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AN INTRINSIC CHARACTERIZATION OF FOLDINGS OF EUCLIDEAN SPACE

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ABSTRACT:

A *folding* $q : R^d \to R^d$ is a nonexpansive mapping that is piecewise isometric. Every such function determines a polyhedral complex whose d-dimensional elements are the maximal sets on which q is an isometry. We characterize the complexes which arise from a folding in this manner.

1. Introduction. The authors have recently introduced a class of mappings, called "foldings", on Euclidean space [5], [6]. A *folding* is a nonexpansive piecewise isometric mapping, with the pieces forming a polyhedral decomposition of the space.

To fix ideas in the simplest case, imagine that a piece of paper is folded several times, translated, flipped, or rotated any finite number of times. These actions define a mapping $q : R^2 \rightarrow R^2$. The creases thereby formed in the page divide the plane into polygonal regions (some of which may be unbounded) called the "folds" of q, such that the restriction of q to each region is an isometry. These folds can be "two-colored", that is, divided into two classes, A and B, such that every pair of polygons sharing an edge belong to opposite classes. (In the paper folding example, these would be the upside-down and the rightside-up polygons). Furthermore, one can easily convince oneself that at each vertex the sum of the angles from incident class A polygons is equal to the angle sum of the incident class B polygons.

There are piecewise isometric nonexpansive mappings on R^2, i.e. "foldings", that are not "paper foldings", so this paradigm should be recognized as only an illustratration.

It is natural to ask whether this angle-sum property characterizes the class of foldings, and if so, whether it generalizes to R^d. Our objective in this paper is to show that this is indeed the case. Theorem 1 states that every folding of R^d determines a polyhedral complex whose d-dimensional faces can be two-colored and for which every face of dimension less than d has the following property: the sum of the class-A inner angles at that face equals the sum of the class-B inner angles at that face. Theorem 2 then shows that this property comprises an intrinsic characterization of the complexes corresponding to foldings; that is, a complex is realized by some folding if, and only if, it has this angle-sum property.

The sufficient condition presented in Theorem 2 for a complex to be realizable by some folding q *seems* to be weaker than the necessary angle-sum condition of Theorem 1: *if merely around every (d-2)-dimensional face* of the complex, the sum of the class-A inner angles equals the sum of the class-B inner angles, then there exists a folding q for which that is the corresponding complex. But the sufficient condition cannot be weaker, and must therefore be equivalent to the necessary condition. Thus, the fact that the angle-sum condition holds for every (d-2)-dimensional face implies that it holds at every face of dimension less than d. Except in the case where d=2, this fact seems to us by no means obvious. For instance, in the case d=3, it says that if one divides the 2-sphere into a finite number of spherically convex polygons (whose sides are arcs of great circles of the sphere), and if these polygons can be two-colored so that around each vertex the sum of the angles of color A equals the sum of the angles of color B, then the sum of the areas of the A-polygons equals the sum of the areas of the B-polygons.

Our original interest in foldings was motivated by our observation, explained in [5] and [6], that for a folding $q : R^d \to R^d$, sequences of iterates of q, $x_{n+1} = q(x_n)$, and sequences of averaged iterates of q, $x_{n+1} = \frac{q(x_n)+x_n}{2}$, exhibit a regular behavior not displayed by the iterates, or averaged iterates, of a general nonexpansive mapping. The behavior exhibited by such sequences is especially striking in the case of averaged iterates. Except for perhaps a finite number of initial terms, such sequences were shown in [5] to behave as if q were globally an isometry. From this observation, the finite termination of certain iterative procedures was deduced. Averaged iterates of a nonexpansive mapping can be regarded also as sequences generated by the "proximal point algorithm" [7], a method for finding a zero of a maximal monotone multifunction. Foldings thus present a case where this algorithm is especially well-behaved. It is hoped that the additional insight provided by our geometric characterization will lead to a better understanding of nonexpansive mappings in general, and of iterative methods for determining their fixed points.

2. Piecewise isometries and their complexes. In this section, the class of "piecewise isometries" on R^m is introduced. The Proposition below gives three equivalent characterizations of this class. It will then be shown that every piecewise isometry on R^m determines a polyhedral complex whose m-cells are the maximal sets on which the mapping is isometric.

