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# MULTIPLE SOLUTIONS OF THE FORCED DOUBLE PENDULUM EQUATION

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## 0. INTRODUCTION

In their paper [7] Capozzi , Fortunato and Salvatore have shown that the double pendulum equation with forcing term has at least two solutions (not differing by a multiple of $2\pi$).

In this paper, we apply the notion of Lusternik-Schnirelman category to the double pendulum equation with forcing term to show that it has at least three solutions (not differing by a multiple of $2\pi$). But if we look at the same equation with a small constant forcing term, it is easy to see that the equation has at least four constant solutions. In order to do that we need a more powerful notion than the one of the Lusternik-Schnirelman category. In fact it is not a notion but rather a family of notions which are called ( Lusternik-Schnirelman) relative categories. In this family, we choose only two for the special properties (which are given in paragraphs 3 and 5) they possess and for their usefulness in critical point theory (a generalisation of the result due to Palais [6], which itself generalises well known results saying that the number of critical points is at least the category of the space).

## 1. LUSTERNIK-SCHNIRELMAN CATEGORY.

Let A be a subset of a topological space X.

The Lusternik-Schnirelman category of A in X, $\mathrm{cat}_X(A)$, is the least integer n such that A can be covered by n closed subsets of X each of which is contractible in X. If no such integer exists, we put $\mathrm{cat}_X(A) = +\infty$. We define $\mathrm{cat}(X) = \mathrm{cat}_X(X)$.

The following properties are easy consequences of the definition.

(1.1) if $X \supset B \supset A$ then $\mathrm{cat}_X(A) \leq \mathrm{cat}_X(B)$,

(1.2) $\mathrm{cat}_X(A \cup B) \leq \mathrm{cat}_X(A) + \mathrm{cat}_X(B)$.

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(1.3) if A is closed and if $h \in C([0, 1] \times A, X)$ is such that $h(0, x) = x$ for every $x \in A$, then $cat_X(A) \leq cat_X(h_1(A))$.

**Definition 1.4:** (Palais-Smale condition; see [6])

A map $\varphi: M \rightarrow \Re$, where M is a $C^1$ Finsler manifold and $\varphi$ is $C^1$, satisfies P-S if for every closed subset S of M such that $\varphi(S)$ is bounded but that $\{ \|\varphi'(s)\| \mid s \in S \}$ is not bounded away from 0 then there exists $s_0 \in S$ such that $\varphi'(s) = 0$.

**Remark 1.5:** P-S is equivalent to the following fact: for every sequence $\{s_n\}$ of elements of M such that $\{\varphi(s_n)\}$ is bounded and $\|\varphi'(s_n)\| \rightarrow 0$ as $n \rightarrow \infty$ then there exists a subsequence $s_{nk} \rightarrow s$ with $\varphi'(s) = 0$.

The following result is due to Palais [6].

**Theorem 1.6**: Let M be a complete $C^2$ Finsler manifold and $\varphi \in C^1(M, \Re)$ a map satisfying the Palais-Smale condition. Then if $\varphi$ is bounded from below $\varphi$ has at least cat(M) critical points.

## 2. THREE SOLUTIONS FOR THE DOUBLE PENDULUM EQUATION.

If a mass m is attached to a very light rod of length l and another mass $m_1$ is attached to it by another such rod of length $l_1$, then with forcing on both masses we get the following system of equations (with periodic boundary conditions) where $\theta$ and $\varphi$ denote the angles made by the rods and the vertical lines, and e and f are the forcing terms of period T:

$$(m+m_1)\, l^2 \;\; \theta'' + m_1 l l_1\, \phi'' \cos(\phi - \theta) - m_1 l l_1\, (\phi')^2 \sin(\phi - \theta) + (m+m_1)\, g l \sin\theta \; = e \quad (2.1.1)$$

$$m_1 l l_1\, \theta'' \;\; \cos(\phi - \theta) + m_1 {l_1}^2\, \phi'' + \; m_1 l l_1 (\theta')^2 \sin(\phi - \theta) + m_1 g l_1 \sin\phi = f \qquad (2.1.2)$$

$$\theta(0) = \theta(T),\;\; \theta'(0) = \theta'(T),\;\; \phi(0) = \phi(T),\;\; \phi'(0) = \phi'(T). \qquad (2.1.3)$$

The solutions of that system of equations are the critical points of the following functional on the Hilbert space $H^1_T \times H^1_T$ where

$$H^1_T = \{\; y\colon [0, T] \rightarrow \Re \mid y' \in L_2 \;, y(0)=y(T)\}$$

with scalar product

$$«x, y» \;=\; \int_0^T (xy \,+\, x'y')\; dt:$$

$$\varphi(\theta, \phi) \;=\; \int_0^T \Big[\; \tfrac{1}{2} \left((m+m_1)\, l^2\, \theta'^2 + 2 m_1 l l_1\, \theta'\phi' \cos(\phi\text{-}\theta) \,+\, m_1 l_1^2 \phi'^2\right)$$

$$+ (m+m_1)\;\; g l \cos\theta + m_1\; g l_1 \cos\phi \;\,+\;\, \theta\, e \;\,+ \phi\, f \;\Big]\; dt \qquad (2.2)$$

We assume that

$$\int_0^T e(t)\; dt \;=\; 0 \;\;\text{and}\;\; \int_0^T f(t)\; dt \;=\; 0, \qquad (2.3)$$

in order that the functional $\varphi$ be bounded from below.

For the sake of simplicity, in our calculations, we shall assume that $m = m_1 = l = l_1 = 1$ and $f = 0$. we get the system

$$2\ \theta'' + \phi'' \cos(\phi - \theta) - (\phi')^2 \sin(\phi - \theta) + 2g \sin\theta = e \tag{2.1.1)'}$$

$$\theta'' \cos(\phi - \theta) + \phi'' + (\theta')^2 \sin(\phi - \theta) + g \sin\phi = 0 \tag{2.1.2)'}$$

$$\theta(0) = \theta(T),\ \theta'(0) = \theta'(T),\ \varphi(0) = \varphi(T),\ \varphi'(0) = \varphi'(T). \tag{2.1.3)'}$$

whose solutions are the critical points of the function

$$\varphi(\theta, \phi) = \int_0^T \left[ \frac{1}{2} \left( 2\theta'^2 + 2\theta'\phi' \cos(\phi - \theta) + \phi'^2 \right) + g(2\cos\theta + \cos\phi) + \theta e \right] dt \tag{2.3)'}$$

where $\theta, \phi \in H^1_T$.

