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Homogenization of almost periodic monotone operators

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ABSTRACT. — We determine some sufficient conditions for the G-convergence of sequences of quasi-linear monotone operators, together with an asymptotic formula for the G-limit. We then prove a homogenization theorem for quasiperiodic monotone operators and, eventually, extend this result to general almost periodic monotone operators using an approximation result and a closure lemma.

Key words : Homogenization, G-convergence, almost periodic functions, monotone operators, quasi-linear equations.

RÉSUMÉ. — Nous déterminons quelques conditions suffisantes pour la G-convergence de suites d'opérateurs monotones quasi linéaires, avec une

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formule asymptotique pour la G -limite. Ensuite nous démontrons un théorème d'homogénéisation pour les opérateurs monotones quasi-périodiques et, enfin, nous étendons ce résultat aux opérateurs monotones presque périodiques en utilisant un résultat d'approximation.

INTRODUCTION

In this paper we consider a class of quasi-linear operators $\mathcal{A} : H_0^{1,p}(\Omega) \rightarrow H^{-1,q}(\Omega)$ of the form

$$\mathcal{A}u = -\operatorname{div}(a(x, Du)),$$

where Ω is a bounded open subset of $\mathbf{R}^n$, $1 < p < +\infty$, $1/p + 1/q = 1$, and the function $a : \Omega \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ satisfies suitable measurability, continuity, and monotonicity assumptions. By M_Ω we denote the set of such functions a . In order to study the behaviour of boundary value problems of the type

$$\begin{cases} -\operatorname{div}(a(x, Du)) = f & \text{on } \Omega, \\ u \in H_0^{1,p}(\Omega) \end{cases}$$

under perturbations of the function $a \in M_\Omega$, a notion of G -convergence has been introduced in [16]. Its definition and main properties are recalled in Section 1.

The main purpose of Section 2 is to determine some conditions on a sequence of functions (a_h) in M_Ω which imply G -convergence. It turns out that one of them is simply the strong convergence in $(L^1(\Omega))^n$ of the sequence $(a_h(\cdot, \xi))$, for every $\xi \in \mathbf{R}^n$ (see Theorem 2.1). A necessary and sufficient condition involves the limit behaviour, as h tends to $+\infty$, of the integrals

$$\int_A a_h(x, Dv_h + \xi) dx$$

for every $\xi \in \mathbf{R}^n$ and for every A in a suitable family of open sets, where the functions v_h are the solutions to the Dirichlet boundary value problems

$$\begin{cases} -\operatorname{div}(a_h(x, Dv_h + \xi)) = 0 & \text{on } A, \\ v_h \in H_0^{1,p}(A) \end{cases}$$

(see Theorem 2.3). The proof of this result relies on a representation formula for functions $a \in M_\Omega$ given in Theorem 2.2.

Section 3 is concerned with the homogenization of operators defined by functions of the class M_Ω ; *i.e.*, the G-convergence of sequences of functions (a_h) of the form

$$a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right),$$

where (ε_h) is a sequence of positive real numbers converging to 0. We suppose that the function $a: \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ satisfies the usual measurability, continuity, and monotonicity assumptions, and a condition of quasiperiodicity with respect to x (see Definition 3.1). By using the results obtained in Section 2, we prove that there exists a monotone operator $b: \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that, for every $f \in H^{-1, q}(\Omega)$, the solutions u_h and the momenta $a\left(\frac{x}{\varepsilon_h}, Du_h\right)$ of the Dirichlet boundary value problems

$$\begin{cases} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon_h}, Du_h\right)\right) = f & \text{on } \Omega, \\ u_h \in H_0^{1, p}(\Omega) \end{cases}$$

converge, as (ε_h) tends to 0, to the solution u and the momentum $b(Du)$ of the homogenized problem

$$\begin{cases} -\operatorname{div}(b(Du)) = f & \text{on } \Omega, \\ u \in H_0^{1, p}(\Omega). \end{cases}$$

Theorem 3.4 gives also an asymptotic formula for the function b ; its proof generalizes a construction used in [36].

In Section 4 we extend the homogenization result of Section 3 to almost periodic operators (in the sense of Besicovitch; see Definition 4.1) defined by functions of the class $M_{\mathbf{R}^n}$ using an approximation result (Lemma 4.4) and a closure lemma (Lemma 4.3).

The notion of G-convergence for second order linear elliptic operators was studied by E. De Giorgi and S. Spagnolo in the symmetric case (see [40], [41], [42], [21]), and then extended to the non-symmetric case by F. Murat and L. Tartar under the name of H-convergence (see [43], [44], and [34]). A further extension to higher order linear elliptic operators can be found in [47] together with an extensive bibliography on this subject. Results for the quasi-linear case are given, among others, in [46], [37], [23], [22] and [16].

For the related problems in homogenization theory under periodicity hypotheses on $a(\cdot, \xi)$, we refer to the books [2], [39], and [1], which contain a wide bibliography on this topic. Homogenization results for quasi-linear operators are obtained in [4], [5], [6], [25], [26], [17], while the almost periodic case for linear equations is studied in [28] and [36]. A corrector result for quasi-linear periodic equations has been obtained in

[18], whereas an analogous theorem for the almost periodic case will appear in [10].

From another point of view, the homogenization of a class of variational integrals of the form

$$F_h(u) = \int_{\Omega} f\left(\frac{x}{\varepsilon_h}, Du\right) dx,$$

which is related to the homogenization of the operators $-\operatorname{div}\left(\partial_{\xi} f\left(\frac{x}{\varepsilon_h}, \xi\right)\right)$, has been studied in [32], [15], and [7], using the techniques of Γ -convergence introduced by E. De Giorgi. Homogenization results for variational integrals under almost periodicity assumptions have been proven in [8], [9], [11].

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1. NOTATIONS AND PRELIMINARY RESULTS

Let p be a real constant, $1 < p < +\infty$, and let q be its dual exponent, $1/p + 1/q = 1$. The Euclidean norm and the scalar product in $\mathbf{R}^n$ are denoted by $|\cdot|$ and $(\cdot, \cdot)$, respectively.

DEFINITION 1.1. — Given four constants α, β, c_1 , and c_2 , such that $c_1 > 0$, $c_2 > 0$, $0 \leq \alpha \leq 1 \wedge (p-1)$, $p \vee 2 \leq \beta < +\infty$, we denote by $M(\alpha, \beta, c_1, c_2)$ the class of all functions $a: \mathbf{R}^n \rightarrow \mathbf{R}^n$ which fulfill the following conditions:

- (i) $|a(0)| \leq c_1$;
- (ii) a satisfies the following inequalities of equicontinuity and strict monotonicity:

$$|a(\xi_1) - a(\xi_2)| \leq c_1 (1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha \quad (1.1)$$

$$(a(\xi_1) - a(\xi_2), \xi_1 - \xi_2) \geq c_2 (1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta \quad (1.2)$$

for every $\xi_1, \xi_2 \in \mathbf{R}^n$.

For every open subset $\mathcal{O}$ of $\mathbf{R}^n$, by $M_{\mathcal{O}}(\alpha, \beta, c_1, c_2)$ we denote the class of all functions $a: \mathcal{O} \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ which satisfy the following conditions:

- (iii) for a. e. $x \in \mathcal{O}$, $|a(x, \cdot)| \in M(\alpha, \beta, c_1, c_2)$;
- (iv) for every $\xi \in \mathbf{R}^n$, $a(\cdot, \xi)$ is Lebesgue measurable.

It is easy to see that (iii) implies that there exist constants $c_3 > 0$, $c_4 > 0$, such that

$$|a(x, \xi)| \leq c_3 (1 + |\xi|)^{p-1}, \quad (1.3)$$

$$|\xi|^p \leq c_4 (1 + (a(x, \xi), \xi)) \quad (1.4)$$

for a.e. $x \in \mathcal{O}$, for every $\xi \in \mathbf{R}^n$. The proof of (1.3) is trivial. As for the proof of (1.4), we just observe that by Young's inequality we have

$$|\xi|^p \leq \frac{p}{\beta} (1 + |\xi|)^{p-\beta} |\xi|^\beta + \frac{\beta-p}{\beta} (1 + |\xi|)^p.$$

In the case $\mathcal{O} = \mathbf{R}^n$, we simply use the notation $M_{\mathbf{R}^n}$ for $M_{\mathbf{R}^n}(\alpha, \beta, c_1, c_2)$.

Let us fix from now on a bounded open subset Ω of $\mathbf{R}^n$. Given $a \in M_\Omega(\alpha, \beta, c_1, c_2)$, it can be proved that for every $f \in H^{-1,q}(\Omega)$ there exists a unique solution $u \in H_0^{1,p}(\Omega)$ to the following Dirichlet boundary value problem

$$\left. \begin{aligned} -\operatorname{div}(a(x, Du)) &= f \quad \text{on } \Omega, \\ u &\in H_0^{1,p}(\Omega). \end{aligned} \right\} \quad (1.5)$$

For a proof we refer, for instance, to [27], Chapter III, Corollary 1.8, or to [31], Chapter 2, Theorem 2.1. The solution to (1.5) satisfies a Meyers' regularity estimate (see [33]) that will be needed in the sequel in the particular case where Ω is a cube, as stated in the following theorem.

THEOREM 1.2. — *Let Q be a cube in $\mathbf{R}^n$ and let $w \in H_0^{1,p}(Q)$ be the weak solution to the equation*

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dw)) &= 0 \quad \text{on } Q, \\ w &\in H_0^{1,p}(Q). \end{aligned} \right\}$$

Then there exists $\eta > 0$ such that $w \in H^{1,p+\eta}(Q)$ and

$$\|Dw\|_{(L^{p+\eta}(Q))^n} \leq C(Q) \|Dw\|_{(L^p(Q))^n}. \quad (1.6)$$

The constant η depends only on c_1, c_2, n, p , while $C(Q)$ depends in addition on Q . Moreover, a simple rescaling argument shows that we can take

$$C(tQ) = t^{-n/r} C(Q)$$

for all $t > 0$, where $\frac{1}{r} + \frac{1}{q} + \frac{1}{p+\eta} = 1$.

In order to study the behaviour of problem (1.5) under perturbations of the function a we make use of the following notion of G-convergence.

DEFINITION 1.3. — We say that a sequence (a_h) in $M_\Omega(\alpha, \beta, c_1, c_2)$ G-converges to $a \in M_\Omega(\alpha, \beta, c_1, c_2)$ if, for every $f \in H^{-1,q}(\Omega)$ and for every sequence (f_h) converging to f strongly in $H^{-1,q}(\Omega)$, the solutions u_h to

the equations

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Du_h)) &= f_h \quad \text{on } \Omega, \\ u_h &\in H_0^{1,p}(\Omega), \end{aligned} \right\} \quad (1.7)$$

satisfy the following conditions:

$$u_h \rightarrow u \quad \text{weakly in } H_0^{1,p}(\Omega), \quad (1.8)$$

$$a_h(x, Du_h) \rightarrow a(x, Du) \quad \text{weakly in } (L^q(\Omega))^n, \quad (1.9)$$

where u is the solution to the equation

$$\left. \begin{aligned} -\operatorname{div}(a(x, Du)) &= f \quad \text{on } \Omega, \\ u &\in H_0^{1,p}(\Omega). \end{aligned} \right\} \quad (1.10)$$

Remark 1.4. — It can be proved that this definition of G-convergence is independent of the boundary condition. More precisely, if $\varphi \in H^{1,p}(\Omega)$, if the sequence $(a_h) \in M_\Omega(\alpha, \beta, c_1, c_2)$ G-converges to $a \in M_\Omega(\alpha, \beta, c_1, c_2)$, (f_h) converges to f strongly in $H^{-1,q}(\Omega)$, and u_h are the solutions in $H^{1,p}(\Omega)$ to the equations

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Du_h)) &= f_h \quad \text{on } \Omega, \\ u_h - \varphi &\in H_0^{1,p}(\Omega), \end{aligned} \right\}$$

then

$$u_h \rightarrow u \quad \text{weakly in } H^{1,p}(\Omega),$$

$$a_h(x, Du_h) \rightarrow a(x, Du) \quad \text{weakly in } (L^q(\Omega))^n,$$

where u is the solution to the equation

$$\left. \begin{aligned} -\operatorname{div}(a(x, Du)) &= f \quad \text{on } \Omega, \\ u - \varphi &\in H_0^{1,p}(\Omega). \end{aligned} \right\}$$

A proof of this fact can be found, for instance, in [16], Theorem 3.8.

