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VARIATIONAL PROBLEMS WITH LIPSCHITZIAN MINIMIZERS

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Abstract. We review the authors' intermediate existence theory for the basic problem in the calculus of variation [5]. Then we show that the extremal growth condition (EGC), under which [5] asserts the existence and Lipschitzian regularity of solutions to the basic problem, is actually implied by various growth conditions used to prove Lipschitzian regularity in the past. Our results unify and extend the class of problems known to have Lipschitzian solutions.

Key Words. Calculus of variations, regularity theory, intermediate existence, Euler equation.

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1. Introduction.

This article concerns the basic problem in the calculus of variations: given a nondegenerate real interval $[a,b]$, a mapping $L:[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, and points x_a, $x_b\in\mathbb{R}^n$, choose an absolutely continuous function $x:[a,b]\to\mathbb{R}^n$ to

$$\text{minimize} \quad \Lambda(x) := \int_a^b L(t,x(t),\dot{x}(t))\,dt \quad \text{subject to} \quad x(a)=x_a,\ x(b)=x_b. \tag{P}$$

Tonelli [9] was the first to describe a satisfactory existence theory for (P). He identified the set of absolutely continuous functions as the smallest class in which a minimizer could reasonably be expected to exist—a contribution so fundamental that it now appears in the very statement of the problem. But while Tonelli's theory represented a great step forward on the basic question of existence, it raised new questions about the traditional necessary conditions. Tonelli conjectured [10], and Clarke and Vinter verified [6], that certain instances of problem (P) satisfying Tonelli's existence hypotheses have minimizers for which the Euler equation fails. This revealed an essential difference between the hypotheses of existence theory and the conditions under which the standard necessary conditions apply. The intermediate existence theory of Clarke and Loewen [5] gives new information about both sides of this gap: on one hand, it offers a new approach to existence theory quite distinct from Tonelli's direct method; on the other, it pertains to a large class of problems whose solutions are Lipschitzian, and hence satisfy the Euler equation (see [2]). The fine points of intermediate existence theory are described in [5]. The purpose of the present paper is to describe its consequences for the Lipschitzian regularity of minimizers under various assumptions.

The plan of the paper is as follows. In Section 2 we review the intermediate existence theory of [5], with particular attention to its intuitive significance and its difference from the direct method of Tonelli. Section 3 describes the theory's consequences for the question of existence in the small, and asserts that under very weak hypotheses, any solution to (P) is locally Lipschitz on an open subset $\Omega \subseteq [a,b]$ of full measure. Global existence and regularity results are given for Lagrangians with slow growth in Section 4 and for coercive Lagrangians in Section 5.

2. Intermediate Existence Theory.

The intermediate existence theory of [5] forms a versatile link between Clarke and Vinter's global regularity theory [7] and their existence and regularity results "in the small" [8]. Better still, it offers new existence and regularity theorems in the global setting, generalizes the local theory in [8], and applies in a variety of intermediate situations. On the local level, the main advance is the removal of Clarke and Vinter's hypothesis that $L(t,x,\cdot)$ be strictly convex for each (t,x). The natural replacement for this assumption enters through (H2), below.

The *Basic Hypotheses*, in force throughout this article, are as follows.

(H1) The Lagrangian $L:[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is locally Lipschitz, and

(H2) For each $(t,x)\in[a,b]\times\mathbb{R}^n$, the function $L(t,x,\cdot)$ is strictly convex at infinity. (A function $\ell:\mathbb{R}^n\to\mathbb{R}$ is called *strictly convex at infinity* if it is convex and its graph contains no rays.)

To appreciate the notion of strict convexity at infinity, let $\ell:\mathbb{R}^n\to\mathbb{R}$ be convex, and let $h(p)=\ell^*(p)=\sup\{\langle p,v\rangle-\ell(v) : v\in\mathbb{R}^n\}$. The graph of ℓ contains a line segment $(v_0,\ell(v_0))+t(v_1-v_0,r)$, $t\in[0,1]$, if and only if the line segment $[v_0,v_1]$ lies inside $\partial h(p)$ for some p obeying $\langle p,v_1-v_0\rangle=r$. Thus the hypothesis that ℓ be strictly convex at infinity is equivalent to the assumption that $\partial h(p)$ is never an unbounded set, just as strict convexity requires that $\partial h(p)$ contains at most one point.

