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AN IMPULSIVE CONTROL PROBLEM WITH STATE CONSTRAINT

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Abstract. We consider an impulsive control problem where state constraints are imposed by minimizing the cost function only over admissible controls such that the controlled diffusion exists from an open set Ω only when no impulse can get it back into Ω.

Then, the optimal cost function satisfies the Quasi-Variational Inequality

$$
(1) \qquad \begin{cases} \operatorname{Max}\,(-\Delta u+\lambda u-f\ ,\ u-Mu) = 0 & \text{in } \Omega \subset \mathbb{R}^N , \\ u|_{\Gamma_o} = \varphi , & \\ u|_{\partial\Omega \setminus \Gamma_o} = Mu , & \end{cases}
$$

where

$$
Mu(x) = k + \inf_{\substack{x+\xi \in \overline{\Omega} \\ \xi \geqslant 0}} \{c_o(\xi) + u(x+\xi)\}
$$

$$
\Gamma_o = \Big\{x \in \partial\Omega\ ,\ \forall\ \xi=(\xi_1,\dots,\xi_N)\ ,\ \xi_i \geqslant 0\ ,\ \forall\ i\ ,\ \exists\ j\ ,\ \xi_j > 0\ ,\ x+\xi \notin \overline{\Omega}\Big\} .
$$

The solution of (1) is not continuous on the boundary and we give a notion of weak solution such that (1) has one and only one solution which is the optimal cost.

Key-words : Impulsive control, State Constraint, Quasi-Variational Inequalities.

Résumé. On considère un problème de contrôle impulsionnel dans lequel on impose une contrainte d'état en minimisant la fonction coût seulement sur les contrôles admissibles tels que la diffusion contrôlée ne sort de l'ouvert de référence Ω , que si aucune impulsion ne peut la ramener dans Ω .

La fonction coût optimal satisfait alors l'inéquation quasi-variationnelle

$$(1) \quad \begin{cases} \operatorname{Max}(-\Delta u+\lambda u-f \, , \, u-Mu) = 0 & \text{dans } \Omega \subset \mathbb{R}^N \\ u|_{\Gamma_o} = \varphi \, , & \\ u|_{\partial\Omega \setminus \Gamma_o} = Mu \, , & \end{cases}$$

où :

$$Mu(x) = k + \inf_{\substack{x+\xi \in \overline{\Omega} \\ \xi \geqslant 0}} \{c_o(\xi) + u(x+\xi)\} \, ,$$

$$\Gamma_o = \left\{ x \in \partial\Omega \, , \, \forall \, \xi=(\xi_1,\ldots,\xi_N) \, , \, \xi_i \geqslant 0 \, , \, \forall \, i \, , \, \exists \, j \, , \, \xi_j > 0 \, , \, x+\xi \notin \overline{\Omega} \right\}$$

La solution de (1) n'est pas continue au bord et nous étudions une notion de solution faible telle que (1) ait une unique solution qui coïncide avec le coût optimal.

Mots clef : Contrôle implusionnel, contraintes d'état, inéquation quasi-variationnelle.

AN IMPULSIVE CONTROL PROBLEM WITH STATE CONSTRAINT

We consider a Quasi-Variational Inequality (Q.V.I. in short) occuring in an impulsive control problem with state constraint. This Q.V.I. may be written as

$$(1)\qquad \begin{cases} \mathrm{Max}\,(-\Delta u+\lambda u-f\ ,\ u-Mu) = 0 & \text{on } \Omega\ , \\ u|_{\Gamma_o} = \varphi_o\ , & \\ u|_{\partial\Omega\setminus\Gamma_o} = Mu\ , & \end{cases}$$

where Ω is some smooth bounded domain of $\mathbb{R}^N$ and

$$(2)\qquad Mu(x) = k + \inf_{\substack{\xi\geqslant 0\\ x+\xi\in\overline{\Omega}}} (c_o(\xi) + u(x+\xi))$$

Here $k>0$ and c_o is a nonnegative subadditive continuous function and $\xi\geqslant 0$ means that $\xi=(\xi_1,\dots,\xi_N)$ with $\xi_i\geqslant 0$ while $\xi>0$ means that $\xi\geqslant 0$ and $\xi_{i_o}>0$ for some i_o. Finally, Γ_o is the part of the boundary defined by

$$(3)\qquad \Gamma_o = \{x\in\partial\Omega\ /\ \forall\ \xi>0\ ,\ \ x+\xi\notin\overline{\Omega}\}\ .$$

On the complementary of Γ_o, the boundary condition is of an implicit type.

These boundary conditions make the main difference between (1) and the classical Q.V.I. introduced by A. Bensoussan and J.L. Lions in [1] (and studied extensively in [3]). They introduce a discontinuity of the solution at the intersection points of Γ_o and $\overline{\partial\Omega\setminus\Gamma_o}$. This will be the main difficulty we have to deal with, since most known results on Q.V.I. use heavily the continuity of solutions.

This paper is organized as follows. In the first section we give a more general version of the equation (1). We define what we will call a solution of (1) and we give the main existence and uniqueness result. In the second section we prove this result. The section III is devoted to prove a regularity result (in $W^{2,\infty}_{loc}$) for some particular unbounded domains. We give also a counterexample which shows that the solution of (1) is only continuous (and not lipschitz continuous) at the boundary points of $\partial\Omega \setminus \Gamma_o$. In section IV, we give the interpretation of the solution in terms of stochastic impulsive control and we check that, despite the discontinuity of u , it is the optimal cost function of the minimization problem. Finally we extend some results of this paper to a more general class of nonlinear equations : namely the Hamilton-Jacobi-Bellman equations. This is achieved in section V.

Finally we would like to emphasize that the problems considered here are closely related to those studied in [15].

I . Main result.

1. Setting the problem.

We will consider a more general formulation of the equation (1) :

$$(4) \qquad \begin{cases} \operatorname{Max}\,(Au-f\ ,\ u-Mu) = 0 & \text{on } \Omega\ , \\ u|_{\Gamma_o} = \varphi_o\ , & \\ u|_{\partial\Omega \setminus \Gamma_o} = Mu\ , & \end{cases}$$

where A is an elliptic second-order differential operator

$$A = -a_{ij}\,\partial^2_{ij} + b_i\,\partial_i + c$$

with

$$(5) \qquad \exists\,\nu > 0 \qquad a_{ij}\xi_i\xi_j \geqslant \nu|\xi|^2 \qquad \forall\,\xi \in \mathbb{R}^N\ ,$$

$$(6) \qquad c \geqslant 0\ ,$$

$$(7) \qquad b_i, c, f \in W^{2,\infty}(\Omega)\ ,\quad a_{ij} \in C^{2,\alpha}(\overline{\Omega}) \quad \text{for some } \alpha > 0\ .$$

The regularity assumed in (7) may be relaxed considerably but we will not bother to do so.

The Q.V.I. with a Dirichlet boundary condition on $\partial\Omega$ has been studied by many authors [1,2,3,4,8,15]. One of the conclusions of these works is that in general the implicit obstacle Mu is not smooth and that we must look after a solution of (4) which is only continuous (and which even does not belong to H^1 since $Mu \notin H^{1/2}(\partial\Omega)$. A convenient way of dealing with such solutions is to adapt Crandall-Lions definition of viscosity solution of first-order Hamilton-Jacobi equation (cf. [5]). This has already been achived in [14,16] but here we must change slightl this definition because of the discontinuity of the solution on the boundary. Inside Ω our definition is the same as the one of [5]. We give this definition in the next section. In the third one we state our main result concerning (4).

