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GUY BOUCHITTÉ

GIUSEPPE BUTTAZZO

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Relaxation for a class of nonconvex functionals defined on measures

by

Guy BOUCHITTÉ

Département de Mathématiques,
Université de Toulon et du Var, BP 132,
83957 La Garde Cedex, France

and

Giuseppe BUTTAZZO

Istituto di Matematiche Applicate,
Via Bonanno, 25/B, 56126 Pisa, Italy

ABSTRACT. — We characterize in a suitable integral form like

$$\bar{F}(\lambda) = \int_{\Omega} \bar{f}\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega \setminus A_{\lambda}} \bar{\varphi}(x, \lambda^s) + \int_{A_{\lambda}} \bar{g}(x, \lambda(x)) d\#$$

the lower semicontinuous envelope $\bar{F}$ of functionals F defined on the space $\mathcal{M}(\Omega; \mathbf{R}^n)$ of all $\mathbf{R}^n$ -valued measures with finite variation on Ω .

RÉSUMÉ. — On établit une représentation intégrale de la forme :

$$\bar{F}(\lambda) = \int_{\Omega} \bar{f}\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega \setminus A_{\lambda}} \bar{\varphi}(x, \lambda^s) + \int_{A_{\lambda}} \bar{g}(x, \lambda(x)) d\#$$

pour la régularisée semicontinue inférieure $\bar{F}$ d'une fonctionnelle F définie sur l'espace $\mathcal{M}(\Omega, \mathbf{R}^n)$ des mesures à variation bornée sur Ω à valeurs dans $\mathbf{R}^n$.

Classification A.M.S. : 49 J 45 (Primary), 46 G 10, 46 E 27.

1. INTRODUCTION

In a previous paper [3] we introduced a new class of nonconvex functionals defined on the space $\mathcal{M}(\Omega; \mathbf{R}^n)$ of all $\mathbf{R}^n$ -valued measures with finite variation on Ω of the form

$$F(\lambda) = \int_{\Omega} f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega \setminus A_\lambda} \varphi(x, \lambda^s) + \int_{A_\lambda} g(x, \lambda(x)) d\# \quad (1.1)$$

where $(d\lambda/d\mu)\mu + \lambda^s$ is the Lebesgue-Nikodym decomposition of λ , A_λ is the set of atoms of λ , $\lambda(x)$ denotes the value $\lambda(\{x\})$, and $\#$ is the counting measure (we refer to Section 2 for further details). For this kind of functionals we proved in [3] (see Theorem 2.4 below), under suitable hypotheses on f , φ , g , a lower semicontinuity result with respect to the weak* $\mathcal{M}(\Omega; \mathbf{R}^n)$ convergence.

In a subsequent paper [4] we characterized all weakly* lower semicontinuous functionals on $\mathcal{M}(\Omega; \mathbf{R}^n)$ satisfying the additivity condition

$$F(\lambda + \nu) = F(\lambda) + F(\nu) \quad \text{for every } \lambda, \quad \nu \in \mathcal{M}(\Omega; \mathbf{R}^n) \quad \text{with } \lambda \perp \nu \quad (1.2)$$

and we proved that they are all of the form (1.1) for suitable integrands f , φ , g .

In the present paper we deal with functionals F of the form

$$F(\lambda) = \begin{cases} \int_{\Omega, +\infty} f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{A_\lambda} g(x, \lambda(x)) d\# \\ \text{if } \lambda^s = 0 \text{ on } \Omega \setminus A_\lambda \quad \text{otherwise} \end{cases}$$

and we consider their (sequential) lower semicontinuous envelope $\bar{F}$ defined by

$$\bar{F} = \sup \{ G : G \leq F, G \text{ sequentially weakly* l.s.c. on } \mathcal{M}(\Omega; \mathbf{R}^n) \}.$$

We prove in Theorem 3.1 that $\bar{F}$ satisfies the additivity condition (1.2) so that, by the results of [4], it can be written in the integral form

$$\bar{F}(\lambda) = \int_{\Omega} \bar{f}\left(x, \frac{d\lambda}{d\bar{\mu}}\right) d\bar{\mu} + \int_{\Omega \setminus A_\lambda} \bar{\varphi}(x, \lambda^s) + \int_{A_\lambda} \bar{g}(x, \lambda(x)) d\#$$

for suitable $\bar{\mu}$, $\bar{f}$, $\bar{\varphi}$, $\bar{g}$. An explicit way to construct $\bar{\mu}$, $\bar{f}$, $\bar{\varphi}$, $\bar{g}$ in terms of μ , f , g is given (see Theorem 3.2), and this is applied in Example 3.4 to the case $f(x, s) = |s|^p$ and $g(x, s) = |s|^q$ with $p \in [1, +\infty]$ and $q \in [0, 1]$.

2. NOTATION AND PRELIMINARY RESULTS

In this section we fix the notation we shall use in the following; we recall them only briefly because they are the same used in Bouchitté &

Buttazzo [3] and [4], to which we refer for further details. In all the paper $(\Omega, \mathcal{B}, \mu)$ will denote a measure space, where Ω is a separable locally compact metric space with distance d , $\mathcal{B}$ is the σ -algebra of all Borel subsets of Ω , and $\mu: \mathcal{B} \rightarrow [0, +\infty[$ is a positive, finite, non-atomic measure. We shall use the following symbols:

– $C_0(\Omega; \mathbf{R}^n)$ is the space of all continuous functions $u: \Omega \rightarrow \mathbf{R}^n$ “vanishing on the boundary”, that is such that for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ with $|u(x)| < \varepsilon$ for all $x \in \Omega \setminus K_\varepsilon$.

– $\mathcal{M}(\Omega; \mathbf{R}^n)$ is the space of all vector-valued measures $\lambda: \mathcal{B} \rightarrow \mathbf{R}^n$ with finite variation on Ω .

– $|\lambda|$ is the variation of $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ defined for every $B \in \mathcal{B}$ by

$$|\lambda|(B) = \sup \left\{ \sum_{h=1}^{\infty} |\lambda(B_h)| : \bigcup_{h=1}^{\infty} B_h \subset B, B_h \text{ pairwise disjoint} \right\}.$$

– $\lambda_h \rightarrow \lambda$ indicates the convergence of λ_h to λ in the weak* topology of $\mathcal{M}(\Omega; \mathbf{R}^n)$ deriving from the duality between $\mathcal{M}(\Omega; \mathbf{R}^n)$ and $C_0(\Omega; \mathbf{R}^n)$.

– $\lambda \ll \mu$ indicates that λ is absolutely continuous with respect to μ , that is $|\lambda|(B) = 0$ whenever $B \in \mathcal{B}$ and $\mu(B) = 0$.