Returning to problem (P), we introduce the *Hamiltonian* $H(t,x,p)=\sup\{\langle p,v\rangle-L(t,x,v) : v\in\mathbb{R}^n\}$. Hypothesis (H2) implies that the partial subgradient $\partial_pH(t,x,p)$ is always a bounded set, although the bound on its magnitude may increase very rapidly with $|p|$. In fact, strict convexity at infinity gives us slightly more than boundedness: it implies that when $\partial_pH(t,x,p)$ contains very large elements, it must be bounded away from the origin. This is the content of Prop. 2.1 below, in which we write (for fixed $R>0$)

$\rho_R(s)=\min\{s, \tilde{\rho}_R(s)\}$, $s\geq 0$, where

$$\tilde{\rho}_R(s) = \inf\{ r\geq 0 : \text{ for some } (t,x,p)\in[a,b]\times R\bar{B}\times\mathbf{R}^n, \text{ one has both } \partial_p H(t,x,p)\cap r\bar{B}\neq\emptyset \text{ and } \partial_p H(t,x,p)\cap(\mathbf{R}^n\backslash sB)\neq\emptyset\}.$$

(Here, and throughout the paper, B denotes the open unit ball of $\mathbf{R}^n$.)

2.1 PROPOSITION. For each fixed $R>0$, the function ρ_R has the following properties:

(a) ρ_R is nondecreasing and obeys $0\leq\rho_R(s)\leq s \quad \forall s>0$;

(b) $\forall(t,x,p)\in[a,b]\times R\bar{B}\times\mathbf{R}^n$, $\forall s>0$,

$\partial_p H(t,x,p)\cap\rho_R(s)B\neq\emptyset \Rightarrow \partial_p H(t,x,p)\subseteq sB$;

(c) $\rho_R(s)\to+\infty$ as $s\to+\infty$;

(d) if $L(t,x,\cdot)$ is strictly convex for each $(t,x)\in[a,b]\times R\bar{B}$, then $\rho_R(s)=s \quad \forall s>0$.

Assertion (b) of Proposition 2.1 makes the role of ρ_R precise, and follows directly from the definition. Assertion (a) is evident, while (d) holds because $\partial_p H(t,x,p)$ contains at most one point in the strictly convex case. The Proposition's most significant assertion is (c), which relies upon strict convexity at infinity, and is proven in [5].

In the theory to be described below, the function ρ_R controls jumps in the derivatives of extremals for (P). Now since the derivative of an arc can be anything in $L^1[a,b]$, this must be understood appropriately. We therefore make the following definition.

2.2 DEFINITION. Let $f:\mathbf{R}\to\mathbf{R}^n$ be measurable. The set of *essential values of* f *at* t, denoted Ess f(t), consists of all $y\in\mathbf{R}^n$ such that

$$\forall\epsilon>0, \qquad m\{\ s\in(t-\epsilon,t+\epsilon) : |y-f(t)|<\epsilon\ \} > 0.$$

It is clear that if $f(t)=g(t)$ a.e. then $\mathrm{Ess}\, f(t)=\mathrm{Ess}\, g(t)\ \forall t$, so that variations in the sets Ess f(t) represent essential changes in f(t). When we speak loosely of jumps in $\dot{x}$ for $x\in AC$, we mean discontinuities in the set-valued map $t\to\mathrm{Ess}\,\dot{x}(t)$. If x happens to be an extremal for (P), we have the following.

2.3 PROPOSITION [5]. Suppose two arcs x and p are given, such that

$$p(t)\in\partial_v L(t,x(t),\dot{x}(t)) \qquad \text{a.e. } t\in[a,b]. \tag{2.1}$$

Then for all t in [a,b], with no exceptions,

$$\overline{co}\ \mathrm{Ess}\,\dot{x}(t) \subseteq \partial_p H(t,x(t),p(t)). \tag{2.2}$$

The inclusion (2.1) in this straightforward result holds whenever x is an extremal for problem (P) in the sense of the Euler inclusion [2], the Hamiltonian inclusion [3, Thm. 4.2.2], or the maximum principle [3, Thm. 5.2.1]. The conclusion (2.2) is to be compared with Proposition 2.1(b): it shows that if $\|x\|_\infty\le R$ and $\mathrm{Ess}\,\dot{x}(t)$ contains any point inside $\rho_R(s)$, then $\mathrm{Ess}\,\dot{x}(t)$ lies entirely inside sB. Informally, it is impossible for $|\dot{x}|$ to jump beyond level s from a level below $\rho_R(s)$.