The following will be used repeatedly:

LEMMA. *If $q : R^m \to R^n$ is nonexpansive, A and B are convex sets in R^m for which $q|A$ and $q|B$ (the restrictions of q to A and*

B) <u>are isometries, and if</u> $A \cap \mathrm{relint}(B) \neq \varnothing$, <u>then the restriction of</u> q <u>to</u> $\mathrm{conv}(A \cup B)$ <u>is an isometry</u>.

Proof: If we can show that q is isometric on $A \cup B$ then, since q is nonexpansive, it will follow that q is also isometric on $\mathrm{conv}(A \cup B)$. For this, it suffices to show that $|q(a)-q(b)| = |a-b|$ whenever $a \in A \setminus B$ and $b \in B \setminus A$.

Without loss of generality, we may suppose that $0 \in A \cap \mathrm{relint}(B)$. Let $t>0$ be chosen small enough so that $-tb \in B$. Since q is nonexpansive and $q|B$ is linear, we have $|q(a)-q(b)| \leq |a-b|$ and $|q(a)+tq(b)| = |q(a)-q(-tb)| \leq |a+tb|$. Hence,

$$(t+1)|q(a)|^2 + t(t+1)|q(b)|^2$$

$$= t|q(a)-q(b)|^2 + |q(a)+tq(b)|^2$$

$$\leq t|a-b|^2 + |a+tb|^2$$

$$= (t+1)|a|^2 + t(t+1)|b|^2$$

$$= (t+1)|q(a)|^2 + t(t+1)|q(b)|^2$$

and the inequality must actually be an equality. Since $t>0$, this forces $|q(a)-q(b)|^2 = |a-b|^2$. ♦ ♦ ♦

For a function $q : R^m \to R^n$ we let $\mathcal{M}_q$ denote the collection of maximal sets F in R^m such that $q|F$ is an isometry.

Clearly, if q is continuous then the elements of $\mathcal{M}_q$ are closed. Also $\bigcup_{F \in \mathcal{M}_q} F = R^m$, and if F and F' have overlapping interiors, $\text{int}(F) \cap \text{int}(F') \neq \varnothing$ -- then (since $q|(F \cup F')$ must be an isometry) F=F'.

PROPOSITION. <u>The following conditions on a function</u> $q : R^m \to R^n$ <u>are equivalent</u>:

(a) <u>the function</u> q <u>is continuous and the family</u> $\mathcal{M}_q$ <u>is locally finite;</u>

(b) <u>there exists a locally finite cover</u> $\mathcal{M}$ <u>of</u> R^m <u>by closed sets</u> F <u>such that</u> $q|F$ <u>is an isometry; and</u>

(c) <u>for each point</u> $x \in R^m$ <u>there is an</u> $\varepsilon(x) > 0$ <u>such that if</u> $y \in B(x,\varepsilon(x))$ (<u>the open ball of radius</u> ε <u>centered at</u> x) <u>then</u> $|q(y)-q(x)| = |y-x|$.

<u>If</u> q <u>is a function satisfying these conditions then</u> q <u>is nonexpansive and the elements of</u> $\mathcal{M}_q$ <u>are full-dimensional, closed, and convex</u>.

Proof: If (a) is satisfied then we may take $\mathcal{M} = \mathcal{M}_q$ in (b), so that (b) holds as well.

If (b) holds, then for each $x \in R^m$ there is $\varepsilon > 0$ such that if $|y-x| \leq \varepsilon$ and $y \in F \in \mathcal{M}$ then $x \in F$. For such an ε, (c) holds.

Suppose henceforth that the function $q : R^m \to R^n$ satisfies (c). Clearly q is continuous. For any $x, y \in R^m$, a compactness argument shows that the interval $[x,y]$ can be divided into a finite number of subintervals on which q is an isometry. The triangle inequality thus shows that $|q(y)-q(x)| \leq |y-x|$ and q is nonexpansive.

A simple argument that the authors have presented in [6] shows that for any nonexpansive function q, the elements of $\mathcal{M}_q$ are not only closed, but convex with nonempty interior.

