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## **On a classical problem of the calculus of variations without convexity assumptions**

by

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**ABSTRACT.** — We show that the functional

$$I(x) = \int_0^T g(t, x(t)) dt + \int_0^T h(t, x'(t)) dt$$

attains a minimum under the condition that  $g$  be concave in  $x$ .

*Key words* : Calculus of variations, normal integrals, convex functionals.

**RÉSUMÉ.** — Nous montrons que le fonctionnel

$$I(x) = \int_0^T g(t, x(t)) dt + \int_0^T h(t, x'(t)) dt$$

atteint le minimum sous la condition de concavité sur  $g$ .

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## INTRODUCTION

We consider the problem of the existence of the minimum for the integral functional  $I(x)$ :

$$I(x) = \int_0^T g(t, x(t)) dt + \int_0^T h(t, x'(t)) dt$$

on the set of functions  $x$  belonging to  $W^{1,p}([0, T], \mathbb{R}^n)$ ,  $p \geq 1$  and satisfying:  $x(0) = a$ ,  $x(T) = b$ ;  $x'(t) \in \Phi(t)$  a. e. on  $[0, T]$ . The set-valued map  $\Phi: [0, T] \rightarrow 2^{\mathbb{R}^n}$  is measurable with non-empty, closed (not necessarily bounded nor convex) values, and each of the functions  $g$  and  $h$  satisfies Carathéodory conditions. Our purpose is to show that, for the existence of the minimum, Tonelli's assumption of convexity of  $h$  with respect to  $x'$  can be replaced by the condition of concavity of  $g$  with respect to  $x$ , all other requirements (e. g. growth conditions) being the same. In particular, we do not impose any regularity on  $g$ ,  $h$  and  $h^{**}$ . Notice that the subset of  $W^{1,p}$  on which the minimum is sought is not weakly closed, due to the lack of convexity of the values of  $\Phi$ .

The problem of avoiding convexity has been considered by: Aubert-Tahraoui [A-T1] and Marcellini [M1] with  $g \equiv 0$  and  $\Phi \equiv \mathbb{R}^1$ ; with  $g$  linear and on a control theory setting, by Olech [O] and Cesari [Ce1]; under different conditions on  $g$  and  $h$  and with  $\Phi \equiv \mathbb{R}^1$ , by Aubert-Tahraoui [A-T1] and Marcellini [M1] (see also the references in [M2] and in [Ce2]). In addition, necessary and sufficient conditions for the existence of minima were given by Ekeland [E] and Raymond [R], under regularity assumptions for the integrands. Our theorem neither contains nor is contained in either Theorem 2 of [M1] or in the results of [A-T1], which concern the case  $n=1$ , while it generalizes Theorem 16.7.i of Cesari [Ce2]. Our main tool is Liapunov's theorem on the range of vector measures as presented in the book of Cesari (§ 16).

## ASSUMPTIONS AND PRELIMINARY RESULTS

We shall assume the following hypothesis.

**HYPOTHESIS (H).** — The set-valued map  $\Phi: [0, T] \rightarrow 2^{\mathbb{R}^n}$  is measurable [C-V] with non-empty closed values. In addition we assume that there exists at least one  $v \in L^p([0, T], \mathbb{R}^n)$  such that  $v(t) \in \Phi(t)$  a. e. and

$$\int_0^T v(t) dt = b - a.$$

The map  $g: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$  is such that

$(g_1)$   $t \rightarrow g(t, x)$  is measurable for each  $x$ ;

(g<sub>2</sub>)  $x \rightarrow g(t, x)$  is continuous for a. e.  $t$ ;

(g<sub>3</sub>)  $x \rightarrow g(t, x)$  is concave for a. e.  $t$ .

Moreover there exist a constant  $\gamma_1$  and a function  $\gamma_2 \in L^1$ , such that

(g<sub>4</sub>)  $g(t, x) \geq -\gamma_1 |x|^p - \gamma_2(t)$ .

The map  $h: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$  is such that

(h<sub>1</sub>)  $t \rightarrow h(t, x')$  is measurable for each  $x'$ ;

(h<sub>2</sub>)  $x' \rightarrow h(t, x')$  is continuous for a. e.  $t$ .

