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Ljusternik-Schnirelman theory with local Palais-Smale condition and singular dynamical systems

by

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ABSTRACT. — We find infinitely many T -periodic solutions to a system $\ddot{u} + \nabla_x V(t, u) = 0$ with a singular, T -periodic potential V , whose behaviour at infinity is subjected to rather weak assumptions. In order to do so, we adapt the Ljusternik-Schnirelman method to handle a functional possibly unbounded from below and which possibly does not satisfy the Palais-Smale condition at any level.

RÉSUMÉ. — Nous trouvons un nombre infini de solutions T -périodiques d'un système $\ddot{u} + \nabla_x V(t, u) = 0$ pour un potentiel singulier, T -périodique V dont le comportement à l'infini est sujet à des hypothèses très faibles. Pour ce faire, nous adaptons la méthode de Ljusternik-Schnirelman pour traiter une fonctionnelle même non bornée inférieurement et ne satisfaisant pas la condition de Palais-Smale à tout niveau.

Mots clés : Ljusternik-Schnirelman theory, singular dynamical systems, periodic solution.

Classification A.M.S. : 58 F 05, 58 E 05, 34 C 25.

0. INTRODUCTION

In this paper we seek T -periodic solutions of second order systems of the type

$$(0.1) \quad \ddot{u} + au + W'(t, u) = 0,$$

where W is singular at $x=0$,

$$W(t+T, x) = W(t, x), \quad \text{and} \quad W'(t, x) = : \nabla_x W(t, x).$$

Problem (0.1) has been studied in [1] under the assumptions:

- (i) $a=0$;
- (ii) $W(t, x), W'(t, x) \rightarrow 0$ as $|x| \rightarrow \infty$ uniformly in t ;
- (iii) W satisfies a "Strong Force condition" (namely $W \simeq -\frac{1}{|x|^\alpha}$, $\alpha \geq 2$, at $x=0$).

(See also [2], [4], [5] for other results in this direction.)

The purpose of this work is to extend the results of [1], retaining condition (iii), but weakening (i) and (ii). More precisely we assume that:

- (j) $a < \left(\frac{\pi}{T}\right)^2$;
- (jj) there exist constants $c, \theta < 2, r > 0$ such that for $|x| \geq r$ and for all $t \in \mathbf{R}$

$$W(t, x) \leq c|x|^\theta, \quad W'(t, x) \cdot x - 2W(t, x) \leq c|x|^\theta,$$

and we show that (0.1) has infinitely many T -periodic solutions u with $u(t) \neq 0 \forall t$.

From the abstract point of view, the solutions of (0.1) are critical points of the action integral

$$(0.1) \quad f(u) = \int_0^T \left\{ \frac{1}{2} |\dot{u}|^2 - \frac{a}{2} |u|^2 - W(t, u) \right\} dt$$

on

$$\Lambda = \{ u \in H^1(S_T^1, \mathbf{R}^N) : u(t) \neq 0, \forall t \in S_T^1 \}.$$

Two difficulties arise in weakening the hypotheses (i), (ii). First, since we made rather weak assumptions on the derivatives of W at infinity, the Palais-Smale condition may possibly fail at any level (while it holds at any level but 0 under the hypotheses (i), (ii); see [1], Lemma 3.1). Second, if $a > 0$ the functional f is no longer bounded from below.

In order to overcome these difficulties we prove in section 2 a Ljusternik-Schnirelman type theorem which establishes the existence of infinitely many critical points (Theorem 2.4). The main features of this theorem

are:

- (a) the Palais-Smale condition is not required on the whole domain of the functional;
- (b) the functional need not be bounded from below;
- (c) a certain control is required on the Ljusternik-Schnirelman category of the sublevel sets of the functional (conditions 2.4.iii and 2.4.iv).

Then in section 3 we show (Theorem 3.5) that if (j), (jj), and (iii) hold, f satisfies the hypotheses of Theorem 2.4. So, whereas checking the Palais-Smale condition (2.4.v) becomes much simpler, more care is needed in verifying conditions 2.4.iii and 2.4.iv. Roughly, the idea is to show that if $f(u) \leq \lambda$, then $\|\dot{u}\|_2 / \inf |u(t)| \leq k(\lambda)$. This allows us to deform the sublevel sets in compact sets (hence with finite category) via a convolution operator.

Theorem 3.5 is completed by two examples. In the former we show a case in which $a=0$, $W(x) \rightarrow 0$ as $x \rightarrow \infty$ and f does not satisfy the usual Palais-Smale condition at any positive level.

In the latter we show that if $a > \left(\frac{\pi}{T}\right)^2$, the category of every sublevel set $\{f \leq \lambda\}$ can actually be infinite, so that Theorem 2.4 cannot be applied.

1. NOTATIONS

If f is a real-valued function on some set Λ and $\lambda \in \mathbf{R}$, $\{f \leq \lambda\}$ denotes the set $\{u \in \Lambda : f(u) \leq \lambda\}$; similar meaning has $\{f \geq \lambda\}$ and so on. If X is a metric space with metric d , and if $x \in X$ and $\rho \in \mathbf{R}$, $B(x, \rho)$ is the ball $\{y \in X : d(x, y) < \rho\}$. If $x, y \in \mathbf{R}^N$, $|x|$ and $x \cdot y$ are respectively the euclidean norm of x and the scalar product of x, y . S_T^1 denotes $\mathbf{R}/T\mathbf{Z}$. Finally, $\|u\|_2 = \left(\int_0^T |u(t)|^2 dt\right)^{1/2}$ and $\|u\|_{1,2} = (\|u\|_2^2 + \|\dot{u}\|_2^2)^{1/2}$ denote respectively the L^2 -norm and the H^1 -norm of $u \in L^2([0, T], \mathbf{R}^N)$, respectively $u \in H^1([0, T], \mathbf{R}^N)$.

Hereafter SF, LS and PS means respectively Strong Force, Ljusternik-Schnirelman, Palais-Smale.

