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On nonhomogeneous elliptic equations involving critical Sobolev exponent

by

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ABSTRACT. — Let $p = \frac{2N}{N-2}$, $N \geq 3$ be the limiting Sobolev exponent and $\Omega \subset \mathbb{R}^N$ open bounded set.

We show that for $f \in H^{-1}$ satisfying a suitable condition and $f \neq 0$, the Dirichlet problem:

$$\begin{cases} -\Delta u = |u|^{p-2}u + f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits two solutions u_0 and u_1 in $H_0^1(\Omega)$.

Also $u_0 \geq 0$ and $u_1 \leq 0$ for $f \geq 0$.

Notice that, in general, this is not the case if $f = 0$ (see [P]).

Key words : Semilinear elliptic equations, critical Sobolev exponent.

RÉSUMÉ. — Soit $p = \frac{2N}{N-2}$ l'exposant de Sobolev critique et $\Omega \subset \mathbb{R}^N$ un domaine borné.

Classification A.M.S. : 35 A 15, 35 J 20, 35 J 65.

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On montre que si $f \in H^{-1}$, $f \neq 0$ satisfait une certaine condition alors le problème de Dirichlet : $\Delta u = |u|^{p-2}u + f$ dans Ω et $u = 0$ dans $\partial\Omega$, admet deux solutions u_0 et u_2 dans $H_0^1(\Omega)$. De plus $u_0 \geq 0$ et $u_1 \geq 0$ si $f \geq 0$.

On remarque que ce n'est pas le cas, en général, si $f = 0$ (voir [P]).

1. INTRODUCTION AND MAIN RESULTS

In a recent paper Brezis-Nirenberg (B.N.1] have considered the following minimization problem:

$$\inf_{u \in H, \|u\|_p = 1} \int_{\Omega} (|\nabla u|^2 - fu) \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$, is a bounded set, $H = H_0^1(\Omega)$, $f \in H^{-1}$ and $p = \frac{2N}{N-2}$, $N \geq 3$ is the limiting exponent in the Sobolev embedding.

It is well known that the infimum in (1.1) is never achieved if $f = 0$ (cf. [B]). In contrast, in [B.N.1] it is shown that for $f \neq 0$ this infimum is always achieved. (See also [C.S.] for previous related results.)

Motivated by this result we consider the functional:

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p} \int_{\Omega} |u|^p - \int_{\Omega} fu, \quad u \in H;$$

whose critical points define weak solutions for the Dirichlet problem:

$$\left. \begin{aligned} -\Delta u &= |u|^{p-2}u + f & \text{on } \Omega \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \right\} \quad (1.2)$$

We investigate suitable minimization and minimax principles of mountain pass-type (cf. [A.R.]), and show how, for suitable f 's, they produce critical values for I in spite of a possible failure of the Palais-Smale condition.

To start, notice that I is bounded from below in the manifold:

$$\Lambda = \{u \in H : \langle I'(u), u \rangle = 0\}$$

[here $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $H = H_0^1(\Omega)$]. Thus a natural question to ask is whether or not I achieves a minimum in Λ .

We show that this is the case if f satisfies the following:

$$\int_{\Omega} fu \leq c_N (\|\nabla u\|_2)^{(N+2)/2} \quad (*)_0$$

$\forall u \in H, \|u\|_p = 1$, where $c_N = \frac{4}{N-2} \left(\frac{N-2}{N+2} \right)^{(N+2)/4}$. More precisely we have:

THEOREM 1. — *Let $f \neq 0$ satisfies $(*)_0$. Then*

$$\inf_{\Lambda} I = c_0 \quad (1.3)$$

is achieved at a point $u_0 \in \Lambda$ which is a critical point for I and $u_0 \geq 0$ for $f \geq 0$.

In addition if f satisfies the more restrictive assumption:

$$\int_{\Omega} f u < c_N (\|\nabla u\|_2)^{(N+2)/2} \quad (*)$$

$\forall u \in H, \|u\|_p = 1$, then u_0 is a *local minimum* for I . \square

Notice that assumption $(*)$ certainly holds if

$$\|f\|_{H^{-1}} \leq c_N S^{N/4}$$

where S is the best Sobolev constant (cf. [T]).

Also if $f=0$ Theorem 1 remains valid and gives the trivial solution $u_0=0$.

Moreover in the situation where u_0 is a local minimum for I , necessarily:

$$\|\nabla u_0\|_2^2 - (p-1) \|u_0\|_p^p \geq 0 \quad (1.4)$$

This suggests to look at the following splitting for Λ :

$$\Lambda^+ = \{u \in \Lambda : \|\nabla u\|_2^2 - (p-1) \|u\|_p^p > 0\}$$

$$\Lambda_0 = \{u \in \Lambda : \|\nabla u\|_2^2 - (p-1) \|u\|_p^p = 0\}$$

$$\Lambda^- = \{u \in \Lambda : \|\nabla u\|_2^2 - (p-1) \|u\|_p^p < 0\}.$$

It turns out that assumption $(*)$ implies $\Lambda_0 = \{0\}$ (see Lemma 2.3 below).

Therefore for $f \neq 0$ and (1.4) we obtain $u_0 \in \Lambda^+$ and consequently

$$c_0 = \inf_{\Lambda} I = \inf_{\Lambda^+} I.$$

So we are led to investigate a second minimization problem. Namely:

$$\inf_{\Lambda^-} I = c_1. \quad (1.5)$$

In this direction we have:

THEOREM 2. — *Let $f \neq 0$ satisfies $(*)$. Then $c_1 > c_0$ and the infimum in (1.5) is achieved at a point $u_1 \in \Lambda^-$ which define a critical point for I .*

Furthermore $u_1 \geq 0$ for $f \geq 0$. \square

Notice that the assumption $f \neq 0$ is *necessary* in Theorem 2. In fact for $f=0$ we have:

$$\inf_{\Lambda^-} I = \inf_{u \neq 0} \frac{1}{N} \left[\frac{\|\nabla u\|_2^2}{\|u\|_p^2} \right]^{N/2} = \frac{1}{N} \left[\inf_{\|u\|_p=1} \|\nabla u\|_2^2 \right]^{N/2}$$

and the infimum in the right hand side is *never* achieved.

The proofs of Theorem 1 and Theorem 2 rely on the Ekeland's variational principle (cf. [A.E.]) and careful estimates inspired by these in [B.N.1].

As an immediate consequence of Theorems 1 and 2 we have the following for the Dirichlet problem (1.2).

THEOREM 3. — Problem (1.2) admits at least *two* weak solutions $u_0, u_1 \in H_0^1(\Omega)$ for $f \neq 0$ satisfying $(*)$; and at least *one* weak solution for f satisfying $(*)_0$.

Moreover $u_0 \geq 0, u_1 \geq 0$ for $f \geq 0$. \square

This result for $f \geq 0$ was also pointed out by Brezis-Nirenberg in [B.N.1]. Their approach however uses in an essential way the fact that f does not change sign. It relies on a result of Crandall-Rabinowitz [C.R.] and techniques developed in [B.N.2].

Furthermore for $f \geq 0$ it is known that (1.2) cannot admit positive solution when $\|f\|_{H^{-1}}$ is too large (see [C.R.], [M.] and [Z]). So our approach necessarily breaks down when $\|f\|_{H^{-1}}$ is large. In fact we suspect that assumptions $(*)_0$ and $(*)$ on f are not only sufficient but also *necessary* to guarantee the statements of Theorems 1 and 2.

By a result of Brezis-Kato [B-K] we know that Theorem 3 gives *classical* solutions if f is sufficiently regular and $\partial\Omega$ is smooth; and for $f \geq 0$, via the strong maximum principle, such solutions are *strictly* positive in Ω .