Moreover:

(h<sub>3</sub>) if  $p = 1$ , there exist: a convex lower semicontinuous monotonic function  $\psi: \mathbb{R}^+ \rightarrow \mathbb{R}$  and a function  $\xi_1(\cdot)$  in  $L^1$  such that

$$h(t, x') \geq \psi(|x'|) - \xi_1(t)$$

and

$$\lim_{r \rightarrow \infty} \frac{\psi(r)}{r} = +\infty.$$

If  $p > 1$ , there exist: a positive constant  $\xi_2$  and a function  $\xi_3(\cdot)$  in  $L^1$  such that  $h(t, x') \geq \xi_2 |x'|^p - \xi_3(t)$  and  $\gamma_1/\xi_2$  is strictly smaller than the best Sobolev constant in  $W_0^{1,p}([0, T])$ . ■

We list some notations and preliminary results. The closed unit ball of  $\mathbb{R}^n$  is  $\bar{B} = \{x \in \mathbb{R}^n : |x| \leq 1\}$ . The characteristic function of a set  $E$  is  $\chi_E(\cdot)$ . Let  $(X, d)$  be a metric space and  $F: X \rightarrow 2^{\mathbb{R}^n}$  be a map from  $X$  into the nonempty compact subsets of  $\mathbb{R}^n$ :  $F$  is called upper semicontinuous on  $X$  if, for every  $x \in X$  and for every  $\varepsilon > 0$ , there exists  $\delta = \delta(x, \varepsilon)$  such that  $d(x, y) < \delta \Rightarrow F(y) \subseteq B(F(x), \varepsilon)$ . A set-valued map  $F$  whose graph is closed and whose values are all contained in a compact set is upper semicontinuous. We also set  $\|F(x)\| = \max\{|y| : y \in F(x)\}$ .

Let  $f^{**}(t, x)$  be the bipolar of the function  $x \rightarrow f(t, x)$ . We have the following

PROPOSITION 1 ([E-T] Prop. I.4.1; Lemma IX.3.3; Prop. IX.3.1). - (a)  $f^{**}(t, x)$  is the largest convex (in  $x$ ) function not larger than  $f(t, x)$ .

(b) Under the growth assumption (h<sub>3</sub>) on  $f$

$$f^{**}(t, x) = \min \left\{ \sum_1^{n+1} \lambda_i f(t, \xi_i) : x = \sum_1^{n+1} \lambda_i \xi_i; \lambda_i \geq 0; \sum_1^{n+1} \lambda_i = 1 \right\}.$$

(c) Let  $x'(\cdot)$  be measurable. Then there exist measurable  $p_i: I \rightarrow [0, 1]$  and measurable  $v_i: I \rightarrow \mathbb{R}^n, i = 1, \dots, n+1$ , such that:

$$\sum_i p_i(t) = 1; \quad x'(t) = \sum_i p_i(t) v_i(t); \quad f^{**}(t, x'(t)) = \sum_i p_i(t) f(t, v_i(t)).$$

The following properties of the subdifferential of a convex function ([E-T], § I.5.1) will be used later.

LEMMA 1. — Let  $f: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfy

- (i)  $f(t, x) \leq k|x|^p + b(t)$  ( $k > 0$ ,  $b \in L^1$ );
- (ii)  $t \rightarrow f(t, x)$  is measurable for every  $x$ ;
- (iii)  $x \rightarrow f(t, x)$  is convex and continuous for almost every  $t$ .

Then, for any continuous  $x: [0, T] \rightarrow \mathbb{R}^n$ , the set valued map

$$t \rightarrow \partial_x f(t, x(t))$$

admits a selection  $\delta(\cdot) \in L^1$ .

*Proof.* — (a) We claim that the map  $t \rightarrow \partial_x f(t, x(t))$  is measurable. In fact, fix  $\Delta > 0$ ; then  $|f(t, x)| \leq k\Delta^p + b(t)$  in  $[0, T] \times \Delta\bar{B}$ . By the Corollary to Proposition 2.2.6 in [C] we have that

$$\|\partial_x f(t, x)\| \leq \frac{2}{\Delta}(k(2\Delta)^p + b(t)) \quad \text{for a.e. } t \in [0, T], \quad \text{for all } x \in \Delta\bar{B}. \quad (1)$$

