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Classical solvability in dimension two of the second boundary-value problem associated with the Monge-Ampère operator

by

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ABSTRACT. — Given two bounded strictly convex domains of $\mathbb{R}^n$ and a positive function on their product, all data being smooth, find a smooth strictly convex function whose gradient maps one domain onto the other with Jacobian determinant proportional to the given function. We solve this problem under the (technical) condition $n=2$.

Key words : Strictly convex functions, prescribed gradient image, Monge-Ampère operator, continuity method, *a priori* estimates.

RÉSUMÉ. — Soit deux domaines bornés strictement convexes de $\mathbb{R}^n$ et une fonction positive définie sur leur produit, ces données étant lisses, trouver une fonction lisse strictement convexe dont le gradient applique un domaine sur l'autre avec déterminant Jacobien proportionnel à la fonction donnée. Nous résolvons ce problème sous la condition (technique) $n=2$.

Classification A.M.S. : 35 J 65, 35 B 45, 53 C 45.

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I. INTRODUCTION

Let D and D^* be bounded C^∞ strictly convex domains of $\mathbb{R}^n$. We denote by $S(D, D^*)$ the subset of $C^\infty(\bar{D})$ consisting of strictly convex real functions ⁽¹⁾ whose gradient maps D onto D^* . Given any $u \in C^\infty(\bar{D})$, we denote by $A(u)$ the Jacobian determinant of the gradient mapping $x \rightarrow du(x)$. The nonlinear second order differential operator A is called the *Monge-Ampère* operator on D . Basic features of A restricted to $S(D, D^*)$ are listed in the preliminary

PROPOSITION 1. — A sends $S(D, D^*)$ into

$$\Sigma := \{f \in C^\infty(\bar{D}), f > 0, \langle f \rangle = |D^*|/|D|\}$$

($\langle f \rangle$ denotes the average of f over D and $|D|$, the Lebesgue measure of D). On $S(D, D^*)$, A is elliptic and its derivative is divergence-like. Given any defining function h^* of D^* , the boundary operator $u \rightarrow B(u) := h^*(du)|_{\partial D}$ is co-normal with respect to A on $S(D, D^*)$. Furthermore, given any $u \in S(D, D^*)$ and any $x \in \partial D$, the co-normal direction at x with respect to the derivative of A at u is nothing but the normal direction of ∂D^* at $du(x)$.

We postpone the proof of proposition 1 till the end of this section. The second boundary-value problem consists in showing that $A : S(D, D^*) \rightarrow \Sigma$ is onto. More generally, we wish to solve in $S(D, D^*)$ two kinds of equations namely

$$\text{Log } A(u) = f(x, du) + \langle u \rangle \quad (1)$$

$$\text{Log } A(u) = F(x, du, u) \quad (2)$$

where $f \in C^\infty(\bar{D} \times \bar{D}^*)$ and $F \in C^\infty(\bar{D} \times \bar{D}^* \times \mathbb{R})$, the latter being uniformly increasing in u . We aim at the following

THEOREM. — Equations (1) and (2) are uniquely solvable in $S(D, D^*)$ provided $n=2$.

The second boundary-value problem was first posed and solved (with $n=2$ but the methods, geometric in nature, extend to any dimension) in a generalized sense in [18] chapter V section 3 (see also [3] theorem 2, where the whole plane is taken in place of D). The elliptic Monge-Ampère operator with a quasilinear Neumann boundary condition is treated in [16], in any dimension, and it is further treated with a quasilinear oblique boundary condition in [21] provided $n=2$. A general study of nonlinear oblique boundary-value problems for nonlinear second order uniformly

⁽¹⁾ Here the meaning of "strictly convex" is restricted to having a positive-definite hessian matrix, which rules out e.g. the strictly convex function $u(x) = |x - y|^4$ near $y \in D$, as pointed out to us by Martin Zerner.

elliptic equations is performed in [15]. Quite recently, the following problem was solved [5]: existence and regularity on a given bounded domain D of $\mathbb{R}^n$ (no convexity assumption, no restriction on n) of a *diffeomorphism* from $\bar{D}$ to itself, reducing to the identity on ∂D , with prescribed positive Jacobian determinant (of average 1 on D).

Remarks. — 1. The restriction $n=2$ is unsatisfactory but we could not draw second order boundary estimates without it. In May 1988, in Granada (Spain), Neil Trudinger informed us that Kai-Sing Tso had treated the problem in *any* dimension; however, from that time on, Tso's preprint has not been available due to a serious gap in his proof, as he himself wrote us [20]. In June 1989, John Urbas visited us in Antibes and he kindly advised us to submit our own 2-dimensional result; it is a pleasure to thank him for his thorough reading of the present paper. This may be the right place to thank also the Referee for pointing out a mistake at the end of the original proof of proposition 2 below, and a few inaccuracies (particularly one in remark 6).

2. We do *not* assume the non-emptiness of $S(D, D^*)$ to prove the theorem; we thus *obtain* it (when $n=2$) as a by-product of our proof. In fact, we found no straightforward way of exhibiting any member of $S(D, D^*)$ —except, of course, if $D=D^*$ —, although we can write down explicitly a $C^\infty(\bar{D})$ convex (but not *strictly* convex) function with gradient image D^* , constructed from any suitable *support function* for D^* . Provided non-emptiness, it is possible to prove that $S(D, D^*)$ is a locally closed Fréchet submanifold of the open subset of strictly convex functions in $C^\infty(\bar{D})$, as the fiber of a submersion.