The next two theorems concern a localization property and a compactness result for G-convergence. Their proofs can be deduced from Theorem 6.1 and Theorem 4.1 in [16] respectively, by using Theorem 7.9 and Corollaries 7.10-7.12 therein.

Let Ω' be an open subset of Ω . For $a \in M_\Omega(\alpha, \beta, c_1, c_2)$ we denote by a' the function of $M_{\Omega'}(\alpha, \beta, c_1, c_2)$ defined by $a' = a|_{\Omega' \times \mathbb{R}^n}$. Then the following localization property holds.

THEOREM 1.5. — *Let (a_h) be a sequence in $M_\Omega(\alpha, \beta, c_1, c_2)$ which G-converges to a in $M_\Omega(\alpha, \beta, c_1, c_2)$. Then (a'_h) G-converges to a' in $M_{\Omega'}(\alpha, \beta, c_1, c_2)$.*

THEOREM 1.6. — *Let (a_h) be a sequence in $M_\Omega(\alpha, \beta, c_1, c_2)$. Then there exist suitable positive constants c'_1, c'_2 and a subsequence $(a_{\sigma(h)})$ of (a_h) which G-converges to a function a of the class $M_\Omega\left(\frac{\alpha}{p-\alpha}, \beta, c'_1, c'_2\right)$.*

In order to simplify the notation, the classes $M_\Omega(\alpha, \beta, c_1, c_2)$ and $M_\Omega\left(\frac{\alpha}{p-\alpha}, \beta, c'_1, c'_2\right)$ given by Theorem 1.6 will be denoted, from now on, by M_Ω and M'_Ω respectively.

Finally, we recall a lemma of compensated compactness type (see [35], [45]) which will be used in the sequel. For its proof see, for example, Lemma 3.4 in [16].

LEMMA 1.7. — *Let (u_h) be a sequence converging to u weakly in $H^{1,p}(\Omega)$. Let (g_h) be a sequence converging to g weakly in $(L^q(\Omega))^n$ with $(\operatorname{div} g_h)$ converging to $\operatorname{div} g$ strongly in $H^{-1,q}(\Omega)$. Then*

$$\int_{\Omega} (g_h, Du_h) \varphi \, dx \rightarrow \int_{\Omega} (g, Du) \varphi \, dx$$

for every $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

2. SUFFICIENT CONDITIONS FOR THE G-CONVERGENCE AND REPRESENTATION FORMULA FOR MONOTONE OPERATORS

In this section we investigate some conditions on a sequence of functions (a_h) in M_Ω which imply G-convergence. Furthermore, we give a representation formula for functions $a \in M_\Omega$ showing that $a(x, \xi)$ can be determined by the knowledge of the solutions v to the Dirichlet boundary value problems

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv + \xi)) &= 0 \quad \text{on } \Omega', \\ v &\in H_0^{1,p}(\Omega'), \end{aligned} \right\}$$

where $\xi \in \mathbf{R}^n$ and Ω' is an open subset of Ω . More precisely, we prove that $a(x, \xi)$ can be calculated by a differentiation process of the set function

$$\Omega' \rightarrow \int_{\Omega'} a(x, Dv(x) + \xi) \, dx$$

along a family of open subsets of Ω . Similar results for minima of variational functionals were proved in [21] and [24] for the quadratic case, and in [19] for the general case.

The following theorem shows that the strong convergence $(L^1(\Omega))^n$ of a sequence (a_h) in M_Ω implies the G-convergence.

THEOREM 2.1. — *Let a_h and $a \in M_\Omega$. Assume that $(a_h(\cdot, \xi))$ converges to $a(\cdot, \xi)$ strongly in $(L^1(\Omega))^n$ for every $\xi \in \mathbf{R}^n$. Then, the sequence (a_h) G-converges to a .*

Proof. — Let (f_h) be a sequence in $H^{-1,q}(\Omega)$ converging to f strongly in $H^{-1,q}(\Omega)$. Let u_h be the solution to the equation

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Du_h)) &= f_h \quad \text{on } \Omega, \\ u_h &\in H_0^{1,p}(\Omega). \end{aligned} \right\}$$

By the definition of G-convergence we have to prove that (1.8), (1.9) and (1.10) are satisfied. Since (f_h) is bounded in $H^{-1,q}(\Omega)$, condition (1.4) implies that (u_h) is bounded in $H_0^{1,p}(\Omega)$, hence $(a_h(\cdot, Du_h(\cdot)))$ is bounded in $(L^q(\Omega))^n$ by (1.3). Therefore, up to a subsequence,

$$\begin{aligned} u_h &\rightarrow u \quad \text{weakly in } H_0^{1,p}(\Omega), \\ a_h(x, Du_h) &\rightarrow g \quad \text{weakly in } (L^q(\Omega))^n, \end{aligned}$$

with $-\operatorname{div} g = f$. We shall show that $g(x) = a(x, Du(x))$ for a.e. $x \in \Omega$, hence u is the unique solution to (1.10). Therefore, the whole sequences (u_h) and $(a_h(\cdot, Du_h(\cdot)))$ converge, and the proof of our assertion is complete. By the strong convergence in $(L^1(\Omega))^n$ of the sequence $(a_h(\cdot, \xi))$ to $a(\cdot, \xi)$ and by the equicontinuity [see Definition 1.1 (iii)], there exists a subsequence, still denoted by (a_h) , such that $(a_h(x, \xi))$ converges to $a(x, \xi)$ for a.e. $x \in \Omega$, for every $\xi \in \mathbf{R}^n$. Furthermore, by taking the equiboundedness condition [Definition 1.1 (iii)] for a_h into account, the dominated convergence theorem implies that

$$a_h(\cdot, \xi) \rightarrow a(\cdot, \xi) \quad \text{strongly in } (L^q(\Omega))^n, \quad \text{for every } \xi \in \mathbf{R}^n.$$

Since $a_h(x, \cdot)$ is monotone for a.e. $x \in \Omega$, we have

$$\int_{\Omega} (a_h(x, Du_h) - a_h(x, \xi), Du_h(x) - \xi) \varphi(x) dx \geq 0$$

for every $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \geq 0$. Passing to the limit as h tends to $+\infty$ we obtain by means of Lemma 1.7 that

$$\int_{\Omega} (g(x) - a(x, \xi), Du(x) - \xi) \varphi(x) dx \geq 0 \quad (2.1)$$

holds for every $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \geq 0$. By a standard density argument, (2.1) implies that

$$(g(x) - a(x, \xi), Du(x) - \xi) \geq 0$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbf{R}^n$ [remind that $a(x, \xi)$ is continuous with respect to ξ by (iii) in Definition 1.1]. By Minty's lemma (see, for example, [27], Chapter III, Lemma 1.5) it follows that

$$(g(x) - a(x, Du(x)), Du(x) - \xi) \geq 0$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbf{R}^n$. Hence, $g(x) = a(x, Du(x))$ for a.e. $x \in \Omega$, which completes the proof. $\square$

In order to state the representation formula for functions in the class M_Ω , given $a \in M_\Omega$, let us define the function

$$\Psi(\xi, \Omega') = \int_{\Omega'} a(x, Dv(x) + \xi) dx \quad (2.2)$$

for every $\xi \in \mathbb{R}^n$, for every open subset Ω' of Ω , where the function v , depending on ξ and Ω' , is the unique solution to

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv + \xi)) &= 0 \quad \text{on } \Omega', \\ v &\in H_0^{1,p}(\Omega'). \end{aligned} \right\} \quad (2.3)$$

THEOREM 2.2. — *Let $a \in M_\Omega$. Then there exists a measurable subset N of Ω with $|N| = 0$ such that*

$$a(x_0, \xi) = \lim_{\rho \rightarrow 0+} \frac{\Psi(\xi, A_\rho(x_0))}{|A_\rho(x_0)|} \quad (2.4)$$

for every $\xi \in \mathbb{R}^n$, $x_0 \in \Omega \setminus N$, where $|\cdot|$ denotes the Lebesgue measure, $A_\rho(x_0) = x_0 + \rho A$, and A is any bounded open subset of $\mathbb{R}^n$.

Proof. — By a standard density argument and by Lebesgue's differentiation theorem (see, for instance, [38], Theorem 8.8) there exists a measurable subset N of Ω with $|N| = 0$ such that

$$\lim_{\rho \rightarrow 0+} \frac{1}{|A_\rho(x_0)|} \int_{A_\rho(x_0)} |a(x, \xi) - a(x_0, \xi)| dx = 0 \quad (2.5)$$

for every $x_0 \in \Omega \setminus N$, for every $\xi \in \mathbb{R}^n$, where $A_\rho(x_0) = x_0 + \rho A$, and A is any bounded open subset of $\mathbb{R}^n$. Given $x_0 \in \Omega \setminus N$, $\rho > 0$, $\xi \in \mathbb{R}^n$ we denote by v the function depending on ξ and $A_\rho(x_0)$ which is the unique solution to the Dirichlet boundary value problem

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v &\in H_0^{1,p}(A_\rho(x_0)). \end{aligned} \right\} \quad (2.6)$$

By performing the change of variables $y = \frac{(x - x_0)}{\rho}$, problem (2.6) becomes

$$\left. \begin{aligned} -\operatorname{div}_y(a(x_0 + \rho y, D_y u(y) + \xi)) &= 0 \quad \text{on } A, \\ u(y) &= \frac{1}{\rho} v(x_0 + \rho y) \in H_0^{1,p}(A). \end{aligned} \right\} \quad (2.7)$$

We may suppose that ρ runs through a sequence (ρ_h) which tends to 0^+ as h tends to $+\infty$. Let us set $a_h(y, \xi) = a(x_0 + \rho_h y, \xi)$ for every $y \in A$, $\xi \in \mathbb{R}^n$. By (2.5) we have

$$a_h(\cdot, \xi) \rightarrow a(x_0, \xi) \quad \text{strongly in } (L^1(A))^n,$$

which guarantees by Theorem 2.1 that

$$(a_h) \text{ G-converges to } a \text{ on } A. \quad (2.8)$$

Since $w=0$ is the unique solution to the Dirichlet problem

$$\left. \begin{aligned} -\operatorname{div}_y(a(x_0, D_y w(y) + \xi)) &= 0 \quad \text{on } A, \\ w &\in H_0^{1,p}(A), \end{aligned} \right\}$$

if u_h denotes the solution to (2.7) corresponding to $\rho = \rho_h$, the G-convergence condition (2.8) and Remark 1.4 imply that (u_h) converges to 0 weakly in $H_0^{1,p}(A)$ and

$$a(x_0 + \rho_h y, D_y u_h(y) + \xi) \rightarrow a(x_0, \xi) \quad \text{weakly in } (L^q(A))^n. \quad (2.9)$$

By (2.9) we have then

$$a(x_0, \xi) = \lim_{h \rightarrow \infty} \frac{1}{|A|} \int_A a(x_0 + \rho_h y, D_y u_h(y) + \xi) dy,$$

which by a change of variables proves (2.4). $\square$

The aim of the next theorem is to obtain a necessary and sufficient condition for the G-convergence of a sequence $a_h \in M_\Omega$ by means of the convergence of the momenta related to the Dirichlet problems

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Dv + \xi)) &= 0 \quad \text{on } A_p(x_0), \\ v &\in H_0^{1,p}(A_p(x_0)), \end{aligned} \right\}$$

where $\xi \in \mathbb{R}^n$.

