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Contingent solutions to the center manifold equation

by

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ABSTRACT. – Given a system of ordinary differential equations with lipschitzian right-hand sides, we state an existence and uniqueness theorem for a "contingent solution" of the first-order system of partial differential equations characterizing center manifolds, as well as the convergence of the viscosity method.

RÉSUMÉ. – *Solution contingente à une équation de variété centrale.* – Étant donné un système d'équations différentielles à second membre lipschitzien, on démontre un théorème d'existence et d'unicité d'une solution "contingente" du système d'équations aux dérivées partielles du premier ordre caractérisant les variétés centrales, ainsi que la convergence de la méthode de viscosité.

INTRODUCTION

Let us consider the system of differential equations

$$\begin{cases} x'(t)=f(x(t),\, y(t)) \\ y'(t)=-\lambda\, y(t)+g(x(t),\, y(t)) \end{cases}$$

where X, Y are finite dimensional vector-spaces, $f: X \times Y \mapsto X$ and $g: X \times Y \mapsto Y$ are lipschitzian maps and $\lambda > 0$.

The problem of finding maps $r: X \mapsto Y$ whose (closed) graphs are viability domains of this system (which can be called *center manifolds*) has been studied in several frameworks: *see* [11], [12] to quote a few and their applications (*see* [4], [8], [7] for instance). Knowing a center manifold and a solution $x(.)$ to

$$x'(t) = f(x(t), r(x(t)))$$

starting at x_0, then the pair $(x(.), y(.))$ is a solution to the system of diffential equations starting at (x_0, y_0) where $y(t) := r(x(t))$.

We can characterize the maps $r: X \mapsto Y$ whose (closed) graphs are center manifolds of this system thanks to Nagumo's Theorem: it states that

$$\forall x \in X, \ (f(x, r(x)), \ -\lambda r(x) + g(x, r(x))) \in T_{\mathrm{graph}\,(r)}(x, r(x))$$

where $T_K(x)$ denotes the contingent cone to K at $x \in K$ defined by

$$v \in T_K(x) \text{ if and only if } \liminf_{h \to 0+} \frac{d(x+hv; K)}{h} = 0.$$

(*See* [1], [5] on viability and invariance domain for instance.) We recall that for any map r, the contingent cone to the graph of r at $(x, r(x))$ is the graph of the *contingent derivative*, which is the set-valued map from X to Y defined by

$$D r(x)(u) := \left\{ v \in Y \mid \liminf_{h \to 0+} \left\| v - \frac{r(x+hu) - r(x)}{h} \right\| = 0 \right\}.$$

Naturally, $D r(x)(u) = r'(x) u$ coincides with the usual derivative whenever r is differentiable at x. It has nonempty values when r is lipschitzian. (*See* [3] for more details on the differential calculus of nonsmooth and set-valued maps.)

Therefore, we can translate this characterization by saying that r is a solution to the quasi-linear first-order system of "contingent" partial differential inclusions.

$$\lambda r(x) \in g(x, r(x)) - D r(x)(f(x, r(x))).$$

One can say that such maps r are *contingent solutions* to the quasi-linear first order system of partial differential equations

$$\forall j = 1, \ldots, m, \quad \lambda r_j(x) = g_j(x, r(x)) - \sum_{i=1}^{n} \frac{\partial r_j(x)}{\partial x_i} f_i(x, r(x))$$

(called the *center manifold equation*), which we shall write in the form

$$\lambda r(x) = g(x, r(x)) - r'(x) f(x, r(x))$$

because $D r(x)(u) = r'(x) u$ whenever r is differentiable.

The classical Center Manifold Theorem states that there exists a *local* $\mathscr{C}^{(2)}$-solution to this system when g is $\mathscr{C}^{(2)}$, vanishes at the origin and when $f(x, y)=\mathrm{A}x+f_0(x, y)$ where the eigenvalues of A have zero real parts, f_0 is $\mathscr{C}^{(2)}$ and vanishes at the origin. The latter requirements are used for the study of the asymptotic properties of the equilibrium.

In this paper, we shall show that there exists a *global* bounded and lipschitzian *contingent solution* to this problem when f and g are only lipschitzian, when

$$\forall x, y, \quad \|g(x, y)\| \leqq c(1+\|y\|)$$

and when λ is large enough. It is unique when λ is even larger.

This type of result is known for equations (*see* [13], [17] for instance) and is announced by P.-L. Lions and Souganidis in more general cases by other methods (private communication.)

Denote by Δr the map of components $\Delta r_i (i=1, \ldots, m)$.

