mathrm{D}_3\,\sigma|^2\,dx$$

$$^{(E)}\mathscr{F}_{1,2}=\int_{|x|\geqq 1+t/2} \tau_-^2\,\tau_+^2\,|D_4\,\mathscr{L}_S\,\underline{\alpha}|^2+\tau_+^4\,|D_3\,\mathscr{L}_S\,\alpha|^2+\tau_+^4\,|D_4\,\mathscr{L}_S\,\rho|^2$$

$$+\int_{|x|\geqq 1+t/2} \tau_-^2\,\tau_+^2\,|D_3\,\mathscr{L}_S\,\rho|^2$$

$$+\tau_+^4\,|D_4\,\mathscr{L}_S\,\sigma|^2+\tau_-^2\,\tau_+^2\,|D_3\,\mathscr{L}_S\,\sigma|^2\,dx$$

$$^{(E)}\mathscr{F}_{1,2}=\int_{|x|\geqq 1+t/2} \tau_-^4\,|D_3\,\mathscr{L}_S\,\underline{\alpha}|^2+\tau_+^4\,|D_4\,\mathscr{L}_S\,\alpha|^2\,dx$$

The main theorem of the section is:

THEOREM 6.1 (The Comparison Theorem). — *Let* F *be a spherically symmetric tensor* $F: R^{3+1} \to \Lambda^2 \mathscr{G}$ *satisfying the Yang-Mills system* 1.1-1.2. *Assume that the energy* $\mathscr{Q}_0$ *is sufficiently small:* $\mathscr{Q}_0 \leqq \varepsilon_0 \leqq 1$. *Then one has the following inequalities:*

(i) $\mathscr{F}_1 \leqq c(\mathscr{Q}_0+\mathscr{Q}_1)$;

(ii) $\mathscr{F}_2 \leqq c(\mathscr{Q}_0+\mathscr{Q}_1+\mathscr{Q}_2)$,

The proof of this proposition consists of showing that all the directional derivatives can be expressed in terms of $\mathscr{L}_S F$ and $\mathscr{L}_S^2 F$. The most important point is to prove that the terms containing angular derivatives can be completely controlled by the basic quantities. This is the only where we use the spherically symmetric Ansatz. More precisely, we have:

PROPOSITION 6.1. — *Let* $F: R^{3+1} \to \Lambda^2 \mathscr{G}$ *a spherically symmetric tensor. It follows that in the exterior region* Σ_e *we have:*

$$|r\,\not{D}_A\,\omega_B|^2 \leqq c(1+|r^2\,\sigma|_{\infty,\,ext}^{1/2})\,|\omega|^2$$

for every component ω_B *of* F *relative to the null frame and every derivative* $\not{D}_A$.

In particular, if $|r^2\,\sigma|_{\infty,\,ext} \leqq 1$ then the angular derivative terms are simply bounded by lower order terms that have been estimated before.

Proof. — We consider initially the SU (2) case. Breaking up the covariant derivative and the gauge-covariant derivative we have:

$$\not{D}_A\,\omega_B = D_A\,\omega_B + \text{lower order terms}$$
$$= \not{\partial}_A\,\omega_B + [\not{A}_A,\,\omega_B] + l.o.t$$

The lower order terms are directly estimated by $r^{-1}|\omega|$. The first term is treated by applying the Lie derivatives $\mathscr{L}_{\underline{O}_{(i)}}$. For an arbitrary component

ω_B we have:

$$\begin{aligned}|r \not{\partial}_A \omega_B|^2 &\leqq \sum_{i=1}^{3} |\mathscr{L}_{O(i)} \omega_B|^2 \\ &= \sum_{i=1}^{3} |-[T_i, \omega_B]|^2 \\ &\leqq c|\omega|^2\end{aligned}$$

The second term requires a more subtle argument. To control the tangential components $\not{A}$ of the potential we go back to the Ansatz 4.9-4.10 and observe that the normal component σ of the curvature is completely constrained by the tangential components of the potential. Indeed, computing the magnetic curvature

$$\begin{aligned}H_i^a = \frac{1}{r^2}(f_1^2+f_2^2-1)\frac{x^i x^a}{r^2} + +\left(\frac{f_2'+f_1\psi}{r}\right)\left(\delta_i^a - \frac{x_i x^a}{r^2}\right) \\ +\left(\frac{-f_1'+f_2\psi}{r}\right)\varepsilon^{iab}\frac{x^b}{r}\end{aligned}$$

In particular:

$$\sigma^a = \frac{1}{r^2}(f_1^2+f_2^2-1)\frac{x^a}{r}$$

and therefore $|r^2\sigma| = |f_1^2+f_2^2-1|$. It follows that one obtains the estimate:

$$|r\not{A}| \leqq c(1+|r^2\sigma|^{1/2})$$

For higher groups this estimate is proved in the same way since it is still true that σ is constrained by $\not{A}$:

$$\sigma = \frac{1}{r^2}\sum_{l=1}^{N}(a_{1l}^2+a_{2l}^2-1)\mathscr{P}_l$$

This completes the proof of the proposition.

PROPOSITION 6.2. — *We have the following bound for the* L^∞ *exterior norm:*

$$|r^2\sigma|_{\infty,\,ext} \leqq c\,\mathscr{Q}_0 + c\,\mathscr{Q}_0^{1/2}$$

To prove this proposition we will need the following form of the Sobolev embedding theorem:

PROPOSITION 6.2 (cf. [3]). — *For any Lie algebra valued tensor* $F: R^{3+1} \to \mathscr{G}$ *for which* $|F|$ *is an ordinary spherically symmetric function* *on* $\mathbb{R}^{3+1}$ *we have:*

(i)

$$|r^{3/2} F|_{\infty,\,\text{ext}} \leqq c_{\text{sob}} \left(\int_{|x| \geqq 1+t/2} |F|^2 + r^2 |D_N F|^2 \, dx \right)^{1/2}$$

(ii)

$$|r \tau_-^{1/2} F|_{\infty,\,\text{ext}} \leqq c_{\text{sob}} \left(\int_{|x| \geqq 1+t/2} |F|^2 + \tau_-^2 |D_N F|^2 \, dx \right)^{1/2}$$

The inequalities follow by applying Kato's argument (4) to corresponding inequalities in [3]. Now, going back to the proof of proposition 6.2:

$$\begin{aligned} |r^2 \sigma|^2_{\infty,\,\text{ext}} &\leqq c \int_{|x| \geqq 1+t/2} r^2 \sigma^2 + \tau_-^2 r^2 |D_N \sigma|^2 \, dx \\ &\leqq c \int_{|x| \geqq 1+t/2} r^2 \sigma^2 + \tau_-^2 r^2 |D_4 \sigma - D_3 \sigma|^2 \, dx \end{aligned}$$

Using the null equations for the components ρ, σ, namely, the equations $D_4 \sigma = -\dfrac{2}{r} \sigma - \not{D} \times \alpha$, $D_3 \sigma = \dfrac{2}{r} \sigma - \not{D} \times \underline{\alpha}$ and proposition 6.1 we find:

$$\begin{aligned} |r^2 \sigma|^2_{\infty,\,\text{ext}} &\leqq c\,\mathcal{Q}_0 + \int_{|x| \geqq 1+t/2} (1 + |r^2 \sigma|_{\infty,\,\text{ext}}) (\tau_-^2 |\underline{\alpha}|^2 + \tau_+^2 |\alpha|^2) \, dx \\ &\leqq c\,\mathcal{Q}_0 + c\,|r^2 \sigma|_{\infty,\,\text{ext}} \mathcal{Q}_0 \end{aligned}$$

From which follows that $|r^2 \sigma|_{\infty,\,\text{ext}} \leqq c\,\mathcal{Q}_0 + c\,\mathcal{Q}_0^{1/2}$.

