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The nonrelativistic limit of the nonlinear Dirac equation

by

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ABSTRACT. — It is proved that there exist solutions of the nonlinear Dirac equation, smooth in time, on a time interval which is independent of c . Moreover after multiplication by a phase factor (dependent on c) these solutions converge to the solution of a coupled system of nonlinear Schrödinger type equations.

Key words : Nonlinear Dirac equations, nonrelativistic limit, Schrödinger equation.

RÉSUMÉ. — *La limite non relativistique de l'équation de Dirac non linéaire.*
— On montre qu'existent des solutions de l'équation de Dirac non linéaire, régulières en temps, sur un intervalle en temps indépendant de c . En plus, après la multiplication avec un facteur de phase (c -dépendant) ces solutions convergent vers la solution d'un système couplé d'équations du type Schrödinger non linéaire.

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INTRODUCTION

We consider the initial value problem for the nonlinear Dirac equation

$$\left. \begin{aligned} i \frac{\partial \Psi}{\partial t} &= -ic \alpha \nabla \Psi + c^2 \beta \Psi + 2\lambda (\beta \Psi | \Psi) \beta \Psi \\ \Psi(0) &= \Psi_{0\varepsilon}, \end{aligned} \right\} \quad (1)$$

where λ and c are positive constants, Ψ is a function from \mathbb{R}^3 into \mathbb{C}^4 , $\alpha \nabla$ stands for $\sum_{j=1}^3 \alpha_j \partial_j$ and α_j and β are 4×4 matrices satisfying $\alpha_j \alpha_k + \alpha_k \alpha_j = 2 \delta_{jk} I$, $\alpha_j \beta + \beta \alpha_j = 0$, β is diagonal and $\beta^2 = I$.

This equation has first been investigated by the physicists; see the references in [1], [2], [3]. L. Vazquez, T. Cazenave *et al.* ([3], [2], [1]) recently started the investigation of the existence of localized (or stationary) solutions of (1).

We are interested in the solutions of the equation with arbitrary (*i. e.* not necessarily radially symmetric as in the above references) initial conditions and their convergence as the speed of light c increases to ∞ .

Introducing $\varepsilon = \frac{1}{2c^2}$, we write (1) in the equivalent form

$$\left. \begin{aligned} i \frac{\partial \Psi}{\partial t} &= -i \sqrt{\frac{1}{2\varepsilon}} \alpha \nabla \Psi + \frac{1}{2\varepsilon} \beta \Psi + 2\lambda (\beta \Psi | \Psi) \beta \Psi \\ \Psi(0) &= \Psi_{0\varepsilon}. \end{aligned} \right\} \quad (2)$$

We first prove the existence of a classical solution of (2) on an interval independent of ε .

In the following theorem as well as in the rest of this paper H^s stands for the Sobolev space $H^s(\mathbb{R}^3)^4$ and L^2 stands for $L^2(\mathbb{R}^3)^4$.

THEOREM 1. — *Let $\varepsilon_0 > 0$. Assume that $\Psi_{0\varepsilon} \in H^2$ ($0 < \varepsilon \leq \varepsilon_0$) and*

$$\sup_{\varepsilon \leq \varepsilon_0} \|\Psi_{0\varepsilon}\|_{H^2} < \infty.$$

Then there exists an interval $J = [-T, T]$ such that for every ε with $0 < \varepsilon \leq \varepsilon_0$ there exists a unique solution

$$\Psi_\varepsilon \in C^2(J, L^2) \cap C^1(J, H^1) \cap C(J, H^2)$$

of the initial value problem (2).

Next we investigate the nonrelativistic limit $\varepsilon \rightarrow 0$ [*i. e.* $c \rightarrow \infty$ in (1)]. Since Ψ_ε cannot be expected to converge, motivated by the linear theory (see [5], [6]), we introduce the new function $\Phi = 2e^{i\beta t/\varepsilon} \beta \Psi$. Inserting this

into (2) we obtain an equivalent initial value problem

$$\left. \begin{aligned} \frac{\partial \Phi}{\partial t} &= \sqrt{\frac{1}{2\varepsilon}} e^{i\beta t/\varepsilon} \alpha \nabla \Phi - \frac{i\lambda}{2} (\beta \Phi | \Phi) \beta \Phi \\ \Phi(0) &= \Phi_{0\varepsilon} \end{aligned} \right\} \quad (3)$$

with $\Phi_{0\varepsilon} = 2\beta\Psi_{0\varepsilon}$.