Observe for any $x \in R^m$ that if $F \in \mathcal{M}_q$ meets $B(x,\varepsilon(x))$, then F contains x. To see this, note that there exists $z \in \text{int}(F) \cap B(x,\varepsilon(x))$. Since q is isometric both on F and on $[x,z]$, it follows by the Lemma that q is isometric on $\text{conv}(F \cup [x,z])$. The maximality of F then implies $x \in F$.

Observe also that if $F \in \mathcal{M}_q$ contains $x \in R^m$, then F meets $S(x,\varepsilon(x))$ (the sphere centered at x with radius $\varepsilon(x)$). For if this were false, then there would be $x \in F \in \mathcal{M}_q$ with $F \cap S(x,\varepsilon(x)) = \varnothing$. Since F is convex, this implies $B(x,\varepsilon(x)) \supset F$. But then for any $y \in \text{int}(F) \setminus \{x\}$, the line segment $[x,y]$ could be extended to meet $S(x,\varepsilon(x))$ at a point z. The restriction of q to this line segment would be an isometry, so the Lemma implies that q is also isometric on $\text{conv}([x,z] \cup F)$. By the maximality of F, we would thus have $F \supset [x,z]$ and F would meet $S(x,\varepsilon(x))$ at z.

To complete the proof of the Proposition, it remains to show only that $\mathcal{M}_q$ is locally finite. We will suppose this to be false and show that this leads to a contradiction. So fix $x_0 \in R^m$ such that $B(x_0,\varepsilon(x_0))$ meets infinitely many members F of $\mathcal{M}_q$. We have already observed that all such F contain x_0 as well.

By an inductive argument, we will next establish the existence of a set $\{x_0, \ldots, x_m\}$ of points in R^m such that for $k=0,\ldots,m,$ the set

$\Delta_k := \mathrm{conv}(\{x_0, ..., x_k\})$ is a k-dimensional simplex that is contained in infinitely many members of $\mathcal{M}_q$. The existence of such a zero-dimensional simplex $\Delta_0 = \{x_0\}$ has already been noted. For each k, we will denote as $\mathcal{M}_k$ the collection of all sets in $\mathcal{M}_q$ that contain Δ_k. Clearly $\mathcal{M}_0 \supset ... \supset \mathcal{M}_m$.

Suppose we have already established the existence of $\Delta_k = \mathrm{conv}(\{x_0, ..., x_k\})$ for some k with $0 \le k < m$ such that $\mathcal{M}_k$ is infinite. Fix $y \in \mathrm{relint}\, \Delta_k$. By the Lemma, the members of $\mathcal{M}_q$ that contain y are precisely the members of $\mathcal{M}_k$. Let N_k be the (m-k)-dimensional affine space orthogonal to Δ_k and containing y. Consider the sphere $S_k := \{w \in N_k : |w-y| = \varepsilon(y)\}$. We have already observed that the collection of sets in $\mathcal{M}_q$ that meet y (i.e., $\mathcal{M}_k$), is the same as the collection of sets in $\mathcal{M}_q$ that meet $S(y,\varepsilon(y))$. We claim that more is true, namely: the collection $\mathcal{M}_k$ is exactly the same as the family, let's call it $\mathcal{S}$, of all sets in $\mathcal{M}_q$ that meet the lower dimensional sphere S_k. Clearly, $\mathcal{M}_k \supset \mathcal{S}$ (because $S(y,\varepsilon(y)) \supset S_k$). Since $\mathcal{S}$ covers S_k, and since every member of $\mathcal{S}$ is a convex set containing Δ_k, it follows that $\cup\mathcal{S}$ contains $\mathrm{conv}(S_k \cup \Delta_k)$, a set that contains y in its interior. Every member of $\mathcal{M}_q$ that contains y (every member of $\mathcal{M}_k$) must therefore intersect the interior of some member of $\mathcal{S}$, and must therefore *be* a member of $\mathcal{S}$. Thus $\mathcal{M}_k = \mathcal{S}$ and, in particular, $\mathcal{S}$ is infinite.