We may consider $(\theta, \phi) \in H^1_T \times H^1_T/\sim\ = M$, where $(\theta, \phi) \sim (\theta_1, \phi_1)$ if and only if $(\exists\, k, l \in \mathbb{Z})\ ((\theta = \theta_1 + 2k\pi)$ and $(\phi = \phi_1 + 2l\pi))$. We have that $(\theta, \phi) \sim (\theta_1, \phi_1)$ implies that $\varphi(\theta, \phi) = \varphi(\theta_1, \phi_1)$.

Let $\theta = \bar{\theta} + \tilde{\theta}$ where $\bar{\theta} = \int_0^T \theta(t)\, dt$, then $M = T \times (\tilde{H}^1_T \times \tilde{H}^1_T) = T \times X$ and $\varphi : M \to \Re$ is well defined, where T is a torus and $\tilde{H}^1_T$ is the Hilbert space of the elements $y \in H^1_T$ satisfying $\int_0^T y(t)\, dt = 0$.

<u>**Proposition 2.4:**</u> $\varphi$ satifies the Palais-Smale condition on H.

<u>**Proof:**</u> Assume that $(\theta_n, \phi_n)$ is a sequence in M such that $\nabla\varphi(\theta_n, \phi_n) \to 0$ and $\varphi(\theta_n, \phi_n)$ is bounded as $n \to \infty$, we have to show that a subsequence

$$(\theta_{n_k}, \phi_{n_k}) \to (\theta_0, \phi_0) \quad \text{for some } (\theta_0, \phi_0) \in M.$$

Note first that from (2.3)', we get that

$$\langle\!\langle \nabla\varphi(\theta, \phi), (s, t) \rangle\!\rangle = < 2\theta' + \phi' \cos(\phi-\theta), s'> + < \theta'\phi' \sin(\phi-\theta) - 2g\sin\theta + e, s >$$
$$< \phi' + \theta' \cos(\phi-\theta), t' > + < \theta'\phi' \sin(\phi-\theta) + g\sin\phi, t > \qquad (2.4.1)$$

for any $\phi, \theta \in M$ and $s, t \in H^1_T$, where $< \ , \ >$ denotes the scalar product in $L_2$.

a) Let us first show that if $\{\varphi(\theta_n, \phi_n)\}$ is bounded then so is $\{(\theta_n, \phi_n)\}$ in M. Since $\theta$ and e are T-periodic we get, by integrating by parts, that

$$\int_0^T \theta\, e\, dt = -\int_0^T \theta' E\, dt \quad \text{where } E(t) = \int_0^t e(t)\, dt,$$

and so

$$\varphi(\theta, \phi) = \int_0^T \frac{1}{2}\left(2\theta'^2 + 2\theta'\phi' \cos(\phi-\theta) + \phi'^2 + 2\theta'E\right) + g\int_0^T (2\cos\theta + \cos\phi) \qquad (2.4.2)$$

Now clearly the second integral is bounded and the first dominates a term of the form

$$K\int_0^T (\theta'^2 + \phi'^2) + 2\int_0^T \theta' E;$$

in fact, for any $\varepsilon$ the expression $\varepsilon^{-2}\theta'^2 + 2\theta'\phi' \cos(\phi-\theta) + \varepsilon^2\phi^2$ is positive, so choose $2^{-1}<\varepsilon<1$ and the choose k small enough. Now if either $\|\theta'\|_2$ or $\|\phi'\|_2$ goes to $\infty$, so does that term, that is so does $\varphi(\theta, \phi)$. We get that if $\{\varphi(\theta_n, \phi_n)\}$ is bounded, then so is $\{(\theta'_n, \phi'_n)\}$ in $L_2$. By the Wirtinger inequality

$$\|\tilde{\theta}\|_2 \leq \omega^{-1}\|\theta'\|_2,$$

where $\omega = 2\pi T^{-1}$, we get that

$\{(\tilde{\theta}_n, \tilde{\phi}_n)\}$ are also bounded in $L_2$. Finally, since **T** is a bounded space, we get that

$\{(\theta_n, \phi_n)\}$ is bounded in M.

b) Let us prove that a subsequence

$$(\theta_{n_k}, \phi_{n_k}) \to (\theta_0, \phi_0) \qquad \text{for some } (\theta_0, \phi_0) \in M.$$

Without loss of generality, by passing to a subsequence if necessary, we may assume that $(\theta_n, \phi_n) \to (\theta_0, \phi_0)$ weakly in $H^1_T \times H^1_T$. Thus $(\theta n, \phi n) \to (\theta_0, \phi_0)$ strongly in $C^0$. It remains to show that $\| \theta'_n - \theta'_0\|_2 \to 0$ and $\| \phi'_n - \phi'_0\|_2 \to 0$.

If in (2.4.1) we replace (s, t) by $(\theta_n - \theta_0, \phi_n - \phi_0)$ we get

$$\langle\langle \nabla\varphi(\theta_n,\phi_n),(\theta_n - \theta_0, \phi_n - \phi_0)\rangle\rangle - \langle\langle \nabla\varphi(\theta_0,\phi_0),(\theta_n - \theta_0, \phi_n - \phi_0)\rangle\rangle =$$

$$2< \theta'_n - \theta'_0, \theta'_n - \theta'_0> + < \phi'_n\cos(\phi_n - \theta_n) - \phi'_0\cos(\phi_0 - \theta_0), \theta'_n - \theta'_0>$$

$$+ < \theta'_n \phi'_n \sin(\phi_n - \theta_n) - \theta'_0 \phi'_0 \sin(\phi_0 - \theta_0), \theta_n - \theta_0> +2g < \sin(\theta_n) - \sin(\theta_0), \theta_n - \theta_0>$$

$$+ <e, \theta_n - \theta_0> + < \phi'_n - \phi'_0, \phi'_n - \phi'_0> + < \theta'_n\cos(\phi_n - \theta_n) - \theta'_0\cos(\phi_0 - \theta_0), \phi'_n - \phi'_0>$$

$$- < \theta'_n \phi'_n \sin(\phi_n - \theta_n) - \theta'_0 \phi'_0 \sin(\phi_0 - \theta_0), \phi_n - \phi_0> - g < \sin(\phi_n) - \sin(\phi_0), \phi_n - \phi_0>.$$