We shall prove futhermore that these solutions can be approximated in the spirit of the "viscosity method" by solutions r_ε to the second-order system

$$\lambda r(x)=\varepsilon \Delta r(x)-r'(x) f(x, r(x))+g(x, r(x))$$

when $\varepsilon \rightarrow 0$. We use for that purpose the fact that the graph of such maps r_ε are stochastic viability domains of the system of stochatic differential equations

$$\begin{cases} dx=f(x, y)\, dt+2\varepsilon\, d\mathrm{W}(t) \\ dy=(-\lambda y+g(x, y))\, dt+k_\varepsilon(x, y)\, d\mathrm{W}(t) \end{cases}$$

where $\mathrm{W}(t)$ is a Wiener process from X to X, provided that $k_\varepsilon(x, r_\varepsilon(x))=\varepsilon r_\varepsilon'(x)$. (We refer to [2] for general results on invariant manifolds by stochastic differential equations.)

The outline of the paper starts with the study of linear contingent partial differential inclusions

$$\lambda r(x) \in \psi(x)-\mathrm{D}r(x)(\varphi(x))$$

where we prove that the solution is still given by the classical formula

$$r(x)=-\int_0^{\infty} e^{-\lambda t} \psi(\mathrm{S}_\varphi(x, t))\, dt$$

where $\mathrm{S}_\varphi(\mathrm{x}, .)$ is the unique solution to the differential equation

$$x'(t)=\varphi(x(t))$$

starting at x at time 0. In the process, we provide *a priori* estimates of the sup-norm (the classical maximum principle) and the Lipschitz and Hölder semi-norms for first and second order systems.

In the second section, we use these results and fixed point theorems to prove the existence of contingent solutions to the first and second order quasi-linear systems and prove the convergence of the viscosity method.

1. THE LINEAR CASE

1.1. Contingent solutions to first order systems

Let $X := \mathbf{R}^n$ and $Y := \mathbf{R}^m$ be given. We introduce two maps $\varphi : X \to X$ and $\psi : X \to Y$.

We shall look for solutions $r : X \to Y$ to the first-order system of partial differential equations

$$\lambda r(x) = \psi(x) - r'(x)\varphi(x). \tag{1}$$

Actually, we shall look for Lipschitz (or even, closed graph) solutions r to this equation. Usually, a Lipschitz map r is not differentiable, but *contingently differentiable* in the sense that its contingent derivative associating to every direction $u \in X$ the subset

$$\mathrm{D}r(x)(u) := \left\{ v \in Y \mid \liminf_{h \to 0+} \left\| v - \frac{r(x+hu) - r(x)}{h} \right\| = 0 \right\}$$

has nonempty values (1). Naturally, $\mathrm{D}r(x)(u) = r'(x)u$ whenever r is differentiable at x. So, we shall provide *contingent solutions* to the first-order system of differential equations (1), which are by definition the solutions to the contingent inclusions

$$\lambda r(x) \in \psi(x) - \mathrm{D}r(x)(\varphi(x)). \tag{2}$$

We recall that the (closed) graphs of solutions to the contingent inclusion (2) are viability domains of the system of differential equations

$$\begin{cases} x'(t) = \varphi(x(t)) \\ y'(t) = -\lambda y(t) + \psi(x(t)) \end{cases}$$

thanks to Nagumo's Theorem.

We shall also consider second order elliptic systems of partial differential equations

$$\lambda r(x) = \varepsilon \Delta r(x) + \psi(x) - r'(x)\varphi(x) \tag{3}$$

(1) We recall that for any map r, the graph of the contingent derivative is the contingent cone to the graph of r at $(x, r(x))$.

which possess unique twice differentiable solutions whenever the functions φ and ψ are lipschitzian.

We shall establish *a priori* estimates independent of ε enjoyed by the solutions to the first and second order systems.

1.2. The maximum principle and other *a priori* estimates

We introduce the Fréchet space $\mathscr{C}:=\mathscr{C}(X, Y)$ of continuous maps $r: X \to Y$ supplied with compact convergence topology, as well as the spaces $\mathscr{C}^{(m)}:=\mathscr{C}^{(m)}(X, Y)$ of m-times continuously differentiable maps. We set

$$\|r\|_\infty := \sup_{x \in X} \|r(x)\| \in [0, \infty]$$

and

$$\|f\|_\alpha := \sup_{x \neq y} \frac{\|r(x)-r(y)\|}{\|x-y\|^\alpha} \in [0, \infty].$$

We recall that on the space $\mathscr{C}^{(1)}$, the semi-norms $\|r\|_\alpha$ are equivalent to the semi-norms (again denoted by)

$$\|r\|_\alpha = \lim_{\eta \to 0+} \sup_{\|x-y\| \leqq \eta} \frac{\|r(x)-r(y)\|}{\|x-y\|^\alpha}$$

and that for $\alpha=1$, they are both equivalent to the norm $\|r'\|_\infty$.

1.2.1. *The maximum principle*

We begin by adapting the maximum principle to the case of systems:

PROPOSITION 1.1. — *Assume that* $r \in \mathscr{C}^{(2)}$. *Then the following* a priori *estimates independent of* $\varepsilon \geqq 0$

$$\forall \lambda > 0, \quad \|r\|_\infty \leqq \frac{1}{\lambda} \|\lambda r + r'.\varphi - \varepsilon \Delta r\|_\infty \tag{4}$$

hold true.