Theorem 1 evidently holds verbatim for Φ_ε . Differentiating (3) with respect to t we conclude that the smooth solutions of (3) (their existence is shown in Theorem 1) is also the solution of the initial value problem

$$\left. \begin{aligned} \frac{\partial^2 \Phi}{\partial t^2} - \frac{1}{2\varepsilon} \left[\Delta \Phi + 2i\beta \frac{\partial \Phi}{\partial t} - \lambda (\beta \Phi | \Phi) \Phi \right] &= F_\varepsilon(\Phi) \\ \Phi(0) &= \Phi_{0\varepsilon}, \quad \frac{\partial \Phi}{\partial t}(0) = \Phi_{1\varepsilon} \end{aligned} \right\} \quad (4)$$

where $\Phi_{1\varepsilon} = \sqrt{\frac{1}{2\varepsilon}} \alpha \nabla \Phi_{0\varepsilon} - \frac{i\lambda}{2} (\beta \Phi_{0\varepsilon} | \Phi_{0\varepsilon}) \beta \Phi_{0\varepsilon}$ and

$$\begin{aligned} F_\varepsilon(t, \Phi) &= -\frac{\lambda^2}{2} (\beta \Phi | \Phi)^2 \Phi \\ &\quad - \frac{i\lambda}{2} \varepsilon^{-1/2} \sum_j \{ e^{i\beta t/\varepsilon} [\partial_j (\beta \Phi | \Phi)] \alpha_j \beta \Phi + 2 \operatorname{Re} (\beta \Phi | e^{i\beta t/\varepsilon} \alpha_j \partial_j \Phi) \beta \Phi \}. \end{aligned}$$

Conversely, as the initial value problem (3) is equivalent to the initial value problem (2) which has a skew-adjoint linear part, standard arguments show that a solution of (4) which is C^2 in time is also a solution of the first order system (3).

This suggests that the solutions Φ_ε of (3) converge to the solution Φ_0 of the nonlinear Schrödinger type equation

$$\left. \begin{aligned} \frac{\partial \Phi}{\partial t} &= \frac{1}{2} i \beta \Delta \Phi - \frac{i\lambda}{2} (\beta \Phi | \Phi) \beta \Phi \\ \Phi(0) &= \Phi_{00}. \end{aligned} \right\} \quad (5)$$

We shall prove an existence result for Φ_0 and the convergence of Φ_ε to Φ_0 :

THEOREM 2. — Assume that $\Phi_{00} \in H^2$. Then there exists an interval $J = [-T, T]$ such that the initial value problem (5) has a unique solution

$$\Phi_0 \in C^1(J, L^2) \cap C(J, H^2).$$

THEOREM 3. — Let $\varepsilon_0 > 0$. Assume that $\Phi_{0\varepsilon} \in H^2$ ($0 \leq \varepsilon \leq \varepsilon_0$) and moreover

$$\sup_{\varepsilon \leq \varepsilon_0} \|\Phi_{0\varepsilon}\|_{H^2} < \infty. \quad (6)$$

Further assume that there exists some $\alpha \in [0, 1]$ such that

$$\lim_{\varepsilon \rightarrow 0} \Phi_{0\varepsilon} = \Phi_{00} \quad \text{in } H^\alpha. \quad (7)$$

Let $J = [-T, T]$ be such that there exists a unique solution $\Phi_\varepsilon \in C^2(J, L^2) \cap C^1(J, H^1) \cap C(J, H^2)$ of the initial value problem (3) for $\varepsilon > 0$ and a unique solution $\Phi_0 \in C^1(J, L^2) \cap C(J, H^2)$ of the initial value problem (5). Then

$$\lim_{\varepsilon \rightarrow 0} \Phi_\varepsilon = \Phi_0 \quad \text{in } C(J, H^\alpha). \quad (8)$$