3. From the proof below, it appears that, given any $\alpha \in (0, 1)$ $C^{2,\alpha}(\bar{D})$ solutions may be derived (by approximation) from the above theorem under the sole regularity assumptions: D and D^* are $C^{2,1}$, f and F are $C^{1,1}$. We did not study further 2-dimensional global regularity refinements as done in [19], [14] for the Dirichlet problem.

4. The uniqueness for (1) shows that, in general, the equation $\text{Log } A(u) = f(x, du)$ is *not* well-posed on $S(D, D^*)$. The idea of introducing in (1) the average term goes back to [6] and it proved to be useful in various contexts ([2], [8], [9], [10]). If $u \in S(D, D^*)$ solves (1), then $v = u + \text{Const.}$ solves in $S(D, D^*)$ the equation $\text{Log } A(v) = f(x, dv) + \langle u \rangle$, while the Legendre transform v^* of v solves in $S(D^*, D)$ the “dual” equation $\text{Log } A(v^*) = -f(dv^*, x) - \langle u \rangle$. In case $f(x, x^*) = f_1(x) - f_2(x^*)$, the value of $\langle u \rangle$ is *a priori* fixed by the constraint (due to the “Jacobian” structure of A)

$$\int_{D^*} e^{f_2(x^*)} dx^* = e^{\langle u \rangle} \int_D e^{f_1(x)} dx.$$

The prescribed Gauss-curvature equation is an example of this type.

Proof of proposition 1. — By its very definition, as the *Jacobian* of the gradient mapping, A readily sends $S(D, D^*)$ into the submanifold Σ .

Let $u \in S(D, D^*)$. In euclidean co-ordinates $(x^1, \dots, x^n)$, $A(u)$ reads

$$A(u) = \det(u_{ij})$$

and the derivative of A at u reads

$$\delta u \in C^\infty(\bar{D}) \rightarrow dA(u)(\delta u) = A^{ij}(\delta u)_{ij}$$

where

$$A^{ij} = A(u) u^{ij}$$

(indices denote partial derivatives, Einstein's convention holds, (u^{ij}) is the matrix inverse of (u_{ij}) and (A^{ij}) , its co-matrix). Since u is strictly convex, A is indeed *elliptic* at u . Furthermore, one easily verifies the following identity: for any $\delta u \in C^\infty(\bar{D})$,

$$A^{ij}(\delta u)_{ij} \equiv [A^{ij}(\delta u)]_j.$$

So, as asserted, $dA(u)$ is *divergence-like*. The co-normal boundary operator associated with A at u is

$$\delta u \in C^\infty(\bar{D}) \rightarrow \beta(\delta u) = A^{ij} N^i (\delta u)_j \in C^\infty(\partial D),$$

N standing for the outward unit normal on ∂D . Fix a defining function h^* for D^* (i. e. $h^* \in C^\infty(\bar{D}^*)$ is strictly convex and vanishes on ∂D^*). Since $u \in S(D, D^*)$, the function $H := h^*(du) \in C^\infty(\bar{D})$ is negative inside D and vanishes on ∂D . Moreover, a straightforward computation yields in D :

$$u^{ij} H_{ij} - u^{ij} [\text{Log } A(u)]_i H_j = u_{ij} h_j^* > 0.$$

Hopf's lemma [12] implies that $H_N > 0$ on ∂D . Since

$$H_i = u_{ij} h_j^*$$

the boundary operators satisfy

$$A(u) dB(u) = H_N \beta.$$

So B is indeed *co-normal* with respect to A at u .

Last, the geometric interpretation of the co-normal direction β given at the end of proposition 1, simply follows from the fact that $dB(u)(x)$ equals the derivative in the direction of $dh^*[du(x)]$ which is precisely (outward) *normal* to ∂D^* at $du(x)$. $\square$

II. THE CONTINUITY METHOD

Fix $(x_0, x_0^*) \in D \times D^*$ and $\lambda \in (0, 1]$ such that the gradient of

$$v_0 = \frac{\lambda}{2} |x - x_0|^2 + x_0^* \cdot x$$

maps $\bar{D}$ into D^* ($|\cdot|$ stands for the standard euclidean norm, $\cdot$ for the euclidean scalar product). Set $u_0 := v_0 - \langle v_0 \rangle$, $D_0 := du_0(D)$. A routine verification shows that D_0 is C^∞ strictly convex. Let $t \in [0, 1] \rightarrow D_t$ be a smooth path of bounded C^∞ strictly convex domains connecting D_0 to $D_1 = D^*$, with $D_t \subset D_{t'}$ for $t < t'$; fix $t \rightarrow h_t$ a smooth path of corresponding defining functions. For each $t \in [0, 1]$, consider in $S(D, D_t)$ the two following equations:

$$\text{Log } A(u) = tf(x, du) + (1-t)n \text{Log } \lambda + \langle u \rangle \quad (1. t)$$

$$\text{Log } A(u) = tF(x, du, u) + (1-t)(u - u_0 + n \text{Log } \lambda). \quad (2. t)$$

By construction u_0 solves both equations for $t=0$, so (for $i=1, 2$) the sets $T_i := \{t \in [0, 1], (i. t) \text{ admits a solution in } S(D, D_t)\}$ are non-empty. Hereafter, we show that they are both relatively open and closed in $[0, 1]$: if so, by connectedness, they coincide with all of $[0, 1]$. The solutions for $t=1$ are those announced in the theorem; their uniqueness is established at the end of this section.