THEOREM 2.3. — *Let (a_h) be a sequence in M_Ω . Let Ψ_h be the function associated to a_h by (2.2). For every $\rho > 0$ and $x_0 \in \mathbb{R}^n$, let $A_p(x_0) = x_0 + \rho A$, where A is any bounded open subset of $\mathbb{R}^n$. Let N be a measurable subset of Ω with $|N| = 0$. Then, the following conditions are equivalent:*

(a) *the limit*

$$\lim_{h \rightarrow \infty} \Psi_h(\xi, A_p(x_0))$$

exists for every $\xi \in \mathbb{R}^n$ and for every $x_0 \in \Omega \setminus N$;

(b) *there exists a function $a \in M'_\Omega$ such that (a_h) G-converges to a .*

Moreover, if the previous conditions are satisfied, then

$$a(x, \xi) = \lim_{\rho \rightarrow 0+} \lim_{h \rightarrow \infty} \frac{\Psi_h(\xi, A_p(x))}{|A_p(x)|} \quad (2.10)$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$.

Proof. — Assume (a). Let $(a_{\tau(h)})$ be a subsequence of (a_h) . By the compactness theorem 1.6 there exist a further subsequence $(a_{\tau(\sigma(h))})$ and $a \in M'_\Omega$ such that

$$(a_{\tau(\sigma(h))}) \text{ G-converges to } a. \quad (2.11)$$

If we show that

$$a(x_0, \xi) = \lim_{\rho \rightarrow 0+} \lim_{h \rightarrow \infty} \frac{\Psi_{\tau(\sigma(h))}(\xi, A_\rho(x_0))}{|A_\rho(x_0)|} \quad (2.12)$$

for a.e. $x_0 \in \Omega$, for every $\xi \in \mathbf{R}^n$, by the existence of the limit in (a) and the Urysohn property of the G-convergence (see [16], Remark 3.7) we may conclude that (b) and (2.10) hold.

Given $\xi \in \mathbf{R}^n$, $x_0 \in \Omega \setminus N$, and $\rho > 0$, we denote by v_h the solution to the Dirichlet problem

$$\left. \begin{aligned} -\operatorname{div}(a_{\tau(\sigma(h))}(x, Dv_h + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v_h &\in H_0^{1,p}(A_\rho(x_0)). \end{aligned} \right\}$$

Since v is the unique solution to the Dirichlet problem

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v &\in H_0^{1,p}(A_\rho(x_0)), \end{aligned} \right\} \quad (2.13)$$

the G-convergence condition (2.11), Theorem 1.5 and Remark 1.4 imply

$$\begin{aligned} v_h &\rightarrow v \quad \text{weakly in } H_0^{1,p}(A_\rho(x_0)) \\ a_{\tau(\sigma(h))}(x, Dv_h + \xi) &\rightarrow a(x, Dv + \xi) \quad \text{weakly in } (L^q(A_\rho(x_0)))^n. \end{aligned}$$

Therefore

$$\Psi_{\tau(\sigma(h))}(\xi, A_\rho(x_0)) \rightarrow \int_{A_\rho(x_0)} a(x, Dv + \xi) dx = \Psi(\xi, A_\rho(x_0)).$$

Now, by the representation Theorem 2.2 we conclude that

$$a(x_0, \xi) = \lim_{\rho \rightarrow 0+} \lim_{h \rightarrow \infty} \frac{\Psi_{\tau(\sigma(h))}(\xi, A_\rho(x_0))}{|A_\rho(x_0)|}$$

for a.e. $x_0 \in \Omega$, for every $\xi \in \mathbf{R}^n$, proving (2.12).

Assume (b). Then, Theorem 1.5 and Remark 1.4 guarantee that the solutions v_h to

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Dv_h + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v_h &\in H_0^{1,p}(A_\rho(x_0)) \end{aligned} \right\}$$

satisfy

$$\begin{aligned} v_h &\rightarrow v \quad \text{weakly in } H_0^{1,p}(A_\rho(x_0)) \\ a_h(x, Dv_h + \xi) &\rightarrow a(x, Dv + \xi) \quad \text{weakly in } (L^q(A_\rho(x_0)))^n, \end{aligned}$$

where v is the unique solution to (2.13). Hence, condition (a) follows immediately. $\square$

Remark 2.4. — Theorem 2.3 provides a simple characterization of G-convergence in the special case of functions a_h in M_Ω satisfying

$$-\operatorname{div}_x(a_h(x, \xi)) = 0 \quad \text{on } \Omega$$

for every $h \in \mathbf{N}$ and $\xi \in \mathbf{R}^n$. In this case (a_h) G-converges to a function $a \in \mathbf{M}_\Omega$ if and only if $(a_h(\cdot, \xi))$ tends to $a(\cdot, \xi)$ weakly in $(L^1(\Omega))^n$, for every $\xi \in \mathbf{R}^n$. According to Theorem 2.3 this condition is clearly sufficient, since in this case

$$\Psi_h(\xi, A_p(x_0)) = \int_{A_p(x_0)} a_h(x, \xi) dx,$$

while its necessity follows easily from the local character of the G-convergence (see Theorem 1.5 and Remark 1.4). In the linear case, the previous characterization was proved in [13]. See also [12] for a similar result in the case $a_h(x, \xi) = \partial_\xi f_h(x, \xi)$ with f_h convex in ξ .

3. HOMOGENIZATION OF QUASIPERIODIC OPERATORS

In this section we give a characterization of the G-limit of a sequence of functions a_h of the form

$$a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right),$$

with $a \in \mathbf{M}_{\mathbf{R}^n}$ verifying suitable hypotheses of quasiperiodicity in the first variable (see Definition 3.1). This result will be used in Section 4 to derive the homogenization theorem for general almost periodic operators.

DEFINITION 3.1 (see [30] 3.3). — A continuous function $f: \mathbf{R}^n \rightarrow \mathbf{R}$ is *quasiperiodic* if there exist $m_1, \dots, m_n \in \mathbf{N}$ and a continuous function $F: \mathbf{R}^N \rightarrow \mathbf{R}$, where $N = m_1 + \dots + m_n$, $N \geq n$, such that

$$f(x_1, \dots, x_n) = F(\underbrace{x_1, \dots, x_1}_{m_1}, \underbrace{x_2, \dots, x_2}_{m_2}, \dots, \underbrace{x_n, \dots, x_n}_{m_n}),$$

and F periodic of periods $\frac{2\pi}{\lambda_1^1}, \frac{2\pi}{\lambda_1^2}, \dots, \frac{2\pi}{\lambda_1^{m_1}}, \frac{2\pi}{\lambda_2^1}, \dots, \frac{2\pi}{\lambda_2^{m_2}}, \dots, \frac{2\pi}{\lambda_n^{m_n}}$, with $\lambda_r^1 \in]0, +\infty[$. It is not restrictive to assume that the frequencies $\lambda_r^1, \dots, \lambda_r^{m_r}$ are linearly independent on $\mathbf{Z}$ for every $r = 1, \dots, n$. This will be done constantly in the sequel. Under this assumption, Kronecker's lemma (see Appendix, Section A) guarantees that F is uniquely determined by f .

Given $m_1, \dots, m_n$ as above and given $\lambda \in \mathbf{R}^{m_1} \times \dots \times \mathbf{R}^{m_n}$, $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_r = (\lambda_r^1, \dots, \lambda_r^{m_r})$ for $r = 1, \dots, n$, we denote by $\text{QP}(\lambda)$ the set of all quasiperiodic functions $f: \mathbf{R}^n \rightarrow \mathbf{R}$ with frequencies λ . Furthermore, by $\text{Trig}(\lambda)$ we indicate the set of all trigonometric polyno-

mials with frequencies λ ; i. e., finite sums of terms of the form

$$P(x) = \operatorname{Re} (c \exp (i \sum_{l,r} k_r^l \lambda_r^l x_r)), \quad (3.1)$$

where $k \in \mathbb{Z}^N$, $c \in \mathbb{C}$.

Remark 3.2. — Trigonometric polynomials are obviously quasiperiodic functions. Moreover, it can be proved that every function f of $\operatorname{QP}(\lambda)$ is the uniform limit of a sequence of trigonometric polynomials belonging to $\operatorname{Trig}(\lambda)$ (see [30] 3. 3).

For every $s > 0$, for every $z \in \mathbb{R}^n$, let $Q_s(z)$ be the cube of side length s and center z . For every $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ we define

$$\oint f(x) dx = \lim_{s \rightarrow \infty} \sup \frac{1}{|Q_s(0)|} \int_{Q_s(0)} f(x) dx.$$

It can be seen easily that for every function f in $\operatorname{QP}(\lambda)$ we have

$$\oint f(x) dx = \lim_{s \rightarrow \infty} \frac{1}{|Q_s(z)|} \int_{Q_s(z)} f(x) dx$$

uniformly with respect to z . This limit is called the *mean value* of f (on $\mathbb{R}^n$) (see [30], 2. 3).

In the sequel it will be useful to consider the function $j: \mathbb{R}^n \rightarrow \mathbb{R}^N$ defined by

$$j(x) = (\underbrace{x_1, \dots, x_1}_{m_1}, \underbrace{x_2, \dots, x_2}_{m_2}, \dots, \underbrace{x_n, \dots, x_n}_{m_n}).$$

Then, introducing the variables

$$(y_1, \dots, y_N) = (y_1^1, \dots, y_1^{m_1}, \dots, y_n^1, \dots, y_n^{m_n})$$

on $\mathbb{R}^N$, for every $v \in \mathcal{C}^1(\mathbb{R}^N)$ and $u(x) = v(j(x))$ we get

$$D_r u(x) = \sum_{l=1}^{m_r} \frac{\partial v}{\partial y_r^l}(j(x)) \quad \text{for every } r = 1, \dots, n,$$

or briefly, $Du(x) = (\partial v) \circ j(x)$, where $\partial = (\partial_1, \dots, \partial_n)$ with $\partial_r = \sum_{l=1}^{m_r} \frac{\partial}{\partial y_r^l}$.