2. Viscosity solutions of (4).

In this section we introduce a notion of weak solutions of the obstacle problem

$$(8)\qquad \begin{cases} \operatorname{Max}\,(Au-f\ ,\ u-\psi) = 0 & \text{on } \Omega\ , \\ u\big|_{\partial\Omega} = \varphi\ , & \end{cases}$$

where we assume

(9) $\exists\, \underline{\varphi}$ lower semicontinuous (l.s.c.), $\exists\, \bar{\varphi}$ upper semi-continuous (u.s.c.) such that $\underline{\varphi} = \bar{\varphi} = \varphi$ a.e. on $\partial\Omega$ (for the N-1 dimensional Lebesgue measure).

(10) $\psi \in C(\bar{\Omega})$, $\psi \geqslant \bar{\varphi}$ on $\partial\Omega$.

We recall and adapt the notion of solution introduced in [14].

<u>Definition.</u> (i) A function $\bar{u} \in C(\Omega)$ which is u.s.c. on $\bar{\Omega}$ and which satisfies $\bar{u}\big|_{\partial\Omega} = \bar{\varphi}$ is a viscosity subsolution of (8) if for any function $\gamma \in C^2(\bar{\Omega})$, $\gamma > \bar{\varphi}$ on $\partial\Omega$ and any x_o such that

$$\operatorname*{Max}_{\bar{\Omega}}\ \{\bar{u}-\gamma\} = (\bar{u}-\gamma)(x_o) = 0\ ,\quad x_o \in \Omega\ ,$$

then

$$\operatorname{Max}\ (A\gamma-f\ ,\ \gamma-\psi)(x_o) \leqslant 0\ .$$

(ii) A function $\underline{u} \in C(\Omega)$ which is l.s.c. and which satisfies $\underline{u}\big|_{\partial\Omega} = \underline{\varphi}$ is a viscosity supersolution of (8) if for any function $\gamma \in C^2(\bar{\Omega})$, $\gamma < \underline{\varphi}$ on $\partial\Omega$ and any x_o such that

$$\operatorname*{Min}_{\bar{\Omega}}\ \{\underline{u}-\gamma\} = (\underline{u}-\gamma)(x_o) = 0\ ,\quad x_o \in \Omega\ ,$$

then

$$\operatorname{Max}\ (A\gamma-f\ ,\ \gamma-\psi)(x_o) \geqslant 0\ .$$

(iii) A fonction $u \in C(\Omega)$ is said to be a viscosity solution of (8) if there exist $\bar{u}$ and $\underline{u}$ with

$$u = \bar{u} = \underline{u} \quad \text{on} \quad \Omega$$

and such that $\bar{u}$ is a viscosity subsolution of (8) and $\underline{u}$ is a viscosity supersolution of (8).

Remarks. 1) As usual it is easy to check that one obtains equivalent formulations if we replace $\gamma \in C^2$ by $\gamma \in C^\infty$, global maximum (or minimum) by global strict, local strict or local maximum (resp. minimum).

2) The reason why it is enough to consider these kinds of boundary conditions which satisfy (9) is clear, the boundary condition in (4) is discontinuous only at points of $\Gamma_o \cap \overline{\partial\Omega \setminus \Gamma_o}$ and satisfies (10) if Mu and φ_o are continuous.

3) One easily checks that this definition implies that $u \in H^1_{loc}(\Omega)$ (see [16]). One could also define a solution of (8) by a variational formula : $u \in H^1_{loc}(\Omega)$, $u \leqslant \psi$, $\limsup_{y \to x,\, y \in \Omega} \text{ess}\ u(y) = \bar{\varphi}(x)$, $\liminf_{y \to x,\, y \in \Omega} \text{ess}\ u(y) = \underline{\varphi}(x)$ and for any $v \in H^1_{loc}(\Omega)$, $v = u$ on a neighborhood of $\partial\Omega$, $v \leqslant \psi$ we have

$$a(u,v-u) \geqslant (f,v-u) \quad ,$$

where $a(\cdot,\cdot)$ is the bilinear form associated to A .

4) In the same way, our definition of viscosity solution can be reduced to : $u \in C(\Omega)$, $\limsup_{y \to x,\, y \in \Omega} u(y) = \bar{\varphi}(x)$, $\liminf_{y \to x,\, y \in \Omega} u(y) = \underline{\varphi}(x)$ and u is a viscosity solution of (8) in Ω .

With this definition we have the

Theorem 1. Under assumptions (5)-(7), (9), (10), there exists a unique viscosity solution of (8).

Proposition 1. Under assumption (5)-(7), let $\bar{\varphi}_n, \underline{\varphi}_n$ be sequences which satisfy (9) and converge uniformly to $\bar{\varphi}, \underline{\varphi}$. Let $\psi_n \in C(\bar{\Omega})$, $\psi_n \geqslant \bar{\varphi}_n$ on $\partial\Omega$ converge uniformly to ψ , then the solution $(\bar{u}_n, \underline{u}_n)$ of (8) for the obstacle ψ_n and the boundary data $(\bar{\varphi}_n, \underline{\varphi}_n)$ converges uniformly to the viscosity solution $(\bar{u}, \underline{u})$ of (8).

The proof of these results is given in section II below.

3. Main results.

In order to guarantee the existence of a solution of (4) we define the operator M_o (first introduced in [15])

$$(11) \qquad M_o\varphi_o(x) = k + \inf_{\substack{\xi \geqslant 0 \\ x+\xi \in \Gamma_o}} \{c_o(\xi) + \varphi_o(x+\xi)\} \quad .$$

Our main result is the following

Theorem 2. Under assumption (5)-(7), let $\varphi_o \in C(\Gamma_o)$ and $M_o\varphi_o \in C(\bar{\Omega})$ then there exists a unique solution u of (4) in the sense that $M\underline{u} \in C(\bar{\Omega})$ and u is a viscosity solution of (8) with $\psi = M\underline{u}$ where $\underline{u}$ is the l.s.c. version of u i.e. $\underline{u} = u$ in Ω, $\underline{u}(x) = \lim \inf_{y \in \Omega, y \to x} u(y)$ $\forall\, x \in \partial\Omega$.

Remark. In particular the function φ defined by $\varphi = \varphi_o$ on Γ_o and $\varphi = M\underline{u}$ on $\partial\Omega \setminus \Gamma_o$ satisfies (9) since $\text{meas}_{N-1}(\partial\Gamma_o) = 0$ and $\partial\Gamma_o$ is smooth.

We give also the proof of this result in the next section.

II . Proof of Theorems 1 and 2.

1. Proof of Theorem 1.

In order to prove Theorem 1 we remark that we can always find, under assumption (9), two functions φ^n, φ_n such that

(i) $\varphi^n \in C(\partial\Omega)$, $\varphi_n \in C(\partial\Omega)$,

(ii) $\varphi^n \searrow \bar{\varphi}$, $\varphi_n \nearrow \underline{\varphi}$ as $n \to +\infty$,

and, for each n , a new obstacle ψ^n such that

(iii) $\psi^n \in C(\bar{\Omega})$, $\psi^n \geqslant \varphi^n$ on $\partial\Omega$, $\psi^n \searrow \psi$ as $n \to +\infty$.

In particular ψ^n converges uniformly to ψ .