2. A THEOREM OF LJUSTERNIK-SCHNIRELMAN TYPE

We first recall some definitions and basic results on Critical Point Theory. Let Λ be a topological space, and let $\mathcal{X}(\Lambda)$ be the family of the closed subsets of Λ which are contractible in Λ ; if $A \subset \Lambda$, the LS category

of Λ relatively to Λ is the number (possibly $+\infty$)

$$\text{Cat}_\Lambda(A) = \inf \left\{ k \in \mathbb{N} : A \subset \bigcup_{i=1}^k X_i \in \mathcal{K}(\Lambda) \right\}.$$

In the following proposition we list some properties of the category.

2.1. PROPOSITION. — *Let Λ be a topological space and $A, B \subset \Lambda$. Then*

$$(2.1) \quad \text{Cat}_\Lambda(A \cup B) \leq \text{Cat}_\Lambda(A) + \text{Cat}_\Lambda(B).$$

If A is closed and there exists a deformation of A in B , i.e., a continuous map $h : [0, 1] \times A \rightarrow \Lambda$ such that $h(0, \cdot) = 1_A$ and $h(1, A) \subset B$ (in particular if $A \subset B$), then

$$(2.2) \quad \text{Cat}_\Lambda(A) \leq \text{Cat}_\Lambda(B).$$

If Λ is regular and locally contractible every compact subset of Λ has finite category.

If Λ is arcwise connected, $\{A_i\}_{i \in I}$ is a locally finite family of pairwise disjoint closed subsets of Λ and $A = \bigcup_{i \in I} A_i$, then

$$(2.3) \quad \text{Cat}_\Lambda(A) = \sup_{i \in I} \text{Cat}_\Lambda(A_i).$$

Proof. — See [7] for the first three properties. Since we have no references for the last, we report here a proof.

We show that $\text{Cat}_\Lambda(A) \leq \sup_{i \in I} \text{Cat}_\Lambda(A_i)$, since the converse inequality follows immediately from (2.2). We can assume $\sup_{i \in I} \text{Cat}_\Lambda(A_i) = m < \infty$, for

otherwise there is nothing to prove. Thus $\forall i = \bigcup_{j=1}^m X_{i,j}$, with $X_{i,j} \in \mathcal{K}(\Lambda)$.

Since Λ is arcwise connected, for every (i, j) there exists a deformation $h_{i,j}$ of $X_{i,j}$ in a common base point $x_0 \in \Lambda$. For any $j \leq m$ set $Y_j = \bigcup_{i \in I} X_{i,j}$,

and let $h_j : [0, 1] \times Y_j \rightarrow \Lambda$ be the map defined by

$$h_j|_{[0,1] \times X_{i,j}} = h_{i,j} \quad \forall i \in I :$$

the definition makes sense because the $\{X_{i,j}\}_{i \in I}$ are pairwise disjoint. Moreover, since $\{X_{i,j}\}_{i \in I}$ is a locally finite family of closed sets, one has that each Y_j is closed and h_j is continuous, whence $Y_j \in \mathcal{K}(\Lambda)$. Therefore $\text{Cat}_\Lambda(A) \leq m$.

Q.E.D.

Now let Λ be an open subset of some Banach space X . For $f \in \mathcal{C}^1(\Lambda)$ we set $Z_f = \{u \in \Lambda : f'(u) = 0\}$ and $\tilde{\Lambda} = \Lambda \setminus Z_f$. In the proof of the main theorem (2.4) we need some technical lemmas. First of all we recall the following proposition

2.2. PROPOSITION. — Let $f \in \mathcal{C}^1(\Lambda)$, and $\alpha \in]0, 1[$: then there exists a locally Lipschitz continuous map $V : \tilde{\Lambda} \rightarrow X$ such that $\forall u \in \tilde{\Lambda}$

$$(2.4) \quad \begin{cases} \|V(u)\| \leq \frac{1}{\alpha} \|f'(u)\|, \\ \langle f'(u), V(u) \rangle \geq \|f'(u)\|^2. \end{cases}$$

Proof. — See [7] or [8] (there $\Lambda = X$ and $\alpha = \frac{1}{2}$, but the same construction works without changes in the case of Λ open subset of X , $\alpha \in]0, 1[$.)

Q.E.D.

Maps like V , the so-called Pseudogradient vector fields, are used to establish a Deformation Lemma (see [7] or [8]). Actually, for our specific purposes, a statement slightly different from the usual ones is needed.

2.3. LEMMA. — Let $\alpha \in]0, 1[$ and let $f \in \mathcal{C}^1(\Lambda)$ be such that

$$(2.5) \quad \forall u_n \rightarrow u \in \partial\Lambda, \quad f(u_n) \rightarrow \infty,$$

and suppose there exists a locally lipschitz map $h : \Lambda \rightarrow \mathbb{R}$ such that $Z_f \subset \{f < h - 1\}$.

Then there exists a continuous map $\eta : [0, \infty[\times \Lambda \rightarrow \Lambda$ such that for any $u \in \Lambda$ one has

- (η i) $\eta(0, u) = u$;
- (η ii) $\eta(\cdot, u)$ is $\mathcal{C}^1$ with $\|\dot{\eta}(t, u)\| \leq 1$;
- (η iii) $f(\eta(\cdot, u))$ is non-increasing;
- (η iv) if $f(\eta(t, u)) \geq h(\eta(t, u))$, then

$$(2.6) \quad \frac{d}{dt} (f(\eta(t, u))) \leq -\alpha \|f'(\eta(t, u))\|.$$

Proof. — Let V be the pseudogradient for f constructed in Proposition 2.2 and let us define a map $F : \Lambda \rightarrow X$ by

$$(2.7) \quad F(u) = \begin{cases} 0, & \text{if } f(u) \leq h(u) - 1; \\ \frac{V(u)}{\|V(u)\|} (f(u) - h(u) + 1), & \text{if } h(u) - 1 \leq f(u) \leq h(u); \\ \frac{V(u)}{\|V(u)\|} & \text{if } f(u) \geq h(u). \end{cases}$$