Fix  $\varepsilon > 0$  and let, by Scorza Dragoni's theorem,  $E_\varepsilon \subseteq [0, T]$  be closed and such that:  $m([0, T] \setminus E_\varepsilon) \leq \varepsilon$ ; the restriction of  $f$  to  $E_\varepsilon \times \Delta\bar{B}$  is continuous as well as the restriction of  $b$  to  $E_\varepsilon$ . We prove first that the map  $(t, x) \rightarrow \partial_x f(t, x)$  is upper semicontinuous on  $E_\varepsilon \times \Delta\bar{B}$ . Let us show that it has closed graph. Let  $(t_n, x_n)$  be in  $E_\varepsilon \times \Delta\bar{B}$ ,  $(t_n, x_n) \rightarrow (t, x)$  and let  $v_n$  be in  $\partial_x f(t_n, x_n)$ ,  $v_n \rightarrow v$ . From

$$f(t_n, x_n) - f(t_n, y) \geq \langle v_n, x_n - y \rangle, \quad y \in \mathbb{R}^n,$$

and the continuity of  $f$ , we have

$$\langle f(t, x) - f(t, y) \rangle \geq \langle v, x - y \rangle, \quad y \in \mathbb{R}^n,$$

so that  $v \in \partial_x f(t, x)$ . By (1) and the boundedness of  $b$  on  $E_\varepsilon$ , the upper semicontinuity follows.

Let  $\Delta$  be such that  $|x(t)| \leq \Delta$ ,  $t \in I$ : then the map  $t \rightarrow \partial_x f(t, x(t))$  is upper semicontinuous on  $E_\varepsilon$ . An application of Lusin's theorem for multi-valued maps yields our claim.

(b) By the theorem of Kuratowski-Ryll Nardzewski (see Theorem III.6 in [C-V]) there exists a measurable selection  $\delta(t) \in \partial_x f(t, x(t))$ . We have

$$\delta(t) \leq \|\partial_x f(t, x(t))\| \leq \frac{2}{\Delta} |k(2\Delta)^p + b(t)|,$$

so that  $\delta \in L^1$ . ■

## MAIN RESULT

THEOREM 1. — Let  $\Phi; f; g$  satisfy hypothesis (H). Then the problem

$$(M) \quad \text{Minimize } \int_0^T g(t, x(t)) dt + \int_0^T h(t, x'(t)) dt$$

on the subset of  $W^{1,p}$  of those  $x(\cdot)$  satisfying:  $x(0) = a, x(T) = b; x'(t) \in \Phi(t)$  a. e. in  $[0, T]$ , admits at least one solution.

*Proof.* — The argument of the proof goes by showing first that the relaxed problem has a solution  $\tilde{x}$ ; then by constructing from  $\tilde{x}$  a different function, a solution to the original problem.

(a) Let us consider the function  $h_\Phi$  defined as

$$h_\Phi(t, x) = \begin{cases} +\infty & \text{for } x \notin \Phi(t) \\ h(t, x) & \text{for } x \in \Phi(t). \end{cases}$$

Then Problem (M) is equivalent to minimizing the functional I, with  $h_\Phi$  replacing  $h$ , on the functions of  $W^{1,p}$  satisfying the boundary conditions.

Set  $h^c$  to be  $h_{\Phi^*}^*$  and consider the problem

$$(MR) \quad \text{Minimize } \int_0^T g(t, x(t)) dt + \int_0^T h^c(t, x'(t)) dt$$

for  $x$  in  $W^{1,p}$ ,  $x(0) = a, x(T) = b$ . By Proposition 1 and the convexity (with respect to  $x'$ ) of the functions appearing in  $(h_3)$ ,  $h^c$  satisfies the same growth condition  $(h_3)$ . Then it is known that problem (MR) has a solution  $\tilde{x}$ . On it,  $h^c(t, \tilde{x}'(t)) < +\infty$  a. e.; by (b) of Proposition 1,  $\tilde{x}'(t)$  belongs to  $\text{co}\Phi(t)$  a. e. and, by (c), there exist measurable functions  $p_i$  and  $v_i$  such that

$$\begin{aligned} \sum_1^{n+1} p_i(t) v_i(t) &= \tilde{x}'(t) \\ \sum_1^{n+1} p_i(t) h_\Phi(t, v_i(t)) &= h^c(t, \tilde{x}'(t)). \end{aligned} \tag{2}$$

Let us remark that any  $v_i(t)$  can be in the complement of  $\Phi(t)$  on a set E of positive measure only if  $p_i \equiv 0$  on it. In this case, we can modify  $v_i$  on E by an arbitrary integrable selection from  $\Phi$  without affecting (2). Hence we can as well assume that  $v_i(t) \in \Phi(t)$  a. e., so that  $h_\Phi(t, v_i(t)) = h(t, v_i(t))$  a. e.

