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Periodic solutions for N-body type problems

by

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ABSTRACT. — In this paper we prove the existence of a T-periodic solution (for any given T) for a class of Hamiltonian systems which includes the N-body one. We also prove that the solution we find is not a simultaneous collision one.

Key words : N-body problem, periodic solutions.

RÉSUMÉ. — Dans l'article on démontre l'existence d'une solution T-périodique (pour chaque $T>0$) pour un ensemble de systèmes hamiltoniens comprenant celui des N-corps. On démontre aussi que la solution ainsi trouvée n'est pas une solution de collision totale.

Mots clés : Problème des N-corps, solutions périodiques.

INTRODUCTION

In the last few years a quite large amount of papers dealing with the existence of periodic solutions for "singular" Hamiltonian systems using

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variational methods has appeared in the literature. Such papers extend to Hamiltonians of the form $\frac{1}{2}|p|^2+V(q)$, where $V(q)$ behaves like $\frac{1}{|q|^\alpha}$ near $|q|=0$, results on periodic solutions of Hamiltonian systems contained in several papers (for example [9]).

We recall here [7], [1], [8], [6], [3], [2] for results on 2-body type problems.

As far as periodic solutions for N-body type problems are concerned, variational methods have just started to be used. We recall here the paper [5], where the variational structure of the problem is used to prove the existence of periodic solutions of the N-body problem in the case one mass is large and all the others very small (actually a bifurcation result), and the paper [4] where the existence of generalized T-periodic solutions ("generalized" means – roughly – that collisions could – *see* paragraph 1 for a precise definition) for the 3-body problem is proved.

Motivated by the paper [4] we have studied the N-body type problem and in this paper we prove the existence of periodic solutions of assigned period T for the following system of ordinary differential equations

$$-m_i x_i'' = \nabla_{x_i} V(x_1, \ldots, x_N), \qquad i=1, \ldots, N \tag{P}$$

where $x_i \in \mathbb{R}^k$, $m_i > 0$ and

$$V(x_1, \ldots, x_N) = \frac{1}{2} \sum_{i \neq j} V_{ij}(x_i - x_j).$$

The N-body problem being our model problem, we assume $V_{ij} \in C^2(\mathbb{R}^k \setminus \{0\}; \mathbb{R})$ and

$$V(x_1, \ldots, x_N) \leqq 0, \qquad \forall (x_1, \ldots, x_N) \in (\mathbb{R}^k)^N;$$
$$V_{ij}(\xi) \to -\infty \quad \text{as } |\xi| \to 0;$$
$$V_{ij}(\xi) = V_{ji}(\xi), \qquad \forall i \neq j, \quad \forall \xi \in \mathbb{R}^k.$$

Under these assumptions we prove, in section 1, the following theorem

THEOREM A. – *Suppose that* V *satisfies the above assumptions. Then* (P) *has*, $\forall T > 0$, *infinitely many generalized solutions. Moreover if* V_{ij} *satisfies*, $\forall i \neq j$, *the following Strong Force condition*

$$\begin{aligned} &\exists U_{ij} \in C^1(\mathbb{R}^k \setminus \{0\}; \mathbb{R}) \quad \textit{such that } U_{ij}(x) \xrightarrow[x \to 0]{} +\infty \\ &V_{ij}(x) \leqq -|\nabla U_{ij}(x)|^2, \qquad \forall x \in \mathbb{R}^k \setminus \{0\}, \quad |x| \textit{ small} \end{aligned} \tag{SF}$$

then (1.1) *has infinitely many non-collision solutions.*

The other two sections of the paper are devoted to the study of simultaneous collision solutions, *i.e.* of generalized solutions of (P) such that all the bodies collide at some time $t=t^*$. The method used to prove existence, based on minimization of a suitable functional, permits us to prove that under the additional assumption

$$-\frac{a}{2}\sum_{1\leqq i\neq j\leqq N}\frac{m_i m_j}{|x_i-x_j|^\alpha}\leqq V(x_1,\ldots,x_N)\leqq -\frac{b}{2}\sum_{1\leqq i\neq j\leqq N}\frac{m_i m_j}{|x_i-x_j|^\alpha}$$

where $1\leqq\frac{a}{b}\leqq\mu$, (here μ is a quantity that can be easily computed; it depends on α and $m_1,\ldots,m_N$; it is >1 in many cases, *See* remark 3.4) the solution we find is not a simultaneous collision solution.

With respect to the existence results contained in [4] we point out that

(*a*) we require $V_{ij}(x)=V_{ji}(x)$ but we have no assumptions (besides boundedness from above) on the behaviour of V at infinity;

(*b*) our method, based on minimization of a suitable functional, is simpler than the one used in [4] and works for any $N\geqq 2$;

(*c*) in the case the strong force condition is not satisfied the result of Theorem A, as that of any theorem proving existence of periodic solution for system like (P), can be obtained just minimizing f on the set of simultaneous collisions (minimum which is achieved whenever it is finite since the action functional is weakly lower semi continuous). To this regard, Theorem 3.3 shows that, in some situations, the solution we find is not a simultaneous collision solution;

(*d*) while in [4] it is proved the existence of infinitely many periodic solutions also in the case in which the potential V depends on time *i.e.* $V=V(t,x_1,\ldots,x_N)$ we can only prove the existence of one solution under the additional assumption

$$V\left(t+\frac{T}{2},-x_1,\ldots,-x_N\right)=V(t,x_1,\ldots,x_N).$$

See Remark 1.2.