Consider the Cauchy problem

$$(2.8) \quad \begin{cases} \frac{\partial \eta}{\partial t} = -F(\eta(t, u)) \\ \eta(0, u) = u, \quad u \in \Lambda. \end{cases}$$

Since V is locally Lipschitz continuous in $\tilde{\Lambda}$ and F vanishes in a neighbourhood of Z_f , F is locally Lipschitz in Λ . In addition $\|F\| \leq 1$ and, from (2.4), there results $\langle f'(u), F(u) \rangle \geq 0$. Hence (2.8) has a unique solution $\eta(t, u)$ for any initial value $u \in \Lambda$; $\eta(\cdot, u)$ is of class $\mathcal{C}^1$ with $\|\dot{\eta}(t, u)\| \leq 1$; $f(\eta(t, u))$ is not increasing in t , because

$$\frac{d}{dt} f(\eta(t, u)) = -\langle f'(\eta(t, u)), F(\eta(t, u)) \rangle \leq 0.$$

Now with standard arguments of o.d.e. we have that $\eta = \eta(t, u)$ is defined and continuous on $[0, \infty[\times \Lambda$. Namely, if for some $u_0 \in \Lambda$ the maximal existence interval $I =]t_0, t_1[$ of $\eta(\cdot, u_0)$ is right-bounded, then there exists the limit u_1 of $\eta(t, u_0)$ as $t \nearrow t_1$. u_1 belongs to Λ , otherwise from (2.5) $\lim_{t \nearrow t_1} f(\eta(t, u_0)) = \infty$, whereas $f(\eta(t, u_0))$ is not increasing. Then η can

be continued for $t > t_1$ and I is not maximal, a contradiction. Thus η verifies (η i), (η ii) and (η iii). Finally suppose that $f(\eta(t, u)) \geq h(\eta(t, u))$. Then from (2.7) one has

$$\begin{aligned} \frac{d}{dt} f(\eta(t, u)) &= -\langle f'(\eta(t, u)), F(\eta(t, u)) \rangle \\ &= -\left\langle f'(\eta(t, u)), \frac{V(\eta(t, u))}{\|V(\eta(t, u))\|} \right\rangle. \end{aligned}$$

Then (η iv) follows, since from (2.4)

$$-\left\langle f'(\eta(t, u)), \frac{V(\eta(t, u))}{\|V(\eta(t, u))\|} \right\rangle \leq -\alpha \|f'(\eta(t, u))\|.$$

Q.E.D.

Lastly we recall the well known Palais-Smale condition. A sequence $\{u_n\} \subset \Lambda$ is a PS sequence iff $f'(u_n) \rightarrow 0$ and $f(u_n)$ is bounded; the PS condition holds in a set $Y \subset \Lambda$ (respectively, at a level $\lambda \in \mathbf{R}$) iff every PS sequence $\{u_n\} \subset Y$ (respectively, with $f(u_k) \rightarrow \lambda$) has a limit point $u \in \Lambda$.

2.4. THEOREM. — *Let X be a Banach space with norm $\|\cdot\|$, Λ an open subset of X , and suppose a functional $f: \Lambda \rightarrow \mathbf{R}$ is given such that the following conditions hold:*

- (i) $\text{Cat}_\Lambda(\Lambda) = +\infty$;
- (ii) $f \in \mathcal{C}^1(\Lambda)$ and $\forall u_n \rightarrow u \in \partial\Lambda, f(u_n) \rightarrow +\infty$;
- (iii) $\forall \lambda \in \mathbf{R}, \text{Cat}_\Lambda(\{f \leq \lambda\}) < +\infty$;

suppose in addition that there exist $g \in \mathcal{C}^1(\Lambda)$, $\beta \in]0, 1[$ and $\lambda_0 \in \mathbf{R}$ such that

- (iv) $\text{Cat}_\Lambda(\{f \leq g\}) < +\infty$;
- (v) *the PS condition holds in the set $\{f \geq g\}$;*
- (vi) $\beta \|f'(u)\| \geq \|g'(u)\|, \forall u \in \{f = g \geq \lambda_0\}$.

Then f has a sequence $\{u_n\} \subset \Lambda$ of critical points such that $f(u_n) \rightarrow +\infty$ and $f(u_n) \geq g(u_n) - 1$.

Proof. — Suppose by contradiction that $Z_f \subset \{f < \max(g, \lambda_*) - 1\}$ for some $\lambda_* \geq \lambda_0$. Let $h = \max(g, \lambda_*)$ and take $\alpha \in]\beta, 1[$: then Lemma 2.3 applies yielding a map η verifying (η i-iv). The set $A = \{f \leq h\}$ is positively invariant for the flow η : indeed, if $u \in \partial A$, either $f(u) = \lambda_*$, or $g(u) = f(u) \geq \lambda^*$. In the former case we have from (η iii) $\eta([0, \infty[, u) \subset \{f \leq \lambda_*\} \subset A$; in the latter one we get from (η iv) and (η ii)

$$\left. \frac{d}{dt}(f-g)(\eta(t, u)) \right|_{t=0} = \left. \frac{d}{dt} f(\eta(t, u)) \right|_{t=0} - \langle g'(u), \dot{\eta}(0, u) \rangle \leq -\alpha \|f'(u)\| + \|g'(u)\|;$$

and from condition (vi) (since $u \in \{f = g \geq \lambda_0\}$)

$$-\alpha \|f'(u)\| + \|g'(u)\| \leq -\alpha \|f'(u)\| + \beta \|f'(u)\| = -(\alpha - \beta) \|f'(u)\|.$$

Note that $f(u) = h(u)$ implies $u \notin Z_f$, since we have assumed $Z_f \subset \{f < h - 1\}$. Therefore

$$\forall u \in \partial A \quad \left. \frac{d}{dt}(f-g)(\eta(t, u)) \right|_{t=0} < 0.$$

Hence $\forall u \in \partial A \exists \varepsilon > 0$ such that $\eta([0, \varepsilon[, u) \subset A$, which proves that A is positively invariant for η .