Then we define $u^n \in C(\bar{\Omega})$, $u_n \in C(\bar{\Omega})$ the unique solution (see [2,11,14]) of the obstacle problems

$$\begin{cases} \operatorname{Max} \ (Au^n - f \ , \ u^n - \psi^n) = 0 \ , \\ u^n\big|_{\partial\Omega} = \varphi^n \ , \end{cases} \tag{12}$$

$$\begin{cases} \operatorname{Max} \ (Au_n - f \ , \ u_n - \psi) = 0 \ , \\ u_n\big|_{\partial\Omega} = \varphi_n \ . \end{cases} \tag{13}$$

It is clear enough that there exist functions $\bar{u}$ (which is u.s.c.) and $\underline{u}$ (which is l.s.c.) such that

$$u^n \searrow \bar{u} \quad , \quad u_n \nearrow \underline{u} \quad \text{as} \quad n \to +\infty \ .$$

Moreover classical estimates (cf. [11,13,15]) show that $\bar{u} \in C(\Omega)$, $\underline{u} \in C(\Omega)$, and thus by standard arguments we see that $\bar{u}$ (resp. $\underline{u}$) is a subsolution (resp. supersolution) of (8). Thus the existence part of Theorem 1 will be proved once we have proved the

Lemma 1. The functions $\underline{u}$ and $\bar{u}$ defined above satisfy

$$\underline{u} = \bar{u} \quad \text{on} \quad \Omega \ .$$

The proof of this Lemma is given in section IV since it uses the stochastic control interpretation which is developed in that section.

Let us turn to the uniqueness of the solution. Thus, let $\bar{v} \in C(\Omega)$, $\bar{v}$ u.s.c., be a subsolution of (8) and assume that

$$\underset{x \in \bar{\Omega}}{\mathrm{Max}} \; (\bar{v}-u^n)(x) = (\bar{v}-u^n)(x_o) = \delta > 0 \quad .$$

Setting

$$\underset{x \in \bar{\Omega}}{\mathrm{Max}} \; (\bar{v}-u^n+\varepsilon|x-x_o|^2)(x) = (\bar{v}-u^n)(x_\varepsilon) + \varepsilon|x_\varepsilon-x_o|^2 \quad ,$$

we may assume that $x_\varepsilon \underset{\varepsilon \to 0}{\to} x_o$. Then $x_o \in \Omega$ and $u^n(x_o) < \psi^n(x_o)$ since $\psi^n \geqslant \psi$. Thus $u^n \in C^2(V)$ where V is some neighborhood of x_o and, by the definition we have

$$A(u^n+\delta-\varepsilon|x-x_o|^2)(x_\varepsilon) \leqslant 0$$

but $Au^n \geqslant 0$ and thus for ε small enough

$$A(u^n+\delta-\varepsilon|x-x_o|^2) > 0 \qquad \text{(in a neighborhood of } x_o \text{)}$$

and thus we have reached a contradiction which proves that $\delta = 0$ and so, that $\bar{v} \leqslant u^n$. Passing to the limit we obtain that

$$\bar{v} \leqslant \bar{u} \quad .$$

In the same way, we could prove that any supersolution $\underline{v}$ of (8) satisfies

$$\underline{v} \geqslant \underline{u}$$

and the uniqueness follows, completing the proof of Theorem 1.

2. Proof of Proposition 1.

Let us first prove the uniform convergence in Proposition 1. It is asserted by the

<u>Lemma 2.</u> With the notations of Proposition 1

$$\sup_x |(\bar{u}_n-\bar{u}_\ell)(x)| \leqslant \mathrm{Max}\left(\|\psi_n-\psi_\ell\|_{L^\infty}, \sup_x \|(\bar{\varphi}_n-\bar{\varphi}_\ell)(x)\|\right) ,$$

$$\sup_x |(\underline{u}_n-\underline{u}_\ell)(x)| \leqslant \mathrm{Max}\left(\|\psi_n-\psi_\ell\|_{L^\infty}, \sup_x \|(\underline{\varphi}_n-\underline{\varphi}_\ell)(x)\|\right) .$$

Again we leave the proof of this Lemma to section IV since it uses stochastic tools. Let us conclude the proof of Proposition 1. With this Lemma we get that

$$\bar{u}_n \underset{n \to +\infty}{\to} \bar{u} \quad , \quad \underline{u}_n \underset{n \to +\infty}{\to} \underline{u} \qquad \text{uniformly on } \bar{\Omega} \quad ,$$

$\bar{u}$ is u.s.c. and $\underline{u}$ is l.s.c. and satisfy

$$\bar{u} = \underline{u} \quad \text{on } \Omega \quad ; \quad \bar{u}, \underline{u} \in C(\Omega) \quad .$$

It is clear that the boundary condition for $\bar{u}$ and $\underline{u}$ is satisfied. Then the viscosity characterisation follows from the classical arguments of [5,12].

3. Proof of Theorem 2.

We prove Theorem 2 with the same argument as in B. Hanouzet and J.L. Joly [8]. Thus we define a decreasing sequence of functions as follows. First we choose a constant C_o large enough $\left(C_o \geqslant \operatorname{Sup} |\varphi_o| + \sup_{|\xi| \leqslant \operatorname{diam}(\Omega)} c_o(\xi)\right)$ and we may solve the equation (which has a unique solution in the sense of the above definition)

$$(14) \qquad \begin{cases} Au^o = f \quad , \\ u^o\big|_{\Gamma_o} = \varphi_o \quad , \\ u^o\big|_{\partial\Omega \setminus \Gamma_o} = C_o \end{cases}$$

Indeed it may be viewed as a particular case of (8) with ψ large enough. By induction we define the solution u^n of

$$(15) \qquad \begin{cases} \operatorname{Max} \; (Au^n - f \; , \; u^n - M\underline{u}^{n-1}) = 0 \quad , \\ u^n\big|_{\Gamma_o} = \varphi_o \quad , \\ u^o\big|_{\partial\Omega \setminus \Gamma_o} = M\underline{u}^{n-1} \quad . \end{cases}$$

Here we denote by $\underline{u}^n$ the l.s.c. version of u^n which existence is asserted by Theorem 1. To apply it, and to prove that the sequence in (15) is well defined, we need to check that $\underline{Mu}^n$ is continuous at each step (indeed (9) clearly holds if $\underline{Mu}^n \in C(\overline{\Omega})$). To do so we use the argument of [15,16]. Let $x_o \in \overline{\Omega}$ and set

$$\underline{Mu}^n(x_o) = k+c_o(\xi_o) + \underline{u}^n(x_o+\xi_o) \quad , \quad \xi_o \geqslant 0 \ .$$

Three cases may occur :

(i) $x_o+\xi_o \in \Omega$, then locally we may write

$$\underline{Mu}^n(x) \leqslant k + c_o(\xi_o) + \underline{u}^n(x+\xi_o) \quad , \quad x+\xi_o \in \Omega$$

and since $\underline{u}^n \in C(\Omega)$ this shows that $\underline{Mu}^n$ is u.s.c. at the point x_o .

(ii) $x_o+\xi_o \in \Gamma_o$, then we have

$$\begin{aligned} \underline{Mu}^n(x_o) &= M_o\varphi(x_o) \\ \underline{Mu}^n(x) &\leqslant M_o\varphi(x) \quad , \quad \forall\, x \in \overline{\Omega} \ , \end{aligned}$$

and again this shows that $\underline{Mu}^n$ is u.s.c. at the point x_o .