1. EXISTENCE OF SOLUTIONS

Let us consider the following system of ordinary differential equations

$$\begin{aligned}-m_i x_i''(t)&=\nabla_{x_i}V(x_1(t),\ldots,x_N(t))\\ x_i(0)&=x_i(T)\\ x_i'(0)&=x_i'(T)\end{aligned}\tag{1.1}$$

where we assume

$$V(x_1,\ldots,x_N)=\frac{1}{2}\sum_{1\leqq i\neq j\leqq N}V_{ij}(x_i-x_j);\tag{V1}$$

$$V_{ij}\in C^2(\mathbb{R}^k\setminus\{0\};\mathbb{R}),\quad V_{ij}(x)=V_{ji}(x),\quad \forall\,1\leqq i\neq j\leqq N,\quad \forall\,x\in\mathbb{R}^k\setminus\{0\};\tag{V2}$$

$$V_{ij}(\xi) \underset{|\xi|\to 0}{\to} -\infty, \qquad \forall\, 1 \leqq i \neq j \leqq N. \tag{V3}$$

$$V(x_1, \ldots, x_N) \leqq 0, \qquad \forall (x_1, \ldots, x_N) \in (\mathbb{R}^k)^N \tag{V4}$$

We will also always assume $m_i > 0$, $\forall\, i$ and set $M = \sum_{i=1}^{N} m_i$.

We will say that a function $X(t) = (x_1(t), \ldots, x_N(t)) \in C^2([0, T]; (\mathbb{R}^k)^N)$ is a *non-collision solution* of (1.1) if $x_i(t) \neq x_j(t)$, $\forall\, i \neq j$, $\forall\, t \in [0, T]$ and if $X(t)$ solves (1.1).

We will say (following [3]) that $X(t) = (x_1(t), \ldots, x_N(t)) \in H^1(S^1; (\mathbb{R}^k)^N)$ is a *generalized solution* of (1.1) if, denoting by $\mathscr{C}$ the set

$$\mathscr{C} = \{ t \in [0, T] \text{ such that } x_i(t) = x_j(t) \text{ for some } i \neq j \}$$

we have that:

(*a*) meas $(\mathscr{C}) = 0$;

(*b*) $X(t) = (x_1(t), \ldots, x_N(t))$ solves

$$-m_i x_i''(t) = \nabla_{x_i} V(x_1(t), \ldots, x_N(t)),$$

$\forall\, t \in [0, T] \setminus \mathscr{C}$;

(*c*) $E = \sum_{i=1}^{N} \frac{m_i}{2} |x_i'(t)|^2 - V(x_1(t), \ldots, x_N(t))$ is constant in $[0, T] \setminus \mathscr{C}$.

Then we prove

THEOREM 1.1. – *Suppose that* V *satisfies* (V1-2-3-4). *Then* (1.1) *has,* $\forall\, T > 0$, *infinitely many generalized solutions. Moreover if* V_{ij} *satisfies,* $\forall\, i \neq j$, *the following Strong Force condition*

$$\begin{aligned} &\exists\, U_{ij} \in C^1(\mathbb{R}^k \setminus \{0\}; \mathbb{R}) \quad \textit{such that } U_{ij}(x) \underset{x\to 0}{\to} +\infty \\ &V_{ij}(x) \leqq -|\nabla U_{ij}(x)|^2, \qquad \forall\, x \in \mathbb{R}^k \setminus \{0\}, \ |x| \textit{ small} \end{aligned} \tag{SF}$$

then (1.1) *has infinitely many non-collision solutions.*

Proof. – We set

$$\Omega = \{ (x_1, \ldots, x_N) \in \mathbb{R}^k \times \ldots \times \mathbb{R}^k \text{ such that } x_i \neq x_j \,\forall\, i \neq j \},$$
$$\Lambda = \{ X(t) = (x_1(t), \ldots, x_N(t)) \text{ such that } X \in H^1(S^1; \Omega) \},$$
$$\Lambda_0 = \left\{ X \in \Lambda \text{ such that } X\left(t + \frac{T}{2}\right) = -X(t) \right\}$$

and define $f : \Lambda \to \mathbb{R}$ as

$$f(x_1, \ldots, x_N) = \sum_{i=1}^{N} \frac{m_i}{2} \int_0^T |x_i'(t)|^2\, dt - \int_0^T V(x_1(t), \ldots, x_N(t))\, dt.$$

It is easy to see that the critical points of f on Λ are non-collision solutions of (1.1).