Now, since $\nabla\varphi(\theta_n,\phi_n) \to 0$ and $(\theta_n - \theta_0, \phi_n - \phi_0) \to 0$ weakly and so are bounded, the first term of the left member of that equation goes to 0 as n goes to infinity; evidently, the second one also goes to 0. Using again the fact that, in any Hilbert space, the product of something bounded and something that goes to zero, itself goes to zero, we get that any term of the right side of the equation that has either $\theta_n - \theta_0$ or $\phi_n - \phi_0$ as an entry of the scalar product must go to zero. We are left with only four terms:

$$0 = \lim_{n \to \infty} \Big[ 2\|\theta'_n - \theta'_0\|^2 + \| \phi'_n - \phi'_0\|^2 + < \phi'_n \sin(\phi_n - \theta_n) - \phi'_0 \sin(\phi_0 - \theta_0), \theta'_n - \theta'_0 >$$
$$+ < \theta'_n \sin(\phi_n - \theta_n) - \theta'_0 \sin(\phi_0 - \theta_0), \phi'_n - \phi'_0 > \Big]$$

But

$$< \phi'_n \sin(\phi_n - \theta_n) - \phi'_0 \sin(\phi_0 - \theta_0), \theta'_n - \theta'_0> = < ( \phi'_n - \phi'_0) \sin(\phi_n - \theta_n), \theta'_n - \theta'_0>$$
$$+ < \phi'_0 (\sin(\phi_n - \theta_n) - \sin(\phi_0 - \theta_0)), \theta'_n - \theta'_0 >$$

and

$$< \phi'_0 (\sin(\phi_n - \theta_n) - \sin(\phi_0 - \theta_0)), \theta'_n - \theta'_0 > = < \sin(\phi_n - \theta_n) - \sin(\phi_0 - \theta_0), \phi'_0 (\theta'_n - \theta'_0) >$$

which goes to zero as n goes to infinity since it is the product of something that goes tozero and a bounded term. We are left with:

$$0 = \lim_{n \to \infty} (2 - \varepsilon^{-2}) \| \theta'_n - \theta'_0 \|^2 + \lim_{n \to \infty} (1 - \varepsilon^2) \| \phi'_n - \phi'_0 \|^2$$

$$+ \lim_{n \to \infty} \int_0^T [\, \varepsilon^{-2} (\theta'_n - \theta'_0)^2 + 2(\theta'_n - \theta'_0)(\phi'_n - \phi'_0) \sin(\phi_n - \theta_n) + \varepsilon^2 (\phi'_n - \phi'_0)^2 \,] \, dt$$

each of which is positive, since the integrand of the last term is a square product, hence they must all be zero and we get the conclusion. ♦

The following proposition is the main result of this paragraph.

**Theorem 2.5:** (2.1)' has at least three solutions (not differing by a multiple of $2\pi(1,1)$).

**Proof:** Evident from (2.4), (1.6) and the well known fact that the torus has category 3. ♦

**Remark 2.6:** a) Since the only thing we need to prove to get at least three solutions is (2.4) and since to prove that it is sufficient that the term in $\theta'\phi'$ be strictly dominated by the ones in $\theta'^2$ and $\phi'^2$, we need only that $(m+m_1)\, l^2 \times m_1 l_1^2 > (m_1 l\, l_1)^2$. Thus if $m > 0$, we get the existence of three solutions (not differing by a multiple of $2\pi$).

b) A .Capozzi ,D. Fortunato and A. Salvatore [7] have previously proved the existence of two solutions.

## 3. RELATIVE CATEGORY

We present two different notions of relative category each of which satisfies a special property stated at the end of this paragraph. Until then let us give a more unified presentation.

**Definition 3.1**: Let X be a topological space et Y a closed subset of X. A closed subset A of X is of the k-th (strong) category relative to Y (we write $Cat_{X,Y}(A) = k$ ) if and only if k is the least positive integer such that

$$A = \bigcup_{i=1}^{n} A_i$$

where for each i, $A_i$ is closed and there exists $h_i$: $A_i \times I \to X$ , where I=[0,1], such that

(1) $h_i(x, 0) = x \qquad \forall\, i \;\forall\, x \in A_i$

(2) $\forall\, i \geq 1$ (a) $\exists\, x_i \in X$ such that $h_i(x, 1) = x_i$

(b) $h_i(A_i \times I) \cap Y = \emptyset$

(3) $i = 0$ (a) $h_0(x, 1) \in Y \qquad \forall\, x \in A_0$

(b) $(h_0(x, t) = y \in Y) \Rightarrow (h_0(x, s) = y) \qquad \forall\, x \in A_0 \;\forall\, s \geq t.$

We say that A is of k-th weak category relative to Y, written $cat_{X,Y}(A) = k$, if k is minimal satisfying conditions (1), (2 a), (3 a) and (3 b') where (3 b') is given by

$$h_0(x, t) = x \qquad \forall\, x \in Y \quad \forall\, t \in I.$$

**Remark 3.2:** (1) We have that $cat_{X,Y}(A) \leq Cat_{X,Y}(A)$ and $cat_{X,Y}(A) \leq Cat_X(A)$.

(2) If $Y = \emptyset$ then $A_0 = \emptyset$ and $cat_{X,\emptyset}(A) = Cat_{X,\emptyset}(A) = Cat_X(A)$.

(3) If one such k does not exist, then $Cat_{X,Y}(A) = \infty$ or $cat_{X,Y}(A) = \infty$.

(4) From (2 b), we get $A \cap Y = A_0 \cap Y \;\; \forall\, i \quad x_i \notin Y$.

(5) Examples:

$$cat_{\Re^2 \setminus \Re \setminus [-1,1]}(\Re^2) = 1, \qquad cat_{\Re^2, \Re}(\Re^2) = 0$$

$$cat_{\Re^2, \{(0,0)\}}(\Re^2) = 0, \qquad cat_{\Re^2, \{(0,0), (0,1)\}}(\Re^2) = 1.$$

(6) There exists an homeomorphism $\varphi$: $X \to X'$ such that $Y' = \varphi(Y)$ and $A' = \varphi(A)$ imply that $cat_{X',Y'}(A') = cat_{X,Y}(A)$ and $Cat_{X',Y'}(A') = Cat_{X,Y}(A)$.

The following is useful for comparing relative categories of different subspaces.

**Definition 3.3:** Let Z, Z' be subsets of X; $Z <_Y Z'$ if and only if there exists $h: Z \times I \to X$ such that

(1) $h_0 = i_Z: Z \to X$ is the inclusion

(2) $Z' \supset h_1(Z)$

(3) if $s \geq t$ then $(h(x, t) = y \in Y \Rightarrow h(x, s) = y)$.

We have the following properties most of which are generalisations of properties of the category itself.