Proof of the Comparison Theorem 6.1. – We start with the Yang-Mills equations in the null form. They contain all derivatives that make up the norm ${}^{(E)}\mathcal{F}_{1,1}$:

$$\begin{aligned} {}^{(E)}\mathcal{F}_{1,1} \leqq c_0 &\int_{|x| \geqq 1+t/2} \\ &\times (\tau_-^2 |\underline{\alpha}|^2 + \tau_+^2 |\alpha|^2 + 1/2 (\tau_-^2 + \tau_+^2)(\rho^2 + \sigma^2)) \, dx \\ &+ c_0 \int_{|x| \geqq 1+t/2} \\ &\times r^2 (\tau_-^2 |\not{D} \underline{\alpha}|^2 + \tau_+^2 |\not{D} \alpha|^2 + \tau_+^2 (\not{D} \rho^2 + \not{D} \sigma^2)) \, dx \end{aligned}$$

---

(4) Kato's inequality asserts that $|\partial |F|| \leqq |DF|$ a.e. for a $\mathcal{G}$-valued function F. More details in [15] and [12].

$$\leqq c\,\mathcal{Q}_0 + c\int_{|x|\geqq 1+t/2} \times (1+|r^2\sigma|_{\infty,e})(\tau_-^2\,|\underline{\alpha}|^2+\tau_+^2\,|\alpha|^2+\tau_+^2\,\rho^2+\tau_+^2\,\sigma^2)\,dx$$
$$\leqq C\,\mathcal{Q}_0$$

The problem consists of the mixed derivatives $D_3\,\underline{\alpha}$ and $D_4\,\alpha$. Using $\mathcal{L}_S$ we write:

$$D_S\,\underline{\alpha}_A=(t-r)\,D_3\,\underline{\alpha}_A+(t+r)\,D_4\,\underline{\alpha}_A=\mathcal{L}_S\,\underline{\alpha}_A-2\,\underline{\alpha}_A$$
$$D_S\,\alpha_A=(t-r)\,D_3\,\alpha_A+(t+r)\,D_4\,\alpha_A=\mathcal{L}_S\,\alpha_A-2\,\alpha_A$$

It follows that:

$$^{(E)}\mathcal{F}_{1,2}\leqq c_0\int_{|x|\geqq 1+t/2} \tau_-^2\,|\mathcal{L}_S\,\underline{\alpha}|^2+\tau_+^2\,|\mathcal{L}_S\,\alpha|^2$$
$$+c_0\int_{|x|\geqq 1+t/2} \tau_-^2\,|\underline{\alpha}|^2+\tau_+^2\,|\alpha|^2+\tau_-^2\,\tau_+^2\,|D_4\,\underline{\alpha}|^2+\tau_+^4\,|D_3\,\alpha|^2\,dx$$
$$\leqq c_0(\mathcal{Q}_0+\mathcal{Q}_1)$$

The second derivatives are estimate similarly. First we commute the Yang-Mills equations with the operator $\mathcal{L}_S$. Using proposition 5.3 we write:

$$D_4(\mathcal{L}_S\,\underline{\alpha}_A)+\frac{1}{r}\mathcal{L}_S\,\underline{\alpha}_A+\not{D}_A(\mathcal{L}_S\,\rho)-\varepsilon_{AB}\,\not{D}_B(\mathcal{L}_S\,\sigma)=\underline{I}_A$$
$$D_3(\mathcal{L}_S\,\alpha_A)+\frac{1}{r}\mathcal{L}_S\,\alpha_A+\not{D}_A(\mathcal{L}_S\,\rho)-\varepsilon_{AB}\,\not{D}_B(\mathcal{L}_S\,\sigma)=I_A$$
$$D_4(\mathcal{L}_S\,\rho)+\frac{2}{r}\mathcal{L}_S\,\rho+\not{D}\,.\,\mathcal{L}_S\,\alpha=K$$
$$D_3(\mathcal{L}_S\,\rho)-\frac{2}{r}\mathcal{L}_S\,\rho-\not{D}\,.\,\mathcal{L}_S\,\underline{\alpha})=\underline{K}$$
$$D_4(\mathcal{L}_S\,\sigma)+\frac{2}{r}\mathcal{L}_S\,\sigma+\not{D}\times(\mathcal{L}_S\,\alpha)=L$$
$$D_3(\mathcal{L}_S\,\sigma)-\frac{2}{r}\mathcal{L}_S\,\sigma+\not{D}\times(\mathcal{L}_S\,\underline{\alpha})=\underline{L}$$

where the error terms are given by:

$$I_A=\frac{1}{r}\underline{\alpha}_A-(t-r)\,[\rho,\,\underline{\alpha}_A]+D_4\,\underline{\alpha}_A$$
$$+(t-r)\,[\underline{\alpha}_A,\,\rho]+(t+r)\,[\alpha_A,\,\rho]$$
$$-\varepsilon_{AB}((t-r)\,[\underline{\alpha}_B,\,\sigma]+(t+r)\,[\alpha_B,\,\sigma])$$

$$\underline{\mathrm{I}}_{\mathrm{A}} = -\frac{1}{r}\alpha_{\mathrm{A}} - (t+r)[\rho, \alpha_{\mathrm{A}}] - \mathrm{D}_3\alpha_{\mathrm{A}}$$
$$-(t-r)[\underline{\alpha}_{\mathrm{A}}, \rho] + (t+t)[\alpha_{\mathrm{A}}, \rho]$$
$$-\varepsilon_{\mathrm{AB}}((t-r)[\underline{\alpha}_{\mathrm{B}}, \sigma] + (t+r)[\alpha_{\mathrm{B}}, \sigma])$$
$$\mathrm{K} = \frac{2}{r}\rho + \mathrm{D}_4\rho + (t-r)[\underline{\alpha}_{\mathrm{A}}, \alpha_{\mathrm{A}}]$$
$$\underline{\mathrm{K}} = -\frac{1}{2}2/r\,\rho - \mathrm{D}_3\rho - (t+r)[\alpha_{\mathrm{A}}, \underline{\alpha}_{\mathrm{A}}]$$
$$= \frac{2}{r}\sigma - (t-r)[\rho, \sigma] + \mathrm{D}_4\sigma$$
$$+(t-r)\varepsilon_{\mathrm{AB}}[\underline{\alpha}_{\mathrm{A}}, \alpha_{\mathrm{B}}] + (t+r)\varepsilon_{\mathrm{AB}}[\alpha_{\mathrm{A}}, \alpha_{\mathrm{B}}]$$
$$\underline{\mathrm{L}} = -\frac{2}{r}\sigma + (t+r)[\rho, \sigma] - \mathrm{D}_3\sigma$$
$$+(t-r)\varepsilon_{\mathrm{AB}}[\underline{\alpha}_{\mathrm{A}}, \underline{\alpha}_{\mathrm{B}}] + (t+r)\varepsilon_{\mathrm{AB}}[\alpha_{\mathrm{A}}, \underline{\alpha}_{\mathrm{B}}]$$

Proceeding as before:

$$^{(\mathrm{E})}\mathscr{F}_{2,1} \leqq c(\mathscr{Q}_0 + \mathscr{Q}_1) + \text{N.L.E.}$$

where N.L.E. denotes:

$$\text{N.L.E.} = \int_{|x| \geqq 1+t/2} \tau_+^2(|\mathrm{I}|^2 + |\mathrm{K}|^2 + |\mathrm{L}|^2) + \tau_-^2(|\underline{\mathrm{I}}|^2 + |\underline{\mathrm{K}}|^2 + |\underline{\mathrm{L}}|^2)\,dx$$

These terms are mostly lower-order terms:

$$\text{N.L.E.} \leqq 2\mathscr{Q}_0 + \mathscr{Q}_1 + 2(|\tau_-\underline{\alpha}|^2_{\infty,\,\mathrm{ext}} + |\tau_+\alpha|^2_{\infty,\,\mathrm{ext}}) \cdot \mathscr{Q}_0$$

Using again 6.2 to estimate the L^∞ norm of the fields we get:

$$|\tau_+\alpha|^2_{\infty,\,\mathrm{ext}} + |\tau_-\underline{\alpha}|^2_{\infty,\,\mathrm{ext}} \leqq 2(\mathscr{Q}_0 + {}^{(\mathrm{E})}\mathscr{F}_1)$$
$$\leqq c(\mathscr{Q}_0 + \mathscr{Q}_1)$$