Proofs

LEMMA 4. — There exists $K > 0$ such that for all $u \in H^2$ following inequalities hold:

$$\|u\|_{L^\infty} \leq K \|u\|_{L^2}^{1/4} \|\Delta u\|_{L^2}^{3/4}, \quad (9)$$

$$\|\nabla u\|_{L^4} \leq K \|u\|_{L^2}^{1/8} \|\Delta u\|_{L^2}^{7/8}. \quad (10)$$

Proof. — Let \hat{u} be the Fourier transform of u . The Plancherel Theorem implies that

$$\begin{aligned} \|u\|_{L^\infty}^2 &\leq \|\hat{u}\|_{L^1}^2 \leq \left(\int_{|x| \leq R} |\hat{u}|^2 \right) \left(\int_{|x| < R} dx \right) \\ &\quad + \left(\int_{|x| \geq R} |x^2 \hat{u}|^2 \right) \left(\int_{|x| \geq R} \frac{dx}{|x|^4} \right) \leq C \left(R^3 \|u\|_{L^2}^2 + \frac{1}{R} \|\Delta u\|_{L^2}^2 \right), \\ \|\nabla u\|_{L^4}^4 &\leq \|x\hat{u}\|_{L^{4/3}}^4 \leq \left(\int |\hat{u}|^2 \right)^2 \left(\int_{|x| < R} |x|^4 dx \right) \\ &\quad + \left(\int_{|x| \geq R} |x|^4 |\hat{u}|^2 \right)^2 \left(\int_{|x| \geq R} \frac{dx}{|x|^4} \right) \leq C \left(R^7 \|u\|_{L^2}^4 + \frac{1}{R} \|\Delta u\|_{L^2}^4 \right). \end{aligned}$$

Setting $R = \left(\frac{\|\Delta u\|_{L^2}}{\|u\|_{L^2}} \right)^{1/2}$ we obtain (9) and (10).

LEMMA 5. — Assume $\varepsilon_0 > 0$ and

$$\sup_{0 < \varepsilon \leq \varepsilon_0} \|\Psi_{0\varepsilon}\|_{H^2} < \infty.$$

Then there exists an interval $J = [-T, T]$ such that if for each $\varepsilon \in (0, \varepsilon_0)$ the initial value problem (2) has a solution $\Psi_\varepsilon \in C^1(J, L^2) \cap C(J, H^2)$ then

$$\sup_{0 < \varepsilon \leq \varepsilon_0} \|\Psi_\varepsilon\|_{C(J, H^2)} < \infty, \quad (11)$$

$$\sup \{ \|\Psi_\varepsilon(t)\|_{L^\infty} : 0 < \varepsilon \leq \varepsilon_0, t \in J \} < \infty. \quad (12)$$

Proof. — It is sufficient to prove (11) since (12) follows from (11) and the embedding theorem. In fact it is sufficient to prove

$$\sup_{0 < \varepsilon \leq \varepsilon_0} \|\Psi_\varepsilon\|_{C([0, T], H^2)} < \infty, \tag{13}$$

since $t \in [-T, 0]$ is treated similarly.

In the proof of (13) we need the conservation of L^2 norm

$$\|\Psi_\varepsilon(t)\|_{L^2} = \|\Psi_{0\varepsilon}\|_{L^2} \quad (t \in J) \tag{14}$$

which follows easily from (2) by scalar multiplication in L^2 by Ψ_ε .

Now we turn to the proof of (13). Apply the operator Δ to both sides of (2). It follows that $\omega = \Delta\Psi$ is the solution of the initial value problem.

$$\left. \begin{aligned} i \frac{\partial \omega}{\partial t} &= -\sqrt{\frac{1}{2\varepsilon}} \alpha \nabla \omega + \frac{1}{2\varepsilon} \beta \omega + 2\lambda (\beta \Psi | \Psi) \beta \omega \\ &\quad + 2\lambda [\Delta (\beta \Psi | \Psi)] \beta \Psi + 2\lambda \sum_j [\partial_j (\beta \Psi | \Psi)] \beta \partial_j \Psi \\ \omega(0) &= \omega_{0\varepsilon} := \Delta \Psi_{0\varepsilon}. \end{aligned} \right\} \tag{15}$$