Since $V_{ij}(x)=V_{ji}(x)$, we have that

$$\begin{aligned} V(-x_1, \ldots, -x_N) &= \frac{1}{2} \sum_{1 \leqq i \neq j \leqq N} V_{ij}(-x_i+x_j) \\ &= \frac{1}{2} \sum_{1 \leqq i \neq j \leqq N} V_{ji}(x_j-x_i) \\ &= V(x_1, \ldots, x_N), \end{aligned}$$

hence one can easily check that the critical points of $f_{|\Lambda_0}$ are actually critical points of f on Λ.

We now prove Theorem 1.1 in three steps.

STEP 1. — $\forall \delta>0$ *we modify* V_{ij} *in* $B_\delta(0)$ *in such a way that the modified potential* V_{ij}^δ *satisfies* (SF).

This can be done, for example, setting

$$V_{ij}^\delta(x)=V_{ij}-\frac{\varphi_\delta(|x|)}{|x|^2}$$

where $\varphi_\delta \in C^\infty(\mathbb{R}^+; \mathbb{R}^+)$, $\varphi_\delta(x)=0$, $\forall x \geqq \delta$, $\varphi_\delta(x)=1$, $\forall x<\frac{\delta}{2}$. Then,

$$V_{ij}^\delta(x) \leqq -\frac{1}{|x|^2}=-|\nabla \log |x||^2, \qquad \forall |x|<\frac{\delta}{2},$$

so that (SF) holds with $U_{ij}(x)=-\log|x|$.

Setting $V^\delta(x_1, \ldots, x_N)=\frac{1}{2} \sum_{1 \leqq i \neq j \leqq N} V_{ij}^\delta(x_i-x_j)$ we define

$$f_\delta(x_1, \ldots, x_N)=\sum_{i=1}^{N} \frac{m_i}{2} \int_0^T |x_i'(t)|^2\,dt-\int_0^T V^\delta(x_1(t), \ldots, x_N(t))\,dt.$$

Clearly, no modification is necessary if V_{ij} already satisfies a (SF) condition.

STEP 2. — *Existence of a minimum for* f_δ.

Let

$$c_\delta=\inf\{f_\delta(X) \text{ where } X \in \Lambda_0\}.$$

Consider a minimizing sequence $(X^{(n)}) \in \Lambda_0$ such that $f_\delta(X^{(n)}) \to c_\delta$. Then, for N large,

$$\sum_{i=1}^{N} \frac{m_i}{2} \int_0^T |x_i^{(n)\prime}(t)|^2\,dt-\int_0^T V^\delta(x_1^{(n)}(t), \ldots, x_N^{(n)}(t))\,dt \leqq c_\delta+\varepsilon.$$

Since $V(x_1^{(n)}, \ldots, x_N^{(n)}) \leqq 0$, we deduce that

$$\int_0^T |x_i^{(n)\prime}(t)|^2\,dt \leqq \frac{2(c_\delta+\varepsilon)}{m_i}, \qquad \forall i.$$

Since $x_i\left(t+\frac{T}{2}\right)=-x_i(t)$, we have that

$$\begin{aligned}|x(t)|&=\frac{1}{2}\left|x(t)-x\left(t+\frac{T}{2}\right)\right|\\&=\frac{1}{2}\left|\int_t^{t+(T/2)}x'(s)\,ds\right|\\&\leqq\frac{1}{2}\sqrt{\frac{T}{2}}\left\{\int_t^{t+(T/2)}|x'(s)|^2\,ds\right\}^{1/2}\\&=\frac{\sqrt{T}}{4}\left\{\int_0^T|x'(t)|^2\,dt\right\}^{1/2},\end{aligned}$$

hence

$$\|x\|_{L^\infty}^2\leqq\frac{T}{16}\int_0^T|x'(t)|^2\,dt,\qquad\forall x\in\Lambda_0,$$

and we deduce, for the minimizing sequence $X^{(n)}=(x_1^{(n)},\ldots,x_N^{(n)})$

$$\|x_i^{(n)}\|_{H^1}\leqq\text{Const.},\qquad\forall 1\leqq i\leqq N,\quad\forall n\text{ sufficiently large.}$$

This implies the existence of $\bar{X}^\delta=(\bar{x}_1^\delta,\ldots,\bar{x}_N^\delta)$ with $\bar{x}_i^\delta\in H^1(S^1;\mathbb{R}^k)$, $\forall 1\leqq i\leqq N$ such that

$$x_i^{(n)}\underset{n\to+\infty}{\longrightarrow}\bar{x}_i^\delta\quad\text{weakly in } H^1(S^1;\mathbb{R}^k),\quad\forall 1\leqq i\leqq N$$

and

$$x_i^{(n)}\underset{n\to+\infty}{\longrightarrow}\bar{x}_i^\delta\quad\text{in } C^0(S^1;\mathbb{R}^k),\quad\forall 1\leqq i\leqq N.$$

It is well known that from (SF) it follows that $f(X^{(n)})\to+\infty$ for every sequence $(X^{(n)})$ such that $X^{(n)}\underset{n\to+\infty}{\longrightarrow}\bar{X}$ weakly in H^1 and strongly in C^0 if $\bar{X}\in\partial\Lambda$ (*see* [7], [1]). This proves that $\bar{X}^\delta\in\Lambda_0$. Since f is weakly lower semicontinuous, it immediately follows that $\bar{X}^\delta\in\Lambda_0$ is a minimum for f_δ on Λ_0. Such a minimum is then a non-collision solution of (1.1) (with V^δ replacing V). In particular we have proved that (1.1) has at least one non-collision solution if (SF) is satisfied (in such a case $V^\delta=V$).