Obviously an equivalent of Theorem 3 holds for the *subcritical* case where one replaces the power $p = \frac{2N}{N-2}$ in (1.2) by $q \in \left(2, \frac{2N}{N-2}\right)$. In such a case more standard compactness arguments apply, and the proof can be consistently simplified. The details are left to the interested reader. Finally going back to the functional I , if f satisfies $(*)$ then Theorem 1 suggests a mountain-pass procedure; which will be carried out as follows.

Take:

$$u_\varepsilon(x) = \frac{\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}} \quad \varepsilon > 0, \quad x \in \mathbb{R}^N \quad (1.6)$$

be an extremal function for the Sobolev inequality in \mathbb{R}^N .

For $a \in \Omega$ let $u_{\varepsilon,a}(x) = u_\varepsilon(x-a)$, and

$$\xi_a \in C_0^\infty(\Omega) \quad \text{with} \quad \xi_a \geq 0 \quad \text{and} \quad \xi_a = 1 \text{ near } a. \quad (1.7)$$

Set

$$\mathcal{F} = \left\{ h : [0, 1] \rightarrow H \text{ continuous, } h(0) = u_0 \right. \\ \left. h(1) = R_0 \xi_a u_{\varepsilon, a} \right\}$$

$R_0 > 0$ fixed.

We have:

THEOREM 4. — For a suitable choice of $R_0 > 0$, $a \in \Omega$ and $\varepsilon > 0$ the value

$$c = \inf_{h \in \mathcal{F}} \max_{t \in [0, 1]} I(h, (t))$$

defines a critical value for I , and $c \geq c_1$. \square

It is not clear whether or not $c = c_1$. So no additional multiplicity can be claimed for (1.2). However, in case $c = c_1$ then it is possible to claim a critical point of mountain-pass type (cf. [H]) for I in Λ^- . This follows by a refined version of the mountain-pass lemma (see [A-R]) obtained by Ghoussoub-Preiss and the fact that Λ^- cannot contain local minima for I (see [G.P., theorem (ter) part a]).

The referee has brought to our attention a paper of O. Rey (See [R.]) where, by a different approach, a result similar to that of Theorem 3 is established when $f \neq 0$, $f \geq 0$ and $\|f\|_{H^{-1}}$ is sufficiently small.

2. THE PROOF OF THEOREM 1

To obtain the proof of Theorem 1 several preliminary results are in order.

We start with a lemma which clarifies the purpose of assumption (*).

LEMMA 2.1. — Let $f \neq 0$ satisfy (*). For every $u \in H$, $u \neq 0$ there exists a unique $t^+ = t^+(u) > 0$ such that $t^+ u \in \Lambda^-$. In particular:

$$t^+ > \left[\frac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p} \right]^{1/(p-2)} := t_{\max}$$

and $I(t^+ u) = \max_{t \geq t_{\max}} I(tu)$

Moreover, if $\int_{\Omega} fu > 0$, then there exists a unique $t^- = t^-(u) > 0$ such that $t^- u \in \Lambda^+$.

In particular,

$$t^- < \left[\frac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p} \right]^{1/(p-2)}$$

and $I(t^- u) \leq I(tu)$, $\forall t \in [0, t^+]$.

Proof. — Set $\varphi(t) = t \|\nabla u\|_2^2 - t^{p-1} \|u\|_p^p$. Easy computations show that φ is concave and achieves its maximum at

$$t_{\max} = \left[\frac{\|\nabla u\|_2^2}{(p-1) \|u\|_p^p} \right]^{1/(p-2)}.$$

Also

$$\varphi(t_{\max}) = \left[\frac{1}{p-1} \right]^{(p-1)/(p-2)} (p-2) \left[\frac{\|\nabla u\|_2^2}{\|u\|_p^p} \right]^{1/(p-2)},$$

that is

$$\varphi(t_{\max}) = c_N \frac{\|\nabla u\|_2^{(N+2)/2}}{\|u\|_p^{N/2}}.$$

Therefore if $\int_{\Omega} fu \leq 0$ then there exists a *unique* $t^+ > t_{\max}$ such that:

$$\varphi(t^+) = \int_{\Omega} fu \quad \text{and} \quad \varphi'(t^+) < 0. \quad \text{Equivalently} \quad t^+ u \in \Lambda^- \quad \text{and} \\ I(t^+ u) \geq I(tu) \quad \forall t \geq t_{\max}.$$

In case $\int_{\Omega} fu > 0$, by assumption $(*)$ we have that necessarily

$$\int_{\Omega} fu < c_N \frac{\|\nabla u\|_2^{(N+2)/2}}{\|u\|_p^{N/2}} = \varphi(t_{\max}).$$

Consequently, in this case, we have unique $0 < t^- < t_{\max} < t^+$ such that

$$\varphi(t^+) = \int_{\Omega} fu = \varphi(t^-)$$

and

$$\varphi'(t^-) > 0 > \varphi'(t^+).$$

Equivalently $t^+ u \in \Lambda^-$ and $t^- u \in \Lambda^+$.

Also $I(t^+ u) \geq I(tu)$, $\forall t \geq t^-$ and $I(t^- u) \leq I(tu)$, $\forall t \in [0, t^+]$.

LEMMA 2.2. — *For $f \neq 0$*

$$\inf_{\|u\|_p=1} \left(c_N \|\nabla u\|_2^{(N+2)/2} - \int_{\Omega} fu \right) =: \mu_0 \quad (2.1)$$

is achieved. In particular if f satisfies $()$, then $\mu_0 > 0$.*

The proof of Lemma 2.2 is technical and a straightforward adaptation of that given in [B.N.1] for an analogous minimization problem.

It will be given in the appendix for the reader's convenience.

Next, for $u \neq 0$ set

$$\psi(u) = c_N \frac{\|\nabla u\|_2^{(N+2)/2}}{\|u\|_p^{N/2}} - \int_{\Omega} fu.$$

Since for $t > 0$, $\|u\|_p = 1$ we have:

$$\psi(tu) = t \left[c_N \|\nabla u\|_2^{(N+2)/2} - \int_{\Omega} fu \right];$$

given $\gamma > 0$, from Lemma 2.2 we derive that

$$\inf_{\|u\| \geq \gamma} \psi(u) \geq \gamma \mu_0. \quad (2.2)$$

In particular if f satisfies $(*)$ then the infimum (2.2) is bounded away from zero.

This remark is crucial for the following:

LEMMA 2.3. — *Let f satisfy $(*)$. For every $u \in \Lambda$, $u \neq 0$ we have*

$$\|\nabla u\|_2^2 - (p-1)\|u\|_p^p \neq 0$$

(i. e. $\Lambda_0 = \{0\}$).

Proof. — Although the result also holds for $f=0$, we shall only be concerned with the case $f \neq 0$.