– $\lambda \perp \mu$ indicates that λ is singular with respect to μ , that is $|\lambda|(\Omega \setminus B) = 0$ for a suitable $B \in \mathcal{B}$ with $\mu(B) = 0$.

– $u\mu$ with $u \in L^1(\Omega; \mathbf{R}^n; \mu)$, is the measure of $\mathcal{M}(\Omega; \mathbf{R}^n)$ (often indicated simply by u) defined by

$$(u\mu)(B) = \int_B u \, d\mu \quad \text{for every } B \in \mathcal{B}.$$

It is well-known that every measure $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ which is absolutely continuous with respect to μ is representable in the form $\lambda = u\mu$ for a suitable $u \in L^1(\Omega; \mathbf{R}^n; \mu)$; moreover, by the Lebesgue-Nikodym decomposition theorem, for every $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ there exists a unique function $u \in L^1_\mu(\Omega; \mathbf{R}^n)$ (often indicated by $d\lambda/d\mu$) and a unique measure $\lambda^s \in \mathcal{M}(\Omega; \mathbf{R}^n)$ such that

$$\begin{cases} \text{(i)} & \lambda = u\mu + \lambda^s \\ \text{(ii)} & \lambda^s \text{ is singular with respect to } \mu. \end{cases}$$

– $u\lambda$ with $u: \Omega \rightarrow \mathbf{R}$ a bounded Borel function and $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$, is the measure of $\mathcal{M}(\Omega; \mathbf{R}^n)$ defined by

$$(u\lambda)(B) = \int_B u \, d\lambda \quad \text{for every } B \in \mathcal{B}.$$

– 1_B with $B \subset \Omega$, is the function

$$1_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \in \Omega \setminus B. \end{cases}$$

- δ_x with $x \in \Omega$, is the measure of $\mathcal{M}(\Omega; \mathbf{R}^n)$

$$\delta_x(\mathbf{B}) = \begin{cases} 1 & \text{if } x \in \mathbf{B} \\ 0 & \text{if } x \in \Omega \setminus \mathbf{B}. \end{cases}$$

- $\mathcal{M}^0(\Omega; \mathbf{R}^n)$ is the space of all non-atomic measures of $\mathcal{M}(\Omega; \mathbf{R}^n)$.
- $\mathcal{M}^\#(\Omega; \mathbf{R}^n)$ is the space of all “purely atomic” measures of $\mathcal{M}(\Omega; \mathbf{R}^n)$, that is the measures of the form

$$\lambda = \sum_{i=1}^{\infty} a_i \delta_{x_i} \quad (x_i \in \Omega, a_i \in \mathbf{R}^n).$$

- $\lambda(x)$ with $x \in \Omega$ and $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$, denotes the quantity $\lambda(\{x\})$.
- A_λ is the set of all atoms of λ , that is

$$A_\lambda = \{x \in \Omega : \lambda(x) \neq 0\}.$$

- $\int_{\mathbf{B}} \varphi(x, \lambda)$ with $\mathbf{B} \in \mathcal{B}$, $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$, and $\varphi : \Omega \times \mathbf{R}^n \rightarrow [0, +\infty]$ a Borel function such that $\varphi(x, \cdot)$ positively 1-homogeneous for every $x \in \Omega$, denotes the quantity

$$\int_{\mathbf{B}} \varphi\left(x, \frac{d\lambda}{d\nu}\right) d\nu$$

which (see for instance Goffman and Serrin [12]) does not depend on ν , when ν varies over all positive measures such that $|\lambda| \ll \nu$.

- f^* with $f : \mathbf{R}^n \rightarrow]-\infty, +\infty]$ proper function, is the usual conjugate function of f

$$f^*(s) = \sup \{sw - f(w) : w \in \mathbf{R}^n\} \quad (s \in \mathbf{R}^n).$$

- f^∞ with $f : \mathbf{R}^n \rightarrow]-\infty, +\infty]$ proper function, is the usual recession function of f

$$f^\infty(s) = \sup \{f(s+t) - f(t) : t \in \mathbf{R}^n, f(t) < +\infty\} \quad (s \in \mathbf{R}^n).$$

It is well-known that when f is convex l.s.c. and proper, f^* is convex l.s.c. and proper too, and we have $f^{**} = f$; moreover, in this case, for the recession function f^∞ the following formula holds (see for instance Rockafellar [16]):

$$f^\infty(s) = \lim_{t \rightarrow +\infty} \frac{f(s_0 + ts)}{t}$$

where s_0 is any point such that $f(s_0) < +\infty$. It can be shown that the definition above does not depend on s_0 , and that the function f^∞ turns out to be convex, l.s.c., and positively 1-homogeneous on $\mathbf{R}^n$.

– $\varphi_{f,\mu}$ with $f: \Omega \times \mathbf{R}^n \rightarrow [0, +\infty]$ a Borel function such that $f(x, \cdot)$ is convex l.s.c. and proper for μ -a.e. $x \in \Omega$, denotes the function

$$\varphi_{f,\mu}(x, s) = \sup \left\{ u(x) : u \in C_0(\Omega; \mathbf{R}^n), \int_{\Omega} f^*(x, u) d\mu < +\infty \right\}$$

defined for every $(x, s) \in \Omega \times \mathbf{R}^n$. The function $\varphi_{f,\mu}(x, s)$ is l.s.c. in (x, s) , convex and positively 1-homogeneous in s , and we have (see for instance Bouchitté and Valadier [5], Proposition 7)

$$\begin{cases} \varphi_{f,\mu}(x, \cdot) \leq f^\infty(x, \cdot) & \text{for } \mu\text{-a.e. } x \in \Omega; \\ \varphi_{f,\mu} \geq f^\infty & \text{if the multimapping } x \rightarrow \text{epi } f^*(x, \cdot) \text{ is l.s.c. on } \Omega. \end{cases}$$

– g^0 with $g: \mathbf{R}^n \rightarrow [0, +\infty]$ a function such that $g(0)=0$, is the function defined by

$$g^0(s) = \limsup_{t \rightarrow 0^+} \frac{g(ts)}{t} \quad (s \in \mathbf{R}^n).$$

– g subadditive with $g: \mathbf{R}^n \rightarrow [0, +\infty]$ a function such that $g(0)=0$, will mean that

$$g(s_1 + s_2) \leq g(s_1) + g(s_2) \quad \text{for every } s_1, s_2 \in \mathbf{R}^n.$$

We remark that g is subadditive if and only if $g^\infty \leq g$, hence $g^\infty = g$ for every subadditive function g with $g(0)=0$.