STEP 3. – $\delta\to 0$.

Clearly $c_\delta<C$, $\forall\delta>0$. This implies

$$\|\bar{x}_i^\delta\|_{H^1}^2\leqq c\int_0^T|\bar{x}_i^{\delta\prime}(t)|^2\,dt\leqq C',\qquad\forall 1\leqq i\leqq N,\quad\forall\delta>0.$$

Then, as before, $\bar{x}_i^\delta\underset{\delta\to 0}{\to}\bar{x}_i$ weakly in H^1 and strongly in C^0. We will show that $\bar{X}=(\bar{x}_1,\ldots,\bar{x}_N)$ is a generalized solution of (1.1).

In fact, set, $\forall i \neq j$, $\mathscr{C}_{ij} = \{ t \in [0, T] \text{ such that } \bar{x}_i(t) = \bar{x}_j(t) \}$. Then each $\mathscr{C}_{ij}$ is a closed set and

$$\overline{x_i^\delta}(t) - \overline{x_j^\delta}(t) \underset{\delta \to 0}{\to} 0 \quad \text{on } \mathscr{C}_{ij} \text{ (uniformly).}$$

Then, if meas $\mathscr{C}_{ij} > 0$

$$c_\delta = f_\delta(\bar{X}^\delta) \geqq - \int_0^T V_{ij}^\delta(\overline{x_i^\delta}(t) - \overline{x_j^\delta}(t))\, dt \underset{\delta \to 0}{\to} +\infty,$$

and we reach a contradiction which proves meas $\mathscr{C}_{ij} = 0$, $\forall i \neq j$.

Let $\mathscr{C} = \bigcup_{1 \leqq i \neq j \leqq N} \mathscr{C}_{ij}$. Then meas $\mathscr{C} = 0$. Take $\forall n \geqq 1\ K_n \subset [0, t] \setminus \mathscr{C}$, K_n compact, $\bigcup_{n \geqq 1} K_n = [0, T] \setminus \mathscr{C}$, $K_n \subset K_{n+1}$. Let $\bar{K}_n = \{ \bar{X}(t) \text{ such that } t \in K_n \}$. Then, $\forall n \geqq 1\ \bar{K}_n$ is compact and $\bar{K}_n \subset \Omega$. Take a neighborhood U_n *of* $\bar{K}_n$ such that the closure of U_n is compact in Ω. Then, $\forall \delta$ sufficiently small we have that $V^\delta \to V$ in $C^1(\bar{U}_n; \mathbb{R})$. Therefore $\nabla_{x_i} V^\delta(\overline{x_1^\delta}(t), \ldots, \overline{x_N^\delta}(t)) \to \nabla_{x_i} V(\bar{x}_1(t), \ldots, \bar{x}_N(t))$ uniformly on K_n. Since

$$-m_i (\overline{x_i^\delta})''(t) = \nabla_{x_i} V^\delta(\overline{x_1^\delta}(t), \ldots, \overline{x_N^\delta}(t))$$

we deduce that

$$\overline{x_i^\delta} \underset{\delta \to 0}{\to} \bar{x}_i \quad \text{in } C^2 \text{ on } K_n$$

and hence $\bar{X}(t) = (\bar{x}_1(t), \ldots, \bar{x}_N(t))$ solves

$$-m_i \bar{x}_i''(t) = \nabla_{x_i} V(\bar{x}_1(t), \ldots, \bar{x}_N(t)), \qquad \forall t \in K_n.$$

Since $\bigcup_{n \geqq 1} K_n = [0, T] \setminus \mathscr{C}$ we have that $\bar{X}$ satisfies (*a*) and (*b*) of the definition of generalized solutions. (*c*) follows noticing that

$$E_\delta(t) \equiv \frac{1}{2} \sum_{i=1}^{N} m_i |\overline{x_i^\delta}'(t)|^2 + V^\delta(\overline{x_1^\delta}(t), \ldots, \overline{x_N^\delta}(t))$$

is a constant of the motion. Moreover, from

$$E_\delta = \frac{1}{T} f_\delta(\overline{x_1^\delta}, \ldots, \overline{x_N^\delta}) - \frac{1}{T} \sum_{i=1}^{N} \frac{m_i}{2} \int_0^T |\overline{x_i^\delta}'(t)|^2\, dt$$

it follows, since $f_\delta(\bar{X}^\delta) = c_\delta$ and $\int_0^T |\overline{x_i^\delta}'(t)|^2\, dt$ are bounded, that E_δ is bounded in $\mathbb{R}$. We can then assume $E_\delta \underset{\delta \to 0}{\to} E$. It follows that,