Arguing by contradiction assume that for some $u \in \Lambda$, $u \neq 0$ we have

$$\|\nabla u\|_2^2 - (p-1)\|u\|_p^p = 0 \quad (2.3)$$

Thus

$$0 = \|\nabla u\|_2^2 - \|u\|_p^p - \int_{\Omega} fu = (p-2)\|u\|_p^p - \int_{\Omega} fu. \quad (2.4)$$

Condition (2.3) implies

$$\|u\|_p \geq \left(\frac{S}{p-1} \right)^{1/(p-2)} := \gamma,$$

and from (2.2) and (2.4) we obtain:

$$\begin{aligned} 0 < \mu_0 \gamma \leq \psi(u) &= \left[\frac{1}{p-1} \right]^{(p-1)/(p-2)} (p-2) \left[\frac{\|\nabla u\|_2^{2(p-1)}}{\|u\|_p^p} \right]^{1/(p-2)} - \int_{\Omega} f u \\ &= (p-2) \left(\left[\frac{1}{p-1} \right]^{(p-1)/(p-2)} \left[\frac{\|\nabla u\|_2^{2(p-1)}}{\|u\|_p^p} \right]^{1/(p-2)} - \|u\|_p^p \right) \\ &= (p-2) \|u\|_p^p \left(\left[\frac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p} \right]^{(p-1)/(p-2)} - 1 \right) = 0 \end{aligned}$$

which yields to a contradiction. \square

As a consequence of Lemma 2.3 we have:

LEMMA 2.4. — *Let $f \neq 0$ satisfy $(*)$. Given $u \in \Lambda$, $u \neq 0$ there exist $\varepsilon > 0$ and a differentiable function $t = t(w) > 0$, $w \in H$, $\|w\| < \varepsilon$ satisfying the following:*

$$t(0) = 1, \quad t(w)(u-w) \in \Lambda, \quad \text{for } \|w\| < \varepsilon,$$

and

$$\langle t'(0), w \rangle = \frac{2 \int_{\Omega} \nabla u \cdot \nabla w - p \int_{\Omega} |u|^{p-2} u w \int_{\Omega} f w}{\|\nabla u\|_2^2 - (p-1)\|u\|_p^p}. \quad (2.5)$$

Proof. — Define $F : \mathbb{R} \times H \rightarrow \mathbb{R}$ as follows:

$$F(t, w) = t \|\nabla(u-w)\|_2^2 - t^{p-1} \|u-w\|_p^p - \int_{\Omega} f(u-w).$$

Since $F(1, 0) = 0$ and $F_t(1, 0) = \|\nabla u\|_2^2 - (p-1)\|u\|_p^p \neq 0$ (by Lemma 2.3), we can apply the implicit function theorem at the point $(1, 0)$ and get the result. \square

We are now ready to give:

The Proof of Theorem 1

We start by showing that I is bounded from below in Λ . Indeed for $u \in \Lambda$ we have:

$$\int_{\Omega} |\nabla u|^2 - \int_{\Omega} |u|^p - \int_{\Omega} f u = 0.$$

Thus:

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p} \int_{\Omega} |u|^p - \int_{\Omega} f u = \frac{1}{N} \int_{\Omega} |\nabla u|^2 - \left(1 - \frac{1}{p}\right) \int_{\Omega} f u \\ &\geq \frac{1}{N} \|\nabla u\|_2^2 - \frac{N+2}{2N} \|f\|_{H^{-1}} \|\nabla u\|_2 \geq -\frac{1}{16N} [(N+2)\|f\|_{H^{-1}}]^2. \end{aligned}$$

In particular

$$c_0 \geq -\frac{1}{16N}[(N+2)\|f\|_{H^{-1}}]^2. \quad (2.6)$$

We first obtain our result for f satisfying $(*)$. The more general situation where f satisfies $(*)_0$ will be subsequently derived by a limiting argument.

So from now on we assume that f satisfy $(*)$.

In order to obtain an upper bound for c_0 , let $v \in H$ be the unique solutions for $-\Delta u = f$. So for $f \neq 0$

$$\int_{\Omega} f v = \|\nabla v\|_2^2 > 0.$$

Set $t_0 = t^-(v) > 0$ as defined by Lemma 2.1.

Hence $t_0 v \in \Lambda^+$ and consequently:

$$\begin{aligned} I(t_0 v) &= \frac{t_0^2}{2} \|\nabla v\|_2^2 - \frac{t_0^p}{p} \|v\|_p^p - t_0 \|\nabla v\|_2^2 \\ &= -\frac{t_0^2}{2} \|\nabla v\|_2^2 + \frac{p-1}{p} t_0^p \|v\|_p^p < -\frac{t_0^2}{N} \|\nabla v\|_2^2 = -\frac{t_0^2}{N} \|f\|_{H^{-1}}^2 \end{aligned}$$

This yields,

$$c_0 < -\frac{t_0^2}{N} \|f\|_{H^{-1}}^2 < 0. \quad (2.7)$$

Clearly Ekeland's variational principle (see [A.E.], Corollary 5.3.2) applies to the minimization problem (1.3). It gives a minimizing sequence $\{u_n\} \subset \Lambda$ with the following properties:

- (i) $I(u_n) < c_0 + \frac{1}{n}$.
- (ii) $I(w) \geq I(u_n) - \frac{1}{n} \|\nabla(w - u_n)\|_2, \forall w \in \Lambda$.

By taking n large, from (2.7) we have:

$$I(u_n) = \frac{1}{N} \int_{\Omega} |\nabla u_n|^2 - \frac{N+2}{2N} \int_{\Omega} f u_n < c_0 + \frac{1}{n} < -\frac{t_0^2}{N} \|f\|_{H^{-1}}^2 \quad (2.8)$$

This implies

$$\int_{\Omega} f u_n \geq \frac{2}{N+2} t_0^2 \|f\|_{H^{-1}}^2 > 0. \quad (2.9)$$

Consequently $u_n \neq 0$, and putting together (2.8) and (2.9) we derive:

$$\frac{2 t_0^2}{N+2} \|f\|_{H^{-1}} \leq \|\nabla u_n\|_2 \leq \frac{N+2}{2} \|f\|_{H^{-1}}. \quad (2.10)$$

Our goal is to obtain $\|I'(u_n)\| \rightarrow 0$ as $n \rightarrow +\infty$.

Hence let us assume $\|I'(u_n)\| > 0$ for n large (otherwise we are done).

Applying Lemma 2.4 with $u = u_n$ and $w = \delta \frac{I'(u_n)}{\|I'(u_n)\|}$ $\delta > 0$ small, we

$$\text{find, } t_n(\delta) := t \left[\delta \frac{I'(u_n)}{\|I'(u_n)\|} \right]$$

such that

$$w_\delta = t_n(\delta) \left[u_n - \delta \frac{I'(u_n)}{\|I'(u_n)\|} \right] \in \Lambda.$$

From condition (ii) we have:

$$\begin{aligned} \frac{1}{n} \|\nabla(w_\delta - u_n)\|_2 &\geq I(u_n) - I(w_\delta) = (1 - t_n(\delta)) \langle I'(w_\delta), u_n \rangle \\ &\quad + \delta t_n(\delta) \langle I'(w_\delta), \frac{I'(u_n)}{\|I'(u_n)\|} \rangle + o(\delta). \end{aligned}$$

Dividing by $\delta > 0$ and passing to the limit as $\delta \rightarrow 0$ we derive:

$$\frac{1}{n} (1 + |t'_n(0)| \|\nabla u_n\|_2) \geq -t'_n(0) \langle I'(u_n), u_n \rangle + \|I'(u_n)\| = \|I'(u_n)\|$$

$$\text{where we have set } t'_n(0) = \left\langle t'(0), \frac{I'(u_n)}{\|I'(u_n)\|} \right\rangle.$$

Thus from (2.10) we conclude:

$$\|I'(u_n)\| \leq \frac{C}{n} (1 + |t'_n(0)|)$$

for a suitable positive constant C .

We are done once we show that $|t'_n(0)|$ is bounded uniformly on n .

From (2.5) and the estimate (2.10) we get:

$$|t'_n(0)| \leq \frac{C_1}{\|\nabla u_n\|_2^2 - (p-1)\|u_n\|_p^p}$$

$C_1 > 0$ suitable constant.

Hence we need to show that $\|\nabla u_n\|_2^2 - (p-1)\|u_n\|_p^p$ is bounded away from zero.