– $\alpha \nabla \beta$ with $\alpha, \beta: \mathbf{R}^n \rightarrow [0, +\infty]$ denotes the inf-convolution

$$(\alpha \nabla \beta)(s) = \inf \{ \alpha(t) + \beta(s-t) : t \in \mathbf{R}^n \}.$$

It is easy to see that

$$\begin{cases} f \nabla f^\infty = f & \text{for every } f: \mathbf{R}^n \rightarrow [0, +\infty] \text{ convex, l.s.c., proper;} \\ g \nabla g = g & \text{for every } g: \mathbf{R}^n \rightarrow [0, +\infty] \text{ subadditive, with } g(0)=0. \end{cases}$$

We also recall some preliminary results which will be used in the following.

PROPOSITION 2.1: (see Bouchitté and Buttazzo [3], Proposition 2.2). – Let $g: \mathbf{R}^n \rightarrow [0, +\infty]$ be a subadditive l.s.c. function, with $g(0)=0$. Then we have:

(i) the function $g^0: \mathbf{R}^n \rightarrow [0, +\infty]$ is convex, l.s.c., and positively 1-homogeneous;

$$(ii) \quad g^0(s) = \sup_{t>0} \frac{g(ts)}{t} = \lim_{t \rightarrow 0^+} \frac{g(ts)}{t} \quad \text{for every } s \in \mathbf{R}^n.$$

PROPOSITION 2.2: (see Bouchitté and Buttazzo [3], Proposition 2.4). – Let $\alpha, \beta: \mathbf{R}^n \rightarrow [0, +\infty]$ be two convex l.s.c. and proper functions, with α

such that

$$\lim_{|s| \rightarrow +\infty} \alpha(s) = +\infty.$$

Then we have:

- (i) $\alpha \nabla \beta$ is l.s.c. and $\alpha \nabla \beta = (\alpha^* + \beta^*)^*$;
- (ii) $\alpha \nabla \beta_h \uparrow \alpha \nabla \beta$ for every sequence $\beta_h: \mathbf{R}^n \rightarrow [0, +\infty]$ of l.s.c. functions with $\beta_h \uparrow \beta$.

PROPOSITION 2.3. — Let $f, g: \mathbf{R}^n \rightarrow [0, +\infty]$ be two subadditive l.s.c. functions with $f(0) = g(0) = 0$. Assume that for a suitable $\alpha > 0$ it is

$$f(s) \geq \alpha |s| \quad \text{for every } s \in \mathbf{R}^n. \quad (2.1)$$

Then we have

$$(f \nabla g)^0 = f^0 \nabla g^0.$$

Proof. — The inequalities $(f \nabla g)^0 \leq f^0$ and $(f \nabla g)^0 \leq g^0$ imply that

$$(f \nabla g)^0 \leq f^0 \nabla g^0.$$

Let us prove the opposite inequality. Let us fix $s \in \mathbf{R}^n$ with $(f \nabla g)^0(s) = C < +\infty$ and for every $t > 0$ let $s_t \in \mathbf{R}^n$ be such that

$$(f \nabla g)(ts) = f(ts_t) + g(ts - ts_t). \quad (2.2)$$

By (2.1) and (2.2) we have for every $t > 0$

$$\alpha |s_t| \leq \frac{f(ts_t)}{t} \leq \frac{(f \nabla g)(ts)}{t} \leq (f \nabla g)^0(s) = C$$

so that we may assume $s_t \rightarrow z$ as $t \rightarrow 0$. For every $\varepsilon > 0$ and $w \in \mathbf{R}^n$ set

$$\begin{aligned} f_\varepsilon(w) &= \sup \{ ww^* : tw^* \leq f(t) \quad \text{for every } |t| \leq \varepsilon \} \\ g_\varepsilon(w) &= \sup \{ ww^* : tw^* \leq g(t) \quad \text{for every } |t| \leq \varepsilon \}. \end{aligned}$$

Fix $\varepsilon > 0$; by Proposition 2.3 of Bouchitté and Buttazzo [3] we have for every t small enough

$$\frac{f(ts_t) + g(ts - ts_t)}{t} \geq f_\varepsilon(s_t) + g_\varepsilon(s - s_t),$$

so that, passing to the $\liminf$ as $t \rightarrow 0$, and taking into account (2.2)

$$(f \nabla g)^0(s) \geq f_\varepsilon(z) + g_\varepsilon(s - z).$$

Finally, passing to the limit as $\varepsilon \rightarrow 0$, by Proposition 2.3 of [3] again, we get

$$(f \nabla g)^0(s) \geq f^0(z) + g^0(s - z) \geq (f^0 \nabla g^0)(s). \quad \blacksquare$$

We shall deal with functionals defined on $\mathcal{M}(\Omega; \mathbf{R}^n)$ of the form

$$F(\lambda) = \int_{\Omega} f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega \setminus A_{\lambda}} \varphi(x, \lambda^s) + \int_{A_{\lambda}} g(x, \lambda(x)) d\# \quad (2.3)$$

For this kind of functionals we proved in [3] a result of lower semicontinuity with respect to the weak* convergence in $\mathcal{M}(\Omega; \mathbf{R}^n)$. More precisely, the following theorem holds.

THEOREM 2.4. — *Let $\mu \in \mathcal{M}(\Omega)$ be a non-atomic positive measure and let $f, \varphi, g: \Omega \times \mathbf{R}^n \rightarrow [0, +\infty]$ be three Borel functions such that*

- (H₁) *$f(x, \cdot)$ is convex and l.s.c. on $\mathbf{R}^n$, and $f(x, 0) = 0$ for μ -a.e. $x \in \Omega$,*
- (H₂) *$f^{\infty}(x, \cdot) = \varphi(x, \cdot) = \varphi_{f, \mu}(x, \cdot)$ for μ -a.e. $x \in \Omega$,*
- (H₃) *g is l.s.c. on $\Omega \times \mathbf{R}^n$, and $g(x, 0) = 0$ for every $x \in \Omega$,*
- (H₄) *$g(x, \cdot)$ is subadditive for all $x \in \Omega$, and $g \leq \varphi_{f, \mu}$ on $\Omega \times \mathbf{R}^n$,*
- (H₅) *$g^0 = \varphi$ on $(\Omega \setminus N) \times \mathbf{R}^n$, where N is a suitable countable subset of Ω ,*

Then the functional F defined in (2.3) is sequentially weakly l.s.c. on $\mathcal{M}(\Omega; \mathbf{R}^n)$.*

Remark 2.5. — The assumption $\varphi = \varphi_{f, \mu}$ on $(\Omega \setminus N) \times \mathbf{R}^n$ with N countable, of Theorem 3.3 of Bouchitté & Buttazzo [3], has been replaced here by the weaker one $\varphi = \varphi_{f, \mu}$ on $(\Omega \setminus M) \times \mathbf{R}^n$ with $\mu(M) = 0$. A careful inspection of our proof shows indeed that this weaker condition is still sufficient to provide the lower semicontinuity of F .