Let us denote by $\operatorname{Trig}_0(\lambda)$ the set of all trigonometric polynomials in $\operatorname{Trig}(\lambda)$ with mean value 0. By Birkhoff's theorem (see Appendix, Theorem A) it is easy to prove that

$$\|u\|_{1,p} = \left(\sum_{r=1}^n \oint |D_r u|^p dx \right)^{1/p}$$

is a norm on $\text{Trig}_0(\lambda)$. Since $(\text{Trig}_0(\lambda), \|\cdot\|_{1,p})$ is not complete, we study its completion. To this aim let us introduce

$$T = \prod_{l,r} \left[0, \frac{2\pi}{\lambda_r^l} \right] \subset \mathbf{R}^N, \quad (3.2)$$

and let us denote by $\text{Trig}(T)$ the set of all trigonometric polynomials in $\mathbf{R}^N$ with period T . Let us also define the set $\text{Trig}_0(T)$ of all functions v in $\text{Trig}(T)$ with mean value zero. On $\text{Trig}_0(T)$ we consider the norm

$$\|v\|_{\mathcal{W}} = \|\partial v\|_{(L^p(T))^n} = \left(\sum_{r=1}^n \frac{1}{|T|} \int_T |\partial_r v|^p dy \right)^{1/p};$$

in fact, if $\partial v = 0$, then $u = v \circ j$ satisfies $Du = 0$. Hence, u is constant. Since v depends uniquely on u (see Definition 3.1), v is constant too; hence $v = 0$. By Birkhoff's theorem (see Appendix, Theorem A) the linear map

$$\text{Re}(c \exp(i \sum_{l,r} k_r^l \lambda_r^l x_r)) \mapsto \text{Re}(c \exp(i \sum_{l,r} k_r^l \lambda_r^l y_r^l))$$

is a bijective isometry between $(\text{Trig}_0(\lambda), \|\cdot\|_{1,p})$ and $(\text{Trig}_0(T), \|\cdot\|_{\mathcal{W}})$, and will be denoted by J .

Let us denote by $\mathcal{W}$ the completion of $\text{Trig}_0(T)$ with respect to the norm $\|\cdot\|_{\mathcal{W}}$, which we can identify with the completion of $\text{Trig}_0(\lambda)$ with respect to the norm $\|\cdot\|_{1,p}$. Finally, let us remark that the isometry

$$\begin{aligned} \partial: \text{Trig}_0(T) &\rightarrow (L^p(T))^n \\ u &\mapsto (\partial_1 u, \partial_2 u, \dots, \partial_n u) \end{aligned}$$

can be extended in a unique way to an isometry between $\mathcal{W}$ and a closed subspace of $(L^p(T))^n$, which makes $\mathcal{W}$ a reflexive Banach space.

Now, let us fix the frequencies λ , with $\lambda_r^1, \dots, \lambda_r^{m_r}$ linearly independent on $\mathbf{Z}$ for every $r = 1, \dots, n$, and let us fix $a \in \mathbf{M}_{\mathbf{R}^n}$ such that

$$a(\cdot, \xi) \in \text{QP}(\lambda) \quad \text{for every } \xi \in \mathbf{R}^n. \quad (3.3)$$

By definition it follows that there exists a unique function $\tilde{a}(\cdot, \xi): \mathbf{R}^N \rightarrow \mathbf{R}^n$ such that $a(x, \xi) = \tilde{a}(j(x), \xi)$ for every $x \in \mathbf{R}^n$ and for every $\xi \in \mathbf{R}^n$, $\tilde{a}(\cdot, \xi)$ is T -periodic and continuous for every $\xi \in \mathbf{R}^n$. It follows that $\tilde{a}(y, \cdot)$ satisfies (1.1) and (1.2) for every $y \in \mathbf{R}^N$. This is obvious for $y = j(x)$, $x \in \mathbf{R}^n$, whereas the conclusion for a general $y \in \mathbf{R}^N$ comes from Kronecker's lemma (see Appendix A) and the continuity of $\tilde{a}(\cdot, \xi)$.

The next proposition is an extension to the case $p \neq 2$ of Lemma 1 in [36].

PROPOSITION 3.3. — *Let $\xi \in \mathbf{R}^n$ be fixed, let $a \in \mathbf{M}_{\mathbf{R}^n}$ satisfying (3.3) and let $\tilde{a}$ be the corresponding T -periodic function. Then, there exists a unique*

solution $w^\xi \in \mathcal{W}$ to the problem

$$\left. \begin{aligned} \frac{1}{|T|} \int_T (\tilde{a}(y, \partial w^\xi + \xi), \partial h) dy &= 0 \quad \text{for every } h \in \mathcal{W}, \\ w^\xi &\in \mathcal{W}. \end{aligned} \right\} \quad (3.4)$$

Furthermore, for any $\delta > 0$ there exist $u_\delta^\xi \in \text{Trig}_0(\lambda)$ and a vector function $g_\delta^\xi \in (\text{QP}(\lambda))^n$ such that

$$-\text{div}(a(x, Du_\delta^\xi + \xi)) = -\text{div } g_\delta^\xi \quad \text{in } \mathcal{D}'(\mathbf{R}^n), \quad (3.5)$$

$$\lim_{\delta \rightarrow 0+} \|J(u_\delta^\xi) - w^\xi\|_{\mathcal{W}} = 0, \quad (3.6)$$

$$\lim_{\delta \rightarrow 0+} \int |g_\delta^\xi|^q dx = 0 \quad (3.7)$$

hold.

Proof. — Given $\xi \in \mathbf{R}^n$, let $\tilde{\mathcal{A}}^\xi: \mathcal{W} \rightarrow \mathcal{W}'$ be the operator defined by

$$\langle \tilde{\mathcal{A}}^\xi w, h \rangle = \frac{1}{|T|} \int_T (\tilde{a}(y, \partial w + \xi), \partial h) dy$$

for every $w, h \in \mathcal{W}$, where $\langle \cdot, \cdot \rangle$ denotes here the duality pairing between $\mathcal{W}$ and its dual space $\mathcal{W}'$. By the coerciveness and continuity properties of $\tilde{\mathcal{A}}^\xi$ on $\mathcal{W}$, the theory of monotone operators on reflexive Banach spaces (see, for example [27]) implies immediately that there exists a unique $w^\xi \in \mathcal{W}$ satisfying $\tilde{\mathcal{A}}^\xi w^\xi = 0$, which implies (3.4). By the density of $J(\text{Trig}_0(\lambda))$ in $\mathcal{W}$ there exists $u_\delta^\xi \in \text{Trig}_0(\lambda)$ such that (3.6) is satisfied. By using (3.4) and the equicontinuity assumption on $\tilde{a}$ we get

$$\begin{aligned} \|\tilde{\mathcal{A}}^\xi J(u_\delta^\xi)\|_{\mathcal{W}'} &= \|\tilde{\mathcal{A}}^\xi J(u_\delta^\xi) - \tilde{\mathcal{A}}^\xi w^\xi\|_{\mathcal{W}'} \\ &\leq c(1 + |\xi| + \|J(u_\delta^\xi)\|_{\mathcal{W}} + \|w^\xi\|_{\mathcal{W}})^{p-1-\alpha} \|J(u_\delta^\xi) - w^\xi\|_{\mathcal{W}}^\alpha. \end{aligned}$$

By (3.6) it follows that

$$\lim_{\delta \rightarrow 0+} \|\tilde{\mathcal{A}}^\xi J(u_\delta^\xi)\|_{\mathcal{W}'} = 0. \quad (3.8)$$

Since $x \mapsto a(x, Du_\delta^\xi + \xi)$ belongs to $(\text{QP}(\lambda))^n$, by Remark 3.2 there exist a function $f_\delta^\xi \in (\text{Trig}(\lambda))^n$ and a quasiperiodic function $h_\delta^\xi \in (\text{QP}(\lambda))^n$ such that

$$a(x, Du_\delta^\xi + \xi) = f_\delta^\xi + h_\delta^\xi, \quad (3.9)$$

and

$$\lim_{\delta \rightarrow 0+} \int |h_\delta^\xi|^q dx = 0. \quad (3.10)$$

Since $a(x, Du_\delta^\xi + \xi) = \tilde{a}(j(x), \partial J(u_\delta^\xi)(j(x)) + \xi)$, by (3.8) and (3.10) we get

$$\lim_{\delta \rightarrow 0+} \left\| \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \right\|_{\mathcal{W}'} = 0, \quad (3.11)$$

where $(F_\delta^\xi)_r$ is $J((f_\delta^\xi)_r)$, and

$$\left\| \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \right\|_{\mathcal{W}'} = \sup_{\substack{v \in \mathcal{W} \\ \|v\|_{\mathcal{W}} \leq 1}} \frac{1}{|T|} \sum_{r=1}^n \int_T (F_\delta^\xi)_r \partial_r v dy.$$

Being $\operatorname{div} f_\delta^\xi \in \operatorname{Trig}_0(\lambda)$, we can write

$$\operatorname{div} f_\delta^\xi(x) = \operatorname{Re} \left(\sum_k c_k \exp \left(i \sum_{l,r} k_r^l \lambda_r^l x_r \right) \right),$$

where the sum runs over a finite set of non-zero vectors $k \in \mathbb{Z}^N$. It turns out that the function

$$w_\delta^\xi(x) = \operatorname{Re} \left(\sum_k b_k \exp \left(i \sum_{l,r} k_r^l \lambda_r^l x_r \right) \right),$$

with $b_k = -c_k / (\sum_r (\sum_l k_r^l \lambda_r^l)^2)$, is the unique solution in $\operatorname{Trig}_0(\lambda)$ to

$$\Delta w_\delta^\xi = \operatorname{div} f_\delta^\xi \quad \text{on } \mathbb{R}^n.$$

If $W_\delta^\xi = J(w_\delta^\xi) \in \operatorname{Trig}(T)$, then

$$\sum_{r=1}^n \partial_r^2 W_\delta^\xi = \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \quad \text{in } \mathbb{R}^N$$

holds. By elliptic regularity (see Appendix, Section B)

$$\left(\frac{1}{|T|} \sum_{r=1}^n \int_T |\partial_r W_\delta^\xi|^q dx \right)^{1/q} \leq c \left\| \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \right\|_{\mathcal{W}'},$$

with $c > 0$ independent of ξ and δ . We have then by Birkhoff's theorem

$$\left(\int |\operatorname{div} w_\delta^\xi|^q dx \right)^{1/q} \leq c \left\| \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \right\|_{\mathcal{W}'}. \quad (3.12)$$

Hence, by setting

$$g_\delta^\xi = \operatorname{div} w_\delta^\xi + h_\delta^\xi$$

we have

$$\operatorname{div} (a(x, Du_\delta^\xi + \xi)) = \operatorname{div} g_\delta^\xi$$

in the sense of distributions on $\mathbb{R}^n$, proving (3.5). Furthermore, by (3.10)-(3.12) we obtain (3.7). $\square$

Let us consider the following Dirichlet boundary value problem

$$\left. \begin{aligned} -\operatorname{div} \left(a \left(\frac{x}{\varepsilon_h}, Du_h \right) \right) &= f \quad \text{on } \Omega, \\ u_h &\in H_0^{1,p}(\Omega), \end{aligned} \right\} \quad (3.13)$$

where $f \in H^{-1,q}(\Omega)$ and (ε_h) is a sequence of positive real numbers converging to 0.