Arguing by contradiction, assume that for a subsequence, which we still call u_n , we have:

$$\|\nabla u_n\|_2^2 - (p-1)\|u_n\|_p^p = o(1). \quad (2.11)$$

From the estimate (2.10) and (2.11) we derive:

$$\|u_n\|_p \geq \gamma \quad (\gamma > 0 \text{ suitable constant})$$

and

$$\left[\frac{\|\nabla u_n\|_2^2}{p-1} \right]^{(p-1)/(p-2)} - [\|u_n\|_p^p]^{(p-1)/(p-2)} = o(1).$$

In addition (2.11), and the fact that $u_n \in \Lambda$ also give:

$$\int_{\Omega} f u_n = (p-2) \|u_n\|_p^p + o(1).$$

This, together with (2.2) implies:

$$\begin{aligned} 0 < \mu_0 \gamma^{(N+2)/2} &\leq \|u_n\|_p^{p/(p-2)} \psi(u_n) \\ &= (p-2) \left[\left[\frac{\|\nabla u_n\|_2^2}{p-1} \right]^{(p-1)/(p-2)} - [\|u_n\|_p^p]^{(p-1)/(p-2)} \right] = o(1), \end{aligned}$$

which is clearly impossible.

In conclusion:

$$\|I'(u_n)\| \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (2.12)$$

Let $u_0 \in H$ be the weak limit in $H_0^1(\Omega)$ of (a subsequence of) u_n .

From (2.9) we derive that:

$$\int_{\Omega} f u_0 > 0$$

and from (2.12) that

$$\langle I'(u_0), w \rangle = 0, \quad \forall w \in H,$$

i.e. u_0 is a weak solution for (1.2).

In particular, $u_0 \in \Lambda$.

Therefore:

$$c_0 \leq I(u_0) = \frac{1}{N} \|\nabla u_0\|_2^2 - \int_{\Omega} f u_0 \leq \lim_{n \rightarrow +\infty} I(u_n) = c_0.$$

Consequently $u_n \rightarrow u_0$ strongly in H and $I(u_0) = c_0 = \inf_{\Lambda} I$. Also from

Lemma 2.1 and (2.12) follows that necessarily $u_0 \in \Lambda^+$.

To conclude that u_0 is a local minimum for I , notice that for every

$u \in H$ with $\int_{\Omega} f u > 0$ we have:

$$\begin{aligned} I(su) &\geq I(t^-u) \\ \text{for every } 0 < s < \left[\frac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p} \right]^{1/(p-2)} \end{aligned} \quad (2.13)$$

(see Lemma 2.1).

In particular for $u = u_0 \in \Lambda^+$ we have:

$$t^- = 1 < \left[\frac{\|\nabla u_0\|_2^2}{(p-1)\|u\|_p^p} \right]^{1/(p-2)}. \quad (2.14)$$

Let $\varepsilon > 0$ sufficiently small to have:

$$1 < \frac{\|\nabla(u_0 - w)\|_2^2}{(p-1)\|u_0 - w\|_p^p}$$

for $\|w\| < \varepsilon$.

From Lemma 2.4, let $t(w) > 0$ satisfy $t(w)(u_0 - w) \in \Lambda$ for every $\|w\| < \varepsilon$.

Since $t(w) \rightarrow 1$ as $\|w\| \rightarrow 0$, we can always assume that

$$t(w) < \left[\frac{\|\nabla(u_0 - w)\|_2^2}{(p-1)\|u_0 - w\|_p^p} \right]^{1/(p-2)}$$

for every $w: \|w\| < \varepsilon$.

Namely, $t(w)(u_0 - w) \in \Lambda^+$ and for $0 < s < \left[\frac{\|\nabla(u_0 - w)\|_2^2}{(p-1)\|u_0 - w\|_p^p} \right]^{1/(p-2)}$

we have,

$$I(s(u_0 - w)) \geq I(t(w)(u_0 - w)) \geq I(u_0).$$

From (2.14) we can take $s = 1$ and conclude:

$$I(u_0 - w) \geq I(w), \quad \forall w \in H, \quad \|w\| < \varepsilon.$$

Furthermore if $f \geq 0$, take, $t_0 = t^-(|u_0|) > 0$ with $t_0|u_0| \in \Lambda^+$.

Necessarily $t_0 \geq 1$, and

$$I(t_0|u_0|) \leq I(|u_0|) \leq I(u_0).$$

So we can always take $u_0 \geq 0$.

To obtain the proof when f satisfies $(*)_0$ we shall apply an approximation argument. To this purpose, notice that if f satisfies $(*)_0$ then $f_\varepsilon = (1 - \varepsilon)f$ satisfies $(*) \forall \varepsilon \in (0, 1)$.

Set

$$I_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p} \int_\Omega |u|^p + (1 - \varepsilon) \int_\Omega fu, \quad u \in H.$$

Let $u_\varepsilon \in \Lambda_\varepsilon^+ = \{u \in H: \langle I'_\varepsilon(u), u \rangle = 0, \|\nabla u\|_2^2 - (p-1)\|u\|_p^p > 0\}$ satisfy:

$$I_\varepsilon(u_\varepsilon) = \inf_{\Lambda_\varepsilon} I_\varepsilon =: c_\varepsilon$$

and

$$\langle I'_\varepsilon(u_\varepsilon), w \rangle = 0, \quad \forall w \in H. \quad (2.15)$$

Clearly $\|\nabla u_\varepsilon\|_2 \leq C_2$, for $0 < \varepsilon < 1$ and $C_2 > 0$ a suitable constant.

Let $u \in \Lambda^+$, necessarily $\int_{\Omega} fu > 0$ and consequently

$$(1 - \varepsilon) \int_{\Omega} fu > 0, \quad 0 < \varepsilon < 1.$$

From Lemma 2.1 applied with $f = f_{\varepsilon}$ we find:

$$0 < t_{\varepsilon}^{-} < \left[\frac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p} \right]^{1/(p-2)}$$

with $t_{\varepsilon}^{-} u \in \Lambda_{\varepsilon}^{+}$.

Since $1 < \frac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p}$, from (2.13) it follows that

$$I_{\varepsilon}(t_{\varepsilon}^{-} u) \leq I_{\varepsilon}(u)$$

and consequently:

$$c_{\varepsilon} \leq I_{\varepsilon}(t_{\varepsilon} u) \leq I_{\varepsilon}(u) \leq I(u) + \varepsilon \|f\|_{H^{-1}} \|\nabla u\|_2 \leq I(u) + \varepsilon C_3$$

(with $C_3 > 0$ a suitable constant).

Estimate (2.6) with $f = f_{\varepsilon}$ and the above inequality imply:

$$-\frac{1}{16N}[(N+2)\|f\|_{H^{-1}}]^2 \leq -\frac{1}{16N}[(N+2)\|f_{\varepsilon}\|_{H^{-1}}]^2 \leq c_{\varepsilon} \leq c_0 + \varepsilon C_3.$$

Let $\varepsilon_n \rightarrow 0$, $n \rightarrow +\infty$ and $u_0 \in H$ satisfy:

(a) $c_{\varepsilon_n} \rightarrow \bar{c} \leq c_0$, $n \rightarrow +\infty$

(b) $u_{\varepsilon_n} \rightarrow u_0$, $n \rightarrow +\infty$ weakly in H .

From (2.15) it follows $\langle I'(u_0), w \rangle = 0$, $\forall w \in H$ (i.e. u_0 is a critical point for I) and $I(u_0) \leq c_0$.

In particular $u_0 \in \Lambda$ and necessarily $I(u_0) = c_0$, (i.e. $u_{\varepsilon_n} \rightarrow u_0$ strongly in H).