By compactness, there is a finite collection $\{w_1, ..., w_N\}$ of points on S_k such that S_k is covered by the union of the open balls $B(w_i,\varepsilon(w_i))$ (i=1,...,N). Every member of $\mathcal{S}$ that meets one of these balls must contain its center as well. Hence at least one of these centers, say w_1, lies in infinitely many members of $\mathcal{M}_k = \mathcal{S}$. The set

$\Delta_{k+1} := \text{conv}(\{w_1\} \cup \Delta_k)$ is thus a (k+1)-simplex such that the corresponding family M_{k+1} is infinite. This completes the induction step.

The m-simplex Δ_m is contained in an infinite number of members of M_q. But the elements of M_q have disjoint interiors. This contradiction completes the proof. ♦ ♦ ♦

Any function q satisfying the equivalent conditions of the Proposition will be called a <u>piecewise isometry</u>. For such q, let K_q denote the collection of convex polyhedra consisting of all faces of elements of M_q. We wish to verify that K_q is a *polyhedral complex subdividing* R^m; that is, we wish to show that

(i) the relative interiors of elements of K_q cover R^m, and

(ii) these relative interiors are pairwise disjoint.

Clearly (i) holds, since the polyhedra in M_q cover R^m and any polyhedral convex set is the union of the relative interiors of its faces. It follows from the Lemma that every $x \in R^m$ lies in a *unique* maximal relatively open convex set on which q is an isometry. But the relative interiors of elements of K_q also have this property of being maximal relatively open convex sets on which q is an isometry. Thus (ii) holds.

3. <u>Characterization of the complexes</u>. We call a piecewise isometry $q : R^d \to R^d$ of a Euclidean space into itself a <u>folding</u>, and the

elements of $\mathcal{M}_q$ we call the folds of q. In this section, we characterize the complexes $\mathcal{K}_q$ corresponding to foldings q.

Suppose F and G are elements of M_q such that $F \cap G$ is of dimension d-1. We call such a pair of folds adjacent. Let α and β be the linear isometries of R^d such that $\alpha|F = q|F$ and $\beta|G = q|G$. Let H be the hyperplane which is the affine span of $F \cap G$ and let $\gamma : R^d \to R^d$ be reflection across H. Clearly $\beta = \alpha\,\gamma$. It follows that if we know the complex M_q and we know $q|F$ then we can determine q on any fold adjacent to F; and, since the graph consisting of folds with the above adjacency relation is connected, we can reconstruct q from M_q and $q|F$.

Note that if F, G, α, β, and γ are as above, then γ reverses orientation; if F and G are adjacent folds, then one of $q|F$ and $q|G$ is orientation preserving and the other is orientation reversing. This determines a partition of the folds into two classes A and B, so that if F and G are adjacent folds then they are in different classes; i.e., the graph of folds is two-colored.

For polyhedra F and G, where F is a nonempty face of G and G is full-dimensional, we denote by $\phi(F,G)$ the *inner angle* of G at F as in [2, Section 14.1]. Intuitively, this is the fraction of R^d filled by the cone generated by G at F. If u is an element of the relative interior of F, S^{d-1} is a sphere of sufficiently small radius centered at u, and if $\mu(S^{d-1}) = 1$, then $\phi(F,G) = \mu(G \cap S^{d-1})$.

THEOREM 1. Let $q : R^d \to R^d$ be a folding, $\{A,B\}$ the two-coloring of $\mathcal{M}_q$ (as described above). Suppose F is a face of the complex $\mathcal{K}_q$ which is of dimension less than d. Then

$$\sum_{\substack{A \in \mathcal{A} \\ A \supseteq F}} \phi(F,A) = \sum_{\substack{B \in \mathcal{B} \\ B \supseteq F}} \phi(F,B).$$

Proof: Let $\iota : R^d \to R^d$ be an isometry such that $q|F = \iota|F$. Then $\iota^{-1}q$ is again a folding, has the same complex as q, and is the identity on F. For this reason, we can replace q with $\iota^{-1}q$ if necessary and assume for simplicity that $q|F$ is the identity mapping.

Fix $u \in \mathrm{relint}(F)$ and let S be the (d-1)-sphere centered at u and having radius $\varepsilon(u)$. Let q_0 denote the restriction of q to S. Clearly q_0 maps S into itself. We will determine the topological degree of the mapping $q_0 : S \to S$.