Let us show that T_1 is relatively open in $[0, 1]$; similar, more standard (due to the monotonicity assumption of F), reasonings hold for T_2 . Fix $\alpha \in (0, 1)$ and denote by $U^{2, \alpha}$ the open subset of $C^{2, \alpha}(\bar{D})$ consisting of strictly convex functions. On $[0, 1] \times U^{2, \alpha}$, consider the smooth map (M, B) defined by

$$\begin{aligned} M(t, u) &:= \text{Log } A(u) - tf(x, du) - (1-t)n \text{Log } \lambda - \langle u \rangle, \\ B(t, u) &:= h_t(du)|_{\partial D}, \end{aligned}$$

and ranging in $C^{0, \alpha}(\bar{D}) \times C^{1, \alpha}(\partial D)$. Let $t_0 \in T$; there thus exists u_0 in $U^{2, \alpha}$ such that $(M, B)(t_0, u_0) = (0, 0)$. The proof is based on the Banach implicit function theorem applied to (M, B) at (t_0, u_0) . We want to show that the map

$$(m, b) := [M_u(t_0, u_0), B_u(t_0, u_0)] : C^{2, \alpha}(\bar{D}) \rightarrow C^{0, \alpha}(\bar{D}) \times C^{1, \alpha}(\partial D)$$

is an *isomorphism*. Record the following expression of (m, b) in euclidean co-ordinates:

$$\begin{aligned} m(\delta u) &= u_0^{ij}(\delta u)_{ij} - t_0 f_{u_i}(x, du_0)(\delta u)_i - \langle \delta u \rangle, \\ b(\delta u) &= (h_t)_i(du_0)(\delta u)_i. \end{aligned}$$

From proposition 1, we know that b is *oblique*; so Hopf's maximum principle [11] combined with Hopf's lemma [12] imply that any $\delta u \in \text{Ker}(m, b)$ is *constant*, hence actually $\langle \delta u \rangle = 0$ and $\delta u \equiv 0$. Therefore (m, b) is *one-to-one*.

Now we fix $(\delta M_0, \delta B_0) \in C^{0, \alpha}(\bar{D}) \times C^{1, \alpha}(\partial D)$ and we look for δu in $C^{2, \alpha}(\bar{D})$ solving: $(m, b)(\delta u_0) = (\delta M_0, \delta B_0)$. Consider the auxiliary map

$$(m', b') := \{A(u_0)(m + \langle \cdot \rangle), [A(u_0)/H_N]b\},$$

where $H = h_t(du_0)$. It follows from proposition 1 that, given any $(\delta M', \delta B') \in C^{0, \alpha}(\bar{D}) \times C^{1, \alpha}(\partial D)$, the function $\delta u' \in C^{2, \alpha}(\bar{D})$ solves:

$$(m', b')(\delta u') = (\delta M', \delta B'), \quad (3)$$

if and only if, for every $\delta v' \in W^{1,2}(D)$,

$$L(\delta u', \delta v') = \int_{\partial D} \delta B' \delta v \, da - \int_D \delta M' \delta v \, dx$$

(da is the measure induced on ∂D by dx), where L is the continuous bilinear form on $W^{1,2}(D)$ given by

$$L(\delta u', \delta v') := \int_D A(u_0) [u_0^{ij} (\delta u')_i (\delta v')_j + t_0 f_{u_i}(x, du_0) (\delta u')_i \delta v'] \, dx.$$

Let us argue on (m', b') as in [6]. Combining the ellipticity of m' and the obliqueness of b' (asserted by proposition 1), with Hopf's maximum principle, Schauder's estimates and Fredholm's theory of *compact* operators, we know that the kernel of the adjoint of (m', b') (formally obtained by varying the first argument of L instead of the second, and by integrating by parts) is *one-dimensional*, let $\delta w \in C^{2,\alpha}(\bar{D})$ span it, and that (3) is solvable up to an additive constant if and only if

$$\int_{\partial D} \delta B' \delta w \, da - \int_D \delta M' \delta w \, dx = 0. \quad (4)$$

Observe that

$$\int_D A(u_0) \delta w \, dx \neq 0$$

since, otherwise, one could solve (3) with $(\delta M', \delta B') = [A(u_0), 0]$ contradicting the maximum principle. We may thus normalize δw by

$$\int_D A(u_0) \delta w \, dx = 1.$$

Then we can solve (3) with right-hand side equals:

$$\left\{ A(u_0) \left[\delta M_0 + \int_{\partial D} [A(u_0)/H_N] \delta B_0 \delta w \, da - \int_D A(u_0) \delta M_0 \delta w \, dx \right], [A(u_0)/H_N] \delta B_0 \right\}$$

since the latter satisfies (4). If $\delta u'_0$ is a solution, then

$$\delta u_0 = \delta u'_0 - \langle \delta u'_0 \rangle + \int_{\partial D} [A(u_0)/H_N] \delta B_0 \delta w \, da - \int_D A(u_0) \delta M_0 \delta w \, dx$$

solves the original equation

$$(m, b)(\delta u_0) = (\delta M_0, \delta B_0).$$

So (m, b) is also *onto*. The implicit function theorem thus implies the existence of a real $\delta > 0$ and of a smooth map

$$t \in (t_0 - \delta, t_0 + \delta) \cap [0, 1] \rightarrow u_t \in U^{2, \alpha}$$

such that $(M, B)(t, u) = (0, 0)$. By proposition 1 and standard elliptic regularity [1], $u_t \in S(D, D_t)$, hence T_1 is relatively open. $\square$