Multiplying by ω and taking the imaginary part we find

$$\begin{aligned} \frac{d}{dt} \|\omega_\varepsilon(t)\|_{L^2}^2 &\leq K \int (|\Psi_\varepsilon|^2 |\omega_\varepsilon|^2 + |\Psi_\varepsilon| |\omega_\varepsilon| |\nabla \Psi_\varepsilon|^2) \\ &\leq K (\|\Psi_\varepsilon\|_{L^\infty}^2 \|\omega_\varepsilon\|_{L^2}^2 + \|\Psi_\varepsilon\|_{L^\infty} \|\nabla \Psi_\varepsilon\|_{L^4}^2 \|\omega_\varepsilon\|_{L^2}). \end{aligned}$$

From (9), (10) and (14) it follows that

$$\frac{d}{dt} \|\omega_\varepsilon(t)\|_{L^2} \leq K \|\omega_\varepsilon(t)\|_{L^2}^{5/2}.$$

From this it follows that if $KT \|\Psi_{0\varepsilon}\|_{H^2}^{3/2} < 1$ then

$$\sup_{0 \leq t \leq T} \|\Delta \Psi_\varepsilon(t)\| \leq \left(\frac{\|\Psi_{0\varepsilon}\|_{H^2}^{3/2}}{1 - KT \|\Psi_{0\varepsilon}\|_{H^2}^{3/2}} \right)^{2/3}.$$

Together with (14) this implies (13).

Let γ be a matrix and $j, k \in \{1, 2, 3\}$. Define the functions

$$\begin{aligned} f_1(u) &:= (\gamma u | u) u, & f_2(u) &:= (\gamma u | u)^2 u, \\ f_3 &:= \partial_j f_1, & f_4 &:= \partial_{jk}^2 f_1, & f_5 &:= \partial_j f_2, & f_6 &:= f_{jk}^2 f_2. \end{aligned}$$

LEMMA 6. — *The functions $f_i, i = 1, \dots, 6$, are locally Lipschitz continuous functions from H^2 to L^2 : for every $K > 0$ there exists $C_K > 0$ such that $\|u\|_{H^2} \leq K, \|u\|_{H^2} \leq K$ imply*

$$\|f_j(u) - f_j(v)\|_{L^2} \leq C_K \|u - v\|_{H^2} \tag{16}$$

Moreover for $\alpha = 0$ and 1

$$\|f_1(u) - f_1(v)\|_{H^\alpha} \leq C_K \|u - v\|_{H^\alpha}. \tag{17}$$

We omit the easy proof (which repeatedly uses Lemma 4 and the Embedding Theorem).

Define the mappings N_ε ($\varepsilon \geq 0$) by

$$\begin{aligned} N_0(\Phi) &:= -\frac{\lambda}{2}(\beta\Phi|\Phi)\Phi \\ N_\varepsilon(\Phi) &:= \varepsilon F_\varepsilon(\Phi) + N_0(\Phi) \quad (\varepsilon > 0). \end{aligned}$$

LEMMA 7. — (a) For $\varepsilon > 0$ the mapping N_ε is a locally Lipschitz continuous map from H^2 to H^1 : for every $K > 0$ there exists $c_K > 0$ such that $\|\Phi\|_{H^2} \leq K$, $\|\Psi\|_{H^2} \leq K$ imply

$$\|N_\varepsilon(\Phi) - N_\varepsilon(\Psi)\|_{H^1} \leq C_K \|\Phi - \Psi\|_{H^2}.$$

(b) The mapping N_0 is a locally Lipschitz continuous map on H^2 .