$\forall t_1, t_2 \in [0, t] \setminus \mathscr{C}$

$$\begin{aligned} E_0(t_1) &\equiv \frac{1}{2} \sum_{i=1}^{N} m_i |\bar{x}_i'(t)|^2 + V(\bar{x}_1(t), \ldots, \bar{x}_N(t)) \\ &= \lim_{\delta \to 0} E_\delta(t_1) \\ &= \lim_{\delta \to 0} E_\delta(t_2) \\ &= E_0(t_2) \end{aligned}$$

and this proves $\bar{X}$ is a generalized. T-periodic solution of (1.1). We deduce that, $\forall T > 0$, (1.1) has at least one generalized solution X_T.

To prove the existence of infinitely many T-periodic solutions we simply remark that X_T cannot be a constant solution of (1.1). Otherwise the symmetry property $X_T\left(t+\frac{T}{2}\right) = -X_T(t)$ would imply that $X_T \equiv 0$, in contradiction with the fact X_T is a generalized solution. Then X_T is a T-periodic, non costant function. Let $\frac{T}{k}$, $k \geqq 1$ be its minimal period. Applying what proved above for $T = \frac{T}{k+1}$ we find a solution $X_{T/(k+1)}$ which is a $\frac{T}{k+1}$-periodic solution of (1.1). Since the problem is autonomous, such a solution is also a T-periodic solution. Iterating such a procedure the theorem follows. □

Remark 1.2. – *If* V *depends on time in a* T-*periodic fashion, satisfies* (V1-4) *and*

$$V\left(t+\frac{T}{2}, x_1, \ldots, x_N\right) = V(t, x_1, \ldots, x_N)$$

then the same proof of Theorem 1.1 *shows that one* T-*periodic solution exists also for the non-autonomous system.* □

2. ESTIMATES ON SIMULTANEOUS COLLISIONS

In this section we want to estimate the infimum of our functional on the generalized solutions of (1.1) which are simultaneous collisions, where by simultaneous collision solution we mean a generalized solution such that it exists a $t^* \in [0, T]$ such that

$$x_i(t^*) = x_j(t^*), \qquad \forall 1 \leqq i, j \leqq N.$$

Such an estimate will be deduced from an estimate on simultaneous collisions for a potential V of the form

$$V(x_1, \ldots, x_N) = -\frac{b}{2} \sum_{1 \leqq i \neq j \leqq N} \frac{m_i m_j}{|x_i - x_j|^\alpha}.$$

We start by proving the following

LEMMA 2.1. – *Let* $X = (x_1, \ldots, x_N) \in \mathbb{R}^k \times \ldots \times \mathbb{R}^k$. *Then*

$$\frac{1}{2} \sum_{1 \leqq i \neq j \leqq N} \frac{m_i m_j}{|x_i - x_j|^\alpha} \geqq \frac{1}{2^{(\alpha+1)/2}} \frac{\left(\sum_{i \neq j} m_i m_j\right)^{(2+\alpha)/2}}{M^{\alpha/2}} \frac{1}{\left(\sum_{i=1}^N m_i |x_i|^2\right)^{\alpha/2}} \tag{2.1}$$

Proof:

$$\begin{aligned}
\sum_{i \neq j} m_i m_j &= \sum_{i \neq j} m_i m_j \frac{|x_i - x_j|^{\alpha/2}}{|x_i - x_j|^{\alpha/2}} \\
&\leqq \left(\sum_{i \neq j} \frac{m_i m_j}{|x_i - x_j|^\alpha}\right)^{1/2} \left(\sum_{i \neq j} m_i m_j |x_i - x_j|^\alpha\right)^{1/2} \\
&\leqq \sqrt{2}\sqrt{V(x_1, \ldots, x_N)}\, \left(\sum_{i \neq j} m_i m_j\right)^{(2-\alpha)/4} \left(\sum_{i \neq j} m_i m_j |x_i - x_j|^2\right)^{\alpha/4} \\
&= \sqrt{2}\sqrt{V(x_1, \ldots, x_N)}\, \left(\sum_{i \neq j} m_i m_j\right)^{(2-\alpha)/4} \\
&\quad \times \left(2M \sum_i m_i |x_i|^2 - 2\left|\sum_i m_i x_i\right|^2\right)^{\alpha/4} \\
&\leqq 2^{(2+\alpha)/4} M^{\alpha/4} \sqrt{V(x_1, \ldots, x_N)}\, \left(\sum_{i \neq j} m_i m_j\right)^{(2-\alpha)/4} \left(\sum_i m_i |x_i|^2\right)^{\alpha/4}
\end{aligned}$$

and the lemma follows. □

Let us introduce the following notation: we set, for $X = (x_1, \ldots, x_N) \in \Lambda$

$$\Phi_b(x_1, \ldots, x_N) = \frac{1}{2} \sum_{i=1}^N m_i \int_0^T |x_i'(t)|^2\, dt + \frac{b}{2} \sum_{1 \leqq i \neq j \leqq N} \int_0^T \frac{dt}{|x_i(t) - x_j(t)|^\alpha}$$

and, for $R \in H_0^1\left(\left[0, \frac{T}{2}\right]; \mathbb{R}^+\right)$

$$\Psi_b(R) = \frac{M}{2} \int_0^{T/2} |R'(t)|^2\, dt + \frac{b}{2^{(\alpha+1)/2}} \frac{\left(\sum_{i \neq j} m_i m_j\right)^{(2+\alpha)/2}}{M^\alpha} \int_0^{T/2} \frac{dt}{R(t)^\alpha}.$$