Remark 2.6. — A slightly more general form of the lower semicontinuity Theorem 2.4 can be given (see Bouchitté and Buttazzo [4]) by requiring, instead of (H₄), that

- (i) the set D_g has no accumulation points,
- (H'₄) (ii) the function g^{∞} is l.s.c. on $\Omega \times \mathbf{R}^n$,
- (iii) $g^{\infty} \leq \varphi_{f, \mu}$ and $g^{\infty} \leq \hat{g}$ on $\Omega \times \mathbf{R}^n$,

where D_g and $\hat{g}$ are defined by

$$D_g = \{x \in \Omega : g(x, \cdot) \text{ is not subadditive}\}$$

$$\hat{g}(x, s) = \liminf_{\substack{(y, t) \rightarrow (x, s) \\ y \neq x}} g(y, t).$$

The fact that all additive sequentially weakly* l.s.c. functionals on $\mathcal{M}(\Omega; \mathbf{R}^n)$ are of the form (2.3) has been shown in [4], where the following result is proved.

THEOREM 2.7: (see Bouchitté and Buttazzo [4], Theorem 2.3). — *Let $F: \mathcal{M}(\Omega, \mathbf{R}^n) \rightarrow [0, +\infty]$ be a functional such that*

- (i) *F is additive (i. e. $F(\lambda + \nu) = F(\lambda) + F(\nu)$ whenever $\lambda \perp \nu$);*
- (ii) *F is sequentially weakly* l.s.c. on $\mathcal{M}(\Omega; \mathbf{R}^n)$.*

Then there exist a non-atomic positive measure $\mu \in \mathcal{M}(\Omega)$ and three Borel functions $f, \varphi, g: \Omega \times \mathbf{R}^n \rightarrow [0, +\infty]$ which satisfy

(H₁) $f(x, \cdot)$ is convex and l.s.c. on $\mathbf{R}^n$, and $f(x, 0) = 0$ for μ -a.e. $x \in \Omega$,

(H₂) $f^\infty(x, \cdot) = \varphi_{f, \mu}(x, \cdot)$ for μ -a.e. $x \in \Omega$,

(H₃) g and g^∞ are l.s.c. on $\Omega \times \mathbf{R}^n$, and $g(x, 0) = 0$ for every $x \in \Omega$,

(H₄) $g^\infty \leq \varphi_{f, \mu}$ and $g^\infty \leq \hat{g}$ on $\Omega \times \mathbf{R}^n$,

(H₅) $g^0 = \varphi = \varphi_{f, \mu}$ on $(\Omega \setminus N) \times \mathbf{R}^n$, where N is a suitable countable subset of Ω , and such that for every $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ the integral representation (2.3) holds.

3. RELAXATION

The main application of Theorem 2.7 consists in representing into an integral form the relaxed functionals associated to additive functionals defined on $\mathcal{M}(\Omega; \mathbf{R}^n)$. More precisely, given a functional $F: \mathcal{M}(\Omega; \mathbf{R}^n) \rightarrow [0, +\infty]$, we consider its relaxed functional $\bar{F}$ defined by

$$\bar{F} = \sup \{ G : G \leq F, G \text{ sequentially weakly* l.s.c. on } \mathcal{M}(\Omega; \mathbf{R}^n) \}.$$

The functional $\bar{F}$ above is sequentially weakly* l.s.c. and less than or equal to F on $\mathcal{M}(\Omega; \mathbf{R}^n)$. We shall apply Theorem 2.7 to $\bar{F}$ thanks to the following result.

THEOREM 3.1. — *Let $F: \mathcal{M}(\Omega; \mathbf{R}^n) \rightarrow [0, +\infty]$ be additive; then $\bar{F}$ is additive too.*

Our goal is to characterize the functional $\bar{F}$ when F is of the form

$$F(\lambda) = \begin{cases} \int_{\Omega} f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{A_\lambda} g(x, \lambda(x)) d\# \\ +\infty & \text{if } \lambda^s = 0 \text{ on } \Omega \setminus A_\lambda \quad \text{otherwise} \end{cases}$$

where $\mu \in \mathcal{M}(\Omega)$ is a non-atomic positive measure and $f, g: \Omega \times \mathbf{R}^n \rightarrow [0, +\infty]$ are two Borel functions satisfying the following assumptions:

$f(x, \cdot)$ is convex and l.s.c. on $\mathbf{R}^n$, and $f(x, 0) = 0$ for μ -a.e. $x \in \Omega$ (3.1)

There exist $\alpha > 0$ and $\beta \in L^1_\mu$ such that:

$$f(x, s) \geq \alpha |s| - \beta(x), \quad \forall (x, s) \in \Omega \times \mathbf{R}^n \quad (3.2)$$

g is l.s.c. on $\Omega \times \mathbf{R}^n$, and $g(x, 0) = 0$ for every $x \in \Omega$ (3.3)

$g(x, \cdot)$ is subadditive for every $x \in \Omega$ (3.4)

$g^0(x, s) \geq \alpha |s|$ for every $(x, s) \in \Omega \times \mathbf{R}^n$. (3.5)

By Theorem 3.1 we may apply the integral representation Theorem 2.7 to $\bar{F}$ and we obtain

$$\bar{F}(\lambda) = \int_{\Omega} \bar{f}\left(x, \frac{d\lambda}{d\bar{\mu}}\right) d\bar{\mu} + \int_{\Omega \setminus A_{\lambda}} \bar{\varphi}(x, \lambda^s) + \int_{A_{\lambda}} \bar{g}(x, \lambda(x)) d\#.$$

for a suitable non-atomic positive measure $\bar{\mu} \in \mathcal{M}(\Omega)$ and suitable Borel functions $\bar{f}, \bar{\varphi}, \bar{g}: \Omega \times \mathbf{R}^n \rightarrow [0, +\infty]$ satisfying conditions (H_1) – (H_5) of Theorem 2.7. In order to characterize these integrands we introduce the functional

$$F_1(\lambda) = \int_{\Omega} f_1\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega \setminus A_{\lambda}} \varphi_1(x, \lambda^s) + \int_{A_{\lambda}} g_1(x, \lambda(x)) d\#$$

where

$$f_1 = f \nabla \varphi_{f, \mu} \nabla g^0, \quad \varphi_1 = \varphi_{f, \mu} \nabla g^0, \quad g_1 = \varphi_{f, \mu} \nabla g.$$

The main result of this paper is the following relaxation theorem.