(c) There exists $C > 0$ such that for all $\Phi, \Psi \in H^2$ and $\alpha \in [0, 1]$ the estimate

$$\|N_0(\Phi) - N_0(\Psi)\|_{H^\alpha} \leq C(\|\Phi\|_{H^2} + \|\Psi\|_{H^2})^2 \|\Phi - \Psi\|_{H^\alpha}. \quad (18)$$

Proof. — (a) and (b) are straightforward consequences of Lemma 6. To prove (c), fix Φ, Ψ and define the linear map $R_{\Phi, \Psi}$ by $R_{\Phi, \Phi}(\theta) = (\beta\Phi|\Phi)\theta + (\beta\Phi|\theta)\Psi + (\theta|\beta\Psi)\Psi$. Then

$$N_0(\Phi) - N_0(\Psi) = R_{\Phi, \Psi}(\Phi - \Psi).$$

It is easy to show that $\|R_{\Phi, \Psi}(\theta)\|_{H^\alpha} \leq C(\|\Phi\|_{H^2} + \|\Psi\|_{H^2})^2 \|\theta\|_{H^\alpha}$ for $\alpha = 0$ and $\alpha = 1$ (in fact such estimate appears in the proof of Lemma 6). Since $R_{\Phi, \Psi}$ is linear, the same estimate holds for all $\alpha \in [0, 1]$.

Proof of Theorem 1. — As mentioned in the Introduction, it is sufficient to prove the statement of Theorem 1 for the solution of the initial value problem (4), since it follows by standard methods that this solution is also solution of (2).

The substitution $U = e^{-i\beta t/2\varepsilon} \Phi$ transforms the initial value problem (4) into

$$\left. \begin{aligned} \varepsilon \frac{\partial^2 U}{\partial t^2} - \frac{1}{2} \Delta U + \frac{1}{4\varepsilon} U &= N_\varepsilon(U) \\ U(0) = U_{0\varepsilon} &:= \Phi_{0\varepsilon}, \quad \frac{\partial U}{\partial t}(0) = U_{1\varepsilon} := \Phi_{1\varepsilon} - \frac{i\beta}{2\varepsilon} \Phi_{0\varepsilon}. \end{aligned} \right\} \quad (19)$$

Converting this equation into an integral equation (cf. [4]) we find that any smooth solution of (4) is also a solution of

$$\Phi_\varepsilon(t) = I_\varepsilon(t) \Phi_{0\varepsilon} + J_\varepsilon(t) \Phi_{1\varepsilon} + \frac{1}{\varepsilon} \int_0^t J_\varepsilon(t-s) N_\varepsilon(\Phi_\varepsilon(s)) ds \quad (20)$$

with

$$I_\varepsilon(t) = e^{i\beta t/2\varepsilon} \left(\cos t A_\varepsilon - \frac{i\beta}{2\varepsilon} A_\varepsilon^{-1} \sin t A_\varepsilon \right), \tag{21}$$

$$J_\varepsilon(t) = e^{i\beta t/2\varepsilon} A_\varepsilon^{-1} \sin t A_\varepsilon \tag{22}$$

and A_ε the positive square root of $\frac{1}{\varepsilon^2} \left(\varepsilon A + \frac{1}{4} \right)$, $A = -\frac{1}{2} \Delta$, naturally defined in L^2 .

We shall use the following well known result: let \mathcal{A} be a m -dissipative operator in a Hilbert space X and let $D(\mathcal{A})$ be its domain endowed with the graph norm. Let F be a locally Lipschitz continuous mapping on $D(\mathcal{A})$. Then for every $x_0 \in D(\mathcal{A})$ the initial value problem

$$\begin{aligned} \frac{dx}{dt} &= \mathcal{A}x + F(x) \\ x(0) &= x_0 \end{aligned}$$

has a maximal classical solution

$$x \in C^1([0, T_{\max}), X) \cap C([0, T_{\max}), D(\mathcal{A})).$$

Moreover if $T_{\max} < \infty$ then

$$\lim_{t \nearrow T_{\max}} (\|\mathcal{A}x(t)\|_X + \|x(t)\|_X) = \infty.$$

Fix $\varepsilon > 0$. We apply the above result to the space $X = D(A_\varepsilon) \times L^2 = H^1 \times L^2$ endowed with the norm

$$\|(u, v)\|_X = (\|A_\varepsilon u\|_{L^2}^2 + \|v\|_{L^2}^2)^{1/2},$$

to the operator A in X defined on $D(\mathcal{A}) = D(A_\varepsilon^2) \times D(A_\varepsilon) = H^2 \times H^1$ by