Consider the set

$$T = \{y \in S : \text{if } q_0(x)=y \text{ then for some } F \in \mathcal{M}_q,\ x \in \mathrm{int}(F)\}$$

of regular values of q_0. Recall (cf. [4], pp.263-7) that we can compute the topological degree of q_0 as follows, using any $y \in T$. Let $x_1, \dots, x_{k+r}$ be the (necessarily finitely many) elements of $q_0^{-1}(y)$ and let $G_1, \dots, G_{k+r}$ be the folds that contain them, where the indices are chosen so that q is orientation-preserving on $G_1, \dots, G_k$ and orientation-reversing on $G_{k+1}, \dots, G_{k+r}$; i.e., $G_1, \dots, G_k \in \mathcal{A}$

and $G_{k+1}, \dots, G_{k+r} \in \mathcal{B}$. Then q_0 is orientation-preserving on $G_1 \cap S, \dots, G_k \cap S$ and orientation-reversing on $G_{k+1} \cap S, \dots, G_{k+r} \cap S$. The degree of q_0 is the difference k-r.

This works, in particular, if y is not in the image of q_0: if the image $q_0(S)$ is a proper subset of S, then the degree is zero. By a result of Freudenthal and Hurewicz [1], a nonexpansive mapping of a compact metric space *onto* itself is an isometry. The mapping $q_0 : S \to S$ is nonexpansive but is not an isometry, so it follows that $q_0(S)$ is a proper subset of S. Therefore the degree of q_0 is zero.

For each fold G containing F, let $C_G : S \to R$ denote the characteristic function of $q_0(G \cap S)$, that is,

$$C_G(x) = \begin{cases} 0 & \text{if} \quad x \notin q_0(G \cap S) \\ 1 & \text{if} \quad x \in q_0(G \cap S) \end{cases}$$

The function

$$\sum_{\substack{G \in \mathcal{A} \\ G \supseteq F}} C_G - \sum_{\substack{G \in \mathcal{B} \\ G \supseteq F}} C_G.$$

is equal to the degree of q_0 (zero) almost everywhere (that is, at every $y \in T$) with respect to a standard rotationally invariant measure on S, normalized so that the measure of S is 1. Integrating over S, and noting that the integral of C_G is $\phi(F,G)$, we get

$$\sum_{\substack{G \in \mathcal{A} \\ G \supseteq F}} \phi(F,G) \; - \; \sum_{\substack{G \in \mathcal{B} \\ G \supseteq F}} \phi(F,G) \; = \; 0,$$

as desired. ♦ ♦ ♦

The next theorem shows that the condition described in Theorem 1 more than suffices to characterize the complexes of the form K_q. First, we present as a lemma a special case which will be useful in the proof of Theorem 2.

LEMMA. Let the polyhedral complex K in R^2 consist of the two-dimensional closed convex cones $C_1, C_2, \ldots, C_{2n}$ together with their faces. Suppose that this complex subdivides R^2 (that is, $C_1 \cup C_2 \cup \ldots \cup C_{2n} = R^2$), that the cones have their common vertex at the origin, and that the condition of Theorem 1 is satisfied with $F = \{0\}$. Then there is a folding $q : R^2 \to R^2$ such that $K_q = K$.

Proof: Let $C_0 = C_{2n}$. We may suppose that the indexing yields a counterclockwise ordering of the cones around the origin. For $1 \leq i \leq 2n$, let $\alpha_i = 2\pi\phi(\{0\},C_i)$. Then α_i is the radian measure of the angle of the cone C_i. The condition of Theorem 1 can be written as

$$\sum_{i=1}^{n} \alpha_{2i} = \sum_{i=1}^{n} \alpha_{2i-1}.$$

Of course, the common value must be π, since the sum of all of the angles is 2π.

For $i = 1, \ldots, 2n$, let $\eta_i : R^2 \to R^2$ be reflection across the line containing the ray $C_{i-1} \cap C_i$. The composition $\eta_i \eta_{i+1}$ of the two reflections η_i and η_{i+1} is rotation through an angle of $2\alpha_i$ radians, so $\eta_1 \eta_2 \cdots \eta_{2n-1} \eta_{2n}$ is rotation through an angle of $2\alpha_1 + 2\alpha_3 + 2\alpha_{2n-1} = 2\pi$ radians -- that is e, the identity transformation.