It follow that:

$$^{(\mathrm{E})}\mathscr{F}_{2,1} \leqq c(\mathscr{Q}_0 + \mathscr{Q}_1) + c\,\mathscr{Q}_0(\mathscr{Q}_0 + \mathscr{Q}_1)$$

The missing derivatives $\mathrm{D}_3\mathscr{L}_{\mathrm{S}}\underline{\alpha}$ and $\mathrm{D}_4\mathscr{L}_{\mathrm{S}}\alpha$ are estimated exactly as before with the help of the scaling vector field S. We conclude then:

THEOREM 6.3 (Exterior Estimates). – *The components* $\{\alpha, \underline{\alpha}, \rho, \sigma\}$ *of the null decomposition of the Yang-Mills tensor* F *satisfy the following* *inequalities in the exterior region:*

$$\sup_{x \in \Sigma_e} \tau_+^1 \tau_-^{3/2} |\underline{\alpha}| \leqq c(\mathcal{Q}_0 + \mathcal{Q}_1)^{1/2}$$

$$\sup_{x \in \Sigma_e} \tau_+^{5/2} |\alpha| \leqq c(\mathcal{Q}_0 + \mathcal{Q}_1)^{1/2}$$

$$\sup_{x \in \Sigma_e} \tau_+^2 \tau_-^{1/2} |\rho| \leqq c(\mathcal{Q}_0 + \mathcal{Q}_1)^{1/2}$$

$$\sup_{x \in \Sigma_e} \tau_+^2 \tau_-^{1/2} |\sigma| \leqq c(\mathcal{Q}_0 + \mathcal{Q}_1)^{1/2}$$

We remark only the maximal rate of decay $r^{-5/2}$ for the component α. This is a consequence of the null equation for $D_3 \alpha_A$ which allowed us to take a higher weight in the norm ${}^{(E)}\mathcal{F}_m$. As we shall see later this will be crucial to get global existence. To complete the proof of theorem 6.1 we have to estimate the interior norms ${}^{(I)}\mathcal{F}_m$. Because the estimates have a completely different flavor we present them separately.

7. INTERIOR ESTIMATES

The null decomposition is not defined in the central line $r=0$ and we have to estimate the fields differently in the interior region $|x| \leqq 1 + t/2$. We use elliptic theory for Hodge systems in three space dimensions.

LEMMA 7.1. – *Let* $U : R^3 \to \mathcal{G}$ *a* 1*-form on* R^3. *Define the operators* $(D \times U)_{ij} = D_i U_j - D_j U_i$ *and* $D . U = D_i U_i$. *It follows that:*

$$\int |DU|^2 dx = \int |D \times U|^2 + \frac{1}{2} |D . U|^2 + U . [F, U] \, dx$$

where $U . [F, U] = U_i . [F_{ij}, U_j]$.

Proof:

$$\int |D \times U|^2 dx = \int (D \times U)_{ij} . (D \times U)^{ij}$$

$$= 2 \int |DU|^2 dx - 2 \int D_j U_i . D^i U^j dx$$

$$= 2 \int |DU|^2 dx + 2 \int U_i . D^j D^i U_j dx$$

$$= 2 \int |DU|^2 + 2 \int U_i . D^i (D^j U_j) + 2 \int U_i . [F^{ji}, U^i] \, dx$$

$$= 2 \int |DU|^2 - 2 \int D^i U_i . D^j U_j - 2 \int U_i . [F_{ij}, U^i] \, dx$$

By a slight modification of the proof we obtain:

LEMMA 7.2. – *Let* E, $H:R^3\to\mathcal{G}$ *two* 1-*forms on* R^3 *and* X *a vector field on* R^3 *such that* $\sup|X|<1$, *where the* sup *is taken over the support of the forms* E and H. *Assume that these forms satisfy the Hodge system:*

$$\begin{aligned} D.E&=\rho_E\\ D\times E-D_X H&=\sigma_E\\ D.H&=\rho_H\\ D\times H-D_X E&=\sigma_H \end{aligned}$$

Then:

$$\int|DE|^2+|DH|^2\,dx\leqq c\int|\rho_E|^2+|\rho_H|^2+|\sigma_E|^2 \\ +|\sigma_H|^2+|F|.(|E|^2+|H|^2)\,dx$$

The interior estimates are achieved by means of truncated fields. Consider the cut-off function:

$$\varphi=\begin{cases}1 & \text{if } |x|\leqq 1+t/2\\ 0 & \text{if } |x|\geqq 1+3/4\,t\end{cases}$$

The function φ can be constructed in such a way to satisfy also the condition $|\nabla\varphi|\leqq c/t$ uniformly in space. We want to prove the following result:

PROPOSITION 7.1. – *If the energy* $\mathcal{Q}_0\leqq\varepsilon_0$ *is small enough, then we obtain the following bounds for the interior norms:*

(i)

$$\int_I|DE|^2+|DH|^2\,dx\leqq ct^{-4}(\mathcal{Q}_0+\mathcal{Q}_1)$$

(ii)

$$\int_I|D^2E|^2+|D^2H|^2\,dx\leqq ct^{-6}(\mathcal{Q}_0+\mathcal{Q}_1+\mathcal{Q}_2+(\mathcal{Q}_0+\mathcal{Q}_1)^2)$$

Let us show how to obtain the first derivative estimates. Truncating the fields as $\tilde{E}=\varphi E$, $\tilde{H}=\varphi H$ and defining the vector field $X=\frac{1}{t}Z$, where Z is the radial vector field $Z=x^i\partial_i$, we obtain:

$$\begin{aligned} D.\tilde{E}&=\tilde{\rho}_E\\ D\times\tilde{E}+D_X\tilde{H}&=\tilde{\sigma}_E\\ D.\tilde{H}&=\tilde{\rho}_H\\ D\times\tilde{H}-D_X\tilde{E}&=\tilde{\sigma}_H \end{aligned}$$

where:

$$\tilde{\rho}_E = (\nabla \varphi) E$$
$$\tilde{\rho}_H = (\nabla \varphi) H$$
$$\tilde{\sigma}_E = -\frac{1}{t}\varphi(\mathscr{L}_S E) + \frac{2}{t}\varphi E + \nabla\varphi \wedge H + (\nabla_X \varphi) E$$
$$\tilde{\sigma}_H = \frac{1}{t}\varphi(\mathscr{L}_S H) - \frac{2}{t}\varphi + \nabla\varphi \wedge E + (\nabla_X \varphi) H$$

Applying now Lemma 7.2 and remarking that on the extended interior region $\hat{I} = \{x \| x | \leqq 1 + 3/4\, t\}$, $t \geqq 8$, we have $|X| \leqq 7/8$, we find:

$$\begin{aligned}
&\int |D\tilde{E}|^2 + |D\tilde{H}|^2\, dx \\
&\leqq c \int |\nabla \varphi|^2 (|E|^2 + |H|^2) + \frac{1}{t^2}(|\mathscr{L}_S E|^2 + |\mathscr{L}_S H|^2) \\
&\quad + \int \frac{4}{t^2}(|E|^2 + |H|^2) + 2|\nabla\varphi|^2(|E|^2 + |H|^2)\, dx \\
&\quad + \int |F|(|\tilde{E}|^2 + |\tilde{H}|^2)\, dx
\end{aligned}$$

It follows that:

$$\int |D\tilde{F}|^2\, dx \leqq c t^{-4}(\mathscr{Q}_0 + \mathscr{Q}_1) + c\int |F|.(|\tilde{E}|^2 + |\tilde{H}|^2)\, dx$$

where $\tilde{F}$ denotes $\tilde{E}$ and $\tilde{H}$. The last term is estimated as follows:

$$\begin{aligned}
\int |F|.(|\tilde{E}|^2 + |\tilde{H}|^2)\, dx &\leqq \|F\|_{L^{3/2}(\hat{I})} \left(\int |\tilde{E}|^6 + |\tilde{H}|^6\, dx\right)^{1/3} \\
&\leqq \|F\|_{L^{3/2}(\hat{I})}\, c_{sob} \int |D\tilde{F}|^2\, dx
\end{aligned}$$