Since Λ can be written as

$$\Lambda = \left(\bigcup_{k \in \mathbf{Z}} \{2k-1 \leq f \leq 2k\} \right) \cup \left(\bigcup_{k \in \mathbf{Z}} \{2k \leq f \leq 2k+1\} \right),$$

and since both $\{\{2k-1 \leq f \leq 2k\}\}_{k \in \mathbf{Z}}$ and $\{\{2k \leq f \leq 2k+1\}\}_{k \in \mathbf{Z}}$ are locally finite families of pairwise disjoint sets, we get, using Proposition 2.1,

$$\begin{aligned} \infty &= \text{Cat}_\Lambda(\Lambda) \\ &= \text{Cat}_\Lambda \left(\bigcup_{k \in \mathbf{Z}} \{2k-1 \leq f \leq 2k\} \right) + \text{Cat}_\Lambda \left(\bigcup_{k \in \mathbf{Z}} \{2k \leq f \leq 2k+1\} \right) \\ &= 2 \sup_{\lambda \in \mathbf{R}} \text{Cat}_\Lambda(\{f \leq \lambda\}). \end{aligned}$$

On the other hand, by (iii) and (iv)

$$\text{Cat}_\Lambda(A) \leq \text{Cat}_\Lambda(\{f \leq g\}) + \text{Cat}_\Lambda(\{f \leq \lambda_*\}) < \infty$$

Thus there exists a $\lambda^* > \lambda_*$ such that

$$(2.9) \quad \text{Cat}_\Lambda(\{f \leq \lambda^*\}) > \text{Cat}_\Lambda(A).$$

Consider the deformations

$$\eta|_{[0, n]} : [0, n] \times \{f \leq \lambda^*\} \rightarrow \Lambda, \quad n \in \mathbf{N}.$$

From (2.2) and (2.9) we infer that $\forall n \in \mathbf{N} \eta(n, \{f \leq \lambda^*\}) \not\subset A$, that is, $\forall n \exists u_n \in \{f \leq \lambda^*\}$ such that $\eta(n, u_n) \in \Lambda \setminus A$; moreover, since A is positively

invariant, we have in fact

$$(2.10) \quad \eta(t, u_n) \in \Lambda \setminus A = \{f > h\} \subset \{f \geq \lambda_*\}, \quad \forall t \in [0, n].$$

By the mean value theorem there exists $t_n \in [0, n]$ such that

$$(2.11) \quad \frac{d}{dt} f(\eta(t_n, u_n)) = \frac{1}{n} (f(\eta(u, u_n)) - f(\eta(0, u_n))).$$

Since from (η iii) and (2.10)

$$\lambda^* \geq f(\eta(0, u_n)) \geq f(\eta(n, u_n)) \geq \lambda_*,$$

(2.11) implies that $\frac{d}{dt} f(\eta(t_n, u_n)) \rightarrow 0$, therefore, again from (2.10) and (η iv), we have

$$f'(\eta(t_n, u_n)) \rightarrow 0.$$

Hence $u_n = \eta(t_n, u_n)$ is a PS sequence in $\{f \geq g\} \cap \{f \geq \lambda_*\}$. By condition (v) we get a critical point $u \in \Lambda$ with $f(u) \geq h(u)$, a contradiction.

Q.E.D.

2.5. *Remark.* — In the case $g = \lambda_0$, a constant, condition (iv) and (vi) are contained in the other ones, while condition (v) reduces to the more standard PS condition

(v') *There exists a $\lambda_0 \in \mathbf{R}$ such that the PS condition holds on $\{f \geq \lambda_0\}$. Namely one has*

2.6. **THEOREM.** — *Let (i), (ii), (iii), (v') hold. Then there exists a sequence $\{u_n\}$ of critical points of f such that $f(u) \rightarrow \infty$.*

The idea of using this principle in Singular Potentials is due to [1] (Rem. 2.15). We introduce conditions (iv)-(vi) because in the applications they allow us to handle a larger and more stable class of potentials than (v').

3. APPLICATION TO T -PERIODIC SOLUTIONS OF SINGULAR TIME-DEPENDENT HAMILTONIAN SYSTEMS

We recall that a potential $W \in \mathcal{C}^1(S_1^1 \times (\mathbf{R}^N \setminus \{0\}))$ satisfies the Strong Force condition [6], if the following holds:

(SF) *There exists a $U \in \mathcal{C}^1(\mathbf{R}^N \setminus \{0\})$ and a $\rho > 0$ such that*

$$\begin{cases} \lim_{x \rightarrow 0} U(x) = \infty \\ W(t, x) \leq -|U'(x)|^2, \quad \forall (t, x) \in S_1^1 \times (\mathbf{R}^N \setminus \{x\}) \quad \text{with } |x| < \rho. \end{cases}$$

Throughout this section we shall deal with a (singular) potential V of the form

$$(V) \quad V(t, x) = \frac{1}{2} a |x|^2 + W(t, x),$$

where

$$(V1) \quad a < \left(\frac{\pi}{T}\right)^2;$$

$$(V2) \quad W \in \mathcal{C}^1(S_T^1 \times (\mathbf{R}^N \setminus \{0\})) \text{ satisfies (SF);}$$

$$(V3) \quad \exists c, \theta < 2, r > 0 \text{ such that } \forall |x| \geq r, \forall t \in S_T^1$$

$$W(t, x) \leq c|x|^\theta, \quad W'(t, x) \cdot x - 2W(t, x) \leq c|x|^\theta$$

If these hypotheses hold we can also assume without loss of generality that

$$(V4) \quad W(t, x) \leq b, \forall x \in \mathbf{R}^N \setminus \{0\}.$$

Indeed, if we take $\tilde{a} \in \left] a, \left(\frac{\pi}{T}\right)^2 \right[$ and pose

$$\tilde{W}(t, x) = -\frac{1}{2}(\tilde{a} - a)|x|^2 + W(t, x),$$

(V) can be written as

$$V(t, x) = \frac{1}{2}\tilde{a}|x|^2 + \tilde{W}(t, x),$$

satisfying (V1)-(V4).