In this section we prove the convergence, as (ε_h) tends to 0^+ , of the solutions u_h of (3.13) to the solution u of the homogenized problem

$$\left. \begin{aligned} -\operatorname{div}(b(Du)) &= f \quad \text{on } \Omega, \\ u &\in H_0^{1,p}(\Omega). \end{aligned} \right\} \quad (3.14)$$

Furthermore, we give an asymptotic formula for the homogenized function b in terms of the solutions v_s^ξ to the following Dirichlet boundary value problems

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv_s^\xi + \xi)) &= 0 \quad \text{on } Q_s(z), \\ v_s^\xi &\in H_0^{1,p}(Q_s(z)). \end{aligned} \right\} \quad (3.15)$$

The convergence result above mentioned will follow from the next two theorems.

THEOREM 3.4. — *Let $a \in M_{\mathbf{R}^n}$ satisfying (3.3) and let $\tilde{a}$ be the corresponding T-periodic function. Let $b: \mathbf{R}^n \rightarrow \mathbf{R}^n$ be the function defined by*

$$b(\xi) = \frac{1}{|T|} \int_T \tilde{a}(y, \partial w^\xi + \xi) dy,$$

where w^ξ is the solution to (3.4). Then, for any family $(z_s)_{s>0}$ in $\mathbf{R}^n$, we have

$$b(\xi) = \lim_{s \rightarrow \infty} \frac{1}{s^n} \int_{Q_s(z_s)} a(x, Dv_s^\xi + \xi) dx,$$

where v_s^ξ is the unique solution to (3.15) with $z = z_s$.

THEOREM 3.5. — *Let $a \in M_{\mathbf{R}^n}$ satisfying (3.3). Let (ε_h) be a sequence of positive real numbers converging to 0 and let $a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right)$ for every $x \in \Omega$ and for every $\xi \in \mathbf{R}^n$. Then (a_h) G-converges to b , where b is the function defined in Theorem 3.4.*

The proofs of these theorems are quite technical and are therefore given at the end of the section as a consequence of the next proposition and some remarks stated in the sequel.

For every $s > 0$, and for every family (z_s) in $\mathbf{R}^n$, let us define

$$g_s(\xi) = \frac{1}{s^n} \int_{Q_s(z_s)} a(x, Dv_s^\xi + \xi) dx, \quad (3.16)$$

where v_s^ξ is the solution to (3.15) on $Q_s(z_s)$. By (iii) and (iv) in Definition 1.1, using Young's and Hölder's inequalities, it turns out that the function v_s^ξ satisfies the following estimates

$$\left. \begin{aligned} \|Dv_s^\xi + \xi\|_{(L^p(Q_s(z_s)))^n}^p &\leq cs^n(1 + |\xi|^p), \\ \|a(x, Dv_s^\xi + \xi)\|_{(L^q(Q_s(z_s)))^n}^q &\leq cs^n(1 + |\xi|^p) \end{aligned} \right\} \quad (3.17)$$

with c independent of z_s . Here and henceforth, we will denote by c any constant depending at most on $c_1, c_2, \alpha, \beta, n, p$, that can change from line to line. By (3.17), we have

$$|g_s(\xi)| \leq c(1 + |\xi|^{p/q}).$$

PROPOSITION 3.6. — *Let $\xi \in \mathbf{R}^n$ be fixed. Let u_s^ξ be as in Proposition 3.3 and let $v_s^\xi \in H_0^{1,p}(Q_s(z_s))$ be the unique solution to (3.15) with z replaced by z_s . Then*

$$\lim_{\delta \rightarrow 0+} \limsup_{s \rightarrow \infty} \frac{1}{s^n} \int_{Q_s(z_s)} |Dv_s^\xi - Du_\delta^\xi|^p dx = 0 \quad (3.18)$$

for every family $(z_s)_{s>0}$ in $\mathbf{R}^n$.

Proof. — Given a family $(z_s)_{s>0}$ we consider

$$\begin{aligned} I_{s,\delta} &\equiv \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Dv_s^\xi + \xi) - a(x, Du_\delta^\xi + \xi), Dv_s^\xi - Du_\delta^\xi) dx \\ &= \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Dv_s^\xi + \xi), Dv_s^\xi) dx \\ &\quad + \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Du_\delta^\xi + \xi), Du_\delta^\xi) dx \\ &\quad - \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Dv_s^\xi + \xi), Du_\delta^\xi) dx \\ &\quad - \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Du_\delta^\xi + \xi), Dv_s^\xi) dx \\ &\equiv I_s^1 + I_{s,\delta}^2 - I_{s,\delta}^3 - I_{s,\delta}^4. \end{aligned}$$

By (3.15), $I_s^1 = 0$. On the other hand,

$$\lim_{s \rightarrow \infty} I_{s,\delta}^2 = \int (a(x, Du_\delta^\xi + \xi), Du_\delta^\xi) dx.$$

By Birkhoff's theorem we get

$$\lim_{s \rightarrow \infty} I_{s,\delta}^2 = \frac{1}{|T|} \int_T (\tilde{a}(y, \partial J(u_\delta^\xi) + \xi), \partial J(u_\delta^\xi)) dy = \langle \tilde{\mathcal{A}}^\xi J(u_\delta^\xi), J(u_\delta^\xi) \rangle,$$

where $\tilde{\mathcal{A}}^\xi$ is the operator defined in the proof of Proposition 3.3. Since by (3.6) we have

$$J(u_\delta^\xi) \rightarrow w^\xi \text{ strongly in } \mathcal{W},$$

the continuity of $\tilde{\mathcal{A}}^\xi$ and the equality $\tilde{\mathcal{A}}^\xi w^\xi = 0$ implies that

$$\lim_{\delta \rightarrow 0+} \lim_{s \rightarrow \infty} I_{s,\delta}^2 = 0.$$

Since

$$\frac{1}{|Q_s(z_s)|} \int_{Q_s(z_s)} |Du_\delta^\xi(x)|^p dx \rightarrow \int |Du_\delta^\xi(x)|^p dx,$$

and $w_s(x) = \frac{1}{s} u_\delta^\xi(sx + z_s)$ converges to 0, as $s \rightarrow +\infty$, we obtain that the sequence (w_s) converges weakly to 0 in $H^{1,p}(Q_1(0))$. Applying Lemma 1.7, we obtain that the sequence of functions

$$\zeta_s(x) = (a(sx + z_s, Dv_s^\xi(sx + z_s) + \xi), Du_\delta^\xi(sx + z_s))$$

converges to 0 in $\mathcal{D}'(Q_1(0))$ as $s \rightarrow +\infty$. On the other hand, by (1.6) and (3.17) we have

$$\|Dv_s^\xi(sx + z_s) + \xi\|_{(L^{p+\eta}(Q_1(0)))^n} \leq c(1 + |\xi|^p)^{(p+\eta)/p};$$

by (1.3) it follows that the sequence of functions

$$x \mapsto a(sx + z_s, Dv_s^\xi(sx + z_s) + \xi)$$

is uniformly bounded in some $(L^\tau(Q_1(0)))^n$ for some $\tau > q$. This implies that there exists $\sigma > 1$ such that

$$\|\zeta_s\|_{L^\sigma(Q_1(0))} \leq c,$$

where c is independent of s . Since (ζ_s) converges to 0 in $\mathcal{D}'(Q_1(0))$, the above inequality implies that (ζ_s) converges to 0 weakly in $L^\sigma(Q_1(0))$, as $s \rightarrow +\infty$. This fact gives then

$$\lim_{s \rightarrow \infty} I_{s,\delta}^3 = 0.$$

Let us show that

$$\lim_{\delta \rightarrow 0^+} \limsup_{s \rightarrow \infty} I_{s,\delta}^4 = 0.$$

Let (g_δ^ξ) be as in Proposition 3.3. By applying Hölder's inequality and (3.17) we obtain

$$\begin{aligned} |I_{s,\delta}^4| &\leq c \left(\frac{1}{s^n} \int_{Q_s(z_s)} |g_\delta^\xi|^q dx \right)^{1/q} \left(\frac{1}{s^n} \int_{Q_s(z_s)} |Dv_s^\xi|^p dx \right)^{1/p} \\ &\leq c \left(\frac{1}{s^n} \int_{Q_s(z_s)} |g_\delta^\xi|^q dx \right)^{1/q} (1 + |\xi|). \end{aligned}$$

Now, by taking the limit first as s tends to $+\infty$ and then as δ tends to 0^+ , (3.7) implies that

$$\lim_{\delta \rightarrow 0} \limsup_{s \rightarrow \infty} |I_{s,\delta}^4| = 0.$$

Since the equicoerciveness assumption in Definition 1.1 (iii) guarantees

$$|I_{s,\delta}| \geq c_2 \frac{1}{s^n} \int_{Q_s(z_s)} (1 + |Dv_s^\xi + \xi| + |Du_\delta^\xi + \xi|)^{p-\beta} |Dv_s^\xi - Du_\delta^\xi|^\beta dx,$$

we obtain immediately

$$c |I_{s,\delta}|^{p/\beta} \left(\frac{1}{s^n} \int_{Q_s(z_s)} (1 + |Dv_s^\xi + \xi| + |Du_\delta^\xi + \xi|)^p dx \right)^{(\beta-p)/\beta} \geq \left(\frac{1}{s^n} \int_{Q_s(z_s)} |Dv_s^\xi - Du_\delta^\xi|^p dx \right).$$

Hence, by passing to the limit first as s tends to $+\infty$, and then as $\delta \rightarrow 0^+$, we get (3.18) and the proof of Proposition 3.6 is accomplished. $\square$

Proof of Theorem 3.5. — Let $x_0 \in \Omega$, and let $\xi \in \mathbb{R}^n$. Consider

$$\Psi_h(\xi, Q_\rho(x_0)) = \int_{Q_\rho(x_0)} a\left(\frac{x}{\varepsilon_h}, Dz_{h,\rho}^\xi(x) + \xi\right) dx,$$

where $z_{h,\rho}^\xi$ is the unique solution to

$$\left. \begin{aligned} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon_h}, Dz_{h,\rho}^\xi + \xi\right)\right) &= 0 \quad \text{on } Q_\rho(x_0), \\ z_{h,\rho}^\xi &\in H_0^{1,p}(Q_\rho(x_0)). \end{aligned} \right\}$$

It follows immediately that

$$\Psi_h(\xi, Q_\rho(x_0)) = (\varepsilon_h)^n \int_{Q_{\rho/\varepsilon_h}(x_0/\varepsilon_h)} a(x, Dw_{h,\rho}^\xi + \xi) dx,$$

where $w_{h,\rho}^\xi$ is the unique solution to

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dw_{h,\rho}^\xi(x) + \xi)) &= 0 \quad \text{on } Q_{\rho/\varepsilon_h}\left(\frac{x_0}{\varepsilon_h}\right), \\ w_{h,\rho}^\xi &\in H_0^{1,p}\left(Q_{\rho/\varepsilon_h}\left(\frac{x_0}{\varepsilon_h}\right)\right). \end{aligned} \right\}$$

By Theorem 3.4 we conclude that

$$\lim_{h \rightarrow \infty} \Psi_h(\xi, Q_\rho(x_0)) = \rho^n b(\xi)$$

for every $\rho > 0$. Hence, the limit

$$\lim_{h \rightarrow \infty} \frac{\Psi_h(\xi, Q_\rho(x_0))}{\rho^n} = b(\xi)$$

is independent on ρ . This implies by Theorem 2.3 that (a_h) G-converges to b and concludes the proof. $\square$

4. HOMOGENIZATION OF ALMOST PERIODIC OPERATORS

In this section we prove the homogenization theorem for general almost periodic monotone operators defined by functions of the class $M_{\mathbf{R}^n}$.