Assuming $n=2$, we shall carry out a $C^{2, \alpha}(\bar{D})$ *a priori* bound, independent of $t \in [0, 1]$, on the solutions in $S(D, D_t)$ of equations (1. t) and (2. t). Provided such a bound exists, the closedness of T_i ($i=1, 2$) follows in a standard way from Ascoli's theorem combined with proposition 1 and elliptic regularity [1].

Last, let us prove that (1) admits *at most one* solution in $S(D, D^*)$; a similar argument holds for (2). By contradiction, let u_0 and u_1 be two distinct solutions of (1) in $S(D, D^*)$. Then, for $t \in [0, 1]$, $u_t := tu_1 + (1-t)u_0 \in S(D, D^*)$ and $u := u_1 - u_0$ solves the linear boundary-value problem:

$$\begin{aligned} \left(\int_0^1 u_t^{ij} dt \right) u_{ij} - \left[\int_0^1 f_{u_t}(x, du_t) dt \right] u_i - \langle u \rangle &= 0 \quad \text{in } D, \\ \left[\int_0^1 (h_1)_i(du_t) dt \right] u_i &= 0 \quad \text{on } \partial D \end{aligned}$$

which is elliptic inside D and oblique on ∂D by proposition 1. The maximum principle implies $u \equiv 0$, contradicting the assumption. $\square$

III. PRELIMINARY *A PRIORI* ESTIMATES

In this section, we do *not* need yet the condition $n=2$. For any $v \in S(D, D_t)$, $dv \in D^*$, hence $|dv|$ is bounded above by $\rho(D^*) := \max_{x^* \in D^*} |x^*|$.

Set $|f|_0 = \max_{\bar{D} \times \bar{D}^*} |f(x, x^*)|$, and let $u \in S(D, D_t)$ solve (1. t), then

$$e^{-|f|_0} A(u) \leq e^{\langle u \rangle} \leq e^{|f|_0 + n |\text{Log } \lambda|} A(u).$$

Integrating this over D yields for $\langle u \rangle$ the pinching:

$$\text{Log} |D_0| - |f|_0 \leq \langle u \rangle \leq \text{Log} |D^*| + |f|_0 + n |\text{Log } \lambda|.$$

Since $|du| \leq \rho(D^*)$, u is *a priori* bounded in $C^1(\bar{D})$ in terms of $|D^*|$, $\rho(D^*)$, $|f|_0$, $|D_0|$, λ and n .

By assumption, there exists $\mu \in (0, 1]$ such that $F_u \geq \mu$ on $\bar{D} \times \bar{D}^* \times \mathbb{R}$. The right-hand side of equation (2. t), let us denote it by

$$f(t, x, du, u),$$

thus satisfies $f_u \geq \mu$ as well, on $[0, 1] \times \bar{D} \times \bar{D}^* \times \mathbb{R}$. Let $u \in S(D, D_t)$ solve (2. t). Set

$$M := \max_{\bar{D}} (u), \quad m := \min_{\bar{D}} (u)$$

$$M_0 := \max_{[0, 1] \times \bar{D} \times \bar{D}^*} [f(t, x, x^*, 0)], \quad m_0 := \min_{[0, 1] \times \bar{D} \times \bar{D}^*} [f(t, x, x^*, 0)].$$

From the mean value theorem, we know that

$$M - m \leq \rho(D^*) \delta(D),$$

$\delta(D)$ standing for the diameter of D . If $M \geq 0$ and $m \leq 0$, it implies $|u| \leq \rho(D^*) \delta(D)$ and we are done. If not, say for instance $M < 0$, then $A(u) = \exp[f(t, x, du, u)] \leq \exp[M_0 + \mu M]$. Integrating this over D yields: $\mu M \geq [\text{Log}(|D_0|/|D|) - M_0]$, hence under our assumption $[\text{Log}(|D_0|/|D|) - M_0] < 0$ and

$$-m = \max_{\bar{D}} |u| \leq \rho(D^*) \delta(D) + [M_0 - \text{Log}(|D_0|/|D|)]/\mu.$$

Similarly, $m > 0$ yields $[\text{Log}(|D^*|/|D|) - m_0] > 0$ and

$$M = \max_{\bar{D}} |u| \leq \rho(D^*) \delta(D) + [\text{Log}(|D^*|/|D|) - m_0]/\mu.$$

In any case, we obtain a $C^1(\bar{D})$ *a priori* bound on u in terms of $|D^*|$, $\rho(D^*)$, $|D|$, $\delta(D)$, $|D_0|$, M_0 , m_0 and μ .