THEOREM 3.2. — *For every $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ we have*

$$\bar{F}(\lambda) = F_1(\lambda).$$

Remark 3.3. — We may consider on g the following weaker assumptions instead of (3.4):

There exists a subset D of Ω , which has no accumulation points, such that $g(x, \cdot)$ is subadditive for every $x \in \Omega \setminus D$, and the function g^{∞} is l.s.c. in (x, s) .

The conclusion will be the same.

Example 3.4. — Let $p \in [1, +\infty]$, $q \in [0, 1]$, and let

$$f(s) = |s|^p, \quad g(s) = |s|^q.$$

In the case $p = +\infty$ we set $f = \chi_{\{|s| \leq 1\}}$ (i.e. the function which is 0 if $|s| \leq 1$ and $+\infty$ otherwise), and in the case $q = 0$ we set $g = 1_{\mathbf{R} \setminus \{0\}}$ (i.e. the function which is 1 if $s \neq 0$ and 0 if $s = 0$). Then we have

$$\begin{aligned} p > 1, \quad q < 1 &\Rightarrow \bar{f} = f, & \bar{g} = g \\ p = 1, \quad q = 1 &\Rightarrow \bar{f} = f, & \bar{g} = g \end{aligned}$$

that is the associated functional F is sequentially weakly* lower semicontinuous. In the remaining cases, F is not sequentially weakly* lower semicontinuous and, after some calculations, one finds

$$\begin{aligned} p > 1, \quad q = 1 &\Rightarrow \bar{g} = g, & \bar{f}(s) = (f \nabla |\cdot|)(s), \\ p = 1, \quad q < 1 &\Rightarrow \bar{f} = f, & \bar{g}(s) = (g \nabla |\cdot|)(s). \end{aligned}$$

It is

$$(f \nabla |\cdot|)(s) = \begin{cases} |s|^p & \text{if } |s| \leq p^{1/(1-p)} \\ |s| + p^{p/(1-p)} - p^{1/(1-p)} & \text{if } |s| > p^{1/(1-p)} \end{cases}$$

$$(g \nabla |\cdot|)(s) = |s| \wedge |s|^q.$$

Of course, in the case $p = +\infty$ and $q = 1$ it is

$$\bar{f}(s) = \begin{cases} 0 & \text{if } |s| \leq 1 \\ |s| - 1 & \text{if } |s| > 1, \end{cases}$$

while, in the case $p = 1$ and $q = 0$ it is

$$\bar{g}(s) = |s| \wedge 1.$$

4. PROOF OF THE RESULTS

In this section we shall prove Theorem 3.1 and Theorem 3.2; some preliminary lemmas will be necessary.

LEMMA 4.1. — *Let $\lambda_h \rightarrow \lambda$, let C be a compact subset of Ω , and for every $t > 0$ let*

$$C(t) = \{x \in \Omega : \text{dist}(x, C) < t\}.$$

Then there exists a sequence $t_h \rightarrow 0$ such that

$$1_{C(t_h)} \lambda_h \rightarrow 1_C \lambda.$$

Proof. — Since $C(r)$ is relatively compact, we have

$$1_{C(r)} \lambda_h \rightarrow 1_{C(r)} \lambda$$

as soon as $\partial C(r)$ is $|\lambda|$ -negligible, hence for all $r \in \mathbf{R}^+ \setminus \mathbf{N}$ with $\mathbf{N}$ at most countable. Choose $r_k \in \mathbf{R}^+ \setminus \mathbf{N}$ with $r_k \rightarrow 0$; then

$$\begin{cases} 1_{C(r_k)} \lambda_h \rightarrow 1_{C(r_k)} \lambda & (\text{as } h \rightarrow \infty) & \text{for every } k \in \mathbf{N}, \\ 1_{C(r_k)} \lambda \rightarrow 1_C \lambda & (\text{as } k \rightarrow \infty). \end{cases}$$

Therefore, the conclusion follows by a standard diagonalization procedure. ■

Remark 4.2. — For every functional $G : \mathcal{M}(\Omega; \mathbf{R}^n) \rightarrow [0, +\infty]$ we define

$$G'(\lambda) = \inf \left\{ \liminf_{h \rightarrow \infty} G(\lambda_h) : \lambda_h \rightarrow \lambda \right\} \quad \text{for every } \lambda \in \mathcal{M}(\Omega; \mathbf{R}^n).$$

It is possible to prove (see for instance Buttazzo [7], Proposition 1.3.2) that if Ξ is the set of all countable ordinals and for every $\xi \in \Xi$ we define

by transfinite induction

$$\begin{aligned} F_0 &= F \\ F_{\xi+1} &= (F_\xi)' \\ F_\xi &= \inf \{ F_\eta : \eta < \xi \} \quad \text{if } \xi \text{ is a limit ordinal,} \end{aligned}$$

we have

$$\bar{F} = \inf \{ F_\xi : \xi \in \Xi \}.$$

LEMMA 4.3. — For every $\varepsilon > 0$ and $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ let us define

$$F_\varepsilon(\lambda) = F(\lambda) + \varepsilon \|\lambda\|. \quad (4.1)$$

Then we have

$$F' = \inf \{ F'_\varepsilon : \varepsilon > 0 \}.$$

Proof. — The inequality $\leq$ is obvious. In order to prove the opposite inequality, fix $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ and $r > 0$; there exists $\lambda_h \rightarrow \lambda$ such that, setting $M = \sup \{ \|\lambda_h\| : h \in \mathbf{N} \}$, it is

$$F'(\lambda) \geq \liminf_{h \rightarrow \infty} F(\lambda_h) = \liminf_{h \rightarrow \infty} [F_\varepsilon(\lambda_h) - \varepsilon \|\lambda_h\|] \geq F'_\varepsilon(\lambda) - \varepsilon M.$$

The conclusion follows by letting $\varepsilon \rightarrow 0$. ■

Proof of Theorem 3.1. — By Remark 4.2 it is enough to show that

$$F \text{ additive} \Rightarrow F' \text{ additive}.$$

Moreover, setting F_ε as in (4.1) and applying Lemma 4.3, it is enough to prove that F'_ε is additive for every $\varepsilon > 0$. By Proposition 1.3.5 and Remark 1.3.6 of Buttazzo [7] it is

$$F'_\varepsilon = \bar{F}_\varepsilon \quad \text{for every } \varepsilon > 0;$$

in particular, F'_ε is weakly* l.s.c. on $\mathcal{M}(\Omega; \mathbf{R}^n)$. We prove first that for every $r > 0$, $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$, and $B_1, B_2 \in \mathcal{B}$ with $B_1 \cap B_2 = \emptyset$ it is

$$r + F'_\varepsilon(1_{B_1 \cup B_2} \lambda) \geq F'_\varepsilon(1_{B_1} \lambda) + F'_\varepsilon(1_{B_2} \lambda). \quad (4.2)$$