DEFINITION 4.1. — A function $f \in L^1_{\text{loc}}(\mathbf{R}^n)$ is *almost periodic* (in the sense of Besicovitch [3]) if there exists a sequence of trigonometric polynomials $P_h: \mathbf{R}^n \rightarrow \mathbf{R}$ such that

$$\lim_{h \rightarrow \infty} \int |P_h(x) - f(x)| dx = 0. \quad (4.1)$$

It is easy to see that for every almost periodic function f and for every $z \in \mathbf{R}^n$ we have

$$\int f(x) dx = \lim_{s \rightarrow \infty} \frac{1}{|Q_s(z)|} \int_{Q_s(z)} f(x) dx.$$

The limit is called the *mean value* of f (on $\mathbf{R}^n$) and is, in general, not uniform with respect to z .

THEOREM 4.2. — Let $a \in M_{\mathbf{R}^n}$ such that $a(\cdot, \xi)$ is almost periodic for all $\xi \in \mathbf{R}^n$ and let (ε_h) be a sequence of positive real numbers converging to 0. Let us define $a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right)$ for every $x \in \Omega$ and $\xi \in \mathbf{R}^n$. Denote by v_s^ξ the solution to

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv_s^\xi + \xi)) &= 0 \quad \text{on } Q_s(0), \\ v_s^\xi &\in H_0^{1,p}(Q_s(0)). \end{aligned} \right\} \quad (4.2)$$

Then, for every $\xi \in \mathbf{R}^n$ there exists the limit

$$b(\xi) = \lim_{s \rightarrow \infty} \frac{1}{s^n} \int_{Q_s(0)} a(x, Dv_s^\xi + \xi) dx. \quad (4.3)$$

Moreover, the map b belongs to $M(\alpha, \beta, c_1, c_2)$ and (a_h) G-converges to b .

The proof of this theorem follows from the homogenization Theorem 3.5 for quasiperiodic functions, by means of an approximation result and a closure lemma, which are stated below.

With a slight change of notation we will write

$$b(x, \xi) = G\text{-}\lim_{\varepsilon \rightarrow 0^+} a\left(\frac{x}{\varepsilon}, \xi\right)$$

meaning that $b(x, \xi)$ is the G-limit of $a\left(\frac{x}{\varepsilon_h}, \xi\right)$ for every sequence (ε_h) which tends to 0^+ .

The following lemma states that homogenization is preserved under passage to the limit in the mean value.

LEMMA 4.3. — *Let (a_h) be a sequence of functions in $M_{\mathbf{R}^n}$, such that for every $h \in \mathbf{N}$ the limit*

$$G - \lim_{\varepsilon \rightarrow 0+} a_h\left(\frac{x}{\varepsilon}, \xi\right) = b_h(\xi) \quad (4.4)$$

exists and is independent of x , and let $a \in M_{\mathbf{R}^n}$ such that for every $R > 0$

$$\limsup_{h \rightarrow \infty} \int \sup_{|\xi| \leq R} |a_h(x, \xi) - a(x, \xi)| dx = 0. \quad (4.5)$$

Then, the limit

$$b(\xi) = \lim_{h \rightarrow \infty} b_h(\xi)$$

exists and

$$G - \lim_{\varepsilon \rightarrow 0+} a\left(\frac{x}{\varepsilon}, \xi\right) = b(\xi).$$

The proof of Theorem 4.2 will be completed by the following approximation result.

LEMMA 4.4. — *Let a be a function of the class $M_{\mathbf{R}^n}$ such that $a(\cdot, \xi)$ is almost periodic for every $\xi \in \mathbf{R}^n$. Then, there exists a sequence (a_h) in $M_{\mathbf{R}^n}$ of quasiperiodic functions satisfying (3.3) (with λ possibly depending on h) such that for every $R \geq 0$ we have*

$$\limsup_{h \rightarrow \infty} \int \sup_{|\xi| \leq R} |a_h(x, \xi) - a(x, \xi)| dx = 0. \quad (4.6)$$

Throughout this section the letter c will denote a positive constant depending at most on $p, n, c_1, c_2, \alpha, \beta$, and possibly on a fixed vector $\xi \in \mathbf{R}^n$. Its value can vary from line to line.

We begin by proving Lemma 4.4.

Proof of Lemma 4.4.

Step 1 (discretization of the function a on bounded sets). — Let (γ_h) be a sequence of natural numbers, and (μ_h) a sequence of positive real numbers. For every $h \in \mathbf{N}$, let us set

$$I_h = \{z = (z_1, \dots, z_n) \in \mathbf{Z}^n : -h\gamma_h \leq z_j < h\gamma_h \text{ for all } j = 1, \dots, n\}.$$

For every $z \in I_h$, let us define

$$\xi_z^h = \frac{1}{\gamma_h} z, \quad Q_z^h = \xi_z^h + \left[0, \frac{1}{\gamma_h}\right]^n, \quad a_z^h(x) = a(x, \xi_z^h),$$

and let us choose a trigonometric polynomial $P_z^h: \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that

$$\int |P_z^h(x) - a_z^h(x)| dx \leq \frac{1}{\mu_h}. \quad (4.7)$$

The choice of the sequences $(\gamma_h) \subset \mathbf{N}$ and $(\mu_h) \subset \mathbf{R}$ will be made in the sequel. Then, taking into account that

$$\bigcup_{z \in I_h} Q_z^h = [-h, h]^n,$$

we define the function $\alpha_h: \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ in the following way:

$$\alpha_h(x, \xi) = \begin{cases} P_z^h(x) & \text{if there exists } z \in I_h \text{ such that } \xi \in Q_z^h, \\ 0 & \text{if } \xi \notin [-h, h]^n. \end{cases}$$

Then, for every $\xi \in \mathbf{R}^n$, the function $\alpha_h(\cdot, \xi)$ is quasiperiodic.

Step 2 (projection of the function α_h on $M_{\mathbf{R}^n}$). — For every $h \in \mathbf{N}$, let us define the function $f_h: [0, +\infty[\rightarrow [0, +\infty[$ by

$$f_h(\rho) = \min(1, e^{h!(h-\rho)}).$$

We also define the Hilbert space L_h^2 of measurable functions $u: \mathbf{R}^n \rightarrow \mathbf{R}^n$, provided with the norm

$$\|u\|_h = \left(\int_{\mathbf{R}^n} |u(\xi)|^2 f_h^2(|\xi|) d\xi \right)^{1/2}.$$

The set $M(\alpha, \beta, c_1, c_2)$ is a closed convex subset of L_h^2 . We can define the projection $\pi_h: L_h^2 \rightarrow M(\alpha, \beta, c_1, c_2)$, and for every $x \in \mathbf{R}^n$ the function

$$a_h(x, \cdot) = \pi_h(\alpha_h(x, \cdot)). \quad (4.8)$$

Step 3 (quasiperiodicity of the function a_h). — Let us consider for every $y \in \mathbf{R}^n$, $N = N(h) \in \mathbf{N}$, the function $\tilde{a}_h(y, \cdot): \mathbf{R}^n \rightarrow \mathbf{R}^n$ defined by

$$\tilde{a}_h(y, \cdot) = \pi_h(\tilde{\alpha}_h(y, \cdot))$$

where, for every $\xi \in \mathbf{R}^n$, $\tilde{\alpha}_h(\cdot, \xi)$ is the periodic function given by the quasiperiodicity of $\alpha_h(\cdot, \xi)$. It follows clearly that

$$a_h(x, \xi) = \tilde{a}_h(j(x), \xi),$$

where j is the function related to the quasiperiodicity of a_h as in Section 3. Since the function $\tilde{\alpha}_h(\cdot, \xi)$ is a periodic function, such is also $\tilde{a}_h(\cdot, \xi)$. Moreover, being $\tilde{a}_h(y, \cdot)$ uniformly continuous with respect to ξ , it remains only to prove that $\tilde{a}_h(\cdot, \xi)$ is continuous for every $\xi \in \mathbf{R}^n$.

Let $\xi, \eta \in \mathbf{R}^n$; for all $u, v \in M(\alpha, \beta, c_1, c_2)$ we have

$$|u(\xi) - v(\xi)| \leq 2c_1(1 + |\xi| + |\eta|)^{p-1-\alpha} |\xi - \eta|^\alpha + |u(\eta) - v(\eta)|.$$

Let $0 < \delta \leq 1$; then an integration over $B_\delta(\xi)$ gives

$$\begin{aligned} |u(\xi) - v(\xi)| |B_\delta(\xi)| &\leq 2c_1(1 + 2|\xi| + \delta)^{p-1-\alpha} \delta^\alpha |B_\delta(\xi)| \\ &\quad + \int_{B_\delta(\xi)} |u(\eta) - v(\eta)| d\eta \\ &\leq c(1 + |\xi|)^{p-1-\alpha} \delta^\alpha |B_\delta(\xi)| \\ &\quad + \left(\int_{B_\delta(\xi)} |u(\eta) - v(\eta)|^2 f_h^2(|\eta|) d\eta \right)^{1/2} \left(\int_{B_\delta(\xi)} (f_h(|\eta|))^{-2} d\eta \right)^{1/2}, \end{aligned}$$

so that we obtain

$$|u(\xi) - v(\xi)| \leq c \left[(1 + |\xi|)^{p-1-\alpha} \delta^\alpha + \frac{\|u - v\|_h}{\delta^{n/2} f_h(|\xi| + 1)} \right].$$

In particular, since π_h is a Lipschitz function with constant 1 in L_h^2 , for all $u, v \in L_h^2$ and for all $\xi \in \mathbf{R}^n$ we obtain

$$|\pi_h u(\xi) - \pi_h v(\xi)| \leq c \left[(1 + |\xi|)^{p-1-\alpha} \delta^\alpha + \frac{\|u - v\|_h}{\delta^{n/2} f_h(|\xi| + 1)} \right]. \quad (4.9)$$

Let us remark that the function $\tilde{\alpha}_h = \tilde{\alpha}_h(y, \xi)$ is uniformly continuous in y , uniformly with respect to ξ . Fixed $\varepsilon > 0$, let $\rho > 0$ such that if $|\tau| < \rho$, then

$$|\tilde{\alpha}_h(y + \tau, \xi) - \tilde{\alpha}_h(y, \xi)| < \varepsilon$$

for all $y \in \mathbf{R}^n$ and $\xi \in \mathbf{R}^n$. Then, for such a τ we have

$$\begin{aligned} \|\tilde{\alpha}_h(y + \tau, \cdot) - \tilde{\alpha}_h(y, \cdot)\|_h &\leq \left(\int_{|y-h, h|^{n/2}} |\tilde{\alpha}_h(y + \tau, \eta) - \tilde{\alpha}_h(y, \eta)|^2 d\eta \right)^{1/2} \leq \varepsilon (2h)^{n/2}, \end{aligned}$$

so that, by (4.9) [with $u(\xi) = \tilde{\alpha}_h(y + \tau, \xi)$ and $v(\xi) = \tilde{\alpha}_h(y, \xi)$]

$$|\tilde{a}_h(y + \tau, \xi) - \tilde{a}_h(y, \xi)| \leq c \left[(1 + |\xi|)^{p-1-\alpha} \delta^\alpha + \frac{\varepsilon (2h)^{n/2}}{\delta^{n/2} f_h(|\xi| + 1)} \right] \quad (4.10)$$

for all $\xi \in \mathbf{R}^n$. Since ε and δ can be chosen independently arbitrarily small, (4.10) proves that $\tilde{a}_h(\cdot, \xi)$ is uniformly continuous.