(b) We consider the integrability of a function that will be used in the remainder of the proof. By Lusin's theorem there exists a sequence  $(K_j)_j$  of disjoint compact subsets of I, and a null set N, such that  $I = N \cup (\cup_j K_j)$  and the restriction of each of the maps  $t \rightarrow h(t, v_i(t))$  to each  $K_j$  is continuous. Set  $S_m = \cup_{j \leq m} K_j$ . We claim: Let  $(E_j^i)_i, i = 1, \dots, n+1$ , be a measurable partition of  $K_j$  with the property that, for every  $j$ ,

$$\int_{K_j} (\sum_i p_i(t) h(t, v_i(t))) dt = \int_{K_j} (\sum_i \chi_{E_j^i}(t) h(t, v_i(t))) dt. \tag{3}$$

Then the map

$$t \rightarrow \sum_j \sum_i^{n+1} \chi_{E_j^i}(t) h(t, v_i(t)) \tag{4}$$

belongs to  $L^1$ . As a consequence, since, for  $p > 1$ ,

$$|\sum_{i,j} \chi_{E_j^i}(t) v_i(t)|^p = \sum_{i,j} \chi_{E_j^i}(t) |v_i(t)|^p \leq \frac{1}{\xi_2} \sum_{i,j} \chi_{E_j^i}(t) (h(t, v_i(t)) + \xi_3(t)),$$

the function  $\sum_{i,j} \chi_{E_j^i} v_i$  belongs to  $L^p$ . Analogously for the case  $p=1$ . To prove the claim, remark that on one hand the map  $t \rightarrow \sum_i p_i(t) h(t, v_i(t))$

is integrable since it equals  $t \rightarrow h^c(t, \tilde{x}'(t))$ . On the other hand the sequence of maps

$$s_m(t) = \sum_{j \leq m} (\sum_i \chi_{E_j^i}(t) v_i(t) (h(t, v_i(t)) + \xi_3(t)))$$

is monotone non decreasing and

$$\int_0^T s_m(t) dt = \sum_{j \leq m} \int_{K_j} \sum_i \chi_{E_j^i}(t) (h(t, v_i(t)) + \xi_3(t)) dt.$$

By (3) the right hand side equals

$$\begin{aligned} \sum_{j \leq m} \int_{K_j} \sum_i p_i(t) (h(t, v_i(t)) + \xi_3(t)) dt \\ = \int_0^T \chi_{S_m}(t) (h^c(t, \tilde{x}'(t)) + \xi_3(t)) dt \\ \leq \int_0^T (h^c(t, \tilde{x}'(t)) + \xi_3(t)) dt < +\infty. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^T (\sum_{i,j} \chi_{E_j^i}(t) (h(t, v_i(t)) + \xi_3(t)) dt \\ = \int_0^T (\lim s_m(t)) dt = \lim \int_0^T s_m(t) dt = \int_0^T (h^c(t, \tilde{x}'(t)) + \xi_3(t)) dt. \end{aligned} \tag{5}$$

(c) Set  $\partial^x g(t, x)$  to be  $-\partial_x(-g(t, x))$  and consider the map  $t \rightarrow \partial^x g(t, \tilde{x}(t))$ .

Lemma 1 shows that there exists an integrable function  $\delta(\cdot)$ , a selection from  $\partial^x g(t, \tilde{x}(t))$ . Set  $B(t)$  to be

$$B(t) = \int_0^t \delta(s) ds.$$

Consider the vector measures  $v_i$  and the scalar measures  $\eta_i, \beta_i$  defined, for  $i = 1, \dots, n + 1$ , by

$$v_i(E) = \int_E v_i(t) dt$$

$$\eta_i(E) = \int_E h(t, v_i(t)) dt$$

$$\beta_i(E) = \int_E \langle v_i(t), B(T) - B(t) \rangle dt.$$

Each of them is a vector valued non-atomic measure on  $[0, T]$ , hence on every  $K_j$ . By an extension of Liapunov's theorem on the range of vector measures ([Ce] 16.1.v) there exists a measurable partition of each  $K_j, (E_j^i)_j, i = 1, \dots, n + 1$ , such that