For simplicity, let us give a unified treatment of higher order *a priori* estimates for equations (1. t) and (2. t) by rewriting these equations into a single general form

$$\text{Log } A(u) = \Gamma(t, x, du, u, \langle u \rangle). \quad (*)$$

Let $u \in S(D, D_t)$ solve (*). In this section, a constant will be said *under control* provided it depends only on the following quantities: $|u|_1$, i. e. the $C^1(\bar{D})$ -norm of u , on the C^2 -norm of Γ on

$$K := [0, 1] \times \bar{D} \times \bar{D}^* \times I \times I,$$

where $I = [-|u|_1, |u|_1]$, on the $C^0([0, 1], C^2)$ -norm of $t \rightarrow h_t$ (the fixed path of defining functions, cf. *supra*), and on the *positive* real

$$\sigma := \min_{t \in [0, 1]} \sigma(t)$$

where $\sigma(t)$ is the smallest eigenvalue of $[(h_t)_{ij}]$ over $\bar{D}_t$.

Since u is convex, a $C^2(\bar{D})$ bound on u follows from a bound on

$$M_2 := \max_{(x, \theta) \in \bar{D} \times S} [u_{\theta\theta}(x)]$$

S standing for the unit sphere of $\mathbb{R}^n$. Set $H := h_t(du)$ and consider

$$Q: (c, \theta, x) \in (0, \infty) \times S \times \bar{D} \rightarrow Q(c, \theta, x) = \text{Log}[u_{\theta\theta}(x)] + c H(x).$$

PROPOSITION 2. — *There exists $C \in (0, \infty)$ under control such that, if $\max_{(\theta, x) \in S \times \bar{D}} [Q(C, \theta, x)]$ occurs at $(z, x_0) \in S \times D$ with x_0 interior to D , then M_2 is under control.*

This proposition does not refer to any boundary condition and constitutes by no means an interior estimate (it is rather the type of argument suited on a compact manifold). A similar proposition (with Δu and $|du|^2$, respectively in place of $u_{\theta\theta}$ and H) is lemma 2 of [13], later (and independently) reproved in [7] (p. 694); a similar argument is used in [4] (p. 398). Here proposition 2 may serve for the higher dimensional theorem, due to the special form of Q ; so for completeness, we provide a detailed proof of it.

Proof. — Fix $(c, \theta) \in (0, \infty) \times S$ and consider Q as a function of x only. Let us record some auxiliary formulae: differentiating twice equation (*) in the θ -direction yields,

$$u^{ij} u_{\theta ij} = (\Gamma)_{\theta} \equiv \Gamma_{\theta} + \Gamma_u u_{\theta} + \Gamma_{u_i} u_{\theta i} \quad (5)$$

$$u^{ij} u_{\theta\theta ij} = (\Gamma)_{\theta\theta} + u^{ik} u^{jm} u_{\theta ij} u_{\theta km} \quad (6)$$

with

$$(\Gamma)_{\theta\theta} \equiv \Gamma_{u_i} u_{\theta\theta i} + \Gamma_{u_i u_j} u_{\theta i} u_{\theta j} + 2(\Gamma_{\theta u_i} + u_{\theta} \Gamma_{uu_i}) u_{\theta i} + \Gamma_u u_{\theta\theta} + [\Gamma_{\theta\theta} + 2u_{\theta} \Gamma_{\theta u} + \Gamma_{uu}(u_{\theta})^2].$$

Differentiating twice H yields (with the subscript t , of h , dropped),

$$H_i = h_k u_{ik} \quad (7)$$

$$H_{ij} = h_k u_{ijk} + h_{km} u_{ik} u_{jm} \quad (8)$$

and similarly for Q ,

$$Q_i = (u_{\theta\theta i}/u_{\theta\theta}) + c H_i \\ Q_{ij} = (u_{\theta\theta ij}/u_{\theta\theta}) - [u_{\theta\theta i} u_{\theta\theta j}/(u_{\theta\theta})^2] + c H_{ij}.$$

Combining (8) with (5) and (7), we get

$$u^{ij} H_{ij} = h_i (\Gamma_i + \Gamma_u u_i) + \Gamma_{u_i} H_i + h_{ij} u_{ij} \quad (9)$$

while from (6) we get,

$$u^{ij} Q_{ij} = [(\Gamma)_{\theta\theta}/u_{\theta\theta}] + (1/u_{\theta\theta}) [u^{ik} u^{jm} u_{\theta ij} u_{\theta km} - (1/u_{\theta\theta}) u^{ij} u_{\theta\theta i} u_{\theta\theta j}] + c u^{ij} H_{ij}.$$

Expanding the square

$$(u_{\theta\theta} u_{\theta ij} - u_{\theta i} u_{\theta\theta j}) (u_{\theta\theta} u_{\theta km} - u_{\theta k} u_{\theta\theta m}) u^{ik} u^{jm}$$

one immediately verifies the identity:

$$u^{ik} u^{jm} u_{\theta ij} u_{\theta km} \geq (1/u_{\theta\theta}) u^{ij} u_{\theta\theta i} u_{\theta\theta j}.$$

So,

$$u^{ij} Q_{ij} \geq [(\Gamma)_{\theta\theta}/u_{\theta\theta}] + c u^{ij} H_{ij}.$$