Proof. — Assume first that there exists $\bar{x}$ achieving the maximum of $\|r(x)\|$. The first-order necessary conditions for a maximum imply that

$$\forall u \in X, \quad \langle r(\bar{x}), r'(\bar{x}) u \rangle = 0.$$

The second-order conditions yield

$$\forall u, v \in X, \quad \langle r(\bar{x}), r''(\bar{x})(u, v) \rangle + \langle r'(\bar{x}) v, r'(\bar{x}) u \rangle \leqq 0.$$

By taking $u=v=e_j$, we infer that

$$\forall i=1, \ldots, m, \qquad r(\bar{x}), \frac{\partial^2 r(\bar{x})}{\partial x_i^2} \leqq 0$$

so that

$$\langle r(\bar{x}), \Delta r(\bar{x})\rangle \leqq 0.$$

Let us consider now a twice differentiable solution r to equation (3). By applying $r(\bar{x})$ to both sides of equation (3), we infer that

$$\left\{\begin{aligned} \lambda \|r\|_\infty^2 = \lambda \|r(\bar{x})\|^2 = \langle r(\bar{x}), \varepsilon \Delta r(\bar{x}) - r'(\bar{x})\varphi(\bar{x}) + \psi(\bar{x})\rangle \\ \leqq \langle r(\bar{x}), \psi(\bar{x})\rangle \leqq \|r\|_\infty \|\psi\|_\infty \end{aligned}\right.$$

so that we derive the *a priori* estimate

$$\forall \lambda > 0, \quad \|r\|_\infty \leqq \frac{1}{\lambda}\|\psi\|_\infty. \tag{5}$$

If the nonnegative bounded function $\chi(.) := \|r(.)\|$ does not achieve its maximum, we use a standard argument which can be found in [10], [19] for instance. One can find approximate maxima x_n such that $\chi(x_n)$ converges to $\sup_{x \in X} \chi(x)$, $\chi'(x_n)$ converges to 0 and $\chi''(x_n)(.,.) \leqq 0$. □

1.2.2. A priori *estimates in* $\mathscr{C}^{(1)}$

Let a: $X \to \mathbf{R}_+$ be a positive twice differentiable function outside of the origin. Denote by

$$\eta(r) := \sup_{x \neq y} \frac{\|r(x) - r(y)\|}{a(x-y)}.$$

In particular, $\eta(r) = \|r\|_\alpha$ when we take $a(z) := \|z\|^\alpha$.

We adapt to the case of systems *a priori* estimates known for equations (*see* [9] for instance):

PROPOSITION 1.2. – *Consider a twice differentiable solution to the second-order equation* (3).

Let us set

$$-\mu(\varphi) := \inf_{x \neq y} \frac{\min(0, \langle a'(x-y), \varphi(x) - \varphi(y)\rangle)}{a(x-y)} \in]-\infty, 0].$$

If $\eta(\psi) < +\infty$ *and* $\mu(\varphi) < +\infty$, *then the* a priori *estimate independent of* ε

$$\forall \lambda > \mu(\varphi), \quad \eta(r) \leqq \frac{\eta(\psi)}{\lambda - \mu(\varphi)} \tag{6}$$

holds true. In particular, if φ *is monotone, then*

$$\forall \lambda > 0, \quad \|r\|_\alpha \leqq \frac{\|\psi\|_\alpha}{\lambda} \tag{7}$$

and if φ *is lipschitzian, then*

$$\forall \lambda > \|\varphi\|_1, \quad \|r\|_\alpha \leqq \frac{\|\psi\|_\alpha}{\lambda - \alpha \|\varphi\|_1} \tag{8}$$

Proof. – Assume first that exists a pair $(\bar{x}, \bar{y})$ achieving the maximum of

$$\|r(x) - r(y)\|^2 / a(x-y)^2.$$

The first-order conditions imply

$$\begin{gathered} \forall u, v \in X, \\ \langle r(\bar{x}) - r(\bar{y}), r'(\bar{x}) u - r'(\bar{y}) v \rangle = \eta(r)^2 a(\bar{x} - \bar{y}) \langle a'(\bar{x} - \bar{y}), u - v \rangle \end{gathered}$$

and the second-order condition that

$$\left\{ \begin{aligned} & \forall u, u_1, v, v_1 \in X, \\ & \langle r'(\bar{x}) u_1 - r'(\bar{y}) v_1, r'(\bar{x}) u - r'(\bar{y}) v \rangle \\ & + \langle r(\bar{x}) - r(\bar{y}), r''(\bar{x})(u_1, u) - r''(\bar{y})(v_1, v) \rangle \\ & \leqq \eta(r)^2 (a(\bar{x} - \bar{y}) a''(\bar{x} - \bar{y})(u_1 - v_1, u - v) \\ & \quad + \langle a'(\bar{x} - \bar{y}), u - v \rangle \langle a'(\bar{x} - \bar{y}), u_1 - v_1 \rangle). \end{aligned} \right.$$