The $L^{3/2}$ norm of F is bounded:

$$\begin{aligned}
\int_{\hat{I}} |F|^{3/2}\, dx = \int_{\hat{I}} |F|^{3/2}.1\, dx &\leqq \left(\int_{\hat{I}} |F|^2\, dx\right)^{3/4} \left(\int_{\hat{I}} 1\, dx\right)^{1/4} \\
&\leqq t^{-3/2}\, \mathscr{Q}_0^{3/4}\, t^{3/4} \\
&\leqq t^{-3/4}\, \mathscr{Q}_0^{3/4}
\end{aligned}$$

and therefore $\|F\|_{L^{3/3}(\hat{I})} \leqq t^{-1/2}\, \mathscr{Q}_0^{1/2}$. We conclude that:

$$\int |D\tilde{F}|^2\, dx \leqq c t^{-4}(\mathscr{Q}_0 + \mathscr{Q}_1) + c_{sob}\, t^{-1/2}\, \mathscr{Q}_0^{1/2}. \int |D\tilde{F}|^2\, dx$$

In particular, if $t \geqq 1$ and $c \mathscr{Q}_0^{1/2} \leqq 1/2$ then

$$\int_{\mathrm{I}} |\mathrm{DE}|^2 + |\mathrm{DH}|^2 \, dx \leqq c t^{-4} (\mathscr{Q}_0 + \mathscr{Q}_1)$$

This completes the estimates of the first derivatives. The second derivatives of F are bounded similarly, by commuting the equations of motion with the operator D_j and applying again Lemma 7.2. We leave the details to the reader since they follow the same pattern of the previous estimates. The following L^∞ bound will follow then:

THEOREM 7.1 (Interior Estimate). – *For a solution* F *of the Yang-Mills system with small energy* $\mathscr{Q}_0 \leqq \varepsilon_0$ *we have the following estimate:*

$$|\mathrm{F}|_{\infty, \mathrm{I}}^2 \leqq c (1+t)^{-5} (\mathscr{Q}_0 + \mathscr{Q}_1 + \mathscr{Q}_2 + (\mathscr{Q}_0 + \mathscr{Q}_1)^2)$$

This will follow from the following version of the classical Sobolev inequality:

THEOREM 7.2. – *For any Lie algebra valued tensor* $\mathrm{F} : \mathrm{R}^{3+1} \to \mathscr{G}$ *we have:*

$$|\mathrm{F}|_{\infty, \mathrm{B}(1+t/2)} \leqq c_{\mathrm{sob}} (1+t)^{-3/2} \times \left(\int_{|x| \leqq 3/2\,(1+t/2)} |\mathrm{F}|^2 + t^2 |\mathrm{DF}|^2 + t^4 |\mathrm{D}^2 \mathrm{F}|^2 \, dx \right)^{1/2}$$

The proof of this result consists first of getting a classical Sobolev inequality in the unit ball:

$$|\mathrm{F}|_{\infty, \mathrm{B}(1)} \leqq c \sum_{i=0}^{2} \| \mathrm{D}^i \mathrm{F} \|_{\mathrm{L}^2 (\mathrm{B}(2))} \tag{7.1}$$

This follow from the inequalities:

$$|\mathrm{F}|_\infty \leqq c (\| \mathrm{F} \|_{\mathrm{L}^6} + \| \mathrm{DF} \|_{\mathrm{L}^6}) \tag{7.2}$$

$$\| \mathrm{F} \|_{\mathrm{L}^6} \leqq c (\| \mathrm{F} \|_{\mathrm{L}^2} + \| \mathrm{DF} \|_{\mathrm{L}^2}) \tag{7.3}$$

The interpolation through L^6 is necessary since Kato's argument applies only to one derivative at a time (*cf.* reference [12]). The proof of theorem 7.2 now follows by a classical scaling argument. Set $\mathrm{R} = 1 + \frac{t}{2}$ and introduce a rescaled variable y by $x = \mathrm{R} y$. Define a new function and a new connection:

$$\tilde{\mathrm{F}}(y) = \mathrm{F}(x)$$

$$\tilde{\mathrm{A}}(y) = \mathrm{RA}(x)$$

The functions $\tilde{\mathrm{F}}$, $\tilde{\mathrm{A}}$ are now functions on the ball of radius 1. Because of the way we have rescaled the gauge connection A we have that $\mathrm{D}_{\tilde{\mathrm{A}}}^k \tilde{\mathrm{F}} = \mathrm{R}_{\mathrm{A}}^k \mathrm{F}$ and then we are in position to apply inequality 7.1. This completes the proof of the theorem.

8. THE ERROR ESTIMATES

We prove here the most important result of the paper. It concerns the bounds on the basic quantities $\mathcal{Q}_k$:

THEOREM 8.1. – *Let* F *a local solution of system* 1.1-1.2 *satisfying the small energy requirement* $\mathcal{Q}_0 \leqq \varepsilon_0$ *and the* a priori *bound* $\mathcal{Q}_k(t) \leqq 1$ *on the domain* $[0, t_*]$ *of time-existence. It follows that the basic quantities satisfy the following bounds:*

$$\mathcal{Q}_k(t) \leqq c\,(\mathcal{Q}_0(0) + \mathcal{Q}_1(0) + \mathcal{Q}_2(0))$$

for $k = 1, 2$.

The proof of this theorem is based on the almost conservation laws 5.2 for $\mathcal{Q}_k$. We use the following notation:

$$\mathcal{Q}_k^* = \sup_{0 \leqq t \leqq t_*} \mathcal{Q}_k(t) \tag{8.1}$$

$$\mathcal{Q}^* = \mathcal{Q}_0^* + \mathcal{Q}_1^* + \mathcal{Q}_2^* \tag{8.2}$$

We will prove that $\mathcal{Q}^* \leqq c\,(\mathcal{Q}_0(0) + \mathcal{Q}_1(0) + \mathcal{Q}_2(0))$. For this matter we need to compute the error terms generated by the commutation with the Lie derivative $\mathcal{L}_{\mathrm{S}}$:

PROPOSITION 8.1 (The Non-linear Error Terms. – *Let* $\mathrm{F} : \mathrm{R}^{3+1} \to \Lambda \mathcal{G}$ *be a solution of solution of system* 1.1-1.2. *The Lie derivatives* $\mathcal{L}_{\mathrm{S}} \mathrm{F}$ *and* $\mathcal{L}_{\mathrm{S}}^2 \mathrm{F}$ *satisfy then the following inhomogeneous equations:*

FOR $\mathcal{L}_{\mathrm{S}}$:

$$(\mathcal{L}_{\mathrm{S}} \mathrm{F})_{\alpha\beta}^{\beta} = [(i_{\mathrm{S}} \mathrm{F})^{\beta},\ \mathrm{F}_{\alpha\beta}]$$
$$(\mathcal{L}_{\mathrm{S}}^* \mathrm{F})_{\alpha\beta}^{;\beta} = [(i_{\mathrm{S}} \mathrm{F})^{\beta},\ {}^*\mathrm{F}_{\alpha\beta}]$$

FOR $\mathcal{L}_{\mathrm{S}}^2$:

$$\begin{aligned}(\mathcal{L}_{\mathrm{S}}^2 \mathrm{F})_{\alpha\beta}^{;\beta} &= [(i_{\mathrm{S}} \mathrm{F})^{\beta},\ \mathcal{L}_{\mathrm{S}} \mathrm{F}_{\alpha\beta}] \\ &\quad + [i_{\mathrm{S}}(\mathcal{L}_{\mathrm{S}} \mathrm{F})^{\beta},\ \mathrm{F})^{\beta},\ \mathrm{F}_{\alpha\beta}] + 2\,[(i_{\mathrm{S}} \mathrm{F})^{\beta},\ \mathrm{F}_{\alpha\beta}] \\ (\mathcal{L}_{\mathrm{S}}^{2*} \mathrm{F})_{\alpha\beta}^{;\beta} &= [(i_{\mathrm{S}} \mathrm{F})^{\beta},\ \mathcal{L}_{\mathrm{S}}^* \mathrm{F}_{\alpha\beta}] \\ &\quad + [i_{\mathrm{S}}(\mathcal{L}_{\mathrm{S}} \mathrm{F})^{\beta},\ \mathrm{F})^{\beta},\ {}^*\mathrm{F}_{\alpha\beta}] + 2\,[(i_{\mathrm{S}} \mathrm{F})^{\beta},\ {}^*\mathrm{F}_{\alpha\beta}]\end{aligned}$$

The proof of this proposition will be a consequence of:

LEMMA 8.1. — *For every 2-form* W *we have:*

$$(\mathscr{L}_S W)_{\alpha\beta}^{;\beta} - \mathscr{L}_S (W_{\alpha\beta}^{;\beta}) = [(i_S F)^\beta, W_{\alpha\beta}] + 2 W_{\alpha\beta}^{;\beta}$$

This follows trivially from the curvature relation $[D_\mu, D_\nu] u = [F_{\mu\nu}, u]$.