Combining the expression of $(\Gamma)_{\theta\theta}$ with that of Q_i and (9) yields,

$$u^{ij} Q_{ij} - \Gamma_{u_i} Q_i \geq ch_{ij} u_{ij} + (1/u_{\theta\theta}) \Gamma_{u_i u_j} u_{\theta i} u_{\theta j} \\ + (2/u_{\theta\theta}) (\Gamma_{\theta u_i} + u_{\theta} \Gamma_{uu_i}) u_{\theta i} + \Gamma_u + ch_i (\Gamma_i + \Gamma_u u_i) \\ + (1/u_{\theta\theta}) [\Gamma_{\theta\theta} + 2 u_{\theta} \Gamma_{\theta u} + \Gamma_{uu} (u_{\theta})^2]. \quad (10)$$

Introducing the constant σ (defined above) we get

$$(1/u_{\theta\theta}) \Gamma_{u_i u_j} u_{\theta i} u_{\theta j} + \frac{1}{3} ch_{ij} u_{ij} \\ \geq (1/u_{\theta\theta}) u_{ik} u_{jm} \left(\theta^k \theta^m \Gamma_{u_i u_j} + \frac{1}{3} c \sigma u_{\theta\theta} \delta_{ij} u^{km} \right) \\ \geq (1/u_{\theta\theta}) u_{\theta i} u_{\theta j} \left(\Gamma_{u_i u_j} + \frac{1}{3} c \sigma \delta_{ij} \right)$$

this last inequality being obtained by noting that, identically for u strictly convex, $u_{\theta\theta} u^{km} \geq \theta^k \theta^m$. Hence our first requirement on c is:

$$\left(\Gamma_{u_i u_j} + \frac{1}{3} c \sigma \delta_{ij} \right) \geq 0$$

in the sense of symmetric matrices, over K . To express our second requirement on c , we first note that the inequality between the arithmetic and the geometric means of n positive numbers applied to the eigenvalues of (u_{ij}) and combined with $(*)$, yields on D :

$$\Delta u \geq n \exp \left(\frac{1}{n} \min_K \Gamma \right) = : \gamma.$$

Then we take c such that

$$2 \min [\Gamma_{yu_y}(r) + u_y(x) \Gamma_{uu_y}(r)] + \frac{1}{3} c \sigma \gamma \geq 0$$

the minimum being taken on $(r, x, y) \in K \times \bar{D} \times S$. From now on, c has a fixed value under control, C , meeting both requirements and we take $(\theta, x) = (z, x_0)$ as defined in proposition 2. In particular, $u_{zz}(x_0)$ is now the *maximum* eigenvalue of $[u_{ij}(x_0)]$; diagonalizing the latter and using the *second* requirement on C , we obtain at x_0 :

$$\frac{1}{3} C h_{ij} u_{ij} + (2/u_{zz}) (\Gamma_{zu_i} + u_z \Gamma_{uu_i}) u_{zi} \geq \frac{1}{3} C \sigma \Delta u + 2 (\Gamma_{zu_z} + u_z \Gamma_{uu_z}) \geq 0.$$

Now (10) yields for $x \rightarrow Q = Q(C, z, x)$ at x_0 ,

$$u^{ij} Q_{ij} - \Gamma_{u_i} Q_i \geq C \left(\frac{1}{3} \sigma u_{zz} - C' \right) - C'' (1/u_{zz}), \quad (11)$$

for some positive constants under control C', C'' . Since $Q(C, z, \cdot)$ assumes its *maximum* at $x_0 \in D$, (11) implies a controlled bound from above on

$u_{zz}(x_0)$, hence also on $(\theta, x) \rightarrow Q(C, \theta, x)$ and on $(\theta, x) \rightarrow u_{\theta\theta}(x)$. Therefore M_2 is under control. $\square$

According to proposition 2, we may assume, without loss of generality, that the point x_0 above lies on ∂D , hence a $C^2(\bar{D})$ *a priori* bound on u follows from an *a priori* bound on $u_{zz}(x_0)$ which, in turn, coincides with

$$\max_{(\theta, x) \in S \times \partial D} [u_{\theta\theta}(x)].$$

IV. A PRIORI ESTIMATES OF SECOND DERIVATIVES ON THE BOUNDARY ($n=2$)

In this section we fix a defining function of D , denoted by k , and we include in the definition of constants *under control* the possible dependence on $|k|_3$, on $\tau := \min_{\partial D} k_N > 0$ and on the minimum over $\bar{D}$ of the smallest eigenvalue of (k_{ij}) , denoted by $s > 0$.

We still let $u \in S(D, D_t)$ solve equation $(*)$. According to proposition 1 $H = h_t(du)$ which vanishes on ∂D , satisfies there $H_N > 0$; moreover, (7) implies on ∂D (dropping the subscript t of h):

$$h_i[du(x)] = H_N u^{ij} N^j(x). \quad (12)$$

In particular, the function on ∂D

$$\varphi(x) := N^i(x) h_i[du(x)]$$

is *positive*. Fix an arbitrary point $x_0 \in \partial D$ and a direct system of euclidean co-ordinates (O, x^1, x^2) satisfying $N(x_0) = \partial/\partial x^2$. Then (12) reads at x_0 ,

$$\left. \begin{aligned} u_{11}(x_0) &= (e^\Gamma/H_N) \varphi(x_0) \\ u_{12}(x_0) &= -(e^\Gamma/H_N)(x_0) h_1[du(x_0)] \end{aligned} \right\} \quad (13)$$

while equation $(*)$ itself provides for $u_{22}(x_0) = u_{NN}(x_0)$,

$$\varphi u_{22}(x_0) = H_N(x) + (e^\Gamma/H_N)(x_0) \{h_1[du(x_0)]\}^2. \quad (14)$$

We thus need positive lower bounds under control on $H_N(x_0)$ and $\varphi(x_0)$, as well as a controlled upper bound on $H_N(x_0)$.