By taking $u = u_1$ and $v = v_1$, this formula yields

$$\left\{ \begin{aligned} & \forall u, v \in X, \\ & \|r'(\bar{x}) u - r'(\bar{y}) v\|^2 + \langle r(\bar{x}) - r(\bar{y}), r''(\bar{x})(u, u) - r''(\bar{y})(v, v) \rangle \\ & \leqq \eta(r)^2 (a(\bar{x} - \bar{y}) a''(\bar{x} - \bar{y})(u - v, u - v) + (\langle a'(\bar{x} - \bar{y}), u - v \rangle)^2) \end{aligned} \right.$$

and consequently, by taking $u = v$, that

$$\forall u \in X, \quad \langle r(\bar{x}) - r(\bar{y}), r''(\bar{x})(u, u) - r''(\bar{y})(u, u) \rangle \leqq 0.$$

In particular, we obtain the following inequality

$$\langle r(\bar{x}) - r(\bar{y}), \Delta r(\bar{x}) - \Delta r(\bar{y}) \rangle \leqq 0$$

by summing up the above inequalities with $u = e_i$, $i = 1, \ldots, n$.

Consider now a twice differentiable solution to the second-order equation (3). By applying $r(\bar{x}) - r(\bar{y})$ to both sides of this equation, we infer that

$$\left\{ \begin{aligned} & \lambda \|r(\bar{x}) - r(\bar{y})\|^2 \\ & = \langle r(\bar{x}) - r(\bar{y}), \varepsilon(\Delta r(\bar{x}) - \Delta r(\bar{y})) - (r'(\bar{x}) \varphi(\bar{x}) - r'(\bar{y}) \varphi(\bar{y})) + \psi(\bar{x}) - \psi(\bar{y}) \rangle \\ & \leqq \|r(\bar{x}) - r(\bar{y})\| \, \|\psi(\bar{x}) - \psi(\bar{y})\| \\ & \qquad\qquad - \eta(r)^2 a(\bar{x} - \bar{y}) \langle a'(\bar{x} - \bar{y}), \varphi(\bar{x}) - \varphi(\bar{y}) \rangle. \end{aligned} \right.$$

Therefore, if we set

$$-\mu(\varphi):=\inf_{x\neq y}\frac{\min(0,\langle a'(x-y),\varphi(x)-\varphi(y)\rangle)}{a(x-y)}\in]-\infty,0]$$

and if we assume that $\mu(\varphi)<+\infty$, we obtain the *a priori* estimate (6) after dividing both sides of this inequality by $a(\bar{x}-\bar{y})^2$.

The $\mathscr{C}^\alpha$ estimates of the solutions are obviously obtained by taking $a(z):=\|z\|^\alpha$, the derivative of which is equal to $a'(z)=\alpha\|z\|^{\alpha-2}z$. In this case, $\mu(\varphi)$ measures the lack of monotonicity since

$$-\mu(\varphi):=\alpha\inf_{x\neq y}\frac{\min(0,\langle\varphi(x)-\varphi(y),x-y\rangle)}{\|x-y\|^2}.$$

We observe that $\mu(\varphi)=0$ whenever φ is monotone (in the sense that $\langle\varphi(x)-\varphi(y),x-y\rangle\geqq 0$ for every pair (x,y)) and that $\mu(\varphi)\leqq\alpha\|\varphi\|_1$ whenever φ is lipschitzian.

When the function $\chi(x,y):=\|r(x)-r(y)\|/a(x-y)$ does not achieve its maximum, the argument used above allows to find approximate maxima x_n such that $\chi(x_n,y_n)$ converges to $\eta(r)$, $\chi'(x_n,y_n)$ converges to 0 and $\chi''(x_n,y_n)(.,.)\leqq 0$. □

1.2.3. A priori *estimates in* $\mathscr{C}^{(2)}$

PROPOSITION 1.3. — *We assume that the functions φ and ψ are continuously differentiable. Consider a solution to the second-order equation* (1). *Then*

$$\forall\lambda>\|\varphi'\|_1+\|\varphi'\|_\infty,\quad\|r'\|_1\leqq\frac{\|\psi'\|_1}{\lambda-\|\varphi'\|_1-\|\varphi'\|_\infty}.\tag{9}$$

Proof. — Actually, we shall prove a more general *a priori* estimate. Let a: $X\to\mathbf{R}_+$ be a positive twice differentiable function outside of the origin. Denote by

$$\eta^1(r):=\sup_{x\neq y,\,u\in B}\frac{\|(r'(x)-r'(y))u\|}{a(x-y)\|u\|}.$$

In particular, $\eta^1(r)=\|r'\|_1$ when we take $a(z):=\|z\|$.