Let $\lambda_h \rightarrow 1_{B_1 \cup B_2} \lambda$ be such that

$$r + F'_\varepsilon(1_{B_1 \cup B_2} \lambda) \geq \liminf_{h \rightarrow \infty} F_\varepsilon(\lambda_h), \quad (4.3)$$

and let $K_i \subset B_i$ be compact sets ($i = 1, 2$). By Lemma 4.1 we have

$$1_{K_i(t_h)} \lambda_h \rightarrow 1_{K_i} \lambda \quad (i = 1, 2)$$

for a suitable sequence $t_h \rightarrow 0$, so that

$$\begin{aligned} \liminf_{h \rightarrow \infty} F_\varepsilon(\lambda_h) &\geq \liminf_{h \rightarrow \infty} F_\varepsilon(1_{K_1(t_h)} \lambda_h) + \liminf_{h \rightarrow \infty} F_\varepsilon(1_{K_2(t_h)} \lambda_h) \\ &\geq F'_\varepsilon(1_{K_1} \lambda) + F'_\varepsilon(1_{K_2} \lambda). \end{aligned} \quad (4.4)$$

Now, (4.2) (hence the superadditivity of F'_ε) follows from (4.3) and (4.4) by taking the supremum as $K_1 \uparrow B_1$ and $K_2 \uparrow B_2$. Finally, we prove that for every $r > 0$, $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$, and $B_1, B_2 \in \mathcal{B}$ with $B_1 \cap B_2 = \emptyset$, it is

$$F'_\varepsilon(1_{B_1 \cup B_2} \lambda) \leq F'_\varepsilon(1_{B_1} \lambda) + F'_\varepsilon(1_{B_2} \lambda) + r. \quad (4.5)$$

Let $\lambda_{1,h} \rightarrow 1_{B_1} \lambda$ and $\lambda_{2,h} \rightarrow 1_{B_2} \lambda$ be such that

$$\liminf_{h \rightarrow \infty} F_\varepsilon(\lambda_{i,h}) \leq F'_\varepsilon(1_{B_i} \lambda) + \frac{r}{2} \quad (i=1, 2), \quad (4.6)$$

and let $K_i \subset B_i$ be compact sets ($i=1, 2$). By Lemma 4.1 we have

$$1_{K_i(t_h)} \lambda_{i,h} \rightarrow 1_{K_i} \lambda \quad (i=1, 2)$$

for a suitable sequence $t_h \rightarrow 0$, so that

$$\begin{aligned} \liminf_{h \rightarrow \infty} [F_\varepsilon(\lambda_{1,h}) + F_\varepsilon(\lambda_{2,h})] &\geq \liminf_{h \rightarrow \infty} [F_\varepsilon(1_{K_1(t_h)} \lambda_{1,h}) + F_\varepsilon(1_{K_2(t_h)} \lambda_{2,h})] \\ &= \liminf_{h \rightarrow \infty} F_\varepsilon(1_{K_1(t_h)} \lambda_{1,h} + 1_{K_2(t_h)} \lambda_{2,h}) \geq F'_\varepsilon(1_{K_1 \cup K_2} \lambda). \end{aligned}$$

Now, (4.5) (hence the subadditivity of F'_ε) follows from (4.6) and (4.7) by taking the supremum as $K_1 \uparrow B_1$ and $K_2 \uparrow B_2$. ■

LEMMA 4.4. — *There exists a countable subset N of Ω such that*

- (i) $\bar{g} \leq g$ on $\Omega \times \mathbf{R}^n$,
- (ii) $\bar{g} \leq \varphi_{f,\mu}$ on $\Omega \times \mathbf{R}^n$,
- (iii) $\bar{\varphi} \leq g^0$ on $(\Omega \setminus N) \times \mathbf{R}^n$,
- (iv) $\bar{\varphi} \leq \varphi_{f,\mu}$ on $(\Omega \setminus N) \times \mathbf{R}^n$.

Proof. — Property (i) follows immediately from the fact that $\bar{F} \leq F$ on $\mathcal{M}(\Omega; \mathbf{R}^n)$.

Let us prove property (ii). Denoting by F_0 the functional

$$F_0(\lambda) = \begin{cases} F(\lambda) & \text{if } \lambda \in \mathcal{M}^0(\Omega; \mathbf{R}^n) \\ +\infty & \text{otherwise,} \end{cases} \quad (4.8)$$

by using Theorem 4 of Bouchitté and Valadier [5] and Proposition 2.2 we have

$$\bar{F}_0(\lambda) = \int_{\Omega} (f \nabla \varphi_{f,\mu}) \left(x, \frac{d\lambda}{d\mu} \right) d\mu + \int_{\Omega} \varphi_{f,\mu}(x, \lambda^s), \quad \forall \lambda \in \widehat{\mathcal{M}}(\Omega; \mathbf{R}^n) \quad (4.9)$$

so that, if $\lambda = s \delta_x$,

$$\bar{g}(x, s) = \bar{F}(s \delta_x) \leq \bar{F}_0(s \delta_x) = \int_{\Omega} \varphi_{f,\mu}(x, s \delta_x) = \varphi_{f,\mu}(x, s).$$

Let us prove property (iii). By the integral representation Theorem 2.7 we have for a suitable countable subset N of Ω

$$\bar{\varphi} = (\bar{g})^0 \quad \text{on } (\Omega \setminus N) \times \mathbf{R}^n,$$

so that (iii) follows from (i).