(iii) $x_o+\xi_o \in \partial\Omega \setminus \Gamma_o$. We show that this is not possible. Indeed this could give

$$\begin{aligned} \underline{Mu}^n(x_o) &= k + c_o(\xi_o) + \underline{u}^n(x_o+\xi_o) \\ &= k + c_o(\xi_o) + \underline{Mu}^{n-1}(x_o+\xi_o) \\ &= 2k + c_o(\xi_o) + c_o(\xi_1) + \underline{u}^{n-1}(x_o+\xi_o+\xi_1) \quad \text{(for some } \xi_1 \geqslant 0 \\ &\geqslant 2k + c_o(\xi_o+\xi_1) + \underline{u}^{n-1}(x_o+\xi_o+\xi_1) \end{aligned}$$

(since c_o is assumed to be subadditive) and finally

$$\underline{Mu}^n(x_o) \geqslant k + \underline{Mu}^{n-1}(x_o)$$

which contradicts the fact that $u^n \leqslant u^{n-1}$. Of course this only holds for $n > 0$. For $n = 0$ the claim is obvious.

Thus we have proved that $\underline{Mu}^n$ is u.s.c. and since $\underline{Mu}$ is always l.s.c. when $\underline{u}$ is l.s.c. (see [15]) we have proved that $\underline{Mu}^n \in C(\bar{\Omega})$ and thus, that the sequence (15) is well defined.

The next step in our proof is the following

Lemma 3. $\exists \mu_o$, $0 < \mu_o \leqslant 1$, such that

$$0 \leqslant \bar{u}^n - \bar{u}^{n+1} \leqslant (1-\mu_o)^n \bar{u}^o \quad ,$$

$$0 \leqslant \underline{u}^n - \underline{u}^{n+1} \leqslant (1-\mu_o)^n \underline{u}^o \quad .$$

Before proving this Lemma, let us conclude the proof of Theorem 2. It shows that

$$0 \leqslant \underline{Mu}^n - \underline{Mu}^{n+1} \leqslant C(1-\mu_o)^n \quad ,$$

and thus, we can use the result of Proposition 1 to get that $\bar{u}^n, \underline{u}^n$ converge uniformly to $\bar{u}, \underline{u}$ solution of (8) with

$$\psi = \underline{Mu} \quad ,$$

and the existence part of Theorem 2 is proved.

The uniqueness is a variant of Lemma 3 and [8] and is left to the reader.

Proof of Lemma 3.

The proof of Lemma 3 uses classical arguments and thus we only sketch it. We prove by induction that if

(16) $$\underline{u}^n - \underline{u}^{n+1} \leqslant \theta\, \underline{u}^n \quad ,$$

then

(17) $$\underline{u}^{n+1} - \underline{u}^{n+2} \leqslant (1-\mu_o)\theta\, \underline{u}^{n+1} \quad .$$

We rewrite (16) as

$$(1-\theta)\underline{u}^n + \theta \cdot 0 \leqslant \underline{u}^{n+1} \quad ,$$

then

$$(1-\theta)M\underline{u}^n + \theta\, M0 \leq M\underline{u}^{n+1} \quad .$$

By monotonicity and concavity arguments which still hold (they may be checked for discontinuous boundary data by regularizing them) we obtain

$$(1-\theta)\underline{u}^{n+1} + \theta\, \underline{v} \leq \underline{u}^{n+2}$$

where v satisfies (in the viscosity sense)

$$\begin{cases} \text{Max } ((Av-f), v-M0) = 0 & \text{on } \Omega \ , \\ v\big|_{\Gamma_o} = \varphi_o \wedge k \ , & \\ v\big|_{\partial\Omega \setminus \Gamma_o} = M0 \ . & \end{cases}$$

For some μ_o , $0 < \mu_o \leq 1$ we have

$$\underline{v} \geq \mu_o \underline{u}_o \quad ,$$

and (17) is proved. Lemma 3 follows directly from (17).

III. Regularity of the solution.

In this section we focuss our attention on the regularity of the solution. In order to simplify the problem we will consider smooth open sets Ω with the property

$$\text{(18)} \qquad \begin{cases} \text{if } \Gamma_o \neq \emptyset & \text{and} \quad \partial\Omega \setminus \Gamma_o \neq \emptyset \ , \\ \text{then } \exists\, \alpha > 0 \ , & d(\Gamma_o \ , \partial\Omega \setminus \Gamma_o) \geq \alpha \ . \end{cases}$$

This property occurs only for unbounded domain (it is achieved for example if Ω is a strip with a good orientation, see the counter-example below) but one easily checks that the existence theory of sections I and II still holds. Moreover, since the discontinuities of the solution of the Q.V.I. only appears on the set $\Gamma_o \cap \overline{\partial\Omega \setminus \Gamma_o}$ it is easy to prove the following variant of Theorem 2 :

Theorem 2'. Let Ω be a smooth open set satisfying (18), let us assume (5)-(7) and that $\exists\, \lambda > 0$, $c(x) \geq \lambda$. Let $\varphi_o \in BUC(\Gamma_o)$, $M_o\varphi_o \in BUC(\bar{\Omega})$, then there exists a unique solution $u \in BUC(\bar{\Omega})$ of (4), in the generalized sense of Theorem 2, and $Mu \in BUC(\bar{\Omega})$.

(Here $BUC(\bar{\Omega})$ denotes the set of bounded uniformly continuous functions on $\bar{\Omega}$).

Here, our goal is not to prove this result (which can be obtained with the arguments of previous section). We will rather show that it can be improved and actually that (with some more assumptions) u belongs to $W^{2,\infty}_{loc}(\Omega)$. This is achieved in the first section. In the second one, we give a counter-example where u is not lipschitz up to a boundary.

1. Interior regularity.

Let us denote by $D^{2,+}(\Omega)$ the cone of semi-concave functions in Ω i.e.

$$D^{2,+}(\Omega) = \left\{ u \in W^{2,\infty}(\Omega) \ , \ \exists C \ , \ \frac{\partial^2 u}{\partial \chi^2} \leq C \ , \ \forall \chi \ , \ |\chi| = 1 \right\} ,$$

and for any set V

$$d(V \ , \ \partial\Omega \setminus \Gamma_o) = \inf_{\substack{y \in V \\ z \in \partial\Omega \setminus \Gamma_o}} \|y-z\|$$

Proposition 2. Under the assumptions of Theorem 2', let $\varphi_o \in C^{2,\alpha}(\Gamma_o)$, let $M_o\varphi_o \in D^{2,+}(\Omega)$, $c_o \in W^{2,\infty}(\mathbb{R}^N)$, and let V be an open subset of Ω such that $d(V \ , \ \partial\Omega \setminus \Gamma_o) > 0$ (this assumption disappears if $\partial\Omega \setminus \Gamma_o = \emptyset$), then $u \in W^{2,\infty}(V)$.