Let us assume that there exists $(\bar{x},\bar{y},\bar{u})$ achieving the maximum of

$$\|(r'(x)-r'(y))u\|^2/a(x-y)^2\|u\|^2.$$

The first conditions imply

$$\left\{\begin{array}{l}\qquad\qquad\forall u,u_1,v,v_1,w\in X,\\ \langle r'(\bar{x})\bar{u}-r'(\bar{y})\bar{u},(r'(\bar{x})-r'(\bar{y}))w\rangle\\ \qquad+\langle r(\bar{x})\bar{u}-r(\bar{y})\bar{u},r''(\bar{x})(u_1,\bar{u})-r''(\bar{y})(v_1,\bar{u})\rangle\\ \quad=\eta^1(r)^2(a(\bar{x}-\bar{y})a'(\bar{x}-\bar{y})(u_1-v_1)\|\bar{u}\|^2+a(\bar{x}-\bar{y})^2\langle\bar{u},w\rangle).\end{array}\right.$$

By taking $u_1 = v_1$ and $w=0$, we infer that

$$\left.\begin{array}{c}\forall u_1 \in X,\\ \langle r'(\bar{x})\bar{u} - r'(\bar{y})\bar{u},\ r''(\bar{x})(u_1, \bar{u}) - r''(\bar{y})(u_1, \bar{u})\rangle = 0.\end{array}\right\} \tag{10}$$

Let r be a $\mathscr{C}^{(2)}$-solution to the linear equation (1): it satisfies the equation

$$\lambda r'(x) u_1 = \psi'(x) u_1 - r'(x) \varphi'(x) u_1 - r''(x)(u_1, \varphi(x)).$$

By applying $(r'(\bar{x})\bar{u} - r(\bar{y})\bar{u})$ to both sides of this equation, we obtain

$$\left\{\begin{array}{l}\lambda \|(r'(\bar{x}) - r'(\bar{y}))\bar{u}\|^2\\ = \langle (r'(\bar{x}) - r(\bar{y}))\bar{u},\ (\psi'(\bar{x}) - (\psi'(\bar{y}))\bar{u} - (r'(\bar{x})\varphi'(\bar{x}) - r'(\bar{y})\varphi'(\bar{y}))\bar{u}\\ \qquad - (r''(\bar{x})(\bar{u}, \varphi(\bar{x})) - r''(\bar{y})(\bar{u}, \varphi(\bar{y})))\rangle\\ \qquad \leqq \|(r'(\bar{x}) - r'(\bar{y}))\bar{u}\| (\|(\psi'(\bar{x}) - \psi'(\bar{y}))\bar{u}\|\\ \qquad\qquad + a(\bar{x} - \bar{y})^2 \|\bar{u}\|^2 (\|r'\|_\infty \eta^1(\varphi) + \eta^1(r)\|\varphi'\|_\infty))\end{array}\right.$$

thanks to (10).

Therefore, after dividing both sides of this inequality by $a(\bar{x} - \bar{y})^2 \|\bar{u}\|^2$, we obtain

$$(\lambda - \|\varphi'\|_\infty)\eta^1(r) \leqq \eta^1(\psi) + \|r'\|_\infty \eta^1(\varphi).$$

If the function

$$(x, y, u) \to \frac{\|(r'(x) - r'(y))u\|}{a(x-y)\|u\|}$$

does not achieve its maximum, we use again the approximation argument. In the case when $a(z) := \|z\|$, we obtain the estimate (9).

1.3. Existence of contingent solutions

When φ is lipschitzian (and thus, with linear growth), we denote by $S_\varphi(x, .)$ the unique solution to the differential equation

$$x'(t) = \varphi(x(t))$$

starting at x at time 0.

PROPOSITION 1.4. — *Assume that φ is lipschitzian and that $\psi \in \mathscr{C}^\alpha$ is bounded. Then for all $\lambda > 0$, the map r defined by*

$$r(x) = -\int_0^\infty e^{-\lambda t} \psi(S_\varphi(x, t))\, dt$$

is the unique solution to the contingent inclusion (2) *and satisfies*

$$\|r\|_\infty \leqq \frac{\|\psi\|_\infty}{\lambda} \quad \text{and} \quad \forall \lambda > \mu(\varphi),\ \|r\|_\alpha \leqq \frac{\|\psi\|_\alpha}{\lambda - \mu(\varphi)}. \tag{11}$$

Proof. — Let ρ be a nonnegative smooth function with compact support and integral to one and set $\rho_h(x) := \frac{1}{h^n}\rho\left(\frac{x}{h}\right)$. We approximate the maps φ and ψ by their convolution products

$$\varphi_h := \rho_h \bigstar \varphi \quad \text{and} \quad \psi_h := \rho_h \bigstar \psi$$

which are smooth functions satisfying

$$\|\varphi_h\|_1 \leqq \|\varphi\|_1, \quad \|\psi_h\|_\alpha \leqq \|\psi\|_\alpha \quad \text{and} \quad \|\psi_h\|_\infty \leqq \|\psi\|_\infty.$$