To estimate the error terms in a more systematic way, we introduce the following notation. Given a vector field X and tensors F, G and H we define the trilinear expression ${}^{(X)}Q(F, G, H) = {}^{(X)}Q_\mu(F, G, H)$ as:

$${}^{(X)}Q_\mu(F, G, H) = F_{\alpha\mu} . [(i_X G)_\beta, H^{\alpha\beta}] + {}^*F_{\alpha\mu} . [(i_X G)_\beta, {}^*H^{\alpha\beta}]$$

The most difficult part of the theorem consists of showing the error terms on the right-hand side of 5.2 are bounded. For this matter one uses L^∞-decay estimates for the curvature F. Here one needs a very precise knowledge of the structure of the error terms, due to the fact that on the wave zone $|x| = 1 + t$ the components of F have a non-uniform behaviour. In particular there exists a slowly decaying component $\underline{\alpha} = O(t^{-1})$. If one does not examine the error terms in detail one can encounter terms which would lead (if present) to logarithmic divergences. The fact that these terms are not present is nothing but the null condition. It follows here from the covariant form of the equations. We state it in the following form:

PROPOSITION 8.2 (Null Condition). — *For every vector* $X = X^3 e_3 + X^4 e_4$ *and tensors* F, G, H *we have:*

$$\begin{aligned} Q_3(F, G, H) = {} & (i_X G)_A . ([\varepsilon_{BA} \sigma(H), \underline{\alpha}_B(F)] - [\varepsilon_{BA} \sigma(F), \underline{\alpha}_B(H)] \\ & - [\underline{\alpha}_A(H), \rho(F)] + [\underline{\alpha}_A(F), \rho(H)] \\ & + (i_X G)_3 . (3 [\rho(F), \rho(H)] - [\sigma(H), \sigma(F)]) \\ & + (i_X G)_4 . (2 [\underline{\alpha}_A(F), \underline{\alpha}_A(H)] + \text{l.o.t} \\ Q_4(F, G, H) = {} & (i_X G)_A . ([\varepsilon_{BA} \sigma(H), \alpha_B(F)] - [\varepsilon_{BA} \sigma(F), \alpha_B(H)] \\ & + [\alpha_A(H), \rho(F)] - [\alpha_A(F), \rho(H)] \\ & + (i_X G)_4 . (3 [\rho(F), \rho(H)] - [\sigma(H), \sigma(F)]) \\ & + (i_X G)_3 . (2 [\alpha_A(F), \alpha_A(H)] + \text{l.o.t} \end{aligned}$$

Here l.o.t denotes lower order terms which are completely harmless. The proof of this proposition consists of a completely tedious but trivial computation, and will therefore be omited. Because we will contract Q with a vector field which has components only in the null directions, only Q_3 and Q_4 are relevant.

The proof of Theorem 8.1 follows now by energy estimates. Let us prove the first bound. Fix an arbitrary time t_0 between 0 and t_* and apply proposition 5.2 with multiplier K_0:

$$\int_{t=t_0} P_0(\mathscr{L}_S F)\,dx = \int_{t=0} P_0(\mathscr{L}_S F)\,dx + \int_0^{t_0} dt \int_{x_0=t} Q_\mu K^\mu\,dx$$

with Q given by:

$$Q_\mu = {}^{(S)}Q_\mu(\mathscr{L}_S F, F, F)$$

All the terms are estimated in the same way. In every term one factor is taken in L^∞ norm and the other two in L^2, using the Cauchy-Schwarz inequality. The most sensitive terms are the terms containing the maximal weights τ_+. To avoid divergences we exploit the improved behavior of the component $\alpha = F_{A4}$ plus a simple integration by parts in space-time.

Let us call by I the second integral on the right-hand side of the previous equation. Calling further $I = I_3 + I_4$, we have:

$$\begin{aligned}
I_3 = & \iint (i_S F)_A \cdot [\varepsilon_{AB}\sigma, \mathscr{L}_S \underline{\alpha}_B]\,\tau_-^2\,dx\,dt \\
& + \iint (i_S F)_A \cdot [\varepsilon_{AB}\underline{\alpha}_B, \mathscr{L}_S \sigma]\,\tau_-^2\,dx\,dt \\
& - \iint (i_S F)_A \cdot [\underline{\alpha}_A, \mathscr{L}_S \rho]\,\tau_-^2\,dx\,dt \\
& + \iint (i_S F)_A \cdot [\mathscr{L}_S \underline{\alpha}_A, \rho]\,\tau_-^2\,dx\,dt \\
& + 3 \iint (i_S F)_3 \cdot [\mathscr{L}_S \rho, \rho]\,\tau_-^2\,dx\,dt \\
& - \iint (i_S F)_3 \cdot [\sigma, \mathscr{L}_S \sigma]\,\tau_-^2\,dx\,dt \\
& + 2 \iint (i_S F)_4 \cdot [\mathscr{L}_S \underline{\alpha}_A, \underline{\alpha}_A]\,\tau_-^2\,dx\,dt + \text{l.o.t} \\
I_4 = & \iint (i_S F)_A \cdot [\varepsilon_{AB}\sigma, \mathscr{L}_S \alpha_B]\,\tau_+^2\,dx\,dt \\
& - \iint (i_S F)_A \cdot [\varepsilon_{AB}\alpha_B, \mathscr{L}_S \sigma]\,\tau_+^2\,dx\,dt \\
& + \iint (i_S F)_A \cdot [\alpha_A, \mathscr{L}_S \rho]\,\tau_+^2\,dx\,dt \\
& - \iint (i_S F)_A \cdot [\mathscr{L}_S \alpha_A, \rho]\,\tau_+^2\,dx\,dt
\end{aligned}$$

$$+3\iint (i_S F)_4 \cdot [\mathscr{L}_S \rho, \rho] \tau_+^2 \, dx\, dt$$

$$-\iint (i_S F)_4 \cdot [\sigma, \mathscr{L}_S \sigma] \tau_+^2 \, dx\, dt$$

$$+2\iint (i_S F)_3 \cdot [\mathscr{L}_S \alpha_A, \alpha_A] \tau_+^2 \, dx\, dt + \text{l.o.t}$$