Finally, let us prove property (iv). If F_0 is the functional defined in (4.8), we have

$$\frac{1}{t} \bar{F}(t\lambda) \leq \frac{1}{t} \bar{F}_0(t\lambda), \quad \forall t > 0, \quad \forall \lambda \in \mathcal{M}^0(\Omega; \mathbf{R}^n).$$

Letting $t \rightarrow +\infty$ and taking (4.9) into account, we get for every $\lambda \in \mathcal{M}^0(\Omega; \mathbf{R}^n)$

$$\begin{aligned} \int_{\Omega} (\bar{f})^{\infty} \left(x, \frac{d\lambda}{d\bar{\mu}} \right) d\bar{\mu} + \int_{\Omega} \bar{\varphi}(x, \lambda^s) &= (\bar{F})^{\infty}(\lambda) \leq (\bar{F}_0)^{\infty}(\lambda) \\ &= \int_{\Omega} (f \nabla \varphi_{f, \mu})^{\infty} \left(x, \frac{d\lambda}{d\mu} \right) d\mu + \int_{\Omega} \varphi_{f, \mu}(x, \lambda^s) = \int_{\Omega} \varphi_{f, \mu}(x, \lambda) \end{aligned}$$

since $\varphi_{f, \mu}(x, \cdot) \leq f^{\infty}(x, \cdot)$ for μ -a.e. $x \in \Omega$. By Theorem 2.7 it is $(\bar{f})^{\infty}(x, \cdot) = \bar{\varphi}(x, \cdot)$ for $\bar{\mu}$ -a.e. $x \in \Omega$, and we obtain

$$\int_{\Omega} \bar{\varphi}(x, \lambda) \leq \int_{\Omega} \varphi_{f, \mu}(x, \lambda), \quad \forall \lambda \in \mathcal{M}^0(\Omega; \mathbf{R}^n),$$

so that (iv) follows from Proposition 3.2 of Bouchitté and Buttazzo [3]. ■

LEMMA 4.5. — *The functional F_1 is sequentially weakly* l.s.c. on $\mathcal{M}(\Omega; \mathbf{R}^n)$ and verifies the inequality $F_1 \leq F$.*

Proof. — The inequality $F_1 \leq F$ is an obvious consequence of the definition of f_1, φ_1, g_1 . We shall apply the lower semicontinuity Theorem 2.4 by showing that the functions f_1, φ_1, g_1 satisfy conditions (H_1) – (H_5) . Conditions (H_1) and (H_3) follow immediately from Proposition 2.2(i), and condition (H_5) follows from Proposition 2.3.

Let us prove condition (H_4) . The subadditivity of $g_1(x, \cdot)$ is an easy consequence of the subadditivity of $g(x, \cdot)$ and $\varphi_{f, \mu}(x, \cdot)$; it remains to prove that $g_1 \leq \varphi_{f_1, \mu}$ on $\Omega \times \mathbf{R}^n$, or equivalently $(g_1)^0 \leq \varphi_{f_1, \mu}$ on $\Omega \times \mathbf{R}^n$. Setting

$$\begin{aligned} \Gamma_f(x) &= \text{dom}(\varphi_{f, \mu})^*(x, \cdot) \\ \Gamma_{f_1}(x) &= \text{dom}(\varphi_{f_1, \mu})^*(x, \cdot) \\ \Gamma_0(x) &= \text{dom}(g^0)^*(x, \cdot) \end{aligned}$$

and using Proposition 2.2(i), it remains to show that

$$\Gamma_0(x) \cap \Gamma_f(x) \subset \Gamma_{f_1}(x), \quad \forall x \in \Omega.$$

Since g^0 is coercive and l.s.c., the multimapping $x \mapsto \Gamma_0(x)$ is l.s.c. and its values are with nonempty interior. The same holds true for $\Gamma_f(x)$ and $\Gamma_{f_1}(x)$. Moreover, by Proposition 6 of Bouchitté and Valadier [5] we have

$$\Gamma_f(x) = \text{cl} \{ s \in \mathbf{R}^n : f^*(\cdot, s) \text{ is locally } \mu\text{-integrable around } x \} \quad (4.10)$$

$$\Gamma_{f_1}(x) = \text{cl} \{ s \in \mathbf{R}^n : (f_1)^*(\cdot, s) \text{ is locally } \mu\text{-integrable around } x \}. \quad (4.11)$$

Let us now fix $x \in \Omega$ and $s \in \text{int}(\Gamma_0(x) \cap \Gamma_f(x))$. The lower semicontinuity of the multimapping Γ_0 implies (see for instance Lemma 15 of [6]) that for a suitable neighbourhood V of x

$$s \in \Gamma_0(y), \quad \forall y \in V.$$

By (4.10) we can choose V such that

$$\int_V f^*(\cdot, s) d\mu < +\infty.$$

Therefore

$$\begin{aligned} \int_V f_1^*(\cdot, s) d\mu &= \int_V [f^*(\cdot, s) + (g^0)^*(\cdot, s) + \varphi_{f, \mu}^*(\cdot, s)] d\mu \\ &= \int_V f^*(\cdot, s) d\mu < +\infty \end{aligned}$$

that is, by (4.11), $s \in \Gamma_{f_1}(x)$. Hence

$$\text{int}(\Gamma_0(x) \cap \Gamma_f(x)) \subset \Gamma_{f_1}(x).$$

The conclusion now follows by recalling that $\Gamma_{f_1}(x)$ is closed, and that $\text{cl}(\text{int } K) = \text{cl } K$ for every convex set $K \subset \mathbf{R}^n$ with nonempty interior.

Finally, let us prove condition (H_2) . Since $f_1 \leq \varphi_1$ on $\Omega \times \mathbf{R}^n$, we have $f_1^\infty \leq \varphi_1^\infty = \varphi_1$ on $\Omega \times \mathbf{R}^n$. By conditions (H_4) and (H_5) already proved, we have for a countable set $N \subset \Omega$

$$\varphi_1 = g_1^0 \leq (\varphi_{f_1, \mu})^0 = \varphi_{f_1, \mu} \quad \text{on } (\Omega \setminus N) \times \mathbf{R}^n.$$

Finally, the inequality

$$\varphi_{f_1, \mu}(x, \cdot) \leq f_1^\infty(x, \cdot) \quad \text{for } \mu\text{-a.e. } x \in \Omega,$$

is a general property of the functions of the form $\varphi_{f, \mu}$ (see Section 2). ■

LEMMA 4.6. — *Setting*

$$E = \{ x \in \Omega : \bar{f}(x, \cdot) \neq \bar{\varphi}(x, \cdot) \}$$

we have that there exists $\alpha \in L_\mu^1(\Omega)$ such that $\alpha\mu = 1_E \bar{\mu}$.