We recall that the explicit solution r_h to the first-order system

$$\lambda r(x) = \psi_h(x) - r'(x)\varphi_h(x)$$

is given by formula

$$r_h(x) = -\int_0^\infty e^{-\lambda t}\psi_h(S_{\varphi_h}(x, t))\,dt.$$

By the *a priori* estimate (4), we infer that

$$\|r_h\|_\infty \leqq \frac{\|\psi\|_\infty}{\lambda}.$$

The graph of r_h is a viability domain of the system of differential equations

$$\begin{cases} x'(t) = \varphi_h(x(t)) \\ y'(t) = -\lambda_y(t) + \psi_h(x(t)). \end{cases}$$

By Nagumo's Theorem, the solutions $(x_h(.), y_h(.))$ to this system of differential equations satisfy

$$\forall t \geqq 0, \quad y_h(t) = r_h(x_h(t)).$$

They remain in a compact subset of $\mathscr{C}(0, \infty; X \times Y)$ because they enjoys the same linear growth, so that a subsequence (again denoted by) $(x_h(.), y_h(.))$ converges uniformly on compact intervals to a function $(x(.), y(.))$. Since the maps φ_h and ψ_h remain in an equicontinuous set, we infer that $(x(.), y(.))$ is a solution to the system

$$\begin{cases} x'(t) = \varphi(x(t)) \\ y'(t) = -\lambda y(t) + \psi(x(t)). \end{cases}$$

On the other hand, it is easy to check that when x_h converges to x, the solutions $S_{\varphi_h}(x_h, .)$ converge to a solution $S_\varphi(x, .)$ uniformly on compact intervals. Since the maps ψ_h remain in an equicontinuous set, we infer that $\psi_h(S_{\varphi_h}(x_h, t))$ converges pointwise to $\psi(S_\varphi(x, .))$. Therefore, the map ψ being bounded, $r_h(x)$ converges also to the map r defined by the theorem, which satisfies

$$\forall t \geqq 0, \quad y(t) = r(x(t)).$$

This amounts to saying that the (closed) graph of a map r is a viability domain of this system of differential equations, that is to say that r is a solution to the contingent inclusion (2).

Uniqueness follows from inequality $\|r\|_{\infty} \leqq \|\psi\|_{\infty}/\lambda$.

Finally, *a priori* estimates (8) imply that r_h are hölderian with

$$\|r_h\|_{\alpha} \leqq \frac{\|\psi\|_{\alpha}}{\lambda - \mu(\varphi)}$$

(These estimates can also be obtained directly from the explicit expression of r_h.) □

2. THE QUASI-LINEAR CASE

We consider now the quasi-linear first-order systems

$$\lambda r(x) = g(x, r(x)) - r'(x) f(x, r(x)) \tag{12}$$

of partial differential equations, and, more generally, their contingent version

$$\lambda r(x) \in g(x, r(x)) - \mathrm{D}r(x)(f(x, r(x))). \tag{13}$$

We recall that the (closed) graphs of solutions to the contingent equation (13) are viability domains of the system of differential equations

$$\left.\begin{aligned} x'(t) &= f(x(t), r(x(t))) \\ y'(t) &= -\lambda y(t) + g(x(t), r(x(t))) \end{aligned}\right\} \tag{14}$$

thanks again to Nagumo's Theorem.

THEOREM 2.1. – *Assume that the maps* $f\colon \mathrm{X} \times \mathrm{Y} \to \mathrm{X}$ *and* $g\colon \mathrm{X} \times \mathrm{Y} \to \mathrm{Y}$ *are lipschitzian and that*

$$\forall x, y, \|g(x, y)\| \leqq c(1 + \|y\|).$$

Then for $\lambda > \max(c, 4\|f\|_1 \|g\|_1)$, *there exists a bounded lipschitzian solution to the contingent inclusion* (13). *It is unique for* λ *large enough.*

Actually, we shall also prove the convergence of the "viscosity method" by introducing the second-order system

$$\lambda r(x) = \varepsilon \Delta r(x) - r'(x) f(x, r(x)) + g(x, r(x)). \tag{15}$$

We shall prove that solutions to the second-order system converge to contingent solutions when $\varepsilon \to 0$.

THEOREM 2.2. – *Assume that the maps* $f\colon \mathrm{X} \times \mathrm{Y} \to \mathrm{X}$ *and* $g\colon \mathrm{X} \times \mathrm{Y} \to \mathrm{Y}$ *are lipschitzian and that*

$$\forall x, y, \|g(x, y)\| \leqq c(1 + \|y\|).$$

Then for $\lambda > \max(c, 4\|\|_1 \|g\|_1)$, *there exists a solution* r_ε *to the second-order system* (15) *converging to a solution to the contingent inclusion* (13) *when* $\varepsilon \to 0$.

Proofs.

1. A priori *estimates.* – We shall denote by H_ε the operator defined on lipschitzian maps s by

$r := H_\varepsilon(s)$ is the solution to $\lambda r(x) = \varepsilon \Delta r(x) - r'(x) f(x, s(x)) + g(x, s(x))$.