This Proposition is nothing but a variant of the similar regularity result of [17], let us only indicate the main steps of its proof. First, using the assumption $M_o\varphi_o \in D^{2,+}(\Omega)$ we can show that $Mu \in D^{2,+}(\Omega)$. Indeed, mimicking the argument of L.A. Cafarelli and A. Friedman [4], let $x_o \in \Omega$ and let $\xi_o \geq 0$ be such that

$$Mu(x_o) = k + c_o(\xi_o) + u(x_o+\xi_o) \quad ,$$

If $x_o+\xi_o \in \Gamma_o$, one has, for h small enough,

$$\frac{Mu(x_o+h\chi)+Mu(x_o-h\chi)-2Mu(x_o)}{h^2} \leqslant \frac{M_o\varphi_o(x_o+h\chi)+M_o\varphi_o(x-h\chi)-2M_o\varphi_o(x_o)}{h^2}$$

$$\leqslant C$$

if $x_o+\xi_o \in \Omega$, then for a neighborhood $\mathcal{O}$ of $x_o+\xi_o$ in $\bar{\Omega}$, one can show that $u \in W^{2,\infty}(\mathcal{O})$ and thus one has

$$\frac{Mu(x_o+h\chi)+Mu(x_o-h\chi)-2Mu(x_o)}{h^2}$$

$$\leqslant \frac{c_o(\xi_o+h\chi)+c_o(\xi_o-h\chi)-2c_o(\xi_o)}{h^2} + \frac{u(x_o+h\chi)+u(x_o-h\chi)-2u(x_o)}{h^2} \leqslant C$$

This proves that $Mu \in D^{2,+}(\Omega)$. Then one easily deduce that $u \in W^{2,\infty}(V)$ at least when $V \cap \Gamma_o = \emptyset$. When $V \cap \Gamma_o \neq \emptyset$ the result is due to R. Jensen [9] (see [17] too).

The end of this section is devoted to give a counter-example to the Lipschitz regularity on $\partial\Omega \setminus \Gamma_o$.

2. A counter-example to $W^{1,\infty}$ regularity at the boundary.

Here, we work in $\mathbb{R}^2$, we make a rotation so $\xi \geqslant 0$ now means $\xi_2 \geqslant |\xi_1|$ and we consider the particular open set Ω :

$$\Omega = \left\{x = (x_1,x_2) \in \mathbb{R}^2 \ , \ x_1 \in \mathbb{R} \ , \ 0 \leqslant x_2 \leqslant 1\right\} \quad .$$

This set is a strip which satisfies (18) but which has been rotated. Thus, to apply the above theory one must change the definition of the implicit obstacle. We set (after a rotation)

$$M'u(x) = k + \inf_{\substack{\xi_2 \geqslant |\xi_1| \\ 0 \leqslant x_2+\xi_2 \leqslant 1}} \{u(x+\xi)\} \quad .$$

Moreover we take the particular example of (4)

$$(19) \quad \begin{cases} \mathrm{Max}\,(-\Delta u+u-f\ ,\ u-M'u) = 0 \ , \\ u\big|_{\Gamma_o} = \varphi_o \ , \\ u\big|_{\Gamma_1} = M'u \ , \end{cases}$$

with

$$\Gamma_o = \{x = (x_1,x_2)\ ,\ x_2 = 1\}$$
$$\Gamma_1 = \{x = (x_1,x_2)\ ,\ x_2 = 0\} \ .$$

From Theorem 2' we deduce that (19) has a unique solution in $BUC(\overline{\Omega})$ and Proposition 2 asserts that $u \in W^{2,\infty}(V)$ for every V with $d(V,\Gamma_1) > 0$.

Our purpose is to prove that, even in this simple situation, this regularity is optimal since $u \notin W^{1,\infty}(\mathbb{R} \times [0,1])$. Therefore we build a counter-example in which Mu is only semi-concave near Γ_1 (and not C^1) and consequently u is not Lipschitz near Γ_1 . Let us recall that in general, the solution of an elliptic equation with a $W^{1,\infty}$ boundary data has a solution which is only holder continuous but not Lipschitz continuous.

To do so, we need to define conveniently f and φ_o . $\varphi_o(x_1)$ will be any smooth function which has the following properties.

$$(20) \quad \begin{cases} \text{for } x_1 \in (-\infty,0) \ , \quad \varphi_o'(x_1) > 0 \ , \\ \lim\limits_{x_1 \to -\infty} \varphi_o(x_1) = 0 \ , \quad \varphi_o(-1) = 1 \ , \quad \varphi_o(0) = 2 \ , \\ \text{for } x_1 \in (0,1) \ , \quad \varphi_o \text{ is strictly decreasing} \ , \\ \text{for } x_1 \geq 1 \ , \quad \varphi_o(x_1) = 1 \ . \end{cases}$$

Then we take

$$(21) \quad f(x_1,x_2) = \varphi_o(x_1) - \varphi_o''(x_1) \in C_b^2(\overline{\Omega}) \ .$$

Let us choose k large enough such that the function

$$\underline{u}(x) = \varphi_o(x_1)$$

satisfies

$$\begin{cases} -\Delta \underline{u} + \underline{u} = (\varphi_o - \varphi_o'')(x_1) & \text{on } \Omega , \\ \underline{u} \leqslant M'\underline{u} , \\ \underline{u}\big|_{\Gamma_o} = \varphi_o . \end{cases}$$

One easily checks that

$$M'\underline{u}(x) = M_o'\varphi_o(x) ,$$

where

$$M_o'\varphi_o(x) = k + \inf_{\xi_1 = \pm(1-x_2)} \varphi_o(x_1+\xi_1)$$

Thus the solution $u(x)$ of the Q.V.I. (19), with "parameters" f, φ_o, k defined above, satisfies

(22) $$M_o'\varphi_o \leqslant M'\underline{u} \leqslant M'u \leqslant M_o'\varphi_o \quad \text{on } \overline{\Omega} ,$$

since

$$\underline{u} \leqslant u .$$

We will deduce from (22) that u is not Lipschitz continuous near Γ_1. Indeed let us introduce the function v solution of.

$$\begin{cases} -\Delta v + v = C_o , & \forall\, x = (x_1, x_2) , \quad x_2 \geqslant 0 , \\ v\big|_{x_2=0} = M_o'\varphi_o . \end{cases}$$

For C_o large enough $\left(C_o \geqslant \|-\varphi''+\varphi\|_{L^\infty}\right)$ and λ large enough $\left(\lambda \geqslant \|v\|_{L^\infty} + \|\varphi_o\|_{L^\infty}\right)$ we have

$$\begin{cases} -\Delta(v+\lambda x_2) + v + \lambda x_2 \geqslant -\varphi'' + \varphi & \text{on } \Omega , \\ v+\lambda x_2\big|_{x_2=0} \geqslant u\big|_{x_2=0} , & \text{(by (22))} \\ v+\lambda x_2\big|_{x_2=1} \geqslant u\big|_{x_2=1} . \end{cases}$$

So that, by the maximum principle,

(23) $$v + \lambda x_2 \geqslant u \quad \text{on } \overline{\Omega} .$$

We conclude our counter-example by proving that at the point $x = 0$ (which belongs to Γ_1) we have

(24) $$\frac{\partial}{\partial n} v+\lambda x_2 \Big|_{x=0} = +\infty \quad ,$$

where n is the outer normal at Ω on Γ_1 (i.e. $n=(0,-1)$) . This implies with (23) that

$$\frac{\partial}{\partial n} u \Big|_{x=0} = +\infty \quad ,$$

and we are done.