We now define $q : R^2 \to R^2$ as follows. For $x \in C_1$, let $q(x) = \eta_1(x)$. Note that q is the identity mapping on the ray $C_0 \cap C_1$. If q has already been defined on $C_1 \cup \ldots \cup C_{i-1}$ with $1 < i < 2n$, then for $x \in C_i \backslash C_{i-1}$, we define $q(x) = \eta_1 \eta_2 \ldots \eta_i(x)$. Since $\eta_1 \eta_2 \ldots \eta_i = \eta_1 \eta_2 \ldots \eta_{i-1}$ on $C_{i-1} \cap C_i$, it follows that $q(x) = \eta_1 \eta_2 \ldots \eta_i(x)$ for all $x \in C_i$, so that $q|C_i$ is an isometry. We continue in this way to define q on $C_1 \cup \ldots \cup C_{2n-1}$. With $C_0 = C_{2n}$ we may do the same, defining $q = \eta_1 \eta_2 \ldots \eta_{2n} = e$, noting that there is no conflict with the definition of q on C_1 since $q = e$ on the ray $C_0 \cap C_1$.

The function q we have defined has the property that $q|C_i$ is an isometry for all i. Criterion (b) of the Proposition thus implies that q is a folding, and clearly $K_q = K$. ♦ ♦ ♦

THEOREM 2. <u>Suppose that K is a polyhedral cell complex subdividing R^d whose d-cells admit a two-coloring $\{A,B\}$, and suppose that for each (d-2)-face F of K the equality</u>

$$\sum_{\substack{A \in A \\ A \supseteq F}} \phi(F,A) = \sum_{\substack{B \in B \\ B \supseteq F}} \phi(F,B).$$

<u>holds. Then there is a folding $q : R^d \to R^d$ such that $K_q = K$.</u>

Proof: We define a folding q as follows. First, choose an arbitrary d-cell G_0, and let $q|G_0$ be the identity transformation: $q(x)=x$ for all $x \in G_0$. Then, for any other d-cell G of K, we find a path G_0, G_1, ..., $G_k = G$ such that consecutive cells are adjacent. We define q on G to be $\gamma_1 \gamma_2 \ldots \gamma_k$, where for each i $(1 \le i \le k)$, γ_i is reflection across the hyperplane spanned by $G_{i-1} \cap G_i$. To verify condition (b) of the Proposition we need only show that q is well-defined in this way.

For this, it suffices to show that if $K_0, K_1, \ldots, K_j = K_0$ is any cycle of d-cells of K, each consecutive pair adjacent, we have $\eta_1 \eta_2 \ldots \eta_j = e$ (the identity transformation), where η_i is reflection across the hyperplane spanned by $K_{i-1} \cap K_i$. If $K_0, \ldots, K_j = K_0$ is a cycle of d-cells containing a particular (d-2)-face F of K then $j=2n$ is even (since cells of A alternate with cells of B around the cycle) and the condition of the theorem gives

$$\sum_{i=1}^{n} \phi(F,K_{2i}) = \sum_{i=1}^{n} \phi(F,K_{2i-1}).$$

By considering the actions of the η_i's on a two-dimensional plane normal to F, we see from the lemma that in this case $\eta_1 \ldots \eta_j = e$.

Completing the proof amounts to a familiar exercise in homotopy theory. Indeed, what we must show is essentially that the fundmental homotopy group of the complement of the (d-2)-skeleton of K in R^d is generated by loops around the (d-2)-faces of K. This involves an argument closely resembling the exposition in [3, pp.235-241] applied to the dual complex of K. For clarity and completeness, we present a complete argument.

By a path we mean a piecewise linear mapping $\phi : [0,1] \to R^d$. We wish to deal with such paths for which $\phi(0)$ and $\phi(1)$ lie interior to d-cells of K, as do the images of all but finitely many points $t_1 < t_2 < \ldots < t_k$ of $(0,1)$. For these, we require that $\phi(t_i)$ lie in the relative interior of some (d-1)-face F_i of K, and that ϕ be linear in a neighborhood of t_i (so that the path cross F_i at $\phi(t_i)$). Let $\eta_i : R^d \to R^d$ denote reflection about the hyperplane containing F_i. With such a path, we associate the affine linear isometry $\eta = \eta_1 \ldots \eta_k$. Note that slightly perturbing the path ϕ does not change the associated isometry.