In all the above terms we break the integrals on two regions (interior and exterior regions) and estimate them trivially with the exception of the two critical terms on I_4 which require an integration by parts first:

$$\begin{aligned}
|I| \leqq |I_c| &+ c\int_0^{t_0} dt \int_{|x| \leqq 1+t/2} |\mathscr{L}_S F| \cdot |F|^2 \cdot (1+t)^3 \, dx \\
&+ |\tau_- \sigma|_{\infty,\,\text{ext}} \cdot \mathscr{Q}_0^{1/2} \cdot \|\tau_- \mathscr{L}_S \underline{\alpha}\| \\
&+ (1+t)^{-1} |\tau_-^2 \underline{\alpha}|_{\infty,\,\text{ext}} \cdot \mathscr{Q}_0^{1/2} \cdot \|\tau_+ \mathscr{L}_S \sigma\| \\
&+ (1+t)^{-1} |\tau_-^2 \underline{\alpha}|_{\infty,\,\text{ext}} \cdot \mathscr{Q}_0^{1/2} \cdot \|\tau_+ \mathscr{L}_S \rho\| \\
&+ |\tau_- \rho|_{\infty,\,\text{ext}} \cdot \mathscr{Q}_0^{1/2} \cdot \|\tau_- \mathscr{L}_S \underline{\alpha}\| \\
&+ 3|\tau_- \rho|_{\infty,\,\text{ext}} \cdot \mathscr{Q}_0^{1/2} \cdot \|\tau_- \mathscr{L}_S \rho\| \\
&+ |\tau_- \sigma|_{\infty,\,\text{ext}} \cdot \mathscr{Q}_0^{1/2} \cdot \|\tau_- \mathscr{L}_S \sigma\| \\
&+ 2|\tau_- \rho|_{\infty,\,\text{ext}} \cdot \mathscr{Q}_0^{1/2} \cdot \|\tau_- \mathscr{L}_S \underline{\alpha}\| \\
&+ |\tau_+ \alpha|_{\infty,\,\text{ext}} \cdot \mathscr{Q}_0^{1/2} \cdot \|\tau_+ \mathscr{L}_S \sigma\| \\
&+ |\tau_+ \alpha|_{\infty,\,\text{ext}} \cdot \mathscr{Q}^{1/2} \cdot \|\tau_+ \mathscr{L}_S \rho\| \\
&+ 3|\tau_- \rho|_{\infty,\,\text{ext}} \cdot \mathscr{Q}_0^{1/2} \cdot \|\tau_+ \mathscr{L}_S \rho\| \\
&+ |\tau_- \rho|_{\infty,\,\text{ext}} \cdot \mathscr{Q}_0^{1/2} \cdot \|\tau_+ \mathscr{L}_S \sigma \\
&+ 2|\tau_+ \alpha|_{\infty,\,\text{ext}} \cdot \mathscr{Q}_0^{1/2} \cdot \|\tau_+ \mathscr{L}_S \alpha\| + \text{l.o.t}
\end{aligned}$$

I_c denote the critical terms:

$$\begin{aligned}
I_c = &\iint (i_S F)_A \cdot [\varepsilon_{AB} \sigma, \mathscr{L}_S \alpha_B] \tau_+^2 \, dx\, dt \\
&- \iint (i_S F)_A \cdot [\mathscr{L}_S \alpha_A, \rho] \tau_+^2 \, dx\, dt
\end{aligned}$$

Using the exterior estimates 6.3 proved in section 6 we have:

$$\begin{aligned}
|I| \leqq |I_c| &+ c\int_0^{t_0} dt \int_{|x| \leqq 1+t/2} |\mathscr{L}_S F| \cdot |F|^2 \cdot (1+t)^3 \, dx \\
&+ c\int_0^{t_0} (1+t)^{-3/2} (\mathscr{Q}_0 + \mathscr{Q}_1)^{1/2} \cdot \mathscr{Q}_0^{1/2} \cdot \mathscr{Q}_1^{1/2} \, dt
\end{aligned}$$

$$\leqq |\mathrm{I}_c| + c\int_0^{t_0} dt\,(1+t)\,|\mathrm{F}|_{\infty,\mathrm{I}} \int_{|x|\leqq 1+t/2} (1+t)\,|\mathscr{L}_{\mathrm{S}}\mathrm{F}|\,.\,(1+t)\,|\mathrm{F}|\,dx$$

$$+ c\int_0^{t_0} (1+t)^{-3/2}\,\mathscr{Q}_0^{1/2}\,.\,\mathscr{Q}^*\,dt$$

$$\leqq |\mathrm{I}_c| + c\int_0^{t_0} dt\,(1+t)^{-3/2}\,\mathscr{Q}^{*1/2}\,\mathscr{Q}_0^{1/2}\,\mathscr{Q}_1^{1/2}$$

$$+ c\int_0^{t_0} (1+t)^{-3/2}\,\mathscr{Q}_0^{1/2}\,.\,\mathscr{Q}^*\,dt$$

We have used the fact that because $\mathscr{Q}_k \leqq 1$, then by theorem 7.1, $|\mathrm{F}|_{\infty,\mathrm{I}} \leqq c\,(1+t)^{-5/2}\,\mathscr{Q}^{*1/2}$. Now, because of the smallness condition $\mathscr{Q}_0 \leqq \varepsilon_0$:

$$|\mathrm{I}| \leqq |\mathrm{I}_c| + c\,\varepsilon_0^{1/2}\,\mathscr{Q}^* \int_0^{t_0} (1+t)^{-3/2}\,dt$$

$$\leqq |\mathrm{I}_c| + c_0\,\varepsilon_0^{1/2}\,\mathscr{Q}^*$$

Let us estimate I_c now. We will estimate only the second summand. The other requires an identical treatment. Consider the expression $(i_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,[\mathscr{L}_{\mathscr{S}}\,\alpha_{\mathscr{A}},\ \rho]$ and rewrite it as:

$$\begin{aligned}(i_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,[\mathscr{L}_{\mathrm{S}}\,\alpha_{\mathrm{A}},\ \rho] &= -(i_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,[\alpha_{\mathrm{A}},\ \mathscr{L}_{\mathrm{S}}\,\rho] + (\mathrm{I}_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,\mathscr{L}_{\mathrm{S}}\,[\alpha_{\mathrm{A}},\ \rho]\\ &= -(i_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,[\alpha_{\mathrm{A}},\ \mathscr{L}_{\mathrm{S}}\,\rho] - i_{\mathrm{S}}(\mathscr{L}_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,[\alpha_{\mathrm{A}},\ \rho]\\ &\quad + \nabla_{\mathrm{S}}((i_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,[\alpha_{\mathrm{A}},\ \rho]) - 2\,(i_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,[\alpha_{\mathrm{A}},\ \rho]\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^{t_0} dt \int_{x_0=t} (i_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,[\mathscr{L}_{\mathrm{S}}\,\alpha_{\mathrm{A}},\ \rho]\,\tau_+^2\,dx\\ = -\int_0^{t_0} dt \int_{x_0=t} (i_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,[\alpha_{\mathrm{A}},\ \mathscr{L}_{\mathrm{S}}\,\rho]\,\tau_+^2\,dx\\ -\int dt \int_{x_0=t} (i_{\mathrm{S}}\,\mathscr{L}_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,[\alpha_{\mathrm{A}},\ \rho]\,\tau_+^2\,dx\\ -2\int_0^{t_0} dt \int_{x_0=t} (i_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,[\alpha_{\mathrm{A}},\ \rho]\,\tau_+^2\,dx\\ +\int_0^{t_0} dt \int_{x_0=t} \nabla_{\mathrm{S}}((i_{\mathrm{S}}\mathrm{F})_{\mathrm{A}}\,.\,[\alpha_{\mathrm{A}},\ \rho])\,.\,\tau_+^2\,dx\end{aligned}$$

The first three terms are estimated as before, taking $\tau_+\,\alpha$ in L^∞ norm. The last one requires one more integration by parts. We are left with a pure space-time divergence term. We only have to estimate:

$$\mathrm{I} = \int_0^{t_0} dt \int_{x_0 = t} \nabla_{\mathrm{S}} ((i_{\mathrm{S}} \mathrm{F})_{\mathrm{A}} \cdot [\alpha_{\mathrm{A}}, \rho] \cdot \tau_+^2) \, dx$$

By straightforward computation:

$$\mathrm{I} = \int_0^{t_0} \left(t \frac{d}{dt} \int_{x_0 = t} (i_{\mathrm{S}} \mathrm{F})_{\mathrm{A}} \cdot [\alpha_{\mathrm{A}}, \rho] \cdot \tau_+^2 \right) dx \Big) dt$$

$$+ \int_0^{t_0} dt \int_{x_0 = t} x_i \, \partial_i ((i_{\mathrm{S}} \mathrm{F})_{\mathrm{A}} \cdot [\alpha_{\mathrm{A}}, \rho]) \cdot \tau_+^2) \, dx$$

$$= -3 \int_0^{t_0} dt \int_{x_0 = t} (i_{\mathrm{S}} \mathrm{F})_{\mathrm{A}} \cdot [\alpha_{\mathrm{A}}, \rho] \cdot \tau_+^2) \, dx$$

$$+ t_0 \int_{x_0 = t_0} (i_{\mathrm{S}} \mathrm{F})_{\mathrm{A}} \cdot [\alpha_{\mathrm{A}}, \rho] \cdot \tau_+^2 \, dx$$