$$\sum_i \int_{K_j} \chi_{E_j^i}(t) dv_i(t) = \sum_i \int_{K_j} p_i(t) dv_i(t); \tag{6}$$

$$\sum_i \int_{K_j} \chi_{E_j^i}(t) d\eta_i(t) = \sum_i \int_{K_j} p_i(t) d\eta_i(t); \tag{7}$$

$$\sum_i \int_{K_j} \chi_{E_j^i}(t) d\beta_i(t) = \sum_i \int_{K_j} p_i(t) d\beta_i(t). \tag{8}$$

(d) We claim that the function  $x : I \rightarrow \mathbb{R}^n$  defined as

$$x'(t) = \sum_j \sum_i \chi_{E_j^i}(t) v_i(t), \quad x(0) = \tilde{x}(0)$$

is a solution to problem (M).

First, let us remark that almost every  $t$  in  $[0, T]$  belongs to exactly one of the  $E_j^i$ , so that, for almost every  $t, x'(t)$  equals one of the  $v_i(t)$  and hence belongs to  $\Phi(t)$ . Moreover

$$h(t, x'(t)) = h(t, \sum_{i,j} \chi_{E_j^i}(t) v_i(t)) = \sum_{i,j} \chi_{E_j^i}(t) h(t, v_i(t)).$$

Hence, by point (b),  $h(t, x'(t))$  is integrable whenever (3) holds, and this follows from the definition of the measures  $\eta_i$  and equality (7). Again by point (b), we have that  $x'(\cdot) \in L^p$ , hence  $x$  is in  $W^{1,p}$ . Moreover,

$$x(T) = \tilde{x}(0) + \int_0^T \sum_{j,i} \chi_{E_j^i}(t) v_i(t) dt = \tilde{x}(0) + \sum_j \int_{K_j} \sum_i \chi_{E_j^i}(t) v_i(t) dt$$

and, from (6), the last integral equals

$$\sum_j \int_{K_j} \sum_i p_i(t) v_i(t) dt = \int_0^T \sum_i p_i(t) v_i(t) dt,$$



so that  $x(T) = \tilde{x}(T)$ .

We claim now that

$$\int_0^T h^c(t, \tilde{x}'(t)) dt = \int_0^T h(t, x'(t)) dt \quad (9)$$

and

$$\int_0^T g(t, \tilde{x}(t)) dt = \int_0^T g(t, x(t)) dt. \quad (10)$$

Ad (9). Again from the definition of the  $\eta_i$ , (5) and (7),

$$\begin{aligned} \int_0^T h^c(t, \tilde{x}'(t)) dt &= \sum_j \int_{K_j} \sum_i p_i(t) h(t, v_i(t)) dt \\ &= \sum_j \sum_i \int_{K_j} \chi_{E_j^i}(t) h(t, v_i(t)) dt \\ &= \sum_j \int_{K_j} h(t, \sum_i \chi_{E_j^i}(t) v_i(t)) dt = \int_0^T h(t, x'(t)) dt. \end{aligned}$$

Ad (10). By the definition of  $\delta(\cdot)$  in (c), for every  $t$  and  $y$ , we have (see [E-T] § I. 5.1)

$$g(t, y) \leq g(t, \tilde{x}(t)) + \langle \delta(t), y - \tilde{x}(t) \rangle. \quad (11)$$

We claim that

$$\int_0^T \langle \delta(t), x(t) - \tilde{x}(t) \rangle dt = 0. \quad (12)$$

Recalling the definition of  $B$  and denoting by  $u_l$  the  $l$ -th component of a vector  $u$ , the above integral can be written as

$$\begin{aligned} &\int_0^T \sum_l \delta_l(t) (x_l(t) - \tilde{x}_l(t)) dt \\ &= \sum_l \int_0^T \delta_l(t) \int_0^t (x'_l(s) - \tilde{x}'_l(s)) ds dt \\ &= \sum_l \int_0^T (x'_l(s) - \tilde{x}'_l(s)) (B_l(T) - B_l(s)) ds \\ &= \int_0^T \langle \sum_{j,i} \chi_{E_j^i}(s) v_i(s) - \sum_i p_i(s) v_i(s), B(T) - B(s) \rangle ds \\ &= \sum_j \int_{K_j} \sum_i (\chi_{E_j^i}(s) - p_i(s)) \langle v_i(s), B(T) - B(s) \rangle ds \\ &= 0, \quad \text{by (8).} \end{aligned}$$