Let us start with $H_N(x_0)$. Aside from (9), H also satisfies in D [still by combining (8), (5), (7)],

$$u^{ij} H_{ij} - u^{ij} (\Gamma)_i H_j = h_{ij} u_{ij}. \quad (15)$$

Set $T = u_{11} + u_{22}$, $T^* = u^{11} + u^{22}$, and note the identity: $T^* = A(u)T$. It implies the existence of positive constants under control, α, β , such that

$$\alpha T^* \leq T \leq \beta T^*, \quad (16)$$

which we simply denote by: $T \simeq T^*$. Consider the function

$$(c, x) \in (0, \infty) \times \bar{D} \rightarrow w(c, x) = H(x) - ck(x).$$

From (9) and $T \geq \gamma$ (cf. *supra*), we infer

$$u^{ij}[w(c, \cdot)]_{ij} \leq -T \left[\frac{1}{2} c(s/\beta) - (u_{ij}/T)(h_{ij} + \Gamma_u h_j) \right] - \left[\frac{1}{2} \gamma c(s/\beta) - h_i(\Gamma_i + \Gamma_u u_i) \right],$$

and there readily exists $c = C > 1$, under control, such that the latter right-hand side is non-positive. Similarly (15) (16) yield:

$$u^{ij}[w(c, \cdot)]_{ij} - u^{ij}(\Gamma)_i[w(c, \cdot)]_j \geq \frac{1}{2} \sigma T - T \max \{0, (c/\alpha) u^{ij}/T^*\} [k_{ij} - k_i(\Gamma_j + \Gamma_u u_j)] + \left(\frac{1}{2} \sigma \gamma + ck_i \Gamma_u \right),$$

(σ was defined at the beginning of section III) and there exists $c \in (0, 1)$ under control such that the right-hand side is nonnegative. Since w identically vanishes on $(0, \infty) \times \partial D$, Hopf's maximum principle [11] implies the following pinching under control on ∂D :

$$c\tau \leq ck_N \leq H_N \leq Ck_N \leq C|k|_1. \quad (17)$$

Combined with (13), it implies a controlled upper bound on $|u_{11}(x_0)| + |u_{12}(x_0)|$. Furthermore, combined with (14), it implies also (the notation $\simeq$ is defined at (16))

$$u_{22}(x_0) \simeq 1/\varphi(x_0). \quad (18)$$

We now turn to a lower bound on $\varphi(x_0)$ and consider the function

$$(c, x) \in (0, \infty) \times \bar{D} \rightarrow P(c, x) = \psi - ck,$$

where

$$\psi(x) := k_i(x) h_i[du(x)].$$

A routine computation using (5) yields in D :

$$u^{ij} \psi_{ij} = k_i h_{ij} (\Gamma_j + \Gamma_u u_j + \Gamma_{u_m} u_{jm}) + 2k_{ij} h_{ij} + k_i h_{ijm} u_{jm} + u^{ij} k_{ijm} h_m.$$

It implies the existence of a constant c_1 under control such that, in D ,

$$u^{ij} P_{ij} \leq c_1 (1 + T) - c(s/\beta) T = - \left[\frac{1}{2} c \gamma (s/\beta) - c_1 \right] - T \left[\frac{1}{2} c (s/\beta) - c_1 \right];$$

let us choose $c = C_0 := 2c_1 \beta / s \min(1, \gamma)$, so that $u^{ij} [P(C_0, \cdot)]_{ij} \leq 0$ in D . By Hopf's maximum principle [11], $P(C_0, \cdot)$ necessarily assumes its *minimum* over $\bar{D}$ at a boundary point y_0 where

$$\psi_N \leq C_0 k_N. \quad (19)$$

Pick a euclidean system of co-ordinates (O, y^1, y^2) such that $N(y_0) = \partial/\partial y^2$. Then $dk(y_0) = k_N \partial/\partial y^2$ while, using (13) (17):

$$|u_{12}(y_0)| \leq C_1 := e^{|\Gamma|_0} |h|_1 / c\tau$$

is under control, and (19) reads:

$$\begin{aligned} u_{22}(y_0)k_N(y_0)h_{22}[du(y_0)] \\ \leq C_0 k_N(y_0) - k_{2i}(y_0)h_i[du(y_0)] \\ - k_N(y_0)u_{12}(y_0)h_{12}[du(y_0)]. \end{aligned}$$

It implies

$$\sigma\gamma u_{22}(y_0) \leq C_0 |k|_1 + |h|_1 |k|_2 + C_1 |k|_1 |h|_2$$

i.e. a controlled bound from above on $u_{22}(y_0)$. Recalling (18), it means a controlled positive bound from below, λ , on $\varphi(y_0)$. Since on ∂D , $P(C_0, \cdot) \equiv k_N \varphi$, and since $P(C_0, \cdot)$ assumes its *minimum* at y_0 , we infer on ∂D :

$$\varphi(x) \geq \lambda k_N(y_0)/k_N(x) \geq \lambda\tau/|k|_1.$$

Using (18) again, we obtain a controlled upper bound on $u_{22}(x_0)$. The second derivatives of u are thus *a priori* bounded on ∂D . $\square$

Remarks. — 5. Proposition 1 and (12) show that the lower bound $\varphi \geq \lambda$ ensures *a priori* the *uniform obliqueness* of the boundary operator at u . Geometrically, it implies another positive lower bound on the scalar product of the outward unit normals, to ∂D at x and to ∂D_t at $du(x)$.