In fact (24) is equivalent to

(24') $$\frac{\partial w}{\partial n} \Big|_{x=0} = +\infty \quad ,$$

where w is the solution of

$$\begin{cases} -\Delta w = 0 \quad , \quad \forall\, x = (x_1,x_2) \quad , \quad x_2 > 0 \quad , \\ w\big|_{x_2=0} = M_o'\varphi_o \quad . \end{cases}$$

But one easily computes $M_o'\varphi_o\big|_{x_2=0}$:

$$\begin{cases} \text{for } x_1 \leq 0 \ , \quad M_o'\varphi_o(x_1,0) = \varphi_o(x_1-1) \quad , \\ \text{for } x_1 \geq 0 \ , \quad M_o'\varphi_o(x_1,0) = 1 \quad . \end{cases}$$

Thus $M_o'\varphi_o\big|_{x_2=0}$ is a smooth function except at the point $x_1=0$ where it is Lipschitz and where it admits left and right derivatives which are respectively $\varphi_o'(-1)$ and 0 .

It is easily checked on the exact formula giving w that (24') holds when $M_o'\varphi_o$ satisfies these properties and this concludes the proof of the

Proposition 2. With the data described above, the viscosity solution of (19) in $BUC(\overline{\Omega})$ is not Lipschitz continuous up to the boundary Γ_1 .

IV . The Stochastic Control Problem.

In this section we give an interpretation of the solution of (4) in terms of impulsive control. We also use this interpretation to prove the lemmas 1 and 2 of section II. The new features in this interpretation are of course the discontinuities of the solution and the boundary condition on $\partial\Omega \setminus \Gamma_o$. For a classical treatment of control of diffusions and impulsive control we refer to [2,3,10,14,16,18].

In order to simplify the notations we only consider here the case of the impulsive control associated to equation (1).

1. Proof of Lemma 1.

Throughout this section we consider a probability space (X,F,F_t,P) with a right-continuous increasing filtration of complete sub- σ fields, and a Wiener process w_t , in $\mathbb{R}^N$, F_t -adapted.

In order to prove Lemma 1 we introduce the stopping time problems associated with the equations

$$(25)\qquad \begin{cases} \operatorname{Max}\,(-\Delta u^n+\lambda u^n-f\ ,\ u^n-\psi^n) = 0 & \text{on } \Omega\ , \\ u^n\big|_{\partial\Omega} = \varphi^n\ , & \end{cases}$$

$$(26)\qquad \begin{cases} \operatorname{Max}\,(-\Delta u_n+\lambda u_n-f\ ,\ u_n-\psi) = 0 & \text{on } \Omega\ , \\ u_n\big|_{\partial\Omega} = \varphi_n\ , & \end{cases}$$

where $\varphi^n \searrow_n \bar{\varphi}$, $\varphi_n \nearrow_n \underline{\varphi}$, $\psi^n \searrow_n \psi$ as $n \to +\infty$, $\psi^n \geqslant \varphi^n$ on $\partial\Omega$, $\varphi^n,\varphi_n \in C(\partial\Omega)$, $\psi^n \in C(\bar{\Omega})$, and φ,ψ satisfy (10),(9).

Thus we introduce the trajectory

$$y_x(t) = x + w_t\ ,$$

and

$$\tau_x = \inf\,\{t \,/\, y_x(t) \notin \Omega\}\ .$$

For a stopping time θ we set

$$(27)\qquad J^n(x,\theta) = E\left\{\int_0^{\theta\wedge\tau_x} f(y_x(s))e^{-\lambda s}ds + \varphi^n(y_x(\tau_x))e^{-\lambda\tau_x} 1_{\tau_x \leq \theta} + \psi^n(y_x(\theta))e^{-\lambda\theta} 1_{\theta < \tau_x}\right\},$$

$$(28)\qquad J_n(x,\theta) = E\left\{\int_0^{\theta\wedge\tau_x} f(y_x(s))e^{-\lambda s}ds + \varphi_n(y_x(\tau_x))e^{-\lambda\tau_x} 1_{\tau_x \leq \theta} + \psi(y_x(\theta))e^{-\lambda\theta} 1_{\theta < \tau_x}\right\}.$$

It is well-known (see [2,11]) that

$$(29)\qquad u^n(x) = \inf_\theta J^n(x,\theta),$$

$$(30)\qquad u_n(x) = \inf_\theta J_n(x,\theta).$$

Now for any $x \in \Omega$, the random variable $y_x(\tau_x)$ has a density on $\partial\Omega$. Thus there exists a function $\mu(x,z) \in L^1_z(\partial\Omega)$ such that

$$0 \leq J^n(x,\theta) - J_n(x,\theta) \leq \int_{\partial\Omega} (\varphi^n-\varphi_n)(z)\,\mu(x,z)dz + \|\psi^n-\psi\|_{L^\infty}.$$

By dominated convergence we obtain, as n goes to $+\infty$, (with the notations of Lemma 1)

$$0 \leq \bar{u}(x) - \underline{u}(x) \underset{x\to+\infty}{\to} 0,$$

and this proves Lemma 1.

2. Stopping time problem with discontinuous boundary data.

On the other hand we may pass to the limit in (27),(28). Since φ^n and φ_n converge monotonically and remains bounded we have

$$(31)\qquad \begin{cases} J^n(x,\theta) \underset{n\to+\infty}{\to} \bar{J}(x,\theta) & \text{for every } x, \\ J_n(x,\theta) \underset{n\to+\infty}{\to} \underline{J}(x,\theta) & \text{for every } x, \end{cases}$$

where

$$(32)\qquad \bar{J}(x,\theta) = E\left\{\int_0^{\theta\wedge\tau_x} f(y_x(s))e^{-\lambda s}ds + \bar{\varphi}(y_x(\tau_x))e^{-\lambda\tau_x}\, 1_{\tau_x \leqslant \theta} + \psi(y_x(\theta))e^{-\lambda\theta}\, 1_{\theta < \tau_x}\right\} ,$$

$$(33)\qquad \underline{J}(x,\theta) = E\left\{\int_0^{\theta\wedge\tau_x} f(y_x(s))e^{-\lambda s}ds + \underline{\varphi}(y_x(\tau_x))e^{-\lambda\tau_x}\, 1_{\tau_x \leqslant \theta} + \psi(y_x(\theta))e^{-\lambda\theta}\, 1_{\theta < \tau_x}\right\} .$$

Moreover $\bar{u}$ is defined as

$$(34)\qquad \bar{u} = \inf_n u^n(x) = \inf_{n,\theta} J^n(x,\theta) = \inf_\theta \bar{J}(x,\theta)$$

Finally we also obtain

$$\bar{J}(x,\theta) \geqslant \underline{J}(x,\theta) \geqslant J_n(x,\theta) \qquad \forall\, x \in \bar{\Omega} ,\ \forall\, \theta ,$$

thus

$$\bar{u}(x) \geqslant \inf_\theta \underline{J}(x,\theta) \geqslant u_n(x) \qquad \forall\, x \in \bar{\Omega} ,$$

and, passing to the limit

$$(35)\qquad \underline{u}(x) = \inf_\theta \underline{J}(x,\theta) \qquad \forall\, x \in \bar{\Omega} ,$$

indeed we may use Lemma 1 for $x \in \Omega$, and this is clear enough for $x \in \partial\Omega$.

We may now prove Lemma 2.