We also set

$m_C = \inf \{ \Phi_b(x_1, \ldots, x_N)$ such that $(x_1(t), \ldots, x_N(t))$ is a simultaneous collision$\}$

Then the following lemma holds

LEMMA 2.2:

$$m_C \geqq 2 \inf \left\{ \Psi_b(\mathrm{R}) \textit{ such that } \mathrm{R} \in \mathrm{H}_0^1 \left(\left[0, \frac{\mathrm{T}}{2} \right]; \mathbb{R}^+ \right) \right\}.$$

Proof. – Suppose $\mathrm{X}(t) = (x_1, \ldots, x_\mathrm{N}) \in \Lambda_0$ is a simultaneous collision. Then, without loss of generality, we can assume that the simultaneous collision occurs at $t=0$, *i.e.* that $x_i(0) = x_j(0)\ \forall i, j$. Set $\eta = x_i(0)$. Then, since $\mathrm{X} \in \Lambda_0$, we have that $x_i\left(\frac{\mathrm{T}}{2}\right) = -\eta$.

On the other hand

$$\mathrm{P}(t) = \sum_{i=1}^{\mathrm{N}} m_i x_i'(t)$$

is constant along solutions; this implies that

$$\mathrm{Z}(t) = \sum_{i=1}^{\mathrm{N}} m_i x_i(t)$$

(which is a continuous function $\forall t$) has the form $\mathrm{Z}(t) = \xi t + \xi_0$. Since $\mathrm{Z}(t)$ is T-periodic we must have $\xi = 0$. Then

$$\xi_0 = \mathrm{Z}(0) = \sum_{i=1}^{\mathrm{N}} m_i x_i(0) = \mathrm{M}\, \eta$$

and

$$\xi_0 = \mathrm{Z}\left(\frac{\mathrm{T}}{2}\right) = \sum_{i=1}^{\mathrm{N}} m_i x_i\left(\frac{\mathrm{T}}{2}\right) = -\mathrm{M}\, \eta,$$

which implies $\xi_0 = \mathrm{M}\, \eta = 0$.

Using Lemma 2.1 and the symmetry property $X\left(t+\frac{T}{2}\right)=-X(t)$ we have that, $\forall X\in\Lambda_0$

$$\Phi_b(x_1, \ldots, x_N)=2\left\{\sum_{i=1}^{N}\frac{m_i}{2}\int_0^{T/2}|x_i'(t)|^2\,dt + \frac{b}{2}\sum_{i\leqq i\neq j\leqq N}\int_0^{T/2}\frac{dt}{|x_i(t)-x_j(t)|^\alpha}\right\}$$
$$\geqq 2\left\{\sum_{i=1}^{N}\frac{m_i}{2}\int_0^{T/2}|x_i'(t)|^2\,dt + \frac{b}{2^{(\alpha+1)/2}}\frac{(\sum_{i\neq j} m_i m_j)^{(2+\alpha)/2}}{M^{\alpha/2}}\int_0^{T/2}\frac{dt}{\left(\sum_{i=1}^{N} m_i|x_i(t)|^2\right)^{\alpha/2}}\right\}$$

Setting now $MR(t)^2=\sum_{i=1}^{N} m_i|x_i(t)|^2$ we have, since X is a simultaneous collision, that $R\in H_0^1\left(\left[0, \frac{T}{2}\right]; \mathbb{R}^+\right)$ and that $MR'(t)^2\leqq\sum_1^N m_i|x_i'(t)|^2$ from which the lemma follows. □

Using the methods of [6] it is possible to give a more explicit estimate of

$$\theta_0(\alpha, b)=\inf\left\{\Psi_b(R)\text{ such that }R\in H_0^1\left(\left[0, \frac{T}{2}\right]; \mathbb{R}^+\right)\right\}$$

In fact if follows from [6], section 2, that

$$\theta_0(\alpha, b)\geqq\theta_1(\alpha, b)$$

where

$$\theta_1(\alpha, b)=T\min_{R\geqq 0}\left\{\omega^2 MR^2+\frac{b}{2^{(\alpha+4)/2}}\frac{(\sum_{i\neq j} m_i m_j)^{(2+\alpha)/2}}{M^\alpha}\frac{1}{R^\alpha}\right\} \tag{2.2}$$

hence

$$\theta_1(\alpha, b)=\left(1+\frac{2}{\alpha}\right)MR_1^2\omega^2 T \tag{2.3}$$

where

$$R_1^{2+\alpha}=\frac{\alpha b\,(\sum_{i\neq j} m_i m_j)^{(2+\alpha)/2}}{2^{(6+\alpha)/2}M^{1+\alpha}\omega^2}. \tag{2.4}$$