A non-collision T-periodic solution of

$$(3.1) \quad \ddot{u} + V'(t, u) = 0$$

is a $u \in \mathcal{C}^2(S_T^1, \mathbf{R}^N \setminus \{0\})$ which solves (3.1). According to the usual notation, we denote by

$$\Lambda = \{u \in H^1(S_T^1, \mathbf{R}^N) : u(t) \neq 0 \forall t \in S_T^1\}$$

the space of H^1 non-collision orbits. It is well known that the non-collision solutions of system (3.1) are the singular points of the action functional $f \in \mathcal{C}^1(\Lambda)$ defined by

$$(3.2) \quad f(u) = \int_0^T \left\{ \frac{1}{2} |\dot{u}|^2 - V(t, u) \right\} dt,$$

whose differential at $u \in \Lambda$ is the linear form

$$(3.3) \quad \langle f'(u), h \rangle = \int_0^T \{ \dot{u} \cdot \dot{h} - V'(t, u) \cdot h \} dt.$$

If $u \in \Lambda$, we denote the pericentrum of the orbit u by

$$(3.4) \quad p(u) = \min_{t \in S_T^1} |u(t)|.$$

Let us draw some consequences of conditions (V1)-(V4).

First of all we have a well known property that motivates the (SF) condition.

3.1. LEMMA. — *Let $\{u_n\} \subset \Lambda$ and $u_n \rightarrow u \in \partial\Lambda$. Then $f(u_n) \rightarrow +\infty$.*

Proof. — See [6].

Q.E.D.

3.2. LEMMA. — *For every $\lambda \in \mathbf{R}$ there exists a constant $k = k(\lambda)$ such that*

$$(3.5) \quad \|\dot{u}\|_2 \leq k(\lambda) p(u), \quad \forall u \in \{f \leq \lambda\}$$

Proof. — By the Poincaré inequality we know that

$$\|v\|_2 \leq \frac{T}{\pi} \|\dot{v}\|_2, \quad \forall v \in H_0^1(0, T; \mathbf{R}^N).$$

Thus if $u \in \Lambda$ and $t_0 \in S_T^1$ is a point where $|u(t)|$ attains its minimum value $p(u)$, since the curve $v(t) = u(t + t_0) - u(t_0)$ is in $H_0^1(0, T; \mathbf{R}^N)$ we obtain

$$(3.6) \quad \|u\|_2 \leq \frac{T}{\pi} \|\dot{u}\|_2 + \sqrt{T} p(u).$$

Condition (V) implies

$$(3.7) \quad f(u) \geq \frac{1}{2} \|\dot{u}\|_2^2 - \frac{a}{2} \|u\|_2^2 - bT, \quad \forall u \in \Lambda,$$

which yields, together with (3.6), to

$$(3.8) \quad f(u) \geq \frac{1}{2} \|\dot{u}\|_2^2 - \frac{a}{2} \left(\frac{T}{\pi} \|\dot{u}\|_2 + \sqrt{T} p(u) \right)^2 - bT.$$

Now if the claim of the lemma is false, then there exists a sequence $\{u_k\} \subset \Lambda$ such that $f(u_k)$ is bounded and

$$(3.9) \quad \|\dot{u}_k\|_2 \geq k p(u_k).$$

Putting (3.9) into (3.8), we get

$$f(u_k) \geq \frac{1}{2} \|\dot{u}_k\|_2^2 \left[1 - a \left(\frac{T}{\pi} + \frac{\sqrt{T}}{k} \right)^2 \right] - bT.$$

Since $a < \left(\frac{\pi}{T} \right)^2$, the term into square brackets is bounded away from zero for large k ; since $f(u_k)$ is bounded we conclude that $\|\dot{u}_k\|_2$ is bounded too. Then from (3.9) $p(u_k)$ tends to zero and, extracting a subsequence as needed, we may suppose that the u_k converge weakly to some $u \in \partial\Lambda$. Due to Lemma 3.1 we have $f(u_k) \rightarrow \infty$, a contradiction which proves the assertion.

Q.E.D.

3.3. LEMMA. — For every $c \in \mathbf{R}$ the set $\Lambda_c = \left\{ u \in \Lambda : \frac{\|\dot{u}\|_2}{p(u)} \leq c \right\}$ is of finite category in Λ .

Proof. — Due to Proposition (2.1) it suffices to give a deformation $h: [0, 1] \times \Lambda_c \rightarrow \Lambda$ such that $h(1, \Lambda_c) \subset \subset \Lambda$. Take $\delta \in]0, T[$ such that $c\sqrt{\delta} \leq \frac{1}{2}$, and define

$$\begin{cases} \varphi(t) = \frac{1}{\delta} & \text{if } t \in [0, \delta]; \\ \varphi(t) = 0, & \text{otherwise.} \end{cases}$$

For any $u \in \Lambda$ let $(u * \varphi)(t)$ be the convolution $\int_0^T u(t-s)\varphi(s)ds$: then we have for any t , by standard inequalities

$$\begin{aligned} |u(t) - (u * \varphi)(t)| &\leq \frac{1}{\delta} \int_0^\delta |u(t) - u(t-s)| ds \\ &\leq \sup_{|s| \leq \delta} |u(t) - u(t-s)| \leq \sqrt{\delta} \|\dot{u}\|_2 = p(u) \left(\frac{\|\dot{u}\|_2}{p(u)} \right) \sqrt{\delta}. \end{aligned}$$

Hence if u is in Λ_c ,

$$(3.10) \quad |u(t) - (u * \varphi)(t)| \leq p(u) c \sqrt{\delta} \leq \frac{1}{2} p(u) \leq \frac{1}{2} |u(t)|,$$

so that $\forall (s, t) \in [0, 1] \times [0, T]$

$$(3.11) \quad (1-s)u(t) + s \frac{(u * \varphi)(t)}{p(u)} \neq 0.$$

Thus the left-hand side of (3.11) defines a homotopy $h: [0, 1] \times \Lambda_c \rightarrow \Lambda$; furthermore $\overline{h(1, \Lambda_c)} \subset \Lambda$. Finally $h(1, \Lambda_c)$ is relatively compact since it is the image of the bounded set $\{u/p(u): u \in \Lambda_c\}$ through the convolution operator $T_\varphi: H^1 \ni u \mapsto u * \varphi \in H^1$, which is compact.