**Proposition 3.4:** Let A, B, Y be closed subsets of X.

i) if $B \supset A$ then $\mathrm{Cat}_{X,Y}(A) \leq \mathrm{Cat}_{X,Y}(B)$

ii) $A <_Y B$ implies $\mathrm{Cat}_{X,Y}(A) \leq \mathrm{Cat}_{X,Y}(B)$

iii) $A <_Y B$ and $B <_Y A$ imply $\mathrm{Cat}_{X,Y}(A) = \mathrm{Cat}_{X,Y}(B)$

iv) if $X \setminus Y \supset B$ then $\mathrm{Cat}_{X,Y}(A \cup B) \leq \mathrm{Cat}_{X,Y}(A) + \mathrm{Cat}_{X \setminus Y}(B)$

v) $\mathrm{Cat}_{X,Y}(X) \geq \mathrm{Cat}_X(X) - \mathrm{Cat}_Y(Y)$

vi) $\mathrm{Cat}_{X,Y}(A) = 0$ if and only if $A <_Y Y$.

Furthermore, in each of the above properties except (vi), we may replace Cat by cat.

**Proof:** ii) Let h satisfying the conditions of definition 3.3 for $A <_Y B$. Let $B_i$ and $H_i$ satisfying the conditions of definition 3.1 for $\mathrm{Cat}_{X,Y}(B) = k$. Then $H_i * h$ satisfies the conditions of definition 3.1 and we find $\mathrm{Cat}_{X,Y}(A)$ where $A_i = (h)_1^{-1}(B_i)$, where $H_i * h: A \times I \to X$ is defined by,

$$H_i * h(x,t) = \begin{cases} h(x,2t) & \text{if } t \leq \frac{1}{2} \\ H_i(h(x,1),2t-1) & \text{if } t \geq \frac{1}{2} \end{cases}$$

v) $\mathrm{Cat}_X(X) \le \mathrm{Cat}_{X,Y}(X) + \mathrm{Cat}_X(Y) \le \mathrm{Cat}_{X,Y}(X) + \mathrm{Cat}_Y(Y)$. ♦

In the case of weak category, iv) has better conditions and vi) turns into the following.

**Proposition 3.5:** Let A, B, Y be closed subsets of X.

iv') $\mathrm{cat}_{X,Y}(A \cup B) \le \mathrm{cat}_{X,Y}(A) + \mathrm{cat}_X(B)$

vi') $\mathrm{cat}_{X,Y}(A) = 0 \Leftrightarrow \exists\, h: A \times I \to X$ such that

(1) $h_0 = i_A: A \to X$ is the inclusion

(2) $Y \supset h_1(A)$

(3) $h(y, t) = y \quad \forall y \in Y$.

**Proof:** Evident. ♦

Let us try to see what happens when X and Y are changed for other subspaces.

**Proposition 3.6:** a) If $X' \supset X \supset A$ and $X \supset Y$ then $\mathrm{Cat}_{X',Y}(A) \le \mathrm{Cat}_{X,Y}(A)$.

b) If $X \supset A$ and $X \supset Y' \supset Y$ and $Y' <_Y Y$ then $\mathrm{Cat}_{X,Y'}(A) \ge \mathrm{Cat}_{X,Y}(A)$.

c) If $X' \supset X \supset A$, $X \supset Y' \supset Y$, $X' \supset A'$ and $r: X' \to X$ is a retraction such that $r^{-1}(Y) = Y'$ and $r^{-1}(A) \supset A' \supset A$, then $\mathrm{Cat}_{X',Y'}(A') \ge \mathrm{Cat}_{X,Y}(A)$.

Furthermore it remains true if we replace Cat by cat.

**Proof:** a) The $A_i$ and $h_i$ for $\mathrm{Cat}_{X,Y}(A)$ satisfy the conditions for $\mathrm{Cat}_{X',Y}(A)$; hence the conclusion.

b) If the $A_i$ and $h_i$ satisfy the conditions for $\mathrm{Cat}_{X,Y'}(A)$ and h satisfies the conditions of definition 3.3 for $Y' <_Y Y$ then $A_i$, $h_i$ $(i \neq 0)$ and $h * h_0$ satisfy the conditions of definition 3.1 for $\mathrm{Cat}_{X,Y}(A)$, where $h * h_0$ is defined as in the proof of 3.4.

c) Let $A_i$ and $h_i$ satisfy the conditions for $Cat_{X',Y'}(A')$ then $r \circ h_i |_{A \times I}$ satisfies the conditions of definition 3.1 for $Cat_{X,Y}(A)$. ♦

The following proposition is not satisfactory as a reciprocal of proposition 3.6.

Proposition 3.7: Suppose that $X' \supset X \supset A, X \supset Y$ and there exists $r: X' \to X$ a retraction such that $r^{-1}(Y) = Y$ then $Cat_{X',Y}(A) = Cat_{X,Y}(A)$. Furthermore it remains true if we replace Cat by cat.

Proof: Taking $Y' = Y$ and $A' = A$ in proposition 3.6 a) and c) we get the conclusion. ♦

The following proposition is the only one we give which is only valid for Cat (the strong relative category).

Proposition 3.8: (Excision)

$$Cat_{X,Y}(A) = Cat_{X\setminus V,\, Y\setminus V}(A\setminus V) \quad \text{where } \overset{\circ}{Y} \supset V.$$

Proof: $\geq$) If the $A_i$ and $h_i$ satisfy the conditions for $Cat_{X,Y}(A)$ then $A_i \setminus V$ and $h_i|_{(A_i \setminus V) \times I}$ satisfy the conditions for $Cat_{X\setminus V,\, Y\setminus V}(A \setminus V)$.

$\leq$) If the $A_i$ and $h_i$ satisfy the conditions for $Cat_{X\setminus V, Y\setminus V}(A \setminus V)$ then $A_0 \cup V$, $A_i$, $h_i$ and

$$h_0^1(x,t) = \begin{cases} h(x,t) & \text{if } x \notin V \\ x & \text{if } x \in V \end{cases}$$

satisfy the conditions of the definition for $Cat_{X,Y}(A)$.

The following proposition is only valid for cat (the weak relative category).

**Proposition 3.9:** Let $X' \supset X \supset A, X \supset Y$ and $X' \supset Y' \supset Y$. If

a) there exists $r: X' \to X$ a retraction $(r(x) = x \ \forall x \in X)$

b) $r(Y') \supset Y$ and $r(Y') <_Y Y$ in X

then $\mathrm{cat}_{X',Y'}(A') \geq \mathrm{cat}_{X,Y}(A)$ for all $r^{-1}(A) \supset A' \supset A$.