This completes the proof. \square

3. THE PROOF OF THEOREMS 2 AND 4

The functional I involves the limiting Sobolev exponent $p = \frac{2N}{N-2}$. This compromises its compactness properties, and a possible failure of the P.S. condition is to be expected.

Our first task is to locate the levels free from this noncompactness effect.

We refer to [B] and [S] for a survey on related problems where such an approach has been successfully used.

In this direction we have:

PROPOSITION 3.1. — *Every sequence $\{u_n\} \subset H$ satisfying:*

$$(a) \ I(u_n) \rightarrow c \text{ with } c < c_0 + \frac{1}{N} S^{N/2}$$

[c_0 as defined in (1.3)].

$$(b) \ \|I'(u_n)\| \rightarrow 0$$

as a convergent subsequence.

Namely the (P.S) condition holds for all level $c < c_0 + \frac{1}{N} S^{N/2}$.

Proof. — It is not difficult to see that (a) and (b) imply that $\|\nabla u_n\|_2$ is uniformly bounded.

Hence for a subsequence of u_n (which we still call u_n), we can find a $w_0 \in H$ such that

$$u_n \rightarrow w_0 \text{ weakly in } H.$$

Consequently from (b) we obtain:

$$\langle I'(w_0), w \rangle = 0, \quad \forall w \in H. \quad (3.1)$$

That is w_0 is a solution in $H_0^1(\Omega)$ for (1.2). In particular $w_0 \neq 0$, $w_0 \in \Lambda$ and $I(w_0) \geq c_0$.

Write $u_n = w_0 + v_n$ with $v_n \rightarrow 0$ weakly in H .

By a Lemma of Brezis-Lieb [B.L.] we have:

$$\|u_n\|_p^p = \|w_0 + v_n\|_p^p = \|w_0\|_p^p + \|v_n\|_p^p + o(1).$$

Hence, for n large, we conclude:

$$\begin{aligned} c_0 + \frac{1}{N} S^{N/2} > I(w_0 + v_n) &= I(w_0) + \frac{1}{2} \|\nabla v_n\|_2^2 - \frac{1}{p} \|v_n\|_p^p + o(1) \\ &\geq c_0 + \frac{1}{2} \|\nabla v_n\|_2^2 - \frac{1}{p} \|v_n\|_p^p + o(1). \end{aligned}$$

which gives:

$$\frac{1}{2} \|\nabla v_n\|_2^2 - \frac{1}{p} \|v_n\|_p^p < \frac{1}{N} S^{N/2} + o(1). \quad (3.2)$$

Also from (b) follows:

$$\begin{aligned} o(1) &= \langle I'(u_n), u_n \rangle = \|\nabla w_0\|^2 - \|w_0\|_p^p - \int_{\Omega} f w_0 + \|\nabla v_n\|_2^2 - \|v_n\|_p^p + o(1) \\ &= \langle I'(w_0), w_0 \rangle + \|\nabla v_n\|_2^2 - \|v_n\|_p^p + o(1); \end{aligned}$$

and taking into account (3.1) we obtain:

$$\|\nabla v_n\|_2^2 - \|v_n\|_p^p = o(1). \quad (3.3)$$

We claim that conditions (3.2) and (3.3) can hold simultaneously only if $\{v_n\}$ admits a subsequence, $\{v_{n_k}\}$ say, which converges strongly to zero, i. e. $\|v_{n_k}\| \rightarrow 0$, $k \rightarrow +\infty$.

Arguing by contradiction assume that $\|v_n\|$ is bounded away from zero. That is for some constant $c_4 > 0$ we have $\|v_n\| \geq c_4$, $\forall n \in \mathbb{N}$.

From (3.3) then it follows:

$$\|v_n\|_p^{p-2} \geq S + o(1),$$

and consequently

$$\|v_n\|_p^p \geq S^{N/2} + o(1).$$

This yields a contradiction since from (3.2) and (3.3) we have:

$$\frac{1}{N} S^{N/2} \leq \frac{1}{N} \|v_n\|_p^p + o(1) = \frac{1}{2} \|\nabla v_n\|_2^2 - \frac{1}{p} \|v_n\|_p^p + o(1) < \frac{1}{N} S^{N/2}$$

for n large.

In conclusion, $u_{n_k} \rightarrow w_0$ strongly. \square

At this point it would not be difficult to derive Theorem 2, if we had the inequality:

$$\inf_{\Lambda^-} I = c_1 < c_0 + \frac{1}{N} S^{N/2}. \quad (3.4)$$

However it appears difficult to derive (3.4) directly.

We shall obtain it by comparison with a mountain-pass value.

To this end, recall that $u_0 \neq 0$. Following [B.N.1] we set $\Sigma \subset \Omega$ to be a set of positive measure such that $u_0 > 0$ on Σ (replace u_0 with $-u_0$ and f with $-f$ if necessary).

$$\text{Set } U_{\varepsilon, a}(x) = \xi_a(x) u_{\varepsilon, a}(x), \quad x \in \mathbb{R}^N;$$

$[u_{\varepsilon, a}$ and ξ_a defined in (1.6) and (1.7)].

LEMMA 3.1. — *For every $R > 0$ and a.e. $a \in \Sigma$, there exists $\varepsilon_0 = \varepsilon_0(R, a) > 0$ such that:*

$$I(u_0 + RU_{\varepsilon, a}) < c_0 + \frac{1}{N} S^{N/2}$$

for every $0 < \varepsilon < \varepsilon_0$.

Proof. — We have:

$$\begin{aligned} I(u_0 + RU_{\varepsilon, a}) = & \int_{\Omega} \frac{|\nabla u_0|^2}{2} + R \int_{\Omega} \nabla u_0 \nabla U_{\varepsilon, a} + \frac{R^2}{2} \int_{\Omega} |\nabla U_{\varepsilon, a}|^2 \\ & - \frac{1}{p} \int_{\Omega} |u_0 + RU_{\varepsilon, a}|^p - \int_{\Omega} f u_0 - R \int_{\Omega} f U_{\varepsilon, a}. \end{aligned} \quad (3.5)$$

A careful estimate obtained by Brezis-Nirenberg (*see* formulae (17) and (22) in [B.N.1]) shows that:

$$\begin{aligned} \|u_0 + R U_{\varepsilon, a}\|_p^p &= \|u_0\|_p^p + R^p \|U_{\varepsilon, a}\|_p^p + p R \int_{\Omega} |u_0|^{p-2} u_0 U_{\varepsilon, a} \\ &\quad + p R^{p-1} \int_{\Omega} U_{\varepsilon, a}^{p-1} u_0 + o[\varepsilon^{(N-2)/2}] \quad \text{for a.e. } a \in \Sigma. \end{aligned}$$

Also from [B.N.2] we have:

$$\|\nabla U_{\varepsilon, a}\|_2^2 = B + O(\varepsilon^{N-2}) \quad \text{and} \quad \|U_{\varepsilon, a}\|_p^p = A + O(\varepsilon^N)$$

where

$$B = \int_{\mathbb{R}^N} |\nabla u_1(x)|^2 dx, \quad A = \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|^2)^N}$$

and

$$S = \frac{B}{A^{2/p}}. \quad (3.6)$$

Substituting in (3.5) and using the fact that u_0 satisfies (1.2) we obtain:

$$\begin{aligned} I(u_0 + R U_{\varepsilon, a}) &= \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 + R \int_{\Omega} \nabla u_0 \cdot \nabla U_{\varepsilon, a} + \frac{R^2}{2} B - \frac{1}{p} \int_{\Omega} |u_0|^p - \frac{R^p}{p} A \\ &\quad - R \int_{\Omega} |u_0| u_0^{p-2} U_{\varepsilon, a} - R^{p-1} \int_{\Omega} U_{\varepsilon, a}^{p-1} u_0 - \int_{\Omega} f u_0 - R \int_{\Omega} f U_{\varepsilon, a} + o[\varepsilon^{(N-2)/2}] \\ &= I(u_0) + \frac{R^2}{2} B - \frac{R^p}{p} A - R^{p-1} \int_{\Omega} U_{\varepsilon, a}^{p-1} u_0 + o[\varepsilon^{(N-2)/2}] \end{aligned}$$

for a.e. $a \in \Sigma$.