6. Let (T, N) and (T^*, N^*) be direct orthonormal moving frames on ∂D and on ∂D_t respectively (N^* stands for the outward unit normal on ∂D_t) and let z_0 be a critical point of: $x \in \partial D \rightarrow N(x) \cdot N^*[du(x)]$. Denote by $J du$ the Jacobian (or differential) of the gradient mapping du . With the help of Frénet's formulae, one verifies that

$$|J du[T(z_0)]| = (R_0^*/R_0), \quad (20)$$

R_0 (resp. R_0^*) standing for the curvature radius of ∂D at z_0 [resp. of ∂D_t at $du(z_0)$]. Equation (*) implies that the area of the parallelogram $[J du(T), J du(N)]$ equals $\exp(\Gamma)$, in particular, it is uniformly bounded *below* by a positive constant. What happens if we drop the *strict* convexity of ∂D at z_0 , but keep that of ∂D_t at $du(z_0)$, *i.e.* if we let R_0 go to infinity and R_0^* remain bounded? From (20), $|J du[T(z_0)]|$ goes to zero hence $|J du[N(z_0)]|$ goes to infinity. In a direct system of euclidean co-ordinates $(0, x^1, x^2)$ such that $N(z_0) = \partial/\partial x^2$, it implies that $|u_{11}(z_0)| + |u_{12}(z_0)|$ goes to zero while $|u_{22}(z_0)|$ blows up like R_0 *i.e.* the control on $u_{NN}(z_0)$ is lost.

V. HIGHER ORDER *A PRIORI* ESTIMATES

Let $u \in S(D, D_t)$ solve equation (*). Fix a generic point $x \in \bar{D}$ and choose a euclidean co-ordinates system which puts $[u_{ij}(x)]$ into a *diagonal* form.

Observe that for each $i \in \{1, \dots, n\}$,

$$u_{ii}(x) = A(u) / \prod_{j \neq i} u_{jj}(x) \geq \gamma / (|u|_2)^{n-1}. \quad (21)$$

In case $n=2$, the $C^2(\bar{D})$ *a priori* estimate drawn on u in the two preceding sections thus implies the controlled *uniform* ellipticity of $d[\text{Log } A(u)]$ on $\bar{D}$. Given $\alpha \in (0, 1)$, a $C^{2,\alpha}(\bar{D})$ *a priori* bound on u now follows from the general theory of [15] (section 6); however, this bound is so straightforward for $n=2$ that we include it for completeness.

First of all, given any interior subdomain D' of D and any $z \in S$, the 2-dimensional regularity theory of [17] applied to u_z , which satisfies (5) in D' , yields a $C^{1,\alpha}(\bar{D}')$ *a priori* bound under control on u_z , hence, since z is arbitrary, a controlled $C^{2,\alpha}(\bar{D}')$ *a priori* bound on u . The theory of [17] also applies to H which satisfies (9) in D and vanishes on ∂D : it yields a $C^{1,\alpha}(\bar{D})$ *a priori* bound under control on H . Solving for u_{11} , u_{12} and u_{22} the 3×3 system given by (7) and equation (*), we get (dropping the subscript t of h):

$$\begin{cases} u_{11} = [(H_1)^2 + (h_2)^2 e^\Gamma] / \Delta \\ u_{12} = (H_1 H_2 - h_1 h_2 e^\Gamma) / \Delta \\ u_{22} = [(H_2)^2 + (h_1)^2 e^\Gamma] / \Delta, \end{cases} \quad (22)$$

where

$$\Delta(x) := H_t(x) h_t[du(x)].$$

Note that (7) and (21) imply

$$\Delta(x) \geq (\gamma / |u|_2) |dh[du(x)]|^2. \quad (23)$$

Given any small enough $\delta \in (0, 1)$, let

$$D_\delta := \{x \in D, \text{dist}(x, \partial D) < \delta\}.$$

From the $C^2(\bar{D})$ *a priori* estimate precedingly drawn on u , it follows that the gradient image $du(D_\delta)$ is contained in $(D_t)_{C\delta}$ for some positive constant C under control. If $\tau^* := \min_{t \in [0, 1]} (\min_{\partial_t D} |dh_t|)$,

then there readily exists $\delta_0 \in (0, 1)$ under control such that, for any $x \in D_{\delta_0}$, $|dh_t[du(x)]| \geq \tau^*/2$. Therefore (22) and (23) imply a $C^\alpha(\bar{D}_{\delta_0})$ *a priori* bound under control on the second derivatives of u . A $C^{2,\alpha}(\bar{D})$ *a priori* bound on u follows.

Actually, a straightforward “bootstrap” argument now provides $C^{k,\alpha}(\bar{D})$ *a priori* bounds on u for each integer $k > 2$.

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