**Proof:** Let H satisfying the conditions of definition 3.3 for $r(Y') <_Y Y$ and let $A'_i$ and $h'_i$ satisfying the conditions of definition 3.1 for $\mathrm{cat}_{X',Y'}(A')$. If $A_i = r(A'_i)$ and, for all $i > 0$,

$h_i = r \circ h'_i \big|_{A_i \times I}$ and $h_0 = H * (r \circ h_0) \big|_{A_0 \times I}$ we have that the conditions of the definition 3.1 for $\mathrm{cat}_{X,Y}(A)$ are satisfied.

## 4. APPLICATION TO CRITICAL POINT THEORY.

Let M be a complete $C^2$ Finsler manifold i.e. a $C^2$ Banach manifold with a Finsler structure on its tangent bundle. (Important examples are complete Riemannian manifolds and Banach spaces.) Let $\varphi \in C^1(M, \mathfrak{R})$. Set

$$\varphi^c = \{ u \in M \mid \varphi(u) \leq c \}$$

$$K_c = \{ u \in M \mid \varphi(u) = c, d\,\varphi(u) = 0 \}.$$

We shall use the following variation of the deformation lemma due to Clark [1] for Banach spaces and to Ni [5] for Finsler manifolds.

**Lemma 4.1:** If $\varphi \in C^1(M, \mathfrak{R})$ satisfies the Palais-Smale condition and if U is an open neighbourhood of $K_c$, then, for every $\varepsilon' > 0$ there exists $\varepsilon \in \,]0, \varepsilon'[$ and a map $f: M \to M$ isotopic to $\mathrm{id}_M$ such that for all $d \in [0, \varepsilon]$, $\varphi^{c-d} \supset f(\varphi^{c+\varepsilon} \setminus U)$.

The following theorem generalises theorem 1.6.

__Theorem 4.2:__ If $\varphi \in C^1(M, \mathfrak{R})$ satisfies the Palais-Smale condition and if $-\infty < a < b < +\infty$ and $K_a = K_b = \emptyset$, then

$$\#\left\{ u \in \varphi^{-1}([a, b]) \mid d\varphi(u) = 0 \right\} \geq \mathrm{Cat}_{M,\, \varphi^a}(\varphi^b).$$

__Proof:__ 1) Let

$$k = \mathrm{Cat}_{M,\, \varphi^a}(\varphi^b).$$

We can assume that $1 \leq k \leq +\infty$. Define, for $j \in N \cap [1, k]$,

$$\mathcal{A}_j = \left\{ A \text{ closed subset of } M \mid \mathrm{Cat}_{M,\varphi^a}(A) \geq j \right\}$$

$$C_j = \inf_{A \in \mathcal{A}_j} \sup_A \varphi.$$

Clearly one has

$$-\infty \leq c_1 \leq c_2 \leq \dots \leq b.$$

By lemma 4.1, with $c = a$ there exists $\varepsilon > 0$ such that $\varphi^a <_Y \varphi^{a+\varepsilon}$ where $Y = \varphi^a$. Thus by proposition 3.4 ii), if $\varphi^{a+\varepsilon} \supset A$ then $\mathrm{Cat}_{M,\, Y}(A) = 0$. It follows that $c_1 \geq a+\varepsilon > a$. Similarly, $c_j < b$ for all $j \in N \cap [1, k]$.

2) In order to prove the theorem, it suffices to show that, if $c = c_i = c_j$, for $1 \leq i \leq j \leq k$, $j < +\infty$, then $\# K_c \geq j - i + 1$. We can assume that $K_c$ contains only a finite number m of critical points, so that $K_c$ is contained in the union of the interiors of m closed contractible in $M \setminus \varphi^a$, sets $B_1, \dots, B_m$.

Let us write $B = \bigcup_{n=1}^{m} B_n$ or, if $m = 0$, $B = \emptyset$ By lemma 4.1 applied to $U = \mathring{B}$ and $0 < \varepsilon' < c - a$, there exists $\varepsilon \in ]0, \varepsilon'[$ and a map $f: M \to M$ given by lemma 4.1 such that

$$\varphi^{c-\varepsilon} \supset f(\varphi^{c+\varepsilon} \setminus \mathring{B}).$$

The definition of $c_j = c$ implies the existence of a closed subset A of $\mathcal{A}_j$ such that

$$\varphi^{c+\varepsilon} = \varphi^{c_j+\varepsilon} \supset A.$$

We obtain by properties i), ii), iv) of the category

$$\begin{aligned} j \le \mathrm{Cat}_{M,\varphi^a}(A) &\le \mathrm{Cat}_{M,\varphi^a}((A\setminus \mathring{B}) \cup B) \\ &\le \mathrm{Cat}_{M,\varphi^a}(A\setminus \mathring{B}) + \mathrm{Cat}_{M\setminus\varphi^a}(B) \\ &\le \mathrm{Cat}_{M,\varphi^a}(f(A\setminus \mathring{B})) + m. \end{aligned}$$

Since $\varphi^{c+\varepsilon} \supset A$, we have $\varphi^{c-\varepsilon} \supset f(A\setminus \mathring{B})$. The definition of $c_i = c$ implies that

$\mathrm{Cat}_{M,\varphi^a}(f(A\setminus \mathring{B})) \le i - 1$, so that $j \le i - 1 + m$, i.e. $\# K_c = m \ge j - i + 1$. ♦

<u>Remark 4.3:</u> i) The proof depends only on properties 3.4 i), ii) and vi) of the category.

ii) If M is compact, we obtain the classical Lusternik-Schnirelman theorem by setting

$$a = \min_M \varphi - 1, \qquad b = \max_M \varphi + 1.$$

## 5. ALGEBRAIC TOPOLOGICAL LEMMA.

In order to calculate the relative category that concerns us in paragraph 6, we need the following proposition of algebraic topology and in particular, its corollary. Let us denote $B_r = \{ x \mid \|x\| \le r\}$ in $\mathfrak{R}^n$ nd $S_r = \partial B_r$.

A standard orientation of a simplex $\{ x_0 \dots x_n \}$ of $\mathfrak{R}^n$ is an ordered sequence $< x_0 \dots x_n >$ such that

$$\det \begin{bmatrix} (x_1 - x_0)_1 & \cdots & (x_n - x_0)_1 \\ \vdots & & \vdots \\ (x_1 - x_0)_n & \cdots & (x_n - x_0)_n \end{bmatrix} > 0$$

where $v_i$ denotes the i-th component of $v \in \Re^n$.

By a standard triangulation of $\Re^n$ we will understand a partition of $\Re^n$ into open simplecies, these open simplecies being the convex hull of a finite set $\sigma$ of affinely independent verticies minus its affine boundary (the latter being the union of the convex hulls of the proper subsets of $\sigma$). The simplecies are ordered by the inclusion of their closures. A finite subpolyhedron P is thus a closed finite union of simplecies; a maximal simplex of P is a simplex of P which is not contained in the closure of any other simplex of P.