Q.E.D.

3.4. LEMMA. — Let $V \in \mathcal{C}^1(S_T^1 \times (\mathbf{R}^N \setminus \{0\}))$ and let SF hold. The functional f verify the PS condition on the bounded sets.

Proof. — Let $\{u_k\}$ be a H^1 -bounded PS sequence. Then, up to a subsequence, it converges weakly in H^1 and strongly in L^∞ to an element u of $H^1(S_T^1, \mathbf{R}^N)$ which belongs to Λ by Lemma (3.1). Hence $V'(t, u_k) \cdot (u - u_k)$ converges uniformly to zero. Since $f'(u_k) \rightarrow 0$ in H^{-1} and

$u - u_k$ is H^1 -bounded we have, from (3.3)

$$\begin{aligned} \|\dot{u}\|_2^2 - \lim_{k \rightarrow \infty} \|\dot{u}_k\|_2^2 &= \lim_{k \rightarrow \infty} \int_0^T \dot{u}_k \cdot (\dot{u} - \dot{u}_k) \\ &= \lim_{k \rightarrow \infty} \left\{ \langle f'(u), u - u_k \rangle + \int_0^T V'(t, u_k) \cdot (u - u_k) \right\} = 0. \end{aligned}$$

Therefore u_k converges to u strongly in H^1 .

Q.E.D.

3.5. THEOREM. — *Let V be a T -periodic time-dependent potential satisfying (V). Then the dynamical system*

$$\ddot{u} + V'(t, u) = 0$$

has infinitely many T -periodic non-collision solutions.

Proof. — We have to check the hypotheses of Theorem 2.4

(i) See [3].

(ii) Lemma 3.1.

(iii) Lemma 3.2 and Lemma 3.3.

Now we shall define $g \in \mathcal{C}^1(\Lambda)$, $\beta \in]0, 1[$ and $\lambda_0 \in \mathbf{R}$ verifying (iv, v, vi). Let k_∞ be a constant such that

$$(3.12) \quad \|u\|_\infty \leq k_\infty \|u\|_{1,2}, \quad \forall u \in H^1(S_T^1, \mathbf{R}^N),$$

[e.g., $k_\infty = (T + T^{-1})^{1/2}$], and choose $\beta \in \left] \frac{\theta}{2}, 1 \right[$. We define

$$g(u) = \gamma \|u\|_{1,2}^\theta, \quad \forall u \in \Lambda,$$

where

$$(3.13) \quad \gamma \geq \frac{\beta c T k_\infty^\theta}{2\beta - \theta}.$$

(iv) We have to show that $\{f \leq g\}$ is a set of finite category in Λ . Let us take $\varepsilon > 0$ such that

$$a_\varepsilon = \frac{a + 2\varepsilon}{1 - 2\varepsilon} < \left(\frac{\pi}{T} \right)^2,$$

$M \in \mathbf{R}$ such that $\forall s \in \mathbf{R} \gamma |s|^\theta \leq \varepsilon s^2 + (1 - 2\varepsilon)M$, and define

$$f_\varepsilon(u) = \int_0^T \left\{ \frac{1}{2} |\dot{u}|^2 - \frac{a_\varepsilon}{2} |u|^2 - \frac{W(t, u)}{1 - 2\varepsilon} \right\} dt.$$

Then

$$\{f \leq g\} \subset \{f \leq \varepsilon \|u\|_{1,2}^2 + (1 - 2\varepsilon)M\} = \{f_\varepsilon \leq M\}.$$

Again we have from Lemma 3.2 that there exists $k \in \mathbf{R}$ such that

$$(3.14) \quad \|\dot{u}\|_2 \leq kp(u), \quad \forall u \in \{f_\varepsilon \leq M\}$$

and by Lemma 3.3,

$$\text{Cat}_\Lambda(\{f \leq g\}) \leq \text{Cat}_\Lambda(\{f \leq M\}) < \infty.$$

(v) For any $\lambda \in \mathbf{R}$ $\{f \geq g\} \cap \{f \leq \lambda\} \subset \{g \leq \lambda\}$ is a bounded set because g is coercive. Therefore by Lemma 3.5 the PS condition holds in $\{f \geq g\}$.

(vi) From (3.6) and (4.14) we find, for some $k_1 > 0$,

$$(3.15) \quad \|u\|_{1,2} \leq k_1 p(u), \quad \forall u \in \{f \leq g\}.$$

We take $\lambda_0 =: \gamma(k_1 r)^\theta$. Then if $u \in \{f = g \geq \lambda_0\}$, there results

$$\|u\|_{1,2} \geq \left(\frac{\lambda_0}{\gamma}\right)^{1/\theta} = k_1 r$$

so we have from (3.12) and (3.15)

$$r \leq p(u) \leq |u(t)| \leq \|u\|_\infty \leq k_\infty \|u\|_{1,2}, \quad \forall t \in S_T^1.$$

Now, taking account of (V3) we get

$$(3.16) \quad \int_0^T \{W'(t, u) \cdot u - 2W(t, u)\} dt \\ \leq T \sup \{W'(t, x) \cdot x - 2W(t, x) : t \in S_T^1, r \leq |x| \leq k_\infty \|u\|_{1,2}\} \\ \leq c T (k_\infty \|u\|_{1,2})^\theta.$$

From (3.2) and (3.3) we get

$$(3.17) \quad \|f'(u)\| \|u\|_{1,2} \geq \langle f'(u), u \rangle = 2f(u) \\ - \int_0^T \{W'(t, u) \cdot u - 2W(t, u)\} dt.$$

From (3.16) and (3.17)

$$\|f'(u)\| \geq (2\gamma - c T k_\infty^\theta) \|u\|_{1,2}^{\theta-1};$$

since $\|g'(u)\| = \gamma \theta \|u\|_{1,2}^{\theta-1}$, we have, from our choice of γ (3.13)

$$\beta \|f'(u)\| - \|g'(u)\| \geq 0, \quad \forall u \in \{f = g \geq \lambda_0\}.$$

Q.E.D.