The first summand is estimated in the same fashion of before. The other one is also bounded:

$$\left| t_0 \int (i_{\mathrm{S}} \mathrm{F}) \cdot [\alpha, \rho] \cdot \tau_+^2 \, dx \right| \leqq t_0 \, | t_0 \mathrm{F} |_{\infty, \mathrm{I}} \int_{|x| \leqq 1 + t/2} t^2 \, | \mathrm{F} |^2 \, dx$$

$$+ t_0 \, | \tau_+ \alpha |_{\infty, \mathrm{ext}} \cdot \int_{|x| \geqq 1 + t/2} | i_{\mathrm{S}} \mathrm{F} | | \rho | \cdot \tau_+ \, dx$$

$$\leqq c (1 + t_0)^{-1/2} \varepsilon_0^{1/2} \cdot \mathcal{Q}^*$$

It follows that $\mathcal{Q}_1$ will satisfy:

$$\mathcal{Q}_1 (t_0) \leqq \mathcal{Q}_1 (0) + c \, \varepsilon_0^{1/2} \, \mathcal{Q}^* + c (1 + t_0)^{-1/2} \, \varepsilon_0^{1/2} \, \mathcal{Q}^*$$

$$\leqq \mathcal{Q}_1 (0) + c \, \varepsilon_0^{1/2} \, \mathcal{Q}^*$$

for all $t_0 \in [0, t_*]$ and henceforth $\mathcal{Q}_1^* \leqq \mathcal{Q}_1 (0) + c \, \varepsilon_0^{1/2} \, \mathcal{Q}^*$.

The estimate of $\mathcal{Q}_2$ is done along the same lines. This completes the proof of theorem 8.1. On the next two sections we analyse the relation between the spherically symmetric Ansatz and the non-linear flow. By applying then a standard continuation argument one can finally conclude the proof of the main theorem in the introduction.

9. THE SPHERICALLY SYMMETRIC FLOW

The space-time description of the spherically symmetric fields given in section 4 is not sufficient for our goal. We have to prove that the symmetric Ansatz needs only to be imposed on the initial data.

DEFINITION 9.1. — *A Cauchy data set* (A, E) *for the* $SU(2)$-*Yang-Mills equations is called spherically symmetric if* A *and* E *are of the form:*

$$A_i^a = \psi \frac{x_i x^a}{r^2} + \frac{f_1}{r}\left(\delta_i^a - \frac{x_i x^a}{r^2}\right) + \frac{f_2 - 1}{r} \varepsilon^{iab} \frac{x^b}{r}$$

$$E_i^a = \xi \frac{x^i x^a}{r^2} + + \left(\frac{g_1}{r}\right)\left(\delta_i^a - \frac{x_i x^a}{r^2}\right) + \left(\frac{g_2}{r}\right) \varepsilon^{iab} \frac{x^b}{r}$$

where all constitutive functions of the Ansatz are functions of t, r alone. Equivalently (cf. *theorem* 4.1)*:*

$$\hat{\mathscr{L}}_{O_{(i)}} A = 0, \qquad \hat{\mathscr{L}}_{O_{(i)}} E = 0$$

A similar definition is valid for higher gauge groups G.

THEOREM 9.1 (Symmetry Invariance under the Flow). — *Assume that a solution* (A, F) *of the Yang-Mills equations, defined in the interval* $[0, t_*]$ *and in the temporal gauge* $A_0 = 0$ *has spherically symmetric initial data* (A(0), E(0)). *It follows that the solution will also satisfy the spherical symmetry conditions* $\hat{\mathscr{L}}_O (A) t = 0$, $\hat{\mathscr{L}}_O F(t) = 0$ *for all later times* $t \in [0, t_*]$.

Proof. — The proof is a simple energy estimate. We prove that for all later times we have:

$$\int_{R^3} |\hat{\mathscr{L}}_O A|^2 + |\hat{\mathscr{L}}_O F|^2 \, dx = 0$$

Using lemma 4.1 and applying proposition 5.2 with multiplier $X = T_0$ we obtain:

$$\frac{d}{dt} \int P^0 (\hat{\mathscr{L}}_O F) \, dx = \int \{ (\hat{\mathscr{L}}_O F)_{\alpha\nu} . [F^{\alpha\beta}, \hat{\mathscr{L}}_O A_\beta] + (\hat{\mathscr{L}}_O^* F)_{\alpha\nu} . [{}^*F^{\alpha\beta}, \hat{\mathscr{L}}_O A_\beta] \} T_0^\nu \, dx$$

Since $P^0 (\hat{\mathscr{L}}_O F) = |\hat{\mathscr{L}}_O F|^2$, then:

$$\frac{d}{dt} \| \hat{\mathscr{L}}_O F \| \leqq c_0 |F|_\infty . \| \hat{\mathscr{L}}_O A \|$$

To control the term $\| \hat{\mathscr{L}}_O A \|$ we apply again lemma 4.1:

$$\hat{\mathscr{L}}_O F_{\mu\nu} = D_\mu (\hat{\mathscr{L}}_O A_\nu) - D_\nu (\hat{\mathscr{L}}_O A_\mu)$$

Taking the *oi* components of this equation, multiplying each of them by $\hat{\mathscr{L}}_O A_i$ and summing over *i* we get:

$$\partial_t \left(\frac{1}{2} |\hat{\mathscr{L}}_O A|^2 \right) = \hat{\mathscr{L}}_O A . \hat{\mathscr{L}}_O F - D_i (\hat{\mathscr{L}}_O A_0) . \hat{\mathscr{L}}_O A_i$$

Since we are in the temporal gauge:

$$\frac{d}{dt}\|\hat{\mathscr{L}}_O \mathrm{A}\| \leqq c\|\hat{\mathscr{L}}_O \mathrm{F}\|$$

We obtain a system of coupled differential inequalties:

$$\frac{d}{dt}\|\hat{\mathscr{L}}_O \mathrm{F}\| \leqq c_0 |\mathrm{F}|_\infty \cdot \|\hat{\mathscr{L}}_O \mathrm{A}\| \tag{9.1}$$

$$\frac{d}{dt}\|\hat{\mathscr{L}}_O \mathrm{A}\| \leqq c\|\hat{\mathscr{L}}_O \mathrm{F}\| \tag{9.2}$$

subject to the initial conditions $\hat{\mathscr{L}}_O \mathrm{A}(0)=0$, $\hat{\mathscr{L}}_O \mathrm{F}(0)=0$. The conclusion follows from Gronwall's lemma.

10. THE BEHAVIOUR AT THE CENTRAL LINE $r=0$

The Ansatz 4.9-4.10 corresponds to potentials which are everywhere regular except possibly at $r=0$. Our aim in this section is to find sufficient conditions which ensure that the curvature tensor is bounded near the central line.

DEFINITION 10.1. – *We call a Cauchy data set* (A, E) *an acceptable initial data set iff:*

(i) *The set* (A, E) *is a spherically symmetric Cauchy data set.*

(ii) *The curvature* F_{A} *of the potential* A *is in the space* $\mathrm{H}^{2,1}$.

The next proposition, whose proof follows from the local existence theorem 3.3 and the invariance theorem 9.1 will show that the global solutions dealt with here form a non-empty set:

THEOREM 10.1. – *Acceptable initial data sets* (A, E) *are preserved by the flow of the Yang-Mills equations.*

The existence of non-trivial solutions to our equations will be then a consequence of:

THEOREM 10.2 (Existence of non-trivial Data). – *There exists a non-trivial class* $\mathscr{S}$ *of acceptable initial data sets. Moreover, the elements of* $\mathscr{S}$ *can be characterized as the spherically symmetric pairs* (A, E) *such that:*

(i) *The constituent functions* $a_\theta(r)$, $a_\varphi(r)$ *of the magnetic potential* A *close an* $su(2)$*-subalgebra* $\{a_\theta(0), a_\varphi(0), \Omega\}$ *sufficiently fast as* $r \to 0$*:*

$$|[a_\theta(r).\ a_\varphi(r)]-\Omega| \leqq cr$$

(*similarly for derivatives*).