(This makes sense since this (linear) elliptic problem does have a $\mathscr{C}^{(2)}$-solution whenever $\varepsilon > 0$ and $f(., s(.))$ and $g(., s(.))$ are lipschitzian.) and by H_0 the operator defined on lipschitzian maps by

$r := H_0(s)$ is the solution to $\lambda r(x) = g(x, s(x)) - r'(x) f(x, s(x))$.

We observe that the functions $\varphi(x) := f(x, s(x))$ and $\psi(x) := g(x, s(x))$ satisfy

$$\|\varphi\|_1 \leqq \|f\|_1 (1 + \|s\|_1),$$
$$\|\psi\|_1 \leqq \|g\|_1 (1 + \|s\|_1) \quad \text{and} \quad \|\psi\|_\infty \leqq \frac{c(1 + \|s\|_\infty)}{\lambda}.$$

By above *a priori* estimates, the map H_ε obeys the inequalities

$$\left.\begin{array}{l} \|H_\varepsilon(s)\|_\infty \leqq \dfrac{c}{\lambda}(1 + \|s\|_\infty) \\ \text{and} \\ \|H_\varepsilon(s)\|_1 \leqq \dfrac{\|g\|_1 (1 + \|s\|_1)}{\lambda - \|f\|_1 (1 + \|s\|_1)} \end{array}\right\} \tag{16}$$

which are independent of $\varepsilon \in [0, \infty[$. We first observe that when $\lambda > c$,

$$\forall \|s\|_\infty \leqq \frac{c}{\lambda - c}, \qquad \|H(s)\|_\infty \leqq \frac{c}{\lambda - c}.$$

When $\lambda > 4\|f\|_1 \|g\|_1$, we denote by

$$\rho(\lambda) := \frac{\lambda - \|f\|_1 - \|g\|_1 - \sqrt{\lambda^2 - 2\lambda(\|f\|_1 + \|g\|_1) + (\|f\|_1 - \|g\|_1)^2}}{2\|f\|_1} > 0$$

the smallest root of the equation

$$\lambda\rho = \|f\|_1 \rho^2 + (\|f\|_1 + \|g\|_1)\rho + \|g\|_1.$$

We observe that for large λ,

$$\lim_{\lambda \to +\infty} \lambda\rho(\lambda) = \|g\|_1$$

We thus infer that

$$\forall \|s\|_1 \leqq \rho(\lambda), \qquad \|H_\varepsilon(s)\|_1 \leqq \rho(\lambda).$$

Let us denote by $A^1_\infty(\lambda)$ the subset of twice continuously differentiable maps defined by

$$A^1_\infty(\lambda) := \left\{ r \in \mathscr{C}^2(X, Y) \,\Big|\, \|r\|_\infty \leqq \frac{c}{\lambda - c} \text{ and } \|r\|_1 \leqq \rho(\lambda) \right\}.$$

We have therefore observed that whenever $\lambda > \max(c, 4\|f\|_1 \|g\|_1)$, the maps H_ε send $A^1_\infty(\lambda)$ to the compact subset $B^1_\infty(\lambda)$ defined by

$$B^1_\infty(\lambda) := \left\{ r \in \mathscr{C}(X, Y) \,\Big|\, \|r\|_\infty \leqq \frac{c}{\lambda - c} \text{ and } \|r\|_1 \leqq \rho(\lambda) \right\}.$$

We now check that they are uniformly equicontinuous on $A^1_\infty(\lambda)$. Indeed, let s_1 and s_2 given in $A^1_\infty(\lambda)$ and let us set $r_1 := H_\varepsilon(s_1)$, $r_2 := H_\varepsilon(s_2)$ and $r := r_1 - r_2$ their difference. The map r is a solution to the equation

$$\begin{aligned} \lambda r(x) &= \varepsilon \Delta r(x) - r'(x) f(x, s_1(x)) \\ &+ g(x, s_1(x)) - g(x, s_2(x)) - r'_2(x)(f(x, s_1(x)) - f(x, s_2(x))). \end{aligned}$$

By the maximum principle, we deduce that

$$\|r\|_\infty \leqq \frac{1}{\lambda}(\|g\|_1 + \|r'_2\|_\infty \|f\|_1) \|s_1 - s_2\|_\infty$$

because

$$\begin{cases} \|f(\cdot, s_1(\cdot)) - f(\cdot, s_2(\cdot))\|_\infty \leqq \|f\|_1 \|s_1 - s_2\|_\infty \\ \|g(\cdot, s_1(\cdot)) - g(\cdot, s_2(\cdot))\|_\infty \leqq \|g\|_1 \|s_1 - s_2\|_\infty \end{cases}$$

Since the semi-norms $\|r'(\cdot)\|_\infty$ and $\|r\|_1$ are equivalent on $\mathscr{C}^{(1)}$, we infer that there exists a constant ν such that $\|r'\|_\infty \leqq \nu \|r\|_1 \leqq \nu\rho(\lambda)$.