Step 4 [proof of (4.6)]. — Fixed $R \geq 0$, let us consider for every $x \in \mathbf{R}^n$ and $h \in \mathbf{N}$ the function

$$g_h(x, R) = \sup_{|\xi| \leq R} |a_h(x, \xi) - a(x, \xi)|. \quad (4.11)$$

Since we are interested in the limit as h tends to $+\infty$, we will suppose $h > R + 1$, so that $f_h(|\xi|) = 1$ for $|\xi| \leq R$.

Let us remark that $a(x, \cdot) \in M(\alpha, \beta, c_1, c_2)$ for almost every $x \in \mathbf{R}^n$, since $a \in M_{\mathbf{R}^n}$, so that

$$a(x, \xi) = \pi_h(a(x, \xi)).$$

for a.e. $x \in \mathbf{R}^n$, for every $\xi \in \mathbf{R}^n$, and for every $h \in \mathbf{N}$. By (4.9), we have for $|\xi| \leq \mathbf{R}$ and $0 < \delta \leq 1$

$$|a_h(x, \xi) - a(x, \xi)| = |\pi_h(\alpha_h(x, \xi)) - \pi_h(a(x, \xi))| \leq c[(1 + \mathbf{R})^{p-1-\alpha} \delta^\alpha + \delta^{-n/2} \|\alpha_h(x, \cdot) - a(x, \cdot)\|_h]. \quad (4.12)$$

Thus, we have to estimate

$$\|\alpha_h(x, \cdot) - a(x, \cdot)\|_h^2 \leq \int_{[-h, h]^n} |\alpha_h(x, \eta) - a(x, \eta)|^2 d\eta + \int_{\{|\eta| > h\}} |\alpha_h(x, \eta) - a(x, \eta)|^2 f_h^2(|\eta|) d\eta \equiv J_h^1 + J_h^2. \quad (4.13)$$

By (1.3) we have

$$J_h^2 \leq c \int_{\{|\eta| > h\}} |\eta|^{2p-2} f_h^2(|\eta|) d\eta \leq c \int_h^{+\infty} \rho^{2p+n-3} f_h^2(\rho) d\rho, \quad (4.14)$$

so that $\lim_{h \rightarrow \infty} J_h^2 = 0$. By the definition of α_h , (1.3), and the Hölder inequality we obtain

$$\begin{aligned} J_h^1 &= \sum_{z \in I_h} \int_{Q_z^h} |\alpha_h(x, \eta) - a(x, \eta)|^2 d\eta = \sum_{z \in I_h} \int_{Q_z^h} |P_z^h(x) - a(x, \eta)|^2 d\eta \\ &\leq \sum_{z \in I_h} \int_{Q_z^h} 2(|P_z^h(x) - a(x, \xi_z^h)|^2 + |a(x, \xi_z^h) - a(x, \eta)|^2) d\eta \\ &\leq \sum_{z \in I_h} \int_{Q_z^h} 2(|P_z^h(x) - a_z^h(x)|^2 + [c_1(1 + |\xi_z^h| + |\eta|)^{p-1-\alpha} |\xi_z^h - \eta|^\alpha]^2) d\eta \\ &\leq \sum_{z \in I_h} 2 \left(\frac{1}{\gamma_h} \right)^n |P_z^h(x) - a_z^h(x)|^2 + ch^{n+2p-2-2\alpha} \left(\frac{1}{\gamma_h} \right)^{2\alpha}. \end{aligned} \quad (4.15)$$

Taking (4.12)-(4.15) into account, we obtain for $|\xi| \leq \mathbf{R}$ and $0 < \delta \leq 1$

$$|a_h(x, \xi) - a(x, \xi)| \leq c(1 + \mathbf{R})^{p-1-\alpha} \delta^\alpha + c \delta^{-n/2} (h^{n/2+p-1-\alpha} (\gamma_h)^{-\alpha} + (\gamma_h)^{-n/2} \sum_{z \in I_h} |P_z^h(x) - a_z^h(x)| + (J_h^2)^{1/2});$$

an integration gives

$$\begin{aligned} \int g_h(x, \mathbf{R}) dx &\leq c(1 + \mathbf{R})^{p-1-\alpha} \delta^\alpha \\ &+ c \delta^{-n/2} \left(h^{n/2+p-1-\alpha} (\gamma_h)^{-\alpha} + (\gamma_h)^{-n/2} \sum_{z \in I_h} \int |P_z^h - a_z^h| dx + (J_h^2)^{1/2} \right) \\ &\leq c(1 + \mathbf{R})^{p-1-\alpha} \delta^\alpha \\ &+ c \delta^{-n/2} \left(h^{n/2+p-1-\alpha} (\gamma_h)^{-\alpha} + h^n \frac{(\gamma_h)^{n/2}}{\mu_h} + (J_h^2)^{1/2} \right). \end{aligned} \quad (4.16)$$

Now, we can choose the sequences (γ_h) and (μ_h) in such a way that

$$\lim_{h \rightarrow \infty} h^{n/2+p-1-\alpha} (\gamma_h)^{-\alpha} = \lim_{h \rightarrow \infty} h^n \frac{(\gamma_h)^{n/2}}{\mu_h} = 0$$

(for example, $\gamma_h = h!$ and $\mu_h = h^{nh}$). Then, (4.16) gives

$$\limsup_{h \rightarrow \infty} \int g_h(x, R) dx \leq c(1+R)^{p-1-\alpha} \delta^\alpha.$$

By the arbitrariness of δ , the proof is completed. $\square$

Proof of Lemma 4.3. — By the representation Theorem 2.3 it is enough to prove that for every $\xi \in \mathbf{R}^n$ and for every cube Q in $\mathbf{R}^n$ the limit

$$\lim_{\varepsilon \rightarrow 0+} \frac{1}{|Q|} \int_Q a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) dx \quad (4.17)$$

exists and is independent of Q , where v_ε is the solution to the following boundary value problem

$$\left. \begin{aligned} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right)\right) &= 0 \quad \text{on } Q, \\ v_\varepsilon &\in H_0^{1,p}(Q). \end{aligned} \right\} \quad (4.18)$$

Given a cube Q in $\mathbf{R}^n$ and $\xi \in \mathbf{R}^n$ let us prove that

$$\lim_{h \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0+} \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx = 0, \quad (4.19)$$

where v_ε^h is the solution to the following boundary value problem

$$\left. \begin{aligned} -\operatorname{div}\left(a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right)\right) &= 0 \quad \text{on } Q, \\ v_\varepsilon^h &\in H_0^{1,p}(Q). \end{aligned} \right\} \quad (4.20)$$

For the sake of simplicity we drop in the notation any explicit dependence on ξ of v_ε and v_ε^h throughout this section. We can write

$$\begin{aligned} & \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx \\ & \leq \left| \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx \right| \\ & \quad + \left| \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx \right| \equiv J_{\varepsilon,h}^1 + J_{\varepsilon,h}^2. \end{aligned}$$

The estimate of $J_{\varepsilon,h}^1$ and $J_{\varepsilon,h}^2$ will be carried out in the following four steps.

Step 1. — Let us fix $\xi \in \mathbf{R}^n$. As in (3.17) one gets

$$\|Dv_\varepsilon^h + \xi\|_{(L^p(Q))^n} \leq c|Q|(1+|\xi|^p).$$

This estimate will be used frequently in the sequel. By the Meyers estimate (1.6) we have then

$$\|Dv_\varepsilon^h + \xi\|_{(L^{p+\eta}(Q))^n} \leq c.$$

Let us fix $R \geq 0$, $h \in \mathbb{N}$ and $\varepsilon > 0$, and let us define the set

$$U_R = \{x \in Q : |Dv_\varepsilon^h(x) + \xi| > R\}.$$

We have then

$$|U_R| R^p \leq \int_{U_R} |Dv_\varepsilon^h + \xi|^p dx,$$

so that, using the Hölder inequality, we obtain the estimate

$$\begin{aligned} \int_{U_R} (1 + |Dv_\varepsilon^h + \xi|^p) dx &\leq |U_R| + |U_R|^{\eta/(p+\eta)} \|Dv_\varepsilon^h + \xi\|_{(L^{p+\eta}(Q))^n}^p \\ &\leq c(R^{-p} + R^{-p\eta/(p+\eta)}). \end{aligned} \quad (4.21)$$

Step 2 (estimate of $J_{\varepsilon,h}^2$). — Given $R > 0$ by (1.3) and (4.12) we have

$$\begin{aligned} (J_{\varepsilon,h}^2)^q &\leq c \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx \\ &= c \int_{U_R} \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx \\ &\quad + c \int_{Q \setminus U_R} \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx \\ &\leq c \int_{U_R} (1 + |Dv_\varepsilon^h + \xi|^p) dx + c \int_{Q \setminus U_R} g_h\left(\frac{x}{\varepsilon}, R\right)^q dx \\ &\leq c \int_{U_R} (1 + |Dv_\varepsilon^h + \xi|^p) dx + c(1+R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx. \end{aligned}$$

By Step 1 we conclude then

$$J_{\varepsilon,h}^2 \leq c \left((R^{-p} + R^{-p\eta/(p+\eta)}) + (1+R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx \right)^{1/q}. \quad (4.22)$$

Step 3 (estimate of $\|Dv_\varepsilon^h - Dv_\varepsilon\|_{(L^p(Q))^n}$). — By (4.18) and (4.20) we have

$$\int_Q \left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right), Dv_\varepsilon - Dv_\varepsilon^h \right) dx = 0,$$

so that by (iii) and (iv) in Definition 1.1 and the Hölder inequality we have

$$0 = \int_Q \left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right), Dv_\varepsilon - Dv_\varepsilon^h \right) dx$$

$$\begin{aligned}
& + \int_Q \left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right), Dv_\varepsilon - Dv_\varepsilon^h \right) dx \\
& \geq c_2 \int_Q (1 + |Dv_\varepsilon + \xi| + |Dv_\varepsilon^h + \xi|)^{p-\beta} |Dv_\varepsilon - Dv_\varepsilon^h|^\beta dx \\
& - \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| |Dv_\varepsilon - Dv_\varepsilon^h| dx \\
& \geq c \left(|Q|^{1/p} + \|Dv_\varepsilon + \xi\|_{(L^p(Q))^n} \right. \\
& \quad \left. + \|Dv_\varepsilon^h + \xi\|_{(L^p(Q))^n} \right)^{p-\beta} \|Dv_\varepsilon - Dv_\varepsilon^h\|_{(L^p(Q))^n}^\beta \\
& - \left(\int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx \right)^{1/q} \\
& \quad \times \left(\int_Q |Dv_\varepsilon - Dv_\varepsilon^h|^p dx \right)^{1/p},
\end{aligned}$$

so that by (4.22) we obtain

$$\begin{aligned}
& \|Dv_\varepsilon^h - Dv_\varepsilon\|_{(L^p(Q))^n}^\beta \\
& \leq c \left((R^{-p} + R^{-p\eta/(p+\eta)}) + (1+R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx \right). \quad (4.23)
\end{aligned}$$