3.6. *Remark.* — Theorem 3.5 can be improved stating that there exists a sequence $\{u_n\} \subset Z_f$ such that $f(u_n) \geq n \|u\|^\theta + n$. This follows at once from Theorem 2.4, for in the definition of the function g we can choose the constant γ arbitrarily large [eq. (3.13)]. We shall use this fact in the following corollary, as a trick to avoid the constant solutions (see also [1], § 7).

3.7. COROLLARY (Autonomous case). — Let $W \in \mathcal{C}^1(\mathbf{R}^N \setminus \{0\})$ be a potential such that (SF) holds, $W'(x) \cdot \frac{x}{|x|^2} \rightarrow +\infty$ as $|x| \rightarrow 0$, and $W'(x) \cdot x - 2W(x) \leq c|x|^\theta$ for $|x| \geq r$, with $\theta < 2$. Then for any $T > 0$ and

for any $a \in \mathbf{R}$, the system

$$(3.18) \quad \ddot{u} + au + W'(u) = 0$$

has infinitely many T -periodic non-constant non-collision solution.

Proof. — The inequality $W'(x) \cdot x - 2W(x) \leq c|x|^0$, $\forall |x| \geq r$ yields by integration $W(x) \leq c_1|x|^2$, $\forall |x| \geq r$. Hence, replacing if needed W with $W - c_1|x|^2$ and a with $a + c_1$, we can suppose without loss of generality that W is bounded from above. We take $k \in \mathbf{N}$ so large that $a < k^2 \left(\frac{\pi}{T} \right)^2$,

and we pose $\tilde{T} = \frac{T}{k}$. Now we look for $\tilde{T}$ -periodic non collision solutions

of system (3.18): Theorem 3.6 applies and we get a sequence $\{u_n\} \subset \Lambda$ of solutions such that $f(u_n) \geq n \|u_n\|_\infty^0 + n$ (Rem. 3.6). Only finitely many of these can be constant: for otherwise (taking the subsequence of the constant solutions) we would get from (3.18), by scalar product with u_n

$$(3.19) \quad a|u_n|^2 + W'(u_n) \cdot u_n = 0,$$

and

$$(3.20) \quad f(u_n) = \int_0^{\tilde{T}} \left\{ -\frac{a}{2}|u_n|^2 - W(u_n) \right\} dt = \frac{\tilde{T}}{2} \{ W'(u_n) \cdot u_n - 2W(u_n) \}$$

Since $f(u_n) \rightarrow \infty$ either $|u_n| \rightarrow 0$ or $|u_n| \rightarrow \infty$. In the former case it follows from our hypothesis on W that $W'(u_n) \cdot \frac{u_n}{|u_n|^2} \rightarrow \infty$, which is in contradiction with (3.19). In the latter one we have from (3.20) that $f(u_n) \leq \frac{\tilde{T}}{2} c|u_n|^0$ for large n , whereas $f(u_n) \geq n|u_n|^0$: a contradiction again.

Q.E.D.

4. FURTHER REMARKS

We emphasize that condition (V) does not imply the usual PS condition (iii)' of Theorem 2.6, even if we assume $\lim_{x \rightarrow \infty} V(t, x) = 0$: we shall show this in Example 4.1. However, if additional hypotheses on V are assumed, such as

$$\limsup_{x \rightarrow \infty} |W(t, x)| + |W'(t, x)| < \infty,$$

then (iii)' holds and Theorem 2.6 applies.

4.1. *Example.* — A potential $V \in \mathcal{C}^1(\mathbb{R}^N)$ satisfying

$$V \leq 0, \quad \lim_{x \rightarrow \infty} V(x) = 0, \quad |V'(x)| \leq |x|^{1/2}$$

(hence also the hypotheses of Theorem 3.6) and such that the corresponding action functional f does not verify the usual PS condition at any positive level.

Let $\{q_n\}_{n \in \mathbb{N}}$ be an enumeration of $\mathbf{Q}^+$ and $\{x_n\}_{n \in \mathbb{N}}$ a sequence in $\mathbb{R}^N$ such that $x_n \rightarrow \infty$, $|x_n| \geq (q_n + 1)^2 + q_n + 1$ and $|x_n - x_m| > q_n + q_m + 2$ if $n \neq m$. For any $n \in \mathbb{N}$ let $\varphi_n \in \mathcal{C}_c^\infty(\mathbb{R}^+)$ be such that

$$\begin{cases} 0 \geq \varphi_n(t) \geq -\frac{1}{n}, & \forall t \geq 0; \\ \varphi_n(t) = 0, & \text{if } t \geq q_n + 1; \\ \varphi'_n(q_n) = q_n; \\ \|\varphi'_n\|_\infty \leq q_n + 1. \end{cases}$$

Define $V_n(x) = \varphi_n(|x - x_n|)$ for every $x \in \mathbb{R}^N$, and let $w \in \mathcal{C}^\infty(S^1_{2\pi}, \mathbb{R}^N)$ satisfy

$$(4.1) \quad \begin{cases} \ddot{w} + w = 0, \\ |w(t)| = 1. \end{cases}$$

Then $u_n = x_n + q_n w$ is a 2π -periodic solution of the system

$$\ddot{u} + V'_n(u) = 0.$$

Since the V_n have disjoint supports it is defined a potential $V = \sum_n V_n$ of class $\mathcal{C}^\infty$ such that $V \leq 0$ and $V(x) \rightarrow 0$ as $x \rightarrow \infty$. Moreover $V(x) \leq |x|^{1/2} \forall x$: if $V(x) \neq 0$, then there exists $n \in \mathbb{N}$ such that $x \in B(x_n, q_n + 1)$, so one has, by the choice of x_n ,

$$|x| \geq |x_n| - (q_n + 1) \geq (q_n + 1)^2$$

and

$$\|V'(x)\| \leq \|V'_n(x)\| \leq \|\varphi'_n\|_\infty \leq q_n + 1 \leq |x|^{1/2}.$$