(ii) *The initial velocity of rotation (conjugation) of the afore-mentioned subalgebra (given by the electric Cauchy data) is zero:*

$$|\dot{a}_\theta(r)|+|\dot{a}_\varphi(r)|\leqq cr$$

(similarly for derivatives).

This picture corresponds in the SU (2) case to the situation where the transverse components f_1, f_2 of the magnetic potential converge sufficiently fast as $r\to 0$ to a point in the circle S^1 and the electric oscillations produce only slow rotations in time of this point.

Proof of Theorem 10.2. – We will treat the SU (2) case first. The problem consists of checking the square-integrability of the curvature and its derivatives near the origin. Consider initially the undifferentiated field F. Computing the curvature from the Ansatz 1.3-1.3:

$$E_i^a=(\psi-\varphi')\frac{x^i x^a}{r^2}+\left(\frac{\dot{f}_1-f_2\varphi}{r}\right)\left(\delta_i^a-\frac{x_i x^a}{r^2}\right)$$
$$+\left(\frac{\dot{f}_2+f_1\varphi}{r}\right)\varepsilon^{iab}\frac{x^b}{r}$$
$$H_i^a=\frac{1}{r^2}(f_1^2+f_2^2-1)\frac{x^i x^a}{r^2}+\left(\frac{f_2'+f_1\psi}{r}\right)\left(\delta_i^a-\frac{x_i x^a}{r^2}\right)$$
$$+\left(\frac{-f_1'+f_2\psi}{r}\right)\varepsilon^{iab}\frac{x^b}{r}$$

The expression for the energy is:

$$\int E^2+H^2\,dx=4\pi\int_0^\infty (E_N^2+\not{E}^2+\not{H}^2+H_N^2)\,r^2\,dr$$

The only term that could possibly be non-integrable near $r=0$ is the radial component of the magnetic field:

$$H_N^a=\frac{1}{r^2}(f_1^2+f_2^2-1)\frac{x^a}{r}$$

Now, if the values of the components, as $r\to 0$, are such that:

$$f_1(0)^2+f_2(0)^2-1=0 \tag{10.1}$$

then the singularity is removed and the field will have finite energy. We also have to check the integrability conditions at the origin for the higher derivatives. For every component u of the vectors E and H we have:

$$|Du|^2=|D_r u|^2+r^2|\not{D}u|^2$$

The last term is harmless and reduces to the previous step. The critical term is $D_r u$. Using remark 2 following proposition 4.1 we take initial data in the gauge $A_r=0$. In this case $D_r u=\partial_r u+[A_r, u]=\partial_r u$ and we need to differentiate only with respect to the radial variable r. This simplifies a lot the computations. By straightforward differentiation we deduce:

$$\int_{|x|\leq 1} |D_r^s E|^2 dx \leq c \sum_{k=0}^{s} \int_0^1 |\chi^{(k)}|^2 + r^{-2s+2k}(|g_1^{(k)}|^2 + |g_2^{(k)}|^2)\, dr \tag{10.2}$$

$$\int_{|x|\leq 1} |D_r^s H|^2 dx \leq c \sum_{k=0}^{s} \int_0^1 r^{-2(s+1)+2k} |(f_1^2+f_2^2-1)^{(k)}|^2 dr + c \sum_{k=0}^{s} \int_0^1 + r^{-2s+2k}(|f_1^{(k+1)}|^2 + |f_2^{(k+1)}|^2)\, dr \tag{10.3}$$

Taking data such that the right-hand sides of 10.2-10.2 are finite we precise the statement of theorem 10.2 in the SU(2) case. One sees that the integrability requirements will be nothing than a reinforcement of condition 10.1, namely that $f_i = f_i(0) + O(r^3)$ and $g_i = O(r^2)$, for $i = 1, 2$. Recalling that g_i plays the role (as Cauchy data) of $\dot{f}_i$, we precise also the velocity condition on the point (f_1, f_2). Similarly, one can find the conditions at $r \to \infty$ which guarantee the decay at spacelike-infinity.

The analysis for higher gauge groups is similar. Here one analyses the fields in the abelian gauge. Recall that the energy is gauge-invariant and the string singularities that occur in this gauge do not manifest at the level of curvature.

Let us consider the energy of the field F:

$$\int E^2 + H^2 \, dx = 4\pi \int_0^\infty (E_N^2 + \not{E}^2 + \not{H}^2 + H_N^2)\, r^2 \, dr$$

As in the SU(2) case, the critical term is:

$$H_N = \frac{1}{r^2 \sin\theta} F_{\theta\varphi} = \frac{1}{r^2}([a_\theta, a_\varphi] - \Omega)$$

Now, if the values of the components a, as $r \to 0$, are such that:

$$[a_\theta(0), a_\varphi(0)] - \Omega = 0$$

Then, the singularity in the integral is removed and the field will have finite energy. This condition means that the set $\{a_\theta(0), a_\varphi(0), \Omega\}$ generate an $su(2)$-subalgebra. (Recall the constraints 4.5-4.7.) The analysis for the higher derivatives is similar to the SU(2) case and one concludes that the su(2) subalgebra closes fast at the line $r = 0$. The freedom of performing a rotation (conjugation) is measured by the electric part of the field. If the flow rotates the subalgebra very fast then this will generate electric oscillations that are not integrable near the origin. This is the explanation of the electric boundary conditions. This completes the proof of theorem 10.2.

11. COMPLETION OF THE PROOF

We complete here the proof of the main theorem. By theorems 3.3 and 9.1 we know already that solutions will exist locally in time and that this local evolution will preserve the class of spherically symmetric solutions. Let us prove now that the local solutions can be extended for all times. Call $\mathcal{Q}_{[2]}(t)$ the expression $\mathcal{Q}_{[2]}(t)=\mathcal{Q}_0(t)+\mathcal{Q}_1(t)+\mathcal{Q}_2(t)$.

Consider the set $\mathcal{S}$ of all times $T \geqq 0$ for which there exists a local solution $(A(t), E(t))$ with regularity as in 3.3, defined for all $t \in [0, T]$ and satisfying the bootstrap assumption $\mathcal{Q}_{[2]}(t) \leqq \varepsilon \leqq 1$.

The number ε is chosen later in an specific way. By restricting the size ε_0 of the initial data we will derive a contradiction. Take initial data so small that:

$$\mathcal{Q}_{[2]}(0) \leqq \varepsilon$$

The set $\mathcal{S}$ will contain then $t=0$ and is therefore not empty. Let $t_* = \sup \mathcal{S}$. If $t_* = +\infty$ then there is nothing else to be proved. If $t_* < +\infty$ then we will get a contradiction. First of all we remark that $t_* \in \mathcal{S}$. This is a consequence of the regularity of the curvature E, $H \in C^0([0, T], H^{2,1})$ and the fact that the set S is closed. In particular:

$$\mathcal{Q}_{[2]}(t_*) \leqq \varepsilon$$

We claim now that $\mathcal{Q}_{[2]}(t_*) = \varepsilon$. Indeed, if we have strict inequality then we can find a slightly larger t for which we have:

$$\mathcal{Q}_{[2]}(t_* + \delta) < \varepsilon \leqq \varepsilon$$

which contradicts the maximality of t_*. On the other hand, in the interval $[0, t_*]$ the norm $\mathcal{Q}_{[2]}(t) \leqq 1$ and we can apply theorem 8.1. We get:

$$\mathcal{Q}_0(t_*)+\mathcal{Q}_1(t_*)+\mathcal{Q}_2(t_*) \leqq 2c(\mathcal{Q}_0(0)+\mathcal{Q}_1(0)+\mathcal{Q}_2(0))^{1/2} \leqq c\,\varepsilon_0^{1/2}$$

Choosing initial data so small that $\varepsilon_0 \leqq \dfrac{\varepsilon^2}{4c^2}$ we get a contradiction and our theorem is proved.

Finally, remark that the initial norm $I(A, E)$ is nothing but the restriction to time $t=0$ of $\mathcal{Q}_0(0)+\mathcal{Q}_1(0)+\mathcal{Q}_2(0)$.

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