Set $u_0 = 0$ outside Ω , it follows:

$$\begin{aligned} \int_{\Omega} U_{\varepsilon, a}^{p-1} u_0 &= \int_{\mathbb{R}^N} u_0(x) \xi_a(x) \frac{\varepsilon^{(N+2)/2}}{(\varepsilon^2 + |x-a|^2)^{(N+2)/2}} dx \\ &= \varepsilon^{(N-2)/2} \int_{\mathbb{R}^N} u_0(x) \xi_a(x) \frac{1}{\varepsilon^N} \psi_1\left(\frac{x}{\varepsilon}\right) dx, \end{aligned}$$

where $\psi_1(x) = \frac{1}{(1 + |x|^2)^{(N+2)/2}} \in L^1(\mathbb{R}^N)$.

Therefore, setting $D = \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|^2)^{(N+2)/2}}$ we derive:

$$\int_{\mathbb{R}^N} u_0(x) \xi_a(x) \frac{1}{\varepsilon^N} \psi_1\left(\frac{x}{\varepsilon}\right) dx \rightarrow u_0(a) D$$

for a.e. $a \in \Sigma$ (*see* [F]).

In other words,

$$\int_{\Omega} U_{\varepsilon, a}^{p-1}(x) u_0(x) dx = \varepsilon^{(N-2)/2} u_0(a) D + o(\varepsilon^{(N-2)/2}).$$

Consequently:

$$I(u_0 + R U_{\varepsilon, a}) = c_0 + \frac{R^2}{2} B - \frac{R^p}{p} A - R^{p-1} u_0(a) D \varepsilon^{(N-2)/2} + o[\varepsilon^{(N-2)/2}].$$

Define:

$$q(s) = \frac{s^2}{2} B - \frac{s^p}{p} A - s^{p-1} u_0(a) D \varepsilon^{(N-2)/2}, \quad s > 0$$

and assume that $q(s)$ achieves its maximum at $s_{\varepsilon} > 0$.

Set

$$S_0 = \left(\frac{B}{A} \right)^{1/(p-2)}.$$

Since s_{ε} satisfies:

$$s_{\varepsilon} B - s_{\varepsilon}^{p-1} A = (p-1) u_0(a) D \varepsilon^{(N-2)/2} s_{\varepsilon}^{p-2} \quad (3.7)$$

necessarily $0 < s_{\varepsilon} < S_0$ and $s_{\varepsilon} \rightarrow S_0$ as $\varepsilon \rightarrow 0$.

Write $s_{\varepsilon} = S_0(1 - \delta_{\varepsilon})$. We study the rate at which $\delta_{\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

From (3.7) we obtain:

$$\left(\frac{B^{p-1}}{A} \right)^{1/(p-2)} (1 - \delta_{\varepsilon} - (1 - \delta_{\varepsilon})^{p-1}) = (p-1) \frac{B}{A} (1 - \delta_{\varepsilon})^{p-2} \varepsilon^{(N-2)/2} u_0(a) D;$$

and expanding for δ_{ε} we derive:

$$(p-2) \left(\frac{B^{p-1}}{A} \right)^{1/(p-2)} \delta_{\varepsilon} = (p-1) \frac{B}{A} u_0(a) D \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}).$$

This implies:

$$\begin{aligned} I(u_0 + R U_{\varepsilon, a}) &\leq c_0 + \frac{s_{\varepsilon}^2}{2} B - \frac{s_{\varepsilon}^p}{p} B - s_{\varepsilon}^{p-1} u_0(a) D \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}) \\ &= c_0 + \frac{S_0^2}{2} B - \frac{S_0^p}{2} A - S_0^2 B \delta_{\varepsilon} + S_0^p A \delta_{\varepsilon} - S_0^{p-1} u_0(a) D \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}) \\ &= c_0 + \frac{1}{N} S^{N/2} - S_0^{p-1} u_0(a) D \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}). \end{aligned}$$

Therefore for $\varepsilon_0 = \varepsilon_0(\mathbf{R}, a) > 0$ sufficiently small we conclude

$$I(u_0 + \mathbf{R}U_{\varepsilon, a}) < c_0 + \frac{1}{N} S^{N/2} \quad (3.8)$$

$\forall 0 < \varepsilon < \varepsilon_0$. \square

Our aim is to state a mountain pass principle that produces a value which is below the threshold $c_0 + \frac{1}{N} S^{N/2}$ but also compares with the value

$$c_1 = \inf_{\Lambda^-} I.$$

To this end observe that under assumption $(*)$, the manifold Λ^- disconnects H in exactly two connected components U_1 and U_2 .

To see this, notice that for every $u \in H$, $\|u\| = \|\nabla u\|_2 = 1$ by Lemma 2.1 we can find a unique $t^+(u) > 0$ such that

$$t^+(u)u \in \Lambda^- \quad \text{and} \quad I(t^+(u)u) = \max_{t \geq t_{\max}} I(tu).$$

The uniqueness of $t^+(u)$ and its extremal property give that $t^+(u)$ is a continuous function of u .

Set

$$U_1 = \left\{ u = 0 \text{ or } u: \|u\| < t^+\left(\frac{u}{\|u\|}\right) \right\}$$

and

$$U_2 = \left\{ u: \|u\| > t^+\left(\frac{u}{\|u\|}\right) \right\}.$$

Clearly $H - \Lambda^- = U_1 \cup U_2$ and $\Lambda^+ \subset U_1$.

In particular $u_0 \in U_1$.

The Proof of Theorem 4

Easy computations show that, for suitable constant $C_5 > 0$ we have:

$$0 < t^+(u) < C_5, \quad \forall u: \|u\| = 1.$$

Set $R_0 = \left(\frac{1}{B} |C_5^2 - \|u_0\|^2| \right)^{1/2} + 1$ and fix $a \in \Sigma$ such that Lemma 3.2 applies, and the estimate (3.8) holds for all $0 < \varepsilon < \varepsilon_0$.

We claim that

$$w_\varepsilon := u_0 + \mathbf{R}_0 \xi_a u_{\varepsilon, a} \in U_2 \quad (3.9)$$

for $\varepsilon > 0$ small.

Indeed

$$\begin{aligned}\|\nabla w_\varepsilon\|_2^2 &= \|\nabla(u_0 + R_0 \xi_a U_{\varepsilon, a})\|_2^2 \\ &= \|u_0\|_2^2 + R_0^2 B + o(1) > C_3^2 \geq \left[t^+ \left(\frac{w_\varepsilon}{\|w_\varepsilon\|} \right) \right]^2,\end{aligned}$$

for $\varepsilon > 0$ small enough.

For such a choice of R_0 and $a \in \Sigma$, fix $\varepsilon > 0$ such that both (3.8) and (3.9) hold.