3. Proof of Lemma 2.

We consider now two obstacles $\psi,\tilde{\psi}$ satisfying (9) and two boundary conditions $\varphi,\tilde{\varphi}$ satisfying (10). With the above notations we have

$$\bar{J}(x,\theta) \leqslant \bar{\tilde{J}}(x,\theta) + E\{|(\bar{\varphi}-\bar{\tilde{\varphi}})(y_x(\tau_x))|\} + \|\psi-\tilde{\psi}\|_{L^\infty} , \qquad \forall\, \theta \text{ stopping time}$$

therefore

$$\bar{u}(x) \leqslant \bar{\tilde{J}}(x,\theta) + \sup_x |(\bar{\varphi}-\bar{\tilde{\varphi}})(x)| + \|\psi-\tilde{\psi}\|_{L^\infty} \qquad \forall\, \theta \text{ stopping time}$$

i.e.

$$\bar{u}-\bar{\tilde{u}} \leqslant \text{Max}\left[\sup_x \quad |(\bar{\varphi}-\bar{\tilde{\varphi}})(x)| \ , \ \|\psi-\tilde{\psi}\|_{L^\infty}\right]$$

and, in the same way, we obtain

$$\underline{u}-\underline{\tilde{u}} \leqslant \text{Max}\left[\sup_x \quad |(\underline{\varphi}-\underline{\tilde{\varphi}})(x)| \ , \ \|\psi-\tilde{\psi}\|_{L^\infty}\right] \quad .$$

This proves Lemma 2.

4. Interpretation of the Q.V.I.

Our purpose is now to give the stochastic interpretation of the solution of (1) in terms of control of diffusion processes. Thus, we consider the impulsive control given by two sequences

$$\begin{cases} \theta^1 \leqslant \theta^2 \leqslant \ldots \leqslant \theta^n \xrightarrow[n \to +\infty]{} +\infty \quad , \\ \xi^n \in \mathbb{R}^N \quad , \quad \xi^n \geqslant 0 \quad , \end{cases}$$

where θ^n are stopping times and ξ^n are random variables F_{θ^n}-measurable. We may solve, with the notations of the preceding sections, the S.D.E.

$$y_x^o(t) = w_t + x \quad , \quad t \geqslant 0 \quad ,$$

and by induction

$$y_x^{n+1}(t) = y_x^n(\theta^{n+1}) + \xi^{n+1} + w_t - w_{\theta^{n+1}} \quad , \quad t \geqslant \theta^{n+1} \quad .$$

Then, we set

$$y_x(t) \equiv y_t \equiv y_x^n(t) \quad , \quad \theta^n \leqslant t < \theta^{n+1} \quad .$$

We denote by

$$\tau = \inf\,\{t \geqslant 0 \ , \ y_x(t) \in \Gamma_o\} \tag{36}$$

and we will say that $(\theta^n,\xi^n)_{n \geqslant 0}$ is an admissible system if $y_x^n(t\wedge\tau) \in \bar{\Omega}$, $\forall\ t \geqslant 0$.

For such a system we define the cost function

$$(37) \qquad J(x,A) = E\left\{\int_0^\tau f(y_s)e^{-\lambda s}\,ds + \sum_{n=1}^{\infty} [k+c_o(\xi^n)]\, e^{-\lambda\theta^n} 1_{\theta^n \leqslant \tau} + \varphi_o(y_x(\tau))e^{-\lambda\tau}\right\}$$

and the optimal cost function

$$(38) \qquad \underline{v}(x) = \inf_{A \text{ adm.}} J(x,A) .$$

Theorem 3. Under assumptions of Theorem 1, the solution u of (1) is such that its l.s.c. representant $\underline{u}$ is equal to $\underline{v}$.

Proof. (i) $\underline{u} \leqslant \underline{v}$.

Let us define a sequence φ_n of functions which belongs to $C(\partial\Omega)$ and such that $\varphi_n = \varphi_o$ on Γ_o , φ_n is increasing with n and converges pointwise to a function larger that $M\underline{u}$ on $\partial\Omega \setminus \Gamma_o$. We consider the equation

$$(39) \qquad \begin{cases} \operatorname{Max}\,(-\Delta u_n+\lambda u_n-f \,,\, u_n-Mu_n) = 0 \,, & u_n \in C(\overline{\Omega}) \,, \\ u_n|_{\partial\Omega} = \varphi_n \wedge Mu_n \,. & \end{cases}$$

We may always assume that

$$k + \inf_{\substack{\xi \geqslant 0 \\ x+\xi \in \partial\Omega}} c_o(\xi) + \varphi_n(x+\xi) = M_o\varphi(x) \in C(\overline{\Omega}) ,$$

and thus we know from [13] that (39) has a unique viscosity solution and that u_n admits a stochastic representation which implies that

$$u_n(x) \leqslant \inf_{A \text{ adm.}} J_n(x,A)$$

$$J_n(x,A) = E\left\{\int_0^\tau f(y_s)e^{-\lambda s}\,dx + \sum_{n=1}^{\infty} (k+c_o(\xi^n))e^{-\lambda\theta^n} 1_{\theta^n \leqslant \tau} + \varphi_n(y_x(\tau))e^{-\lambda\tau}\right\} .$$

Since J_n and J take the same values for admissible systems satisfying (36) we have

$$u_n(x) \leqslant \underline{v}(x) .$$

On the other hand it is easy to prove analytically that $u_n \nearrow \underline{u}$ as $n \to +\infty$. Indeed u_n is obtained, using the decreasing iterative process

$$(15') \quad \begin{cases} \operatorname{Max}\,(-\Delta u_n^k + \lambda u_n^k - f \;,\; u_n^k - M u_n^{k-1}) = 0 \;, \\ u_n^k{}_{|\partial\Omega} = \inf\,(M u_n^{k-1}, \varphi_n) \;, \end{cases}$$

and u_n^k converges uniformly to u_n with

$$|u_n^k - u_n| \leqslant C(1-\mu_o)^k \;,$$

for some $0 < \mu_o \leqslant 1$ independent of φ_n. Since it is clear that u_n^k converges to $\underline{u}^k$ solution of (15) we obtain that $u_n \to \underline{u}$ and thus (i) is proved.

(ii) $\underline{v \leqslant \underline{u}}$.

As in the proof of (i) we introduce functions $\varphi^n \in C(\partial\Omega)$, but now we impose that φ^n is large enough on $\partial\Omega \setminus \Gamma_o$, φ^n is decreasing to a function which is equal to $\bar{\varphi}$ on Γ_o. Again we may assume that

$$k + \inf_{\substack{\xi \geqslant 0 \\ x+\xi \in \partial\Omega}} \{c_o(\xi) + \varphi^n(x+\xi)\} \in C(\bar{\Omega}) \;,$$

and thus we may solve

$$\begin{cases} \operatorname{Max}\,(-\Delta u^n + \lambda u^n - f \;,\; u^n - M u^n) = 0 \;, \quad u^n \in C(\bar{\Omega}) \;, \\ u^n{}_{|\partial\Omega} = \varphi^n \wedge M u^n \;, \end{cases}$$

and one easily checks that $u^n \searrow_n \bar{u}$ as $n \to \infty$. u^n admits the stochastic representation

$$u^n = \inf_A J^n(x,A) \;,$$

$$J^n(x,A) = E\left\{\int_o^{\tau'} f(y_s) e^{-\lambda s} ds + \sum_{n=1}^{\infty} (k + c_o(\xi^n)) e^{-\lambda\theta^n} 1_{\theta^n \leqslant \tau'} + \varphi^n(y_x(\tau')) e^{-\lambda\tau'}\right\}$$

where

$$\tau' = \inf \{t \geq 0 , y_x(t) \notin \bar{\Omega}\} \quad .$$

Let us prove that $u^n \geq \underline{v}$. Thus for a system A let us choose $\tilde{A}$ in the following way. On the set

$$A = \{\tau' < +\infty , \quad y_x(\tau') \in \partial\Omega \setminus \Gamma_o\}$$

there exists a unique n_o such that

$$\theta^{n_o} \leq \tau' < \theta^{n_o+1} \quad .$$

We define $\tilde{A}$ by

$$\begin{cases} \tilde{\theta}^n = \theta^n \quad , \quad \tilde{\xi}^n = \xi^n \quad , \quad \forall\, n \leq n_o \; , \\ \tilde{\theta}^{n_o+1} = \tau' \quad , \\ \tilde{\xi}^{n_o+1} \text{ is such that } y_x(\tau) + \tilde{\xi}^{n_o+1} \in \Gamma_o \quad , \\ \tilde{\tau} = \tau' \quad . \end{cases}$$