To complete our proof, we show that if $\phi(0) = \phi(1)$ then $\eta = e$. Let S denote the unit square, $S = \{(x,y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$ in R^2. For each integer n, $n \geq 1$, denote by T_n the triangulation of S having triangles

$$\{(x,y) : \frac{i}{n} \le x,\ \frac{j}{n} \le y,\ x+y \le \frac{i+j+1}{n}\}$$

and

$$\{(x,y) : \frac{i+j+1}{n} \le x+y,\ \ x \le \frac{i+1}{n},\ \ y \le \frac{j+1}{n}\}$$

for integers i, j such that $0 \le i, j \le n-1$ (so that there are $2n^2$ triangles in T_n altogether). Let $\tau : [0,1] \to S$ be a piecewise linear parametrization of the boundary of S such that $\tau(0) = \tau(1) = (0,0) \in R^2$, and $\tau(x) \neq \tau(y)$ if $x < y < 1$. For n large enough there is a piecewise linear function ϕ_0 mapping the boundary ∂S of S into R^d which is linear on each edge of T_n in ∂S and such that $\phi(t) = \phi_0(\tau(t))$ for $0 \le t \le 1$. The function ϕ_0 can be extended to a function $\bar{\phi} : S \to R^d$ which is linear on each triangle in T_n. By slightly perturbing $\bar{\phi}$ and considering a sufficiently large multiple n' of n, so that $T_{n'}$ is a subdivision of T_n, we can arrange that:

(i) No vertex of $T_{n'}$ is mapped into a (d-1)-cell of K, the image of no edge has non-empty intersection with a (d-2)-cell of K, and the image of no triangle of T has nonempty intersection with a (d-3)-cell of K ; and

(ii) For each triangle T of $T_{n'}$, one of three possibilities holds: (a) $\bar{\phi}(T)$ is contained in the interior of a d-cell of K ; (b) $\bar{\phi}(T)$ has nonempty intersection with a (d-1)-cell F of K , and is contained in the interior of the union of the two d-cells which contain F; and (c) $\bar{\phi}(T)$ has nonempty intersection with a (d-2)-cell of K and is contained in the interior of the union of the d-cells which contain it.

For any edge E of the triangulation and choice of direction of the edge, the function $\bar{\phi}$ determines a path in R^d and to this path is associated an isometry, as above. In this way, we can define a function $\bar{\eta}$ which maps directed edges of $T_{n'}$ to isometries of R^d. Clearly it remains to show that the product $\bar{\eta}(E_1) \ldots \bar{\eta}(E_p)$ of the isometries corresponding to the edges around the boundary of S (taken in their natural order) is e. It is not hard to see that this will hold if merely the product of the isometries corresponding to directed edges around each triangle of $T_{n'}$ is e. So, let T be a triangle of $T_{n'}$. One of the three possibilities (a)-(c) above holds. Let E_1, E_2, and E_3 be the edges of T, with counterclockwise direction. If (a) holds, then clearly $\bar{\eta}(E_1) = \bar{\eta}(E_2) = \bar{\eta}(E_3) = e$, and the product is e. If (b) holds then if F is the (d-1)-face that $\bar{\phi}(T)$ intersects and γ is reflection about the hyperplane containing F, two of $\bar{\eta}(E_1)$, $\bar{\eta}(E_2)$, and $\bar{\eta}(E_3)$ equal γ and the third is e, so the product is e. If (c) holds, and F is the (d-2)-face intersecting $\bar{\phi}(T)$, then in traversing E_1, E_2, and E_3, one intersects each of the (d-1)-faces containing F in their natural cyclic order around F. Then $\bar{\eta}(E_1)\, \bar{\eta}(E_2)\, \bar{\eta}(E_3)$ is the product of the corresponding reflections, and we have already observed that this product is e. ♦ ♦ ♦

We leave open the problem of similarly characterizing the complexes K_q, where $q : R^m \to R^n$ is a piecewise isometry, and $m<n$.

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