Hence, for every $\varepsilon \geqq 0$, s_1 and s_2 given in $A^1_\infty(\lambda)$, we obtain the inequality

$$\|H_\varepsilon(s_1) - H_\varepsilon(s_2)\|_\infty \leqq \frac{\|g\|_1 + \nu\rho(\lambda)\|f\|_1}{\lambda} \|s_1 - s_2\|_\infty. \tag{17}$$

Therefore, we can extend by continuity these maps H_ε to maps (again denoted by) H_ε mapping the ball $B^1_\infty(\lambda)$ to itself and satisfying the same inequalities.

2. *Existence and uniquenes.* – Assume that $\lambda > \max(c, 4\|f\|_1 \|g\|_1)$. Since the ball $B^1_\infty(\lambda)$ is compact and convex and H_ε is continuous from this ball to itself, there exists a fixed point $r_\varepsilon \in B^1_\infty(\lambda)$ of the map H_ε by the Brouwer-Schauder-Fan Fixed-Point Theorem in locally convex spaces, *i. e.*, a solution to the equation (15).

For $\varepsilon = 0$, $r_0 = H_0(r_0)$ is a solution to the contingent inclusion (13).

Uniqueness is guaranteed for large enough λ's, actually when

$$\|g\|_1 - \nu\rho(\lambda)\|f\|_1 < \lambda$$

This is possible because $\lim_{\lambda \to +\infty} \lambda\rho(\lambda) = \|g\|_1$. □

3. *Convergence of the viscosity method.* – Since the maps r_ε remain in an equicontinuous and pointwise bounded subset, which is compact, a subsequences (again denoted by) r_ε converges to a map $\bar{r}$. It remains to prove that $\bar{r}$ is a solution to the contingent inclusion (13).

For that purpose, we observe that the graph of a solution to the second-order equation (15) is a viability domain of the system of stochastic differential equations

$$\left.\begin{aligned} dx&=f(x, y)\,dt+2\varepsilon\,dW(t)\\ dy&=(-\lambda y+g(x, y))\,dt+k_\varepsilon(x, y)\,dW(t)\end{aligned}\right\}\tag{18}$$

where $W(t)$ is a Wiener process from X to X, provided that $k_\varepsilon(x, r_\varepsilon(x))=\varepsilon\, r'_\varepsilon(x)$.

Indeed, if the graph of a $\mathscr{C}^{(2)}$-map r is a viability domain of this system of stochastic differential equations, then, for any initial state $(x_0, r(x_0)$, the solution to

$$\left\{\begin{aligned} x(t)&=x_0+\int_0^t f(x(s), y(s))\,ds+2\varepsilon\, W(t)\\ y(t)&=y_0+\int_0^t (g(x(s), y(s))-\lambda y(s))\,ds\\ &\qquad+\int_0^t k_\varepsilon(x(s), y(s))\,dW(s)\end{aligned}\right.$$

satisfies $y(t)=r(x(t))$ for every $t\geqq 0$, so that Ito's formula implies that r is a solution to (15).

Conversely, Ito's formula implies that for any $\mathscr{C}^{(2)}$-solution r to the system (15) satisfying $\varepsilon\, r'(x)=k_\varepsilon(x, r(x))$, we have

$$\left\{\begin{aligned} r(x(t))&=r(x_0)+\int_0^t (r'(x(s))f(x(s), r(x(s)))\\ &\quad-\varepsilon\,\Delta r(s))\,ds+\varepsilon\int_0^t r'(x(s))\,dW(s)\\ &=r(x_0)+\int_0^t (g(x(s), r(x(s)))\\ &\quad-\lambda\, r(x(s)))\,ds+\varepsilon\int_0^t r'(x(s))\,dW(s)\end{aligned}\right.$$

so that $(x(t), r(x(t)))$ is a solution to the system of stochastic differential equations.

Let $(x_\varepsilon(.), y_\varepsilon(.))$ be solutions to the system

$$\left.\begin{aligned} dx&=f(x, y)\,dt+2\varepsilon\,dW(t)\\ dy&=(-\lambda y+g(x, y))\,dt+k_\varepsilon(x, y)\,dW(t)\end{aligned}\right\}\tag{19}$$

satisfying

$$\forall\, t \geqq 0, \quad r_\varepsilon(x_\varepsilon(t)) = y_\varepsilon(t).$$

Since $\| r'_\varepsilon \|_\infty \leqq \rho(\lambda)$, it is well known that when ε converges to 0, the solutions $(x_\varepsilon(.), y_\varepsilon(.))$ converges almost surely to a solution $(x(.), y(.))$ to the system of differential (14). Since the maps r_ε remain in an equicontinuous subset, they satisfy

$$\forall\, t \geqq 0, \quad r(x(t)) = y(t).$$

This shows that r is a solution to the quasi-linear contingent inclusion (13). □

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