Lemma 5.1: Let T be a standard triangulation of $\Re^n$. Let P be a finite subpolyhedron whose maximal simplices are all of dimension n and let

$$a_p = \sum_{\sigma \in O_p} \sigma \text{ where } O_p = \left\{ \sigma \mid \sigma \text{ is an n-simplex of P with the standard orientation of } \Re^n \right\}.$$

then the support of $\partial a_p$ is $\partial P$, the boundary of P.

Proof: (For the methods used here, see [2, lemmas 1.5 and 1.6]).

First notice that the support of $\partial a_p$ is contained in the union of the convex hulls of the (n-1) simplices of P.

Case 1: If we have an interior simplex then it belongs to two simplices of dimension n of P, one on each side of the hyperplane generated by its vertices; its coefficient in the boundary of these two n-simplices will be alternately 1 and -1 hence its coefficient in the boundary of the sum will be 0.

Case 2: If we have an exterior simplex then it is in the boundary of only one n-simplex of P and so its coefficient is $\pm 1$ in $\partial a_p$. Furthermore it is included in $\partial P$. (Notice that the coefficient will be +1 if it has the orientation inherited from the n-simplex of P containing it.) ♦

Proposition 5.2: Let A be the (closed) ring defined by

$$A = \bar{B}_R \setminus B_r, \text{ where } 0 < r < R.$$

Let $A_r$ and $A_R$ be two disjoint closed subsets of A such that $A_r \supset S_r$ and $A_R \supset S_R$. Then if

$A_0 = A_r \cup A_R$ and i: $A \setminus A_0 \rightarrow A$ is the inclusion, we get that $i_{*\,n-1}(H_{n-1}(A \setminus A_0))$ is not trivial where H denotes the singular homology.

**Proof:** Let T be a standard triangulation of $\mathfrak{R}^n$ for which each of the vertices has coordinates of the form z/m where z is an integer (for some integer m big enough) and the simplices have diameter smaller or equal to the square routh of $n/m^2$. Without loss of generality, by a homeomorphism of $\mathfrak{R}^n$, we may assimilate A to a subpolyhedra of $\mathfrak{R}^n$ and $\bar{B}_r$ to another subpolyhedra B of $\mathfrak{R}^n$. (Notice that $A \cup B$ is an acyclic polyhedron since it is homeomorphic to $\bar{B}_R$. Now let $3d = d(A_r, A_R)$. By subdividing, we may also suppose that the triangulation T has a mesh smaller than d. Let P be the smallest subpolyhedron of $\mathfrak{R}^n$ containing $B \cup A_r$ within its interior. (Notice that the maximal simplices of P are all of dimension n.)

Let $a_p = \Sigma\ \{\sigma \mid \sigma$ is an n-simplex of P with the standard orientation of $\mathfrak{R}^n\}$, we have that the support of $\partial a_p$ is $\partial P$ (by lemma 5.1); hence $\partial a_p$ is included in $N_{2d}(B \cup A_r) \setminus (B \cup A_r)$ which is a subset of $A \setminus (A_R \cup A_r) = A \setminus A_0$.

Thus we can assimilate $\partial a_p$ with a singular cycle of $A \setminus A_0$; it suffices to show that its inclusion in the set of singular cycles of A is not a boundary. Since the singular homology of A is naturally isomorphic to the simplicial homology of the polyhedron A, it suffices to show that the cycle $\partial a_p$ is not a boundary in the simplicial chain complex of A.

But if there were a $b \in C_nA$ such that $\partial b = \partial a_p$ then we would have that $b \neq a_p$ and $\partial(a_p - b) = 0$ and so that $a_p - b$ is a cycle of $C_n(A \cup B)$ which cannot be a boundary because $C_{n+1}(A) = 0$; hence $H_n(A \cup B) \neq 0$ which contradicts the fact that $A \cup B$ is acyclic. Hence $\partial a_p$ has a non-trivial class of homology in A. ♦

The following corollary will be used in paragraph 6 to prove the existence of a fourth solution of the double pendulum equation under certain conditions.

Corollary 5.3: Let A be the (closed) ring defined by

$$A = \bar{B}_R \setminus B_r .$$

Let $A_0$ be a closed subset of A such that $A_0 \supset S_r \cup S_R$ and there exists h: $A_0 \times I \to A$ such that $S_r \cup S_R \supset h_0(A_0)$, $h_0(x) = x$ for all $x \in A_0$ and for all $s \in S_r \cup S_R$, we have $h_t(s) = s$ $\forall t \in I$.

Then $i_{*\,n-1}(H_{n-1}(A \setminus A_0))$ is non trivial, where i: $A \setminus A_0 \to A$ is the inclusion and H denotes the singular homology.

Proof: By lemma 5.1, it suffices to show that $A_0 = A_r \cup A_R$ where $A_r$ and $A_R$ are two disjoint closed subsets such that $A_r \supset S_r$ and $A_R \supset S_R$. Set $A_r = h_1^{-1}(S_r) \supset S_r$ and $A_R = h_1^{-1}(S_R) \supset S_R$; then we get the conclusion. ♦

## 6. FOUR SOLUTIONS FOR THE DOUBLE PENDULUM EQUATION.

In paragraph 2 we proved the existence of at least 3 solutions for the forced pendulum equation, but in the special case of the non forced problem we get the existence of 4 different constant solutions (none of which is another plus a multiple of $2\pi$). In this paragraph we shall show under some conditions the existence of a fourth solution. We may assume that there is a finite number of critical points, otherwise there are more than four solutions.First let us localise 2 of the 3 known solutions.

Proposition 6.1: We have that $\varphi$ has at least 2 critical points in $\varphi^{-gT}$.

Proof: If $\theta = \pi$ and $\phi = c$ a constant, then

$$\varphi(\pi,c) = \int_0^T g(-2+\cos c)\, dt + \pi \int_0^T e\, dt = gT(\cos c - 2) + 0 \le gT\,(1-2) = -gT.$$

But $\{(\pi,c) \mid c \in \Re \text{ modulo } 2\pi\}$ forms a non contractible circle in $\mathbf{T} \times \{(0,0)\}$ and so in M, moreover it is included in $\varphi^{-gT}$. Now if -gT is not a critical value, put s=1; if -gT is a critical value, it is an isolated one so -sgT is not a critical value for 0<s<1 and s close enough to 1,further more $\varphi^{-sgT} \supset \varphi^{-gT}$. By Theorem 1.6 and the classical caracterisation of critical values, we have that $\mathrm{Cat}\ \varphi^{-sgT} \geq 2$ and so $\varphi$ has at least 2 critical points in $\varphi^{-sgT}$, for some $0<s\leq 1$ as close to 1 as we want. Since $\varphi$ has no critical value greater than -gT and smaller or equal to -sgT, we get the conclusion. ♦

**Remark 6.2:** In the general case, we get that $\varphi(\pi,c) \geq gT\,[m_1 l_1-(m+m_1)l] = L$ and so $\mathrm{Cat}\ \varphi^{L} \geq 2$ and $\varphi$ has 2 critical points in $\varphi^{L}$.