Step 4 (estimate of $J_{\varepsilon,h}^1$). — By the equicontinuity condition of a and Hölder's inequality we have

$$\begin{aligned}
J_{\varepsilon,h}^1 & = \left| \int_Q \left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right) dx \right| \\
& \leq c_1 \int_Q (1 + |Dv_\varepsilon + \xi| + |Dv_\varepsilon^h + \xi|)^{p-1-\alpha} |Dv_\varepsilon - Dv_\varepsilon^h|^\alpha dx \\
& \leq c_1 |Q|^{1/p} \left(\int_Q (1 + |Dv_\varepsilon + \xi| + |Dv_\varepsilon^h + \xi|)^p dx \right)^{(p-1-\alpha)/p} \\
& \quad \times \left(\int_Q |Dv_\varepsilon - Dv_\varepsilon^h|^p dx \right)^{\alpha/p}.
\end{aligned}$$

Hence, by (4.23) we get

$$J_{\varepsilon,h}^1 \leq c \left((R^{-p} + R^{-p\eta/(p+\eta)}) + (1+R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx \right)^{\alpha/(\beta-1)}. \quad (4.24)$$

Taking into account that there exists $s > 0$ such that $Q \subset Q_s(0)$ and

$$\begin{aligned}
\int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx & \leq \int_{Q_s(0)} g_h\left(\frac{x}{\varepsilon}, R\right) dx \\
& \leq \varepsilon^n \int_{Q_{s/\varepsilon}(0)} g_h(x, R) dx \leq \frac{c}{|Q_{s/\varepsilon}(0)|} \int_{Q_{s/\varepsilon}(0)} g_h(x, R) dx,
\end{aligned}$$

by passing in (4.22) and (4.24) first to the limit as ε tends to 0^+ , then as h tends to $+\infty$, and eventually as R tends to $+\infty$ we obtain (4.19). By (4.4) and Theorem 2.3 this implies that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{|Q|} \int_Q a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) dx \\ = \lim_{h \rightarrow \infty} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{|Q|} \int_Q a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) dx \\ = \lim_{h \rightarrow \infty} b_h(\xi) = b(\xi). \end{aligned} \quad (4.25)$$

Since the first limit does not depend on Q , the proof of Lemma 4.3 can be concluded by applying again the representation Theorem 2.3. $\square$

Proof of Theorem 4.2. — Let $a \in M_{\mathbf{R}^n}$ such that $a(\cdot, \xi)$ is almost periodic for every $\xi \in \mathbf{R}^n$. By Lemma 4.4 there exists a sequence (a_h) in $M_{\mathbf{R}^n}$ satisfying (3.3) (with λ possibly depending on h) such that condition (4.5) is satisfied. By the homogenization Theorem 3.5 we obtain that for every $h \in \mathbf{N}$ the limit

$$G\text{-}\lim_{\varepsilon \rightarrow 0^+} a_h\left(\frac{x}{\varepsilon}, \xi\right) = b_h(\xi)$$

exists and is independent of x . Hence, by Lemma 4.3 the limit

$$G\text{-}\lim_{\varepsilon \rightarrow 0^+} a\left(\frac{x}{\varepsilon}, \xi\right) = b(\xi)$$

exists. Finally, the representation formula follows from (4.25). $\square$

APPENDIX

A. Birkhoff's theorem for quasiperiodic functions

Before we give a direct proof of Birkhoff's Ergodic theorem in the quasiperiodic case, we state Kronecker's lemma (see, for instance, [30] 3.1).

KRONECKER'S LEMMA. — *The set of vectors which are equivalent modulo T to vectors of the form $j(x)$, with $x \in \mathbf{R}^n$, is dense in $\mathbf{R}^N$.*

THEOREM A. — *Let $f: \mathbf{R}^n \rightarrow \mathbf{R}$ be a quasiperiodic function in $QP(\lambda)$ (see Section 3). Then*

$$\oint f(x) dx = \frac{1}{|T|} \int_T F(y) dy,$$

where T is defined by (3.2).

Proof. — Let $X = j(\mathbf{R}^n)$ and $Z = X^\perp$. Let us identify $\mathbf{R}^N = X \oplus Z$. Moreover, let us introduce

$$P_s^N(0) = j(Q_s(0)) \times B_s^Z(0),$$

where $B_s^Z(0)$ is the ball in Z of radius s and center 0, and define the set S by $S = \{z \in B_s^Z(0) : z \text{ is equivalent modulo } T \text{ to vectors of the type } j(\tau) \text{ with } \tau \in \mathbf{R}^n\}$. By Kronecker's lemma the set S is dense in $B_s^Z(0)$. Then, given $z \in S$ there exists $\tau = \tau(z) \in \mathbf{R}^n$ such that

$$F(j(x) + z) = f(x + \tau)$$

for every $x \in \mathbf{R}^n$. Since the limit

$$\lim_{s \rightarrow +\infty} \frac{1}{s^n} \int_{Q_s(\tau)} f(x) dx = \int f(x) dx$$

exists uniformly with respect to τ (see [30] 2.3), given $\varepsilon > 0$ we have

$$\left| \frac{1}{s^n} \int_{Q_s(0)} f(x + \tau) dx - \int f(x) dx \right| < \varepsilon$$

for every $\tau \in \mathbf{R}^n$, for s sufficiently large, so that for $z \in S$

$$\left| \frac{1}{s^n} \int_{Q_s(0)} F(j(x) + z) dx - \int f(x) dx \right| < \varepsilon \quad (\text{A.1})$$

for s sufficiently large. By the uniform continuity of F and the density of S , the estimate (A.1) holds for every $z \in B_s^Z(0)$. By Fubini's theorem we have

$$\begin{aligned} \frac{1}{|P_s^N(0)|} \int_{P_s^N(0)} F(y) dy &= \frac{1}{s^n} \int_{Q_s(0)} \left(\frac{1}{|B_s^Z(0)|} \int_{B_s^Z(0)} F(j(x) + z) dz \right) dx \\ &= \frac{1}{|B_s^Z(0)|} \int_{B_s^Z(0)} \left(\frac{1}{s^n} \int_{Q_s(0)} F(j(x) + z) dx \right) dz, \end{aligned}$$

which by (A.1) yields that

$$\left| \frac{1}{|P_s^N(0)|} \int_{P_s^N(0)} F(y) dy - \int f(x) dx \right| < \varepsilon.$$

This implies together with the periodicity of F that

$$\frac{1}{|T|} \int_T F(y) dy = \lim_{s \rightarrow +\infty} \frac{1}{|P_s^N(0)|} \int_{P_s^N(0)} F(y) dy = \int f(x) dx$$

and concludes the proof. $\square$

B. Regularity of the quasiperiodic solutions of $\Delta u = \operatorname{div} f$

Remark B.1. — Let $\mathcal{W}$ be the completion of $\operatorname{Trig}_0(T)$ with respect to the norm $\|\cdot\|_{\mathcal{W}}$ introduced in Section 3. Let us denote by $\mathcal{W}'$ its dual. It turns out that for every $S \in \mathcal{W}'$ there exists $G \in (L^q(T))^n$ such that

$$\langle S, v \rangle = \frac{1}{|T|} \sum_{r=1}^n \int_T G_r \partial_r v dy \quad \text{for every } v \in \mathcal{W}. \quad (\text{B.1})$$

In fact, since by the map ∂ the space $\mathcal{W}$ is isometric to a subspace of $(L^p(T))^n$, by the Hahn-Banach and Riesz theorems there exists a function $G \in (L^q(T))^n$ such that (B.1) and

$$\|S\|_{\mathcal{W}'} = \left(\frac{1}{|T|} \sum_{r=1}^n \int_T |G_r|^q dy \right)^{1/q}$$

hold. We will write $\|S\|_{\mathcal{W}'} = \left\| \sum_{r=1}^n \partial_r G_r \right\|_{\mathcal{W}'}$.

THEOREM B. — Let $F \in (\operatorname{Trig}(T))^n$ and let $W \in \operatorname{Trig}_0(T)$ be the unique solution to

$$\sum_{r=1}^n \partial_r^2 W = \sum_{r=1}^n \partial_r F_r \quad \text{in } \mathbf{R}^N. \quad (\text{B.2})$$

Then,

$$\left(\frac{1}{|T|} \sum_{r=1}^n \int_T |\partial_r W|^q dy \right)^{1/q} \leq c \left\| \sum_{r=1}^n \partial_r F_r \right\|_{\mathcal{W}'}, \quad (\text{B.3})$$

with $c = c(T) > 0$ and $\partial_r = \sum_{l=1}^{m_r} \frac{\partial}{\partial y_r^l}$ as defined in Section 3.

Proof. — Let us define $S \in \mathcal{W}'$ by

$$\langle S, v \rangle = \frac{1}{|T|} \sum_{r=1}^n \int_T F_r \partial_r v dy \quad \text{for every } v \in \mathcal{W}.$$

By Remark B.1 there exists $G \in (L^q(T))^n$ such that

$$\langle S, v \rangle = \frac{1}{|T|} \sum_{r=1}^n \int_T G_r \partial_r v dy \quad \text{for every } v \in \mathcal{W}$$

and

$$\|G\|_{(L^q(T))^n} = \|S\|_{\mathcal{W}'}. \quad (\text{B.4})$$

By extending G on $\mathbf{R}^N$ by periodicity we have that

$$\sum_{r=1}^n \partial_r^2 W = \sum_{r=1}^n \partial_r G_r \quad \text{in } \mathcal{D}'(\mathbf{R}^N). \quad (\text{B.5})$$

Let us introduce on $\mathbf{R}^N$ a new orthogonal base $(E_l)_{1 \leq l \leq N}$ with

$$\begin{aligned} E_1 &= (\underbrace{1, \dots, 1}_{m_1}, 0, \dots, 0) \\ &\vdots \\ E_n &= (0, \dots, 0, \underbrace{1, \dots, 1}_{m_n}, 0, \dots, 0). \end{aligned}$$

We shall denote the new coordinates by

$$(x, z) = (x_1, \dots, x_n, z_1, \dots, z_{N-n}) \in \mathbf{R}^N.$$

It turns out that

$$D_{x_l} u(x, z) = \partial_l u(y) \quad \text{for every } u \in \mathcal{W}, \quad \text{for every } l = 1, \dots, n,$$

where $y = (x, z)$. Hence by (B.5) the function $W \in \mathcal{C}^\infty(\mathbf{R}^n \times \mathbf{R}^{N-n})$ satisfies for almost every $z \in \mathbf{R}^{N-n}$

$$\Delta_x W(\cdot, z) = \operatorname{div}_x G(\cdot, z) \quad \text{in } \mathcal{D}'(\mathbf{R}^n).$$

By elliptic regularity (*see*, for instance [14]) for every $A' \subset \subset A \subset \subset \mathbf{R}^n$ there exists a constant $c = c(A', A)$, such that

$$\int_{A'} |D_x W(x, z)|^q dx \leq c \int_A |G(x, z)|^q dx.$$

By Fubini's theorem we get

$$\int_B \int_{A'} |D_x W(x, z)|^q dx dz \leq c \int_B \int_A |G(x, z)|^q dx dz$$

for every $B \subset \subset \mathbf{R}^{N-n}$. Choosing A' , A and B such that

$$T \subset B \times A' \subset B \times A \subset \prod_{l,r} \left[-\frac{2\pi}{\lambda_r^l}, \frac{4\pi}{\lambda_r^l} \right],$$

and using finally the periodicity of G , we get

$$\int_T |D_x W(x, z)|^q dx dz \leq c \int_T |G(x, z)|^q dx dz.$$

By (B.4) we obtain (B.3), and we can conclude the proof. $\square$

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