$$\mathcal{A}(u, v) = (v, -A_\varepsilon^2 u),$$

and to the mapping F defined by

$$F(u, v) = (0, N_\varepsilon(u)).$$

Then \mathcal{A} is skew-adjoint and F is locally Lipschitz continuous on $D(\mathcal{A})$ by lemma 7. We conclude that there exists $T_\varepsilon > 0$ and a unique classical solution U_ε of (19) on $[0, T_\varepsilon)$ and moreover that

$$\lim_{t \nearrow T_\varepsilon} \left(\|U_\varepsilon(t)\|_{H^2} + \left\| \frac{\partial U_\varepsilon}{\partial t} \right\|_{H^1} \right) = \infty$$

if $T_\varepsilon < \infty$.

Since (19) and (4) are equivalent, it follows that there exists a unique classical solution Φ_ε of (4) on $[0, T_\varepsilon)$ and moreover that if $T_\varepsilon < \infty$ then

$$\lim_{t \nearrow T_\varepsilon} \left(\|\Phi_\varepsilon(t)\|_{H^2} + \left\| \frac{\partial \Phi_\varepsilon(t)}{\partial t} - \frac{i\beta}{2\varepsilon} \Phi_\varepsilon(t) \right\|_{H^1} \right) = \infty.$$

Since Φ_ε is the solution of (3) and since (3) is equivalent to (2), we can apply lemma 5 and conclude that if $T_\varepsilon < T$ (T is the number in that lemma) then $\lim_{t \nearrow T_\varepsilon} \|\Phi_\varepsilon(t)\|_{H^2} < \infty$, consequently $\lim_{t \nearrow T_\varepsilon} \left\| \frac{\partial \Phi_\varepsilon(t)}{\partial t} \right\|_{H^1} = \infty$. However the equation (2) implies $\left\| \frac{\partial \Phi_\varepsilon(t)}{\partial t} \right\|_{H^1} \leq C \|\Phi_\varepsilon(t)\|_{H^2}$ which leads to a contradiction.

This implies $T_\varepsilon \geq T$. From the symmetry of the equation (19) with respect to t we conclude that the maximal existence for Φ_ε (hence also for Ψ_ε) is in fact $(-T_\varepsilon, T_\varepsilon)$.

Proof of Theorem 2. – Using the notation from the proof of Theorem 1, the initial value problem (5) can be written as

$$\left. \begin{aligned} \frac{d\Phi}{dt} &= -i\beta A \Phi + i\beta N_0(\Phi) \\ \Phi(0) &= \Phi_{00}, \end{aligned} \right\} \quad (23)$$

Since $-i\beta A$ is a skew adjoint operator in L^2 and N_0 is a locally Lipschitz map on $D(A) = H^2$ by Lemma 7, the conclusions of Theorem 2 are a straightforward application of the classical existence result that was used in the proof of Theorem 1.

Proof of Theorem 3. – Denote

$$I_0(t) = e^{-i\beta A t}.$$

Then the solution of the initial value problem (23) [and consequently (5)] satisfies also the integral equation

$$\Phi_0(t) = I_0(t) \Phi_{00} + i \int_0^t \beta I_0(t-s) N_0(\Phi_0(s)) ds; \quad (24)$$

note that β , A and $I_0(t)$ are diagonal commuting.

Using the representations (20) and (24) of the solutions Φ_ε and Φ_0 , their difference can be written as

$$\Phi_\varepsilon(t) - \Phi_0(t) = \sum_{i=1}^6 L_\varepsilon^{(i)}(t),$$

where

$$\begin{aligned} L_\varepsilon^{(1)}(t) &= [I_\varepsilon(t) - I_0(t)] \Phi_{00} \\ L_\varepsilon^{(2)}(t) &= I_\varepsilon(t) (\Phi_{0\varepsilon} - \Phi_{00}) \\ L_\varepsilon^{(3)}(t) &= J_\varepsilon(t) \Phi_{1\varepsilon} \\ L_\varepsilon^{(4)}(t) &= \int_0^t \left[\frac{1}{\varepsilon} J_\varepsilon(t-s) - i\beta I_0(t-s) \right] N_0(\Phi_0(s)) ds \end{aligned}$$

$$L_\varepsilon^{(5)}(t) = \frac{1}{\varepsilon} \int_0^t J_\varepsilon(t-s) [N_\varepsilon(\Phi_\varepsilon(s)) - N_0(\Phi_\varepsilon(s))] ds$$

$$L_\varepsilon^{(6)}(t) = \frac{1}{\varepsilon} \int_0^t J_\varepsilon(t-s) [N_0(\Phi_\varepsilon(s)) - N_0(\Phi_0(s))] ds$$