Remark 2.3. — *In the paper* [6] *it is also proved that*

$$\varphi(\alpha)=\frac{\theta_0(\alpha, b)}{\theta_1(\alpha, b)}$$

is a lower semicontinuous function with values in $[1, +\infty]$ *such that*

1. $\varphi(1)=1$;
2. $1<\varphi(\alpha)<+\infty, \forall \alpha>1$;
3. $\varphi(\alpha)=+\infty, \forall \alpha \geqq 2$.

To sumarize the results of this section, we can state the following

PROPOSITION 2.4. — *Suppose*

$$V(x_1, \ldots, x_N) \leqq -\frac{b}{2} \sum_{1 \leqq i \neq j \leqq N} \frac{m_i m_j}{|x_i - x_j|^\alpha}. \tag{V 5}$$

Then

Inf $\{f(X)$ *such that* X is a simultaneous collision$\}$

$$\geqq 2\theta_0(\alpha, b) \geqq 2\theta_1(\alpha, b).$$

3. EXISTENCE OF SOLUTIONS WHICH ARE NOT SIMULTANEOUS COLLISIONS

In this section we will prove that, under suitable assumptions of V,

$\inf\{f(X) \mid X \in \Lambda_0\} < \inf\{f(X) \mid X$ is a simultaneous collision solution$\}$

and from this it will follow that the generalized solution found via Theorem 2.1 is not a simultaneous collision.

Let us first of all estimate the infirmum of f on Λ_0.

LEMMA 3.1. — *Suppose*

$$V(x_1, \ldots, x_N) \geqq -\frac{a}{2} \sum_{1 \leqq i \neq j \leqq N} \frac{m_i m_j}{|x_i - x_j|^\alpha}. \tag{V 6}$$

Then, denoting by X_0 *one of the solution found via Theorem* 1.1, *we have that*

$$f(X_0) \leqq \left(1+\frac{2}{\alpha}\right) \frac{R_2^2 \omega^2 T}{M} \sigma(m_1, \ldots, m_N)$$

where

$$R_2^{\alpha+2}=\frac{\alpha a M}{2^{2+\alpha}\omega^2} \frac{\rho(\alpha, m_1, \ldots, m_N)}{\sigma(m_1, \ldots, m_N)}, \tag{3.2}$$

$$\rho(\alpha, m_1, \ldots, m_N) = \sum_{1 \leq i \neq j \leq N} \frac{m_i m_j}{\left|\sin \frac{\pi(i-j)}{N}\right|^\alpha} \tag{3.3}$$

and

$$\sigma(m_1, \ldots, m_N) = \sum_{1 \leq i \neq j \leq N} m_i m_j \sin^2 \frac{\pi(i-j)}{N} \tag{3.4}$$

Proof. – Take $\xi, \eta \in \mathbb{R}^k$ such that $|\xi|^2 = |\eta|^2 = 1$, $(\xi, \eta) = 0$ and define

$$\bar{x}_i(t) = R \left\{ \xi \left[\cos\left(\omega t + \frac{2\pi i}{N}\right) - \frac{1}{M} \sum_1^N m_l \cos\left(\omega t + \frac{2\pi l}{N}\right) \right] + \eta \left[\sin\left(\omega t + \frac{2\pi i}{N}\right) - \frac{1}{M} \sum_1^N m_l \sin\left(\omega t + \frac{2\pi l}{N}\right) \right] \right\}$$

where $R \in \mathbb{R}$ is to be determined. Then

$$\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\bar{x}_i'(t)|^2 dt = \frac{1}{2}\left(M - \frac{1}{M} \sum_{i,j} m_i m_j \cos \frac{2\pi(i-j)}{N}\right) R^2 \omega^2 T = \frac{R^2 \omega^2 T}{M} \sum_{i,j} m_i m_j \sin^2 \frac{\pi(i-j)}{N}$$

while

$$|\bar{x}_i(t) - \bar{x}_j(t)|^2 = 4 R^2 \sin^2 \frac{\pi(i-j)}{N}.$$

This implies

$$-V(\bar{x}_1(t), \ldots, \bar{x}_N(t)) \leq \frac{a}{2} \sum_{1 \leq i \neq j \leq N} \frac{m_i m_j}{|\bar{x}_i(t) - \bar{x}_j(t)|^\alpha} = \frac{a}{2^{1+\alpha} R^\alpha} \sum_{1 \leq i \neq j \leq N} \frac{m_i m_j}{\left|\sin \frac{\pi(i-j)}{N}\right|^\alpha}$$

We deduce that, $\forall R > 0$

$$f(\bar{x}_1, \ldots, \bar{x}_N) \leq \frac{R^2 \omega^2 T}{M} \sigma(m_1, \ldots, m_N) + \frac{aT}{2^{1+\alpha} R^\alpha} \rho(\alpha, m_1, \ldots, m_N).$$