It remains for us to show that $\varphi$ has at least 2 other critical points in M- $\varphi^{-gT}$. For that we will show that

$$\mathrm{Cat}_{M,\varphi^{-gT}}(M) \geq 2.$$

Now we need to find a condition under which there remains a non contractible circle in M- $\varphi^{-gT}$ in order to prove the above inequality. That is the purpose of the following four items.

**Lemma 6.3:** If $\theta$ is of mean value zero and if $\varphi(\theta,\phi) \leq -sgT$ for some 0<s<1, then

$$\sqrt{T}\,(\|e\|_2+2g\sqrt{T}) \geq 2\pi\sqrt{2g(1+s)}\,. \tag{6.3.1}$$

**Proof:** We have that

$$-sgT \geq \varphi(\theta,\phi) = \int_0^T \Big[\ \frac{1}{2}\left(2\theta'^2+2\theta'\phi'\cos(\phi-\theta)+\phi'^2\right)+g(2\cos\theta+\cos\phi)+\theta\, e\ \Big]\, dt$$

and since $\theta'^2+2\theta'\phi'\cos(\phi-\theta)+\phi'^2 \geq 0$, we obtain that

$$\begin{aligned}-sgT &\geq \int_0^T \Big[\frac{1}{2}\theta'^2+\ g\,(2\cos 0+\cos\phi)+\theta\, e\ \Big]\, dt\ -\ 2g\int_0^T [\cos 0-\cos\theta]\ dt\\ &\geq \int_0^T \Big[\ \frac{1}{2}\theta'^2\ \ g\,(2-|\cos\phi|)+\tilde{\theta}\, e\ \Big]\, dt\ -\ 2g\int_0^T |\,\tilde{\theta}\,|\ |\sin\alpha\,(\tilde{\theta})\,|\ dt\end{aligned} \tag{6.3.2}$$

where $0 \le |\alpha(\tilde{\theta})(t)| \le |\tilde{\theta}(t)|$ for all t. But $|\sin \alpha(\tilde{\theta})| \le 1$, so by the Schwarz and Wirtinger inequalities, we get that

$$-sgT \ge \frac{1}{2} \|\theta'\|_2^2 + gT - \|\tilde{\theta}\|_2 \|e\|_2 - 2g \|\tilde{\theta}\|_2 \sqrt{T}$$

and

$$0 \ge \frac{1}{2} \|\theta'\|_2^2 - \|\theta'\|_2 \frac{T}{2\pi} \left( \|e\|_2 + 2g \sqrt{T} \right) + (1+s)\, gT. \tag{6.3.3}$$

Now the right member of (6.3.3) is a second degree polynomial in the variable $\|\theta'\|_2$ and it is negative or zero, so

$$\left[ \frac{T}{2\pi} \left( \|e\|_2 + 2g \sqrt{T} \right) \right]^2 - 4 \left[ \frac{1}{2} \right] \left[ (1+s) gT \right] \ge 0$$

hence we get the conclusion. ♦

<u>Remark 6.4:</u> In the general case, (6.3.3) would become

$$\begin{aligned} 0 \ge & \left( (m + m_1) l^2 - \varepsilon^{-2} m_1 l^2 \right) \|\theta'\|_2^2 + m_1 l_1^2 (1 - \varepsilon^{-2}) \|\phi'\|_2^2 \\ & + (1+s) \left[ (m + m_1) l - m_1 l_1 \right] gT \\ & - \frac{T}{2\pi} \left( \|\theta'\|_2 \|e\|_2 + \|\phi'\|_2 \|f\|_2 \right) - \frac{gT}{\pi} \sqrt{T}\, \|\theta'\|_2 \end{aligned} \tag{6.4.1}$$

which implies, since

$$0 \le m_1 l_1^2 (1 - \varepsilon^2) \|\phi'\|_2^2 - \frac{T}{2\pi} \|f\|_2 \|\phi'\|_2 + \frac{T^2}{16\pi^2} \frac{\|f\|_2}{m_1 l_1 (1 - \varepsilon^2)}$$

that

$$\begin{aligned} 0 \ge & \left( (m + m_1) l^2 - \varepsilon^{-2} m_1 l^2 \right) \|\theta'\|_2^2 - \|\theta'\|_2 \left( \frac{gT\sqrt{T}}{\pi} + \frac{T \|e\|_2}{2\pi} \right) \\ & + (1+s) \left[ (m + m_1) l - m_1 l_1 \right] gT - \frac{T^2}{16\pi^2} \frac{\|f\|_2}{m_1 l_1 (1 - \varepsilon^2)} \end{aligned} \tag{6.4.2}$$

which must imply that

$$\left[\frac{T}{2\pi}\left(2g\sqrt{T}+\|e\|_2\right)\right]^2 - 4\left[(m+m_1)\,l^2-\varepsilon^{-2}m_1\,l^2\right]\left[(1+s)\left[(m+m_1)\,l-m_1\,l_1\right]gT-\frac{T^2}{16\pi^2}\,\frac{\|f\|_2}{m_1\,l_1\,(1-\varepsilon^2)}\right]\geq 0$$

If B denotes the second term of the left member of the precedeing inequality, then we get

$$\sqrt{T}\left(\|e\|_2+2g\sqrt{T}\right)\geq 2\pi\,(-B)^{\frac{1}{2}} \tag{6.4.3}$$

for any $0<\varepsilon<1$.

The following proposition gives the desired condition.

**Proposition 6.5:** If

$$\sqrt{T}\,\|e\|_2+2gT<4\pi\sqrt{g} \tag{6.5.1}$$

then there exists $s<1$ such that $\varphi(\theta,\phi)\geq -sgT$ for all $(\theta,\phi)$ such that $\dot{\theta}=0$.