By (11) this proves that

$$\int_0^T g(t, x(t)) dt \leq \int_0^T g(t, \tilde{x}(t)) dt. \quad (12)$$

Since  $\tilde{x}$  is a solution of the problem (MR), by (a) in Proposition 1,

$$\int_0^T g(t, x(t)) dt + \int_0^T h(t, x'(t)) dt \geq \int_0^T g(t, \tilde{x}(t)) dt + \int_0^T h^c(t, \tilde{x}'(t)) dt.$$

On the other hand, by (9) and (12), (10) follows. This proves that  $x$  is a solution to the problem (M). ■

*Remark.* — In case  $g(t, \cdot)$  is strictly concave for almost every  $t$ , i.e. if there exists a selection  $\delta$  from  $\partial^x g(t, \tilde{x}(t))$  such that the inequality sign in (11) is strict for  $y \neq \tilde{x}(t)$ , the functions  $x(t)$  and  $\tilde{x}(t)$  have to coincide, otherwise the integral functional would assume on  $x$  a value strictly less than its infimum. Therefore, in this case, every  $x$  which is a minimizer for the relaxed problem (MR) is also a minimizer for the original problem (M).

## REFERENCES

- [A-T1] G. AUBERT and R. TAHRAOUI, Théorèmes d'existence pour des problèmes du calcul des variations du type :  $\text{Inf} \int_0^L f(x, u'(x)) dx$  et  $\text{Inf} \int_0^L f(x, u(x), u'(x)) dx$ , *J. Diff. Eq.*, Vol. 33, 1979, pp. 1-15.
- [A-T2] G. AUBERT and R. TAHRAOUI, Théorèmes d'existence en optimisation non convexe, *Appl. Anal.*, Vol. 18, 1984, pp. 75-100.
- [C-V] C. CASTAING and M. VALADIER, Convex analysis and measurable multifunctions, *Lecture Notes in Math.*, Springer-Verlag, Berlin, 1977.
- [Ce1] L. CESARI, An existence theorem without convexity conditions, *S.I.A.M. J. Control*, Vol. 12, 1974, pp. 319-331.
- [Ce2] L. CESARI, *Optimization-Theory and Applications*, Springer-Verlag, New York, 1983.
- [C] F. H. CLARKE, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.
- [E] I. EKELAND, Discontinuités de champs hamiltoniens et existence de solutions optimales en calcul des variations, *Publications Mathématiques de l'I.H.E.S.*, Vol. 47, 1977, pp. 5-32.
- [E-T] I. EKELAND and R. TEMAM, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.
- [M1] P. MARCELLINI, Alcune osservazioni sull'esistenza del minimo di integrali del calcolo delle variazioni senza ipotesi di convessità, *Rend. di Matem.*, (2), Vol. 13, 1980, pp. 271-281.
- [M2] P. MARCELLINI, A Relation Between Existence of Minima for Non Convex Integrals and Uniqueness for Non Strictly Convex Integrals of the Calculus of Variations, *Proc. of Congress on Mathematical Theories of Optimization*, S. Margherita Ligure, J. P. CECCONI and T. ZOLEZZI Ed., *Lecture Notes in Math.*, Vol. 979, Springer-Verlag, Berlin, 1983.

- [M3] P. MARCELLINI, Some Remarks on Uniqueness in the Calculus of Variations, *Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar*, Vol. IV, Pitman, Boston, 1984.
- [O] C. OLECH, Integrals of Set-Valued Functions and Linear Optimal Control Problems, *Colloque sur la Théorie Mathématique du Contrôle Optimal*, C.B.R.M., Vander, Louvain, 1970, pp. 109-125.
- [R] J. P. RAYMOND, Conditions nécessaires et suffisantes d'existence de solutions en calcul des variations, *Ann. Inst. H. Poincaré, Analyse non linéaire*, Vol. 4, 1987, pp. 169-202.

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