We shall prove

$$\lim_{\varepsilon \rightarrow 0} L_\varepsilon^{(i)} = 0 \quad \text{in } C([0, T]; H^\alpha) \tag{25_i}$$

for $i = 1, \dots, 5$.

Since $\Phi_{0\varepsilon} \in H^\alpha$ and $N_0(\Phi_0(\cdot)) \in C([0, T]; H^2)$ by Theorem 2, the statements (25₁) and (25₄) follow directly from Lemma 2.2 in [5]. Noting the estimate

$$\sup_{\substack{\varepsilon > 0 \\ t \in \mathbb{R}}} \left(\|I_\varepsilon(t)\|_{\mathcal{L}(\mathbb{L}^2)} + \left\| \frac{1}{\varepsilon} J_\varepsilon(t) \right\|_{\mathcal{L}(\mathbb{L}^2)} \right) < \infty, \tag{26}$$

we see that (25₂) is a consequence of (7).

Similarly, the estimate

$$\begin{aligned} \varepsilon \|\Phi_{1\varepsilon}\|_{H^1} &\leq C(\varepsilon^{1/2} \|\Phi_{0\varepsilon}\|_{H^2} + \varepsilon \|N_0(\Phi_{0\varepsilon})\|_{H^1}) \\ &\leq C(\varepsilon^{1/2} \|\Phi_{0\varepsilon}\|_{H^2} + \varepsilon \|\Phi_{0\varepsilon}\|_{L^\infty}^2 \|\Phi_{0\varepsilon}\|_{H^1}) \end{aligned}$$

together with (6) and the Sobolev embedding theorem imply $\lim_{\varepsilon \rightarrow 0} \varepsilon \Phi_{1\varepsilon} = 0$

in H^1 , therefore (25₃) follows from (26). Next

$$\|F_\varepsilon(\Phi)\|_{H^1} \leq C(\varepsilon \|\Phi\|_{L^\infty}^4 + \varepsilon^{1/2} \|\Phi\|_{L^\infty}^2 \|\Phi\|_{H^2} + \varepsilon^{1/2} \|\Phi\|_{W^{1,4}}^2 \|\Phi\|_{L^\infty}),$$

hence it follows from (26) and Lemma 4 that

$$\|L_\varepsilon^{(5)}(t)\|_{H^\alpha} \leq C\varepsilon \int_0^t \|F_\varepsilon(\Phi_\varepsilon(s))\|_{H^\alpha} ds \leq C\varepsilon \int_0^t \|F_\varepsilon(\Phi_\varepsilon(s))\|_{H^1} ds$$

Using Lemma 4 and Lemma 5 [recall that in (11) and (12) Ψ_ε can be replaced by Φ_ε] it follows that $\|L_\varepsilon^{(5)}(t)\|_{H^\alpha} \leq C t \varepsilon^{1/2}$, hence the equality (24₅) is also proved. Using (26) once again we see that

$$\|L_\varepsilon^{(6)}(t)\|_{H^\alpha} \leq C \int_0^t \|\Phi_\varepsilon(s) - \Phi_0(s)\|_{H^\alpha} ds.$$

It follows that

$$\|\Phi_\varepsilon(t) - \Phi_0(t)\|_{H^\alpha} \leq C \left(a_\varepsilon + \int_0^t \|\Phi_\varepsilon(s) - \Phi_0(s)\|_{H^\alpha} ds \right) \quad (0 \leq t \leq T)$$

with $\lim_{\varepsilon \rightarrow 0} a_\varepsilon = 0$. This implies (8) on $[0, T]$; the proof for $[-T, 0]$ is identical.

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