In the case when $L(t,x,\cdot)$ is strictly convex, inclusion (2.2) implies that $\mathrm{Ess}\,\dot{x}(t)$ contains at most one point for each $t\in[a,b]$. If x is known to be Lipschitz, then it follows that $\mathrm{Ess}\,\dot{x}(t)$ contains exactly one point for each $t\in[a,b]$, and consequently $\dot{x}$ is actually continuous.

We now consider two fundamental quantities associated with solutions to problem (P).

2.4 DEFINITION. Let $R>0$ and $0<r<s$ be given. We define

$$\Delta_R(r,s) = \inf\{t_1 - t_0\},$$

$$\bullet \quad V_R(r,s) = \sup\{\int_{t_0}^{t_1} L(t,x(t),\dot{x}(t))\, dt\},$$

where both extrema are taken over all triples (t_0,t_1,x) for which $[t_0,t_1]\subseteq[a,b]$ is nondegenerate, $x\in AC[t_0,t_1]$, and the following conditions hold:

(a) Ess $\dot{x}(\tau)\cap r\bar{B} \neq \emptyset$ for some $\tau\in[t_0,t_1]$;

(b) Ess $\dot{x}(\sigma)\cap(\mathbb{R}^n\backslash\rho_R(s)B) \neq \emptyset$ for some $\sigma\in[t_0,t_1]$;

(c) Ess $\dot{x}(t) \subseteq sB$ for all $t\in(t_0,t_1)$;

(d) $|x(t)|<R$ for all $t\in[t_0,t_1]$;

(e) x solves the following problem on $[t_0,t_1]$:

$$\min\{\int_{t_0}^{t_1} L(t,y,\dot{y})\, dt : y(t_i)=x(t_i),\ |y(t)|<R\ \ \forall t,\ |\dot{y}(t)|\leq s \text{ a.e.}\}.$$

(Conditions (a) and (b) are understood to involve the appropriate one-sided essential values in cases where either σ or τ falls at an endpoint of $[t_0,t_1]$.)

These numbers have natural interpretations. The first one, $\Delta_R(r,s)$, describes the shortest time interval over which an arc solving the auxiliary problem (e) can exhibit an increase in velocity from $|\dot{x}(\tau)|\leq r$ to $|\dot{x}(\sigma)|\geq s$. (The appearance of $\rho_R(s)$ in condition (b) accounts for the fact that a jump beyond level s must start above level $\rho_R(s)$, as we have seen.) For very large values of s, such an increase may be impossible, and conditions (a)–(e) may be inconsistent. This forces $\Delta_R(r,s)=+\infty$, a condition we regard as desirable. As for $V_R(r,s)$, it measures the largest possible objective value in the auxiliary problem (e). Of course, the supremum may be approximated by a triple (t_0,t_1,x) completely unrelated to the triples for which t_1-t_0 approximates $\Delta_R(r,s)$. When there are no triples obeying (a)-(e), one has $V_R(r,s)=-\infty$.

The key properties of Δ_R and V_R are described in the next two results. The first of these is a combination of [5, Prop. 2.11] with [5, Prop. 6.1]; the second is new.

2.5 PROPOSITION. Fix any $r, R>0$. Then for any S obeying $\rho_R(S)\geq r$, the mapping $s\rightarrow\Delta_R(r,s)$ is nondecreasing on $[S,+\infty)$. Also, there exist $\epsilon>0$ and $S>0$ such that

$$\rho_R(S)\geq r \qquad \text{and} \qquad \Delta_R(r,s)>\epsilon \qquad \forall s\geq S.$$

2.6 PROPOSITION. For every choice of $r, R>0$, there is a constant $M\geq 0$ such that $V_R(r,s)\leq M(b-a)$ for all s sufficiently large.