(the existence of such a $\tilde{\xi}^{n_o+1}$ is clear enough). Then $\tilde{A}$ satisfies (36) and we have

$$\begin{aligned} J^n(x,A) &= E\left\{\int_0^{\tau'} f(y_s)e^{-\lambda s}ds + \sum_{n=1}^{\infty} (k+c_o(\xi^n))e^{-\lambda\theta^n} 1_{\theta^n \leq \tau'} + \right. \\ &\qquad \left. + 1_{A^C}\, \varphi^n(y_x(\tau'))e^{-\lambda\tau'} + 1_A\, \varphi^n(y_x(\tau'))e^{-\lambda\tau'}\right\} \\ &\geq J(x,\tilde{A}) + E\left\{1_A\, \varphi^n(y_x(\tilde{\tau}))e^{-\lambda\tilde{\tau}} - 1_A\, [k+c_o(\tilde{\xi}^{n_o+1})]\; e^{\lambda\tilde{\tau}} + \right. \\ &\qquad \left. - 1_A\, \varphi(y_x(\tilde{\tau}))e^{-\lambda\tilde{\tau}}\right\} \\ &\geq \underline{v}(x) \end{aligned}$$

for φ^n large enough on $\partial\Omega \setminus \Gamma_o$. This proves that

$$u^n \geq \underline{v} \qquad \text{on } \bar{\Omega} \quad ,$$

passing to the limit we obtain

$$\bar{u} \geq \underline{v} \qquad \text{on } \bar{\Omega} \quad .$$

Finally (ii) is proved since $\bar{u} = \underline{u}$ on Ω , and the result is obvious on $\partial\Omega$.

V . Extension to Hamilton-Jacobi-Bellman equation.

When considering a more general control problem we are led to the Hamilton-Jacobi-Bellman equation. In this section we extend some results of the previous sections to the H.J.B. equation. This equation reads, for example, in our case,

$$
(40)\qquad \begin{cases} \operatorname{Max}\left[\max\limits_{1 \leq i \leq m} (A^i u - f^i)\ ,\ u - Mu\right] = 0 & \text{in } \Omega\ , \\ u\big|_{\Gamma_o} = \varphi\ , & \\ u\big|_{\partial\Omega \setminus \Gamma_o} = Mu\ . & \end{cases}
$$

Here A^i are elliptic second-order differential operators

$$
A^i = -a^i_{jk}\,\partial^2_{jk} + b^i_j\,\partial_j + c^i
$$

satisfying (5)-(7) with the same constants λ,ν,α .

The H.J.B. equation (the non-linear second-order part of (40)) with a smooth boundary condition has been studied in [6,7,10,13,14], while the Q.V.I. associated to H.J.B. equations has been studied in [15-18]. But it seems difficult to extend this results to (40) as it has been done in section I,II for $m = 1$. Nevertheless, in the case of open sets satisfying (18), the methods of the section III may be applied without any difficulty and we have the

Theorem 4. Let Ω be a smooth open set satisfying (18), let the operators A^i satisfy (5)-(7) and let $\varphi_o \in BUC(\Gamma_o)$, $M_o\varphi_o \in BUC(\overline{\Omega})$, then there exists a unique solution $u \in BUC(\overline{\Omega})$ of (40) and $Mu \in BUC(\overline{\Omega})$. Moreover, if $\varphi_o \in C^{2,\alpha}(\Gamma_o)$, $M_o\varphi_o \in D^{2,+}(\Omega)$, $c_o \in W^{2,\infty}(\mathbb{R}^N)$ and V is an open subset of Ω such that $d(V\ ,\ \partial\Omega\setminus\Gamma_o) > 0$, then $u \in W^{2,\infty}(V)$.

The interested reader is refered to the paper quoted above for this extension. Let us only notice that, since we only use continuous solutions of (40), the definition of solutions used in Theorem 4 may be the definition of section I but is exactly the one of [5,14].

Let us conclude this section with the interpretation of (40) in terms of optimal stochastic control. Here we have to consider a mixed control of continuous and impulsive type. Thus, with the notations of section IV, we consider two sequences

$$\theta^1 < \theta^2 < \dots < \theta^n \quad \dots \quad ,$$
$$\xi^1 \; , \; \xi^2 \; , \; \dots \; , \; \xi^n \; , \; \dots$$

where θ^i is an increasing sequence of stopping time and ξ^n a random variable F_{θ_n}-adapted, and we consider a progressively measurable process $v(t) \in \{1,\dots,m\}$. Then, let σ^i be the positive square root of a^i. We may solve the S.D.E.

$$\begin{cases} d\, y_x^o(t) = \sigma^{v(t)}\,(y_x^o(t))\, dw_t - b^{v(t)}\,(y_x^o(t))\, dt \quad , \\ y_x^o(0) = x \quad , \end{cases}$$

$$\begin{cases} d\, y_x^n(t) = \sigma^{v(t)}\,(y_x^n(t))\, dw_t - b^{v(t)}\,(y_x^n(t))\, dt \quad , \\ y_x^n(\theta^n) = y_x^{n-1}(\theta^n) + \xi^n \quad . \end{cases}$$

Then $y_x(t)$ is the process defined as

$$y_x(t) = y_x^n(t) \quad , \quad \theta^n \leq t < \theta^{n+1} \quad . \tag{41}$$

Following the section IV, we call admissible system the data of sequences $(\theta^n)_{n \geq 1}$; $(\xi^n)_{n \geq 1}$ and of a continuous control $v(t)$ such that $y_x^{n+1}(\theta^{n+1}) \in \bar{\Omega}$ whenever $y_n^n(\theta^n) \in \bar{\Omega}$. Denoting τ the first exit time of y_x from $\bar{\Omega}$, we set for an admissible system A such that $y_x(\tau) \in \Gamma_o$:

$$(42) \qquad J(x,A) = E\left\{\int_0^{\tau} f^{v(s)}(y_x(s))e^{-\lambda s}ds + \sum_{n=1}^{\infty} [k+c_o(\xi^n)]e^{-\lambda\theta^n} 1_{\theta^n \leq \tau} + \varphi(y_x(\tau))e^{-\lambda\tau}\right\} .$$

Again, we easily deduce from the above references and from the argument of section IV that the optimal cost function is characterized by the

Proposition 3. Under the assumptions of Theorem 4, the solution u of (40) satisfies

$$u(x) = \inf_{\substack{A \text{ adm.} \\ y_x(\tau) \in \Gamma_o}} J(x,A) .$$

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