**Proof:** Otherwise, we would have that for every s such that $0<s<1$, there is a $(\theta,\phi)$ such that $\dot{\theta}=0$ and $\varphi(\theta,\phi)\leq -sgT$. Thus, by lemma 6.3, $\sqrt{T}\,\|e\|_2+2gT\geq 2\pi\sqrt{2g(1+s)}$ and at the limit, as s goes to 1, we get

$\sqrt{T}\,\|e\|_2+2gT\geq 2\pi\sqrt{4g}=4\pi\sqrt{g}$, contradicting (6.5.1). ♦

**Remark 6.6:** In the general case, (6.5.1) is replaced by : there exist $0<\varepsilon<1$ such that

$$\left(\|e\|_2+2g\sqrt{T}\right)<2\pi\,(-B)^{\frac{1}{2}} \tag{6.6.1}$$

where

$$-B=4\left[(m+m_1)\,l^2-\varepsilon^{-2}m_1\,l^2\right]\left[(1+s)\left[(m+m_1)\,l-m_1\,l_1\right]gT-\frac{T^2}{16\pi^2}\,\frac{\|f\|_2}{m_1\,l_1\,(1-\varepsilon^2)}\right].$$

Let us now use our condition to get the lower bound for the relative category.

**Proposition 6.7:** If (6.5.1) is satisfied, then

$$\mathrm{Cat}_{M,\varphi^{-sgT}}(M) \geq 2, \text{ for some } 0<s<1.$$

Hence $\varphi$ has two critical points in $M \setminus \varphi^{-gT}$.

Proof: Let r: M $\rightarrow$ T be the map defined by

$$r(\theta,\phi) = (\bar{\theta}, \bar{\phi}).$$

If $C = \{ (\bar{\theta}, \bar{\phi}) \mid \bar{\theta} = \pi \}$ and if we identify T with $T \times \{ (0,0) \}$, then we have that a) r is a retraction; b) $\varphi^{-gT} \supset C$ by the proof of proposition 6.1 and, since C is compact, for any 0<s<1

$$\exists \delta > 0 \text{ such that } \varphi^{-sgT} \supset \{ (\bar{\theta}, \bar{\phi}) \mid \pi - \delta \leq \bar{\theta} \leq \pi + \delta \} = C' ;$$

c) by proposition 6.5, if s is as in proposition 6.5 and if

$$D = \{ (\bar{\theta}, \bar{\phi}) \mid \bar{\theta} = 0 \},$$

then we have that $r^{-1}(D) \cap \varphi^{-sgT} = \emptyset$, i.e. $T \setminus D \supset r(\varphi^{-sgT}) \supset C'$ and $T \setminus D <_{C'} C'$. Hence, by proposition 3.9 and remark 3.2 (1), we get

$$\mathrm{Cat}_{M,\varphi^{-sgT}}(M) \geq \mathrm{cat}_{M,\varphi^{-sgT}}(M) \geq \mathrm{cat}_{T,C'}(T).$$

But we know, by lemma 6.8, that $\mathrm{cat}_{T,C'}(T) \geq 2$, hence

$$\mathrm{Cat}_{M,\varphi^{-sgT}}(M) \geq 2.$$

And so, by theorem 4.2, we get the conclusion, since all of the above can be done replacing M by $\varphi^{3gT}$. ♦

In order to complete the proof of proposition 6.7, we need the following lemma.

Lemma 6.8: $\mathrm{cat}_{T,C'}(T) \geq 2$.

Proof: It suffices to show that if

$$T = \bigcup_{0}^{k} A_i,$$

where the $A_i$ and $h_i$ satisfy the conditions of definition 3.1, then $i_* H_1(T \setminus A_0)$ is non trivial in

$H_1(T)$. However to show this, it suffices to show that $i_* H_1(T \setminus A_0)$ is non trivial in $H_1(T \setminus \mathring{C}')$.

But since $T \setminus \mathring{C}'$ is homeomorphic to $\bar{B}_R \setminus B_r$ in $\Re^2$, by corollary 5.3, it suffices to show that

$$T \setminus \mathring{C} \supset h_0((A_0 \setminus \mathring{C}') \times I). \tag{6.8.1}$$

Fortunately, by lemma 6.9, we may assume that $h_0$ satisfies (6.8.1). ♦

The following result is used in lemma 6.8.

**Lemma 6.9:** We may assume that $h_0$ satisfies (6.8.1).

**Proof:** Let

$$p: T \setminus \mathring{C}' \to [\pi + \delta, 3\pi - \delta]$$

the continuous projection defined by

$$p(\bar{\theta}, \bar{\varphi}) = \bar{\theta} + 2k_{\bar{\theta}}\pi .$$

Let $p_i: T \to S^1$ the projection on the i-th component, for $i = 1, 2$. Let $p_1 \circ h_0(x, t)$ be considered a map of t whose image is in $S^1$, the unit circle in the complex plane and let $g(x, t)$ be a continuous representation of the argument of $p_1 \circ h_0(x, t)$. Set

$$h'(x, t) = (g(x, t),\ p_2 \circ h_0(x, t))$$

(where $h'(x, t)$ represents the same point as $h_0(x, t)$ on the torus).

(In the usual way, one shows that g and h' are continuous in x, hence in (x, t) by showing that, if $| x - x' | < \delta$,

$$| g(x, t) - g(x', t) | = | p_1 \circ h_0(x, t) - p_1 \circ h_0(x', t) | .)$$

Set $h(x, t) = (\rho g(x, t),\ p_2 \circ h_0(x, t))$ where $\rho: \Re \to [\pi - \delta, 3\pi - \delta]$ is the obvious retraction. We have that h satisfies (6.8.1), that $h(\_,0) = h'(\_,0) = h_0(\_,0) = 1_{A_0}$ and that

$\partial C' \supset h_1(A_0)$ , since $C' \supset h_0(A_0 \times \{1\})$ [that is $p_1 \circ h_0(x,1) \geq 3\pi - \delta$ or $p_1 \circ h_0(x,1) \leq \pi + \delta$ ], and finaly that $h(x,t) = h_0(x,t) = x \ \ \forall t \in [0,1] \ \ \forall x \in \partial C'$. ♦

Theorem 6.10: If (6.5.1) is satisfied, then (2.1)' has at least 4 solutions (not differing by a multiple of $2\pi(1,1)$).

Proof: By (6.1) $\varphi$ has at least 2 critical points in $\varphi^{-gT}$; by (6.7) $\varphi$ has at least 2 critical points in $M \setminus \varphi^{-gT}$; so $\varphi$ has at least 4 critical points and since the critical points of $\varphi$ are the solutions of (2.1) we get the conclusion. ♦

Remark 6.11: In the general case we get the same conclusion, for m>0, provided that the condition (6.6.1) is satisfied.

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