Each u_n solves

$$\begin{cases} \ddot{u} + V'(u) = 0 \\ u(t) = u(t + 2\pi), \end{cases}$$

Thus for any $n \in \mathbb{N}$

$$\begin{aligned} f'(u_n) &= 0, \\ f(u_n) &= \pi q_n^2 - 2\pi \varphi_n(q_n) \\ \|u_n\|_{1,2} &\rightarrow \infty. \end{aligned}$$

Since $\varphi_n(q_n) \rightarrow 0$ as $n \rightarrow \infty$, one has that for any $\lambda \in \mathbb{R}^+$ there exists a subsequence of $\{u_n\}$ which is a non-compact PS sequence at the level λ

for f . Of course the same example can be done for a singular potential, simply adding to V a singular perturbation with compact support.

4.2. *Remark.* — Notice that if V is autonomous no assumptions on the coefficient a are needed in order to get infinitely many T -periodical solution of (3.1). In the following example we show that if we drop condition $a < \left(\frac{\pi}{T}\right)^2$, (iv) and (v) in general fail to hold.

4.3. *Example.* — A potential $V \in \mathcal{C}^1(\mathbf{R}^N \setminus \{0\})$ such that the corresponding action functional f does not verify conditions (iv) and (v).

Let $a > \left(\frac{\pi}{T}\right)^2$, and let $V \in \mathcal{C}^1(\mathbf{R}^N \setminus \{0\})$ be such that $V(x) \geq \frac{1}{2}a|x|^2$, $\forall x$ with $|x| \geq 1$. We show that for any λ_0 and ρ_0 ,

$$\text{Cat}_\Lambda(\{f \leq \lambda_0\} \setminus B(0, \rho_0)) = \infty;$$

in order to do this it is sufficient to exhibit a deformation of a set of infinite category, e. g., $A = \{u \in \Lambda : |u(t)| = 1 \forall t\}$, in $\{f \leq \lambda_0\} \setminus B(0, \rho_0)$.

Choose T^* in $\left[\frac{\pi}{\sqrt{a}}, T\right]$, and define the functions $[0, 1] \times [0, T] \rightarrow \mathbf{R}$

$$g(s, t) = \begin{cases} 0, & \text{if } 0 \leq t \leq sT^* \\ T \frac{t - sT^*}{T - sT^*}, & \text{if } sT^* < t \leq T, \end{cases}$$

$$l(s, t) = \begin{cases} s \sin\left(\frac{\pi t}{sT^*}\right), & \text{if } 0 \leq t \leq sT^* \\ 0, & \text{if } sT^* < t \leq T. \end{cases}$$

Consider the homotopy $h: [0, 1] \times A \rightarrow \Lambda$:

$$h(s, u) = u \circ g(s, \cdot),$$

and set $B = h(1, A)$: clearly every $u \in B$ is constant on $[0, T^*]$. For $r \in \mathcal{C}(B)$, $r \geq 0$ consider the homotopy $k: [0, 1] \times B \rightarrow \Lambda$:

$$k(s, u) = u + r(u)u(0)l(s, \cdot),$$

We shall choose r in such a way that $k(1, B) \subset \{f \leq \lambda_0\} \setminus B(0, \rho_0)$. In order to do this, we note that

$$(4.2) \quad \|k(1, u)\|_{1,2} \geq C \|k(1, u)\|_\infty \geq C \left| k(1, u) \left(\frac{T^*}{2} \right) \right|$$

$$= C \left| u \left(\frac{T^*}{2} \right) + r(u)u(0) \right| = C(r(u) + 1)|u(0)| \geq Cr(u)|u(0)|.$$

Thus $\|k(1, u)\|_{1,2} \geq \rho_0$ whenever $r(u) \geq \frac{\rho_0}{C}$. Furthermore, making the positions

$$r_1(u) = \int_{T^*}^T \left\{ \frac{1}{2} |\dot{u}|^2 - V(u) \right\} dt$$

and

$$\mu = -\frac{1}{2} \int_0^{T^*} \left\{ \left(\frac{\pi}{T^*} \right)^2 \cos^2 \left(\frac{\pi}{T^*} t \right) - a \sin^2 \left(\frac{\pi}{T^*} t \right) \right\} dt = \frac{T^*}{4} \left[a - \left(\frac{\pi}{T^*} \right)^2 \right],$$

there results

$$\begin{aligned} (4.3) \quad & f(k(1, u)) \\ & \leq \frac{1}{2} \int_0^{T^*} \left\{ \left(\frac{\pi}{T^*} \right)^2 r(u)^2 \cos^2 \left(\frac{\pi}{T^*} t \right) - a \left[r(u) \sin \left(\frac{\pi}{T^*} t \right) + 1 \right]^2 \right\} dt + r_1(u) \\ & \leq \frac{1}{2} r(u)^2 \int_0^{T^*} \left\{ \left(\frac{\pi}{T^*} \right)^2 \cos^2 \left(\frac{\pi}{T^*} t \right) - a \sin^2 \left(\frac{\pi}{T^*} t \right) \right\} dt + r_1(u) \\ & = -\mu r(u)^2 + r_1(u). \end{aligned}$$

Since $\mu > 0$ and $r_1 \in \mathcal{C}(B)$, if we take

$$r(u) = \max \left(\frac{\rho_0}{C}, \sqrt{\frac{|r_1(u) - \lambda_0|}{\mu}} \right)$$

we have from (4.2) and (4.3) that $\|k(1, u)\|_{1,2} \geq \rho_0$ and $f(k(1, u)) \leq \lambda_0$. Then we have

$$\text{Cat}_\Lambda(\{f \leq \lambda_0\} \setminus B(0, \rho_0)) \geq \text{Cat}_\Lambda(B) = \text{Cat}_\Lambda(h(1, A)) \geq \text{Cat}_\Lambda(A) = \infty.$$

Q.E.D.

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