Proof. — Let us consider $\lambda \in \mathcal{M}^0(\Omega; \mathbf{R}^n)$ with $\lambda \perp \mu$; taking into account that $F_1 \leq \bar{F}$ (by Lemma 4.6) and $\bar{\varphi} \leq \varphi_1$ (by Lemma 4.5) we have

$$\bar{F}(\lambda) \geq F_1(\lambda) = \int_\Omega \varphi_1(x, \lambda) \geq \int_\Omega \bar{\varphi}(x, \lambda) = (\bar{F})^\infty(\lambda).$$

Since $\bar{F} \leq (\bar{F})^\infty$ on $\mathcal{M}(\Omega; \mathbf{R}^n)$, we obtain

$$\bar{F}(\lambda) = (\bar{F})^\infty(\lambda) \quad \text{for every } \lambda \in \mathcal{M}^0(\Omega; \mathbf{R}^n) \quad \text{with } \lambda \perp \mu. \quad (4.12)$$

Consider now the Lebesgue-Nikodym decomposition of $1_E \bar{\mu}$ with respect to μ

$$1_E \bar{\mu} = \alpha \mu + \nu \quad \text{with } \alpha \in L^1_\mu(\Omega), \quad \nu \perp \mu,$$

and let

$$\lambda = u 1_E \nu \quad \text{with } u \in L^1_\nu(\Omega).$$

We have, by (4.12)

$$\int_E \bar{f}(x, u) d\nu = \bar{F}(\lambda) = (\bar{F})^\infty(\lambda) = \int_E \bar{\varphi}(x, \lambda) = \int_E \bar{\varphi}(x, u) d\nu.$$

Since $u \in L^1_\nu(\Omega)$ is arbitrary, we get

$$\bar{f}(x, \cdot) = \bar{\varphi}(x, \cdot) \quad \nu\text{-a.e. on } E,$$

and, by definition of E , this implies $\nu(E) = 0$, that is $\nu = 0$. ■

Proof of Theorem 3.2. — By Lemma 4.5 it is enough to show that

$$\bar{F} \leq F_1 \quad \text{on } \mathcal{M}(\Omega; \mathbf{R}^n),$$

that is

$$\bar{g} \leq g_1 \quad \text{on } \Omega \times \mathbf{R}^n \quad (4.13)$$

$$\bar{\varphi} \leq \varphi_1 \quad \text{on } (\Omega \setminus N) \times \mathbf{R}^n \quad (4.14)$$

$$1_E \bar{\mu} = \alpha \mu \quad (4.15)$$

$$f_1(x, s) \geq \begin{cases} \alpha(x) \bar{f}\left(x, \frac{s}{\alpha(x)}\right) & \text{if } \alpha(x) \neq 0 \\ \bar{\varphi}(x, s) & \text{if } \alpha(x) = 0 \end{cases} \quad \text{on } (\Omega \setminus M) \times \mathbf{R}^n \quad (4.16)$$

where N is a suitable countable subset of Ω , M is a suitable Borel subset of Ω with $\mu(M) = 0$, and α is a suitable function in $L^1_\mu(\Omega)$.

Conditions (4.13) and (4.14) follow from Lemma 4.4, whereas (4.15) follows from Lemma 4.6. Let us now prove (4.16). Take $u \in L^1_\mu(\Omega; \mathbf{R}^n)$ and $\lambda = u \mu$. We have

$$1_{\{\alpha \neq 0\} \cap E} \lambda = \frac{u}{\alpha} 1_{\{\alpha \neq 0\} \cap E} \bar{\mu} \quad \text{so that}$$

$$\bar{F}(1_{\{\alpha \neq 0\} \cap E} \lambda) = \int_{\{\alpha \neq 0\}} \alpha \bar{f}\left(x, \frac{u}{\alpha}\right) d\mu \quad (4.17)$$

$1_{\{\alpha \neq 0\} \setminus E} \lambda = 0$ because $\alpha = 0$ μ -a.e. on $\Omega \setminus E$, hence

$$\bar{F}(1_{\{\alpha \neq 0\} \setminus E} \lambda) = 0 \quad (4.18)$$

$1_{\{\alpha \neq 0\} \cap E} \lambda \perp \bar{\mu}$ because $\bar{\mu}(\{\alpha = 0\} \cap E) = 0$, hence

$$\bar{F}(1_{\{\alpha = 0\} \cap E} \lambda) = \int_{\{\alpha = 0\} \cap E} \bar{\varphi}(x, \lambda) \quad (4.19)$$

$\bar{f} = \bar{\varphi}$ on $(\Omega \setminus E) \times \mathbf{R}^n$ so that

$$\bar{F}(1_{\{\alpha = 0\} \setminus E} \lambda) = \int_{\{\alpha = 0\} \setminus E} \bar{\varphi}(x, \lambda). \quad (4.20)$$

Collecting (4.17)-(4.20) we get

$$\begin{aligned} \int_{\Omega} f(x, u) d\mu &= F(\lambda) \geq \bar{F}(\lambda) \\ &= \bar{F}(1_{\{\alpha \neq 0\} \cap E} \lambda) + \bar{F}(1_{\{\alpha \neq 0\} \setminus E} \lambda) + \bar{F}(1_{\{\alpha = 0\} \cap E} \lambda) + \bar{F}(1_{\{\alpha = 0\} \setminus E} \lambda) \\ &= \int_{\{\alpha \neq 0\}} \alpha \bar{f}\left(x, \frac{u}{\alpha}\right) d\mu + \int_{\{\alpha = 0\}} \bar{\varphi}(x, u) d\mu. \end{aligned}$$

Since $u \in L^1_{\mu}(\Omega; \mathbf{R}^n)$ was arbitrary, we obtain for a suitable $B \in \mathcal{B}$ with $\mu(M) = 0$

$$f(x, s) \geq \begin{cases} \alpha(x) \bar{f}\left(x, \frac{s}{\alpha(x)}\right) & \text{if } \alpha(x) \neq 0 \\ \bar{\varphi}(x, s) & \text{if } \alpha(x) = 0 \end{cases} \quad (4.21)$$

for every $(x, s) \in (\Omega \setminus M) \times \mathbf{R}^n$. Now, (4.16) comes out easily from (4.21). Indeed, for μ -a.e. $x \in \Omega$ with $\alpha(x) = 0$, we have, using (4.14) and (4.21):

$$\bar{\varphi}(x, \cdot) \leq \inf\{\varphi_1(x, \cdot), f(x, \cdot)\} \leq \varphi_1(x, \cdot) \vee f(x, \cdot) = f_1(x, \cdot).$$

On the other hand, by Theorem 2.7 and (4.14) we get

$$\bar{f}(x, \cdot) \leq (\bar{f})^{\infty}(x, \cdot) \leq \bar{\varphi}(x, \cdot) \leq \varphi_1(x, \cdot)$$

$\bar{\mu}$ -a.e. on Ω , hence μ -a.e. on $\{\alpha \neq 0\}$, so that by (4.21):

$$\alpha(x) \bar{f}\left(x, \frac{s}{\alpha(x)}\right) \leq \inf\{\varphi_1(x, s), f(x, s)\} \leq f_1(x, s)$$

on $(\Omega \setminus M) \times \mathbf{R}^n$ with $\mu(M) = 0$. Therefore (4.16) is proved, and the proof of Theorem 3.2 is completely achieved. ■

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