Set

$$\mathcal{F} = \left\{ h: [0, 1] \rightarrow H \text{ continuous, } \begin{aligned} h(0) &= u_0 \\ h(1) &= R_0 \xi_a u_{\varepsilon, a} \end{aligned} \right\}$$

Clearly $h: [0, 1] \rightarrow H$ given by $h(t) = u_0 + t R_0 \xi_a u_{\varepsilon, a}$ belongs to \mathcal{F} . So by Lemma 2.3 we conclude:

$$c = \inf_{h \in \mathcal{F}} \max_{t \in [0, 1]} I(h(t)) < c_0 + \frac{1}{N} S^{N/2} \quad (3.10)$$

Also, since the range of any $h \in \mathcal{F}$ intersect Λ^- , we have

$$c \geq c_1 = \inf_{\Lambda^-} I. \quad (3.11)$$

At this point the conclusion of Theorem 4 follows by Lemma 3.1 and a straightforward application of the mountain-pass lemma (cf. [A.R.]). \square

The Proof of Theorem 2

Analogously to the proof of Theorem 1, one can show that the Ekeland's variational principle gives a sequence $\{u_n\} \subset \Lambda^-$ satisfying:

$$\begin{aligned}I'(u_n) &\rightarrow c_1 \\ \|I'(u_n)\| &\rightarrow 0\end{aligned}$$

But from (3.10) and (3.11), we have:

$$c_1 < c_0 + \frac{1}{N} S^{N/2}.$$

Thus, by Lemma 3.1, we obtain a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u_1 \in H$ such that:

$$u_{n_k} \rightarrow u_1 \text{ strongly in } H.$$

Consequently u_1 is a critical point for I , $u_1 \in \Lambda^-$ (since Λ^- is closed) and $I(u_1) = c_1$.

Finally to see that $f \geq 0$ yields $u_1 \geq 0$, let $t^+ > 0$ satisfy

$$t^+ |u_1| \in \Lambda^-.$$

From Lemma 2.1 we conclude:

$$I(u_1) = \max_{t \geq t_{\max}} I(tu_1) \geq I(t^+ u_1) \geq I(t^+ |u_1|).$$

So we can always take $u_1 \geq 0$. \square

4. APPENDIX

The Proof of Lemma 2.2

Let $\{u_n\}$ be a minimizing sequence for (2.1) such that for $u_0 \in H$ we have $u_n \rightarrow u_0$ weakly in H and $u_n \rightarrow u_0$ pointwise a.e. in Ω .

In general $\|u_0\|_p \leq 1$. We are done once we show $\|u_0\|_p = 1$.

To obtain this, we shall argue by contradiction and assume

$$\|u_0\|_p < 1.$$

Hence write $u_n = u_0 + w_n$ where $w_n \rightarrow 0$ weakly in H .

We have

$$\begin{aligned} \mu_0 + o(1) &= c_n \|\nabla u_n\|^{(N+2)/2} - \int_{\Omega} f u_n = c_n (\|\nabla u_0\|_2^2 + \|\nabla w_n\|_2^2)^{(N+2)/4} \\ &\quad - \int_{\Omega} f u_0 + o(1) \quad (4.1) \end{aligned}$$

On the other hand,

$$1 = \|u_0 + w_n\|_p^p = \|u_0\|_p^p + \|w_n\|_p^p + o(1)$$

(see [B.L.]), which gives:

$$\|w_n\|_p^2 = (1 - \|u_0\|_p^{2/p})^{2/p} + o(1).$$

So from (4.1) we conclude:

$$\begin{aligned} \mu_0 + o(1) &= c_n (\|\nabla u_0\|_2^2 + \|\nabla w_n\|_2^2)^{(N+2)/4} - \int_{\Omega} f u_0 \\ &\geq c_n [\|\nabla u_0\|_2^2 + S(1 - \|u_0\|_p^{2/p} + o(1))]^{(N+2)/4} - \int_{\Omega} f u_0, \end{aligned}$$

That is,

$$c_n [\|\nabla u_0\|_2^2 + S(1 - \|u_0\|_p^{2/p} + o(1))]^{(N+2)/4} - \int_{\Omega} f u_0 \leq \mu_0. \quad (4.2)$$

Following [B.N.1] for every $u \in H$, $\|u\|_p < 1$ and $a \in \Omega$ let $c_\varepsilon = c_\varepsilon(a) > 0$ satisfy the following:

$$\|u + c_\varepsilon \mathbf{U}_{\varepsilon, a}\|_p = 1$$

[recall $U_{\varepsilon, a}(x) = \xi_a(x) u_{\varepsilon, a}(x)$ with ξ_a and $u_{\varepsilon, a}$ given in (1.6) and (1.7)].

We have:

$$\begin{aligned} \|\nabla(u + c_\varepsilon U_{\varepsilon, a})\|_2^2 &= \|\nabla u\|_2^2 + c_\varepsilon^2 \|\nabla U_{\varepsilon, a}\|_2^2 + o(1) \\ &= \|\nabla u\|_2^2 + c_\varepsilon^2 B + o(1) \end{aligned} \quad (4.3)$$

and

$$1 = \|u + c_\varepsilon U_{\varepsilon, a}\|_p^p = \|u\|_p^p + c_\varepsilon^p \|U_{\varepsilon, a}\|_p^p + o(1) = \|u\|_p^p + c_\varepsilon^p A + o(1)$$

[A, B as given in (3.6)].

Thus

$$c_\varepsilon^2 = \frac{1}{A^{2/p}} (1 - \|u\|_p^p)^{2/p} + o(1). \quad (4.4)$$

Substituting in (4.3) we obtain:

$$\begin{aligned} \|\nabla(u + c_\varepsilon U_{\varepsilon, a})\|_2^2 &= \|\nabla u\|_2^2 + \frac{B}{A^{2/p}} (1 - \|u\|_p^p)^{2/p} + o(1) \\ &= \|\nabla u\|_2^2 + S (1 - \|u\|_p^p)^{2/p} + o(1). \end{aligned}$$

This yields:

$$\begin{aligned} \mu_0 &\leq c_N \|\nabla(u + c_\varepsilon U_{\varepsilon, a})\|_2^{(N+2)/2} - \int_{\Omega} f(u + c_\varepsilon U_{\varepsilon, a}) \\ &= c_N (\|\nabla u\|_2^2 + S (1 - \|u\|_p^p)^{2/p})^{(N+2)/4} - \int_{\Omega} fu + o(1), \end{aligned}$$

and passing to the limit as $\varepsilon \rightarrow 0$, we derive:

$$\mu_0 \leq c_N [\|\nabla u\|_2^2 + S (1 - \|u\|_p^p)^{2/p}]^{(N+2)/4} - \int_{\Omega} fu, \quad \forall u \in H, \quad \|u\|_p < 1.$$

Therefore from (4.2) we conclude:

$$c_N [\|\nabla u_0\|_2^2 - S (1 - \|u_0\|_p^p)^{2/p}]^{(N+2)/4} - \int_{\Omega} fu = \mu_0 \quad (4.5)$$

and that for every $w \in H$ necessarily:

$$\frac{d}{dt} \left[c_N [\|\nabla(u_0 + tw)\|_2^2 + S (1 - \|u_0 + tw\|_p^p)^{2/p}]^{(N+2)/4} - \int_{\Omega} f(u_0 + tw) \right]_{t=0} = 0.$$

That is:

$$\begin{aligned} \frac{N+2}{2} c_N \left[\|\nabla u_0\|_2^2 + S (1 - \|u_0\|_p^p)^{2/p} \right]^{(N-2)/4} \\ \times \left[\int_{\Omega} \nabla u_0 \cdot \nabla w - S (1 - \|u_0\|_p^p)^{(2-p)/p} \int_{\Omega} |u_0| u_0^{p-2} w \right] \\ - \int_{\Omega} fw = 0, \quad \forall w \in H. \end{aligned}$$

So setting $\sigma_0 = \frac{N+2}{2} c_N \left[\|\nabla u_0\|_2^2 + S(1 - \|u_0\|_p^p)^{2/p} \right]^{(N-2)/4} > 0$

and

$$\lambda_0 = \frac{S}{(1 - \|u_0\|_p^p)^{(p-2)/p}}$$

we obtain that u_0 weakly satisfies:

$$-\Delta u_0 = \lambda_0 |u_0|^{p-2} u_0 + \frac{1}{\sigma_0} f. \quad (4.5)$$

Since $f \neq 0$, in particular, we have that $u_0 \neq 0$.