Minimizing the right hand side, we find

$$f(\bar{x}_1, \ldots, \bar{x}_N) \leq \left(1 + \frac{2}{\alpha}\right) \frac{R_2^2 \omega^2 T}{M} \sigma(m_1, \ldots, m_N)$$

where

$$R_2^{\alpha+2} = \frac{\alpha a M}{2^{2+\alpha}\omega^2} \frac{\rho(\alpha, m_1, \ldots, m_N)}{\sigma(m_1, \ldots, m_N)}.$$

Now, let X_0 be the (generalized) solution of equation (1.1) which we have denoted X_T in the proof of Theorem 1.1. Then $X_0 = \lim_{\delta \to 0} X_\delta$. Since

$$f_\delta(X_\delta) = \min \{f_\delta(X) \text{ such that } X \in \Lambda_0\}$$

we have that

$$f_\delta(X_\delta) \leqq f_\delta(\bar{X}).$$

Since $V_\delta \equiv V$ outside a δ-neighborhood of the singularity set, we have that

$$f_\delta(X_\delta) \leqq f_\delta(\bar{X}) = f(\bar{X}).$$

On the other hand it follows from the lower semi-continuity of f that

$$f(X_0) \leqq \liminf_{\delta \to 0} f(X_\delta) \leqq \liminf_{\delta \to 0} f_\delta(X_\delta) \leqq f(\bar{X})$$

and the lemma follows. □

Setting

$$\mu = \frac{2^\alpha \left(\sum_{1 \leqq i \neq j \leqq N} m_i m_j\right)^{(2+\alpha)/2}}{\rho(\alpha, m_1, \ldots, m_N)\, \sigma(m_1, \ldots, m_N)^{\alpha/2}} \tag{3.5}$$

we can now prove

THEOREM 3.3. — *Suppose that* (V 1-6) *hold and that*

$$\mu > 1. \tag{3.6}$$

Then, if

$$1 \leqq \frac{a}{b} \leqq \mu \tag{3.7}$$

(1.1) *has, for every* $T > 0$, *at least one* T-*periodic solution which is not a simultaneous collision solution.*

Proof. — Take X_0 to be one of the solutions of (1.1) for which Lemma 3.1 hold. Suppose that X_0 is a simultaneous collision solution. Then from Proposition 2.4 it follows that

$$2\theta_1(\alpha, b) \leqq f(X_0).$$

On the other hand from Lemma 3.1 we deduce that

$$f(X_0) \leqq \left(1 + \frac{2}{\alpha}\right) \frac{R_2^2 \omega^2 T}{M} \sigma(m_1, \ldots, m_N)$$

where R_2 is given by (3.2). This implies

$$2\theta_1(\alpha, b) \leqq \left(1+\frac{2}{\alpha}\right)\frac{R_2^2\omega^2 T}{M}\sigma(m_1, \ldots, m_N)$$

i.e.

$$2R_1^2 \leqq R_2^2\frac{\sigma(m_1, \ldots, m_N)}{M^2}$$

which reduces to

$$\frac{a}{b} \geqq \frac{2^\alpha(\sum_{i\neq j} m_i m_j)^{(2+\alpha)/2}}{\rho(\alpha, m_1, \ldots, m_N)\,\sigma(m_1, \ldots, m_N)^{\alpha/2}},$$

contradiction which proves the theorem. □

Remark 3.4. – Theorem 3.3 states that among the functions satisfying the symmetry property $X\left(t+\frac{T}{2}\right) = -X(-t)$, the total collisions solutions of (1.1) have action (*i.e.* value of the functional), greater than the one of a particular planar and uniformly rotating function, provided V satisfies condition (3.7). We remark that our condition (3.7) is not, in general, optimal. In fact we know that it exists a planar and uniformly rotating solution of (1.1), with $V_{ij}(\xi) = \frac{m_i m_j}{|\xi|}$, for which the value of Φ_1 is smaller or equal than the one of our test funtion $\bar{X}$ (coinciding with it when all the masses are equal). Moreover we do not know if our estimate on the minimum of f on Λ_0 could be improved considering particular non-planar solutions. Such a problem seems interesting since it could indicate the existence of periodic, non-planar solutions. Another remark concerns the estimates on simultaneous collisions. While such an estimate proves to be optimal (in the sense that we can construct solutions whose action is equal to the estimated ones) for $N=3$ and $k \geqq 2$, it becomes less and less precise for k fixed as N increases.

One can check that condition (3.6) holds for a large class of choices of masses and α (even if not for all). For example, if $m_i = m_j\ \forall i, j$, we have that $\mu > 1$ for all values of $N \leqq 68$. Moreover, μ increases the more different the masses are. For example, for $N=16$, we have that, if $m_i = 1$, then $\mu \approx 1.39$, while, if $m_i = i$, then $\mu \approx 1.43$ and, if $m_i = e^i$, then $\mu \approx 1.64$. □

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