Hence for a set of positive measure $\Sigma \subset \Omega$ we have:

$$u_0(a) > 0, \quad \forall a \in \Sigma,$$

(replace u_0 with $-u_0$ and f with $-f$ if necessarily).

Let $a \in \Sigma$ and $c_\varepsilon = c_\varepsilon(a)$ satisfy:

$$\|u_0 + c_\varepsilon U_{\varepsilon, a}\|_p = 1.$$

We will reach a contradiction by showing that

$$I(u_0 + c_\varepsilon U_{\varepsilon, a}) < \mu_0$$

for a suitable choice of $a \in \Sigma$ and $\varepsilon > 0$ small enough.

To this end, let $c_\varepsilon^p = \frac{1 - \|u_0\|_p^p}{A}$. From (4.4) it follows that $c_\varepsilon \nearrow c_0$ as $\varepsilon \rightarrow 0$. Set $c_\varepsilon = c_0(1 - \delta_\varepsilon)$, $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. In [B.N.1], Brezis-Nirenberg have obtained a precise rate at which $\delta_\varepsilon \rightarrow 0$, by showing that, for a.e. $a \in \Sigma$, one has:

$$\delta_\varepsilon A c_0^p = \varepsilon^{(N-2)/2} \left[c_0 \int_{\Omega} |u_0(x)| |u_0^{p-2}(x)| \xi_a(x) \frac{dx}{|x-a|^{N-2}} + c_0^{p-1} u_0(a) D \right] + o(\varepsilon^{(N-2)/2}) \quad (4.7)$$

with

$$D = \int_{\mathbb{R}^N} \frac{dx}{(\varepsilon^2 + |x|^2)^{(N+2)/2}}. \quad (\text{See formula (2.9) in [B.N.1].})$$

Now fix $a \in \Sigma$ for which (4.7) holds and

$$\int_{\Omega} \frac{|u_0|^{p-2} u_0 \xi_a}{(\varepsilon^2 + |x-a|^2)^{(N-2)/2}} \rightarrow \int_{\Omega} \frac{|u_0|^{p-2} u_0 \xi_a}{|x-a|^{N-2}} \quad \text{as } \varepsilon \rightarrow 0. \quad (4.8)$$

Using (4.5), (4.7) and the definition of c_0 we obtain:

$$\begin{aligned}
 I(u_0 + c_0 U_{\varepsilon, a}) &= c_N \left[\|\nabla u_0\|_2^2 + 2c_\varepsilon \int_{\Omega} \nabla u_0 \cdot \nabla U_{\varepsilon, a} + c_\varepsilon^2 \|\nabla U_{\varepsilon, a}\|_2^2 \right]^{(N+2)/4} \\
 &\quad - \int_{\Omega} f u_0 - c_\varepsilon \int_{\Omega} f U_{\varepsilon, a} \\
 &= c_N \left[\|\nabla u_0\|_2^2 + 2c_0 \int_{\Omega} \nabla u_0 \cdot \nabla U_{\varepsilon, a} + c_0^2 (1 - 2\delta_\varepsilon) B + o[\varepsilon^{(N-2)/2}] \right]^{(N+2)/4} \\
 &\quad - \int_{\Omega} f u_0 - c_\varepsilon \int_{\Omega} f U_{\varepsilon, a} = c_N \left[\|\nabla u_0\|_2^2 + c_0^2 B \right]^{(N+2)/4} - \int_{\Omega} f u_0 \\
 &\quad + \frac{N+2}{4} c_N \left[\|\nabla u_0\|_2^2 + c_0^2 B \right]^{(N-2)/4} \left[2c_0 \int_{\Omega} \nabla u_0 \cdot \nabla U_{\varepsilon, a} \right. \\
 &\quad \left. - 2c_0^2 \delta_\varepsilon B \right] - c_0 \int_{\Omega} f U_{\varepsilon, a} \\
 &\quad + o[\varepsilon^{(N-2)/2}] = \mu_0 + c_0 \left[\sigma_0 \int_{\Omega} \nabla u_0 \cdot \nabla U_{\varepsilon, a} \right. \\
 &\quad \left. - \int_{\Omega} f U_{\varepsilon, a} \right] - \sigma_0 c_0^2 B \delta_\varepsilon + o[\varepsilon^{(N-2)/2}].
 \end{aligned}$$

Thus from equation (4.6) we derive:

$$I(u_0 + c_\varepsilon U_{\varepsilon, a}) = \mu_0 + \sigma_0 \lambda_0 c_0 \int_{\Omega} |u_0|^{p-2} u_0 U_{\varepsilon, a} - \delta_0 c_0^2 B \delta_\varepsilon + o[\varepsilon^{(N-2)/2}].$$

On the other hand from (4.8) we have:

$$\int_{\Omega} |u_0|^{p-2} u_0 U_{\varepsilon, a} = \varepsilon^{(N-2)/2} \int_{\Omega} \frac{|u_0(x)|^{p-2} u_0(x)}{|x-a|^{N-2}} \xi_a(x) dx + o[\varepsilon^{(N-2)/2}].$$

Therefore:

$$\begin{aligned}
 I(u_0 + c_\varepsilon U_{\varepsilon, a}) &= \mu_0 + \sigma_0 \left[\varepsilon^{(N-2)/2} \lambda_0 \int_{\Omega} \frac{|u_0(x)|^{p-2} u_0(x)}{|x-a|^{N-2}} \xi_a - c_0^2 B \delta_\varepsilon \right] + o[\varepsilon^{(N-2)/2}] \\
 &= \mu_0 + \sigma_0 \left[\frac{S \varepsilon^{(N-2)/2}}{(1 - \|u_0\|_p^p)^{(p-2)/2}} c_0 \int_{\Omega} \frac{|u_0|^{p-2} u_0}{|x-a|^{N-2}} \xi_a - B c_0^2 \delta_\varepsilon \right] + o(\varepsilon^{(N-2)/2}) \\
 &= \mu_0 + \sigma_0 \left[\frac{S}{A^{(p-2)/p} c_0^{p-2}} \varepsilon^{(N-2)/2} c_0 \right. \\
 &\quad \left. \times \int_{\Omega} \frac{|u_0|^{p-2} u_0}{|x-a|^{N-2}} \xi_a - B c_0^2 A \delta_\varepsilon \right] + o[\varepsilon^{(N-2)/2}] \\
 &= \mu_0 + \sigma_0 \frac{B}{A c_0^{p-2}} \left[\varepsilon^{(N-2)/2} c_0 \int_{\Omega} \frac{|u_0|^{p-2} u_0}{|x-a|^{N-2}} \xi_a - c_0^p A \delta_\varepsilon \right] + o[\varepsilon^{(N-2)/2}].
 \end{aligned}$$

Finally, from (4.7) we conclude:

$$I(u_0 + c_\varepsilon U_{\varepsilon, a}) = \mu_0 - \sigma_0 \frac{B}{A} c_0 u_0(a) D \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}) < \mu_0$$

for $\varepsilon > 0$ sufficiently small.

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