Proof. Fix $r, R>0$. According to Prop. 2.5, there are positive constants ϵ and S_0 such that

$$s\geq S_0 \quad \Rightarrow \quad \Delta_R(r,s)>\epsilon. \tag{2.3}$$

Let us define $S=\max\{S_0, 2R/\epsilon\}$, and set

$$M=\max\{\, L(t,x,v) : (t,x)\in[a,b]\times R\bar{B},\ |v|\leq S\,\}.$$

We will show that

$$s \geq S \quad \Rightarrow \quad V_R(r,s) \leq M(b-a). \tag{2.4}$$

Indeed, fix any $s \geq S$ and let x on $[t_0,t_1]$ be any Lipschitz arc satisfying conditions 2.4(a)-(e). By definition of $\Delta_R(r,s)$, (2.3) implies $t_1-t_0 \geq \epsilon$. Consequently the straight-line arc $\bar{x}$ joining $(t_0,x(t_0))$ to $(t_1,x(t_1))$ obeys

$$|\dot{\bar{x}}(t)| = |x(t_1)-x(t_0)|/(t_1-t_0) \leq 2R/\epsilon \leq S \qquad \forall t \in [t_0,t_1],$$

so that $\bar{x}$ is an admissible arc in problem 2.4(e). But x solves this problem, so we must have

$$\int_{t_0}^{t_1} L(t,x(t),\dot{x}(t))\,dt \leq \int_{t_0}^{t_1} L(t,\bar{x}(t),\dot{\bar{x}}(t))\,dt \leq M(t_1-t_0) \leq M(b-a).$$

Now $V_R(r,s)$ is the supremum of this inequality's left-hand sides, so (2.4) follows. ////

We may now quote the intermediate existence theorem [5, Thm. 3.2]. We present only its Lagrangian formulation; an equivalent Hamiltonian formulation appears in [5, Thm. 3.1], where it is used to give new sufficient conditions for the existence of periodic trajectories of Hamiltonian systems. The theorem concerns not problem (P), but a related problem involving a state constraint:

$$\min \{ \Lambda(x) := \int_a^b L(t,x(t),\dot{x}(t))\,dt \ : \ x(a)=x_a,\ x(b)=x_b,\ \|x\|_\infty < R \}. \tag{P_R}$$

2.7 THEOREM (INTERMEDIATE EXISTENCE). Fix $R>0$. Assume the Basic Hypotheses, as well as

(H3) for some $\bar{\alpha}>0$, one has $L(t,x,v)\geq\bar{\alpha}|v|$ $\forall(t,x,v)\in[a,b]\times R\bar{B}\times\mathbb{R}^n$.

Suppose that for some α in $(0,\bar{\alpha})$ and some Lipschitz arc $\bar{x}$ admissible for (P_R), one has

(H4) $\Lambda(\bar{x})/\alpha + \min\{|x_a|,|x_b|\} < R$,

(H5) $b-a\leq\Delta_R(\bar{r},\bar{s})$ for some $\bar{s}>\rho_R(\bar{r})$, where $\bar{r}:=\Lambda(\bar{x})/(\alpha(b-a))$.

Then the set of solutions to problem (P_R) is nonempty, and consists entirely of Lipschitz arcs.

To highlight the difference between Thm. 2.7 and classical existence results based on the direct method, we refer to the instance of problem (P_R) with $n=1$, $L(t,x,v)=(1+|v|^2)^{1/2}-x^2/8$, $R=2$, $(a,x_a)=(0,0)$, and $(b,x_b)=(b,0)$. It is shown in [5, Example 3.2.1] that for any $b>0$ sufficiently small, hypotheses (H1)-(H5) hold and hence Thm. 2.7 applies to assert the existence of a Lipschitzian solution for (P_R). However, for any fixed $b>0$, and any $\lambda>\inf(P)$, $\epsilon>0$, the level set

$$\{\ x\in AC[0,b] :\ \Lambda(x)\leq\lambda,\ \|x\|_\infty\leq\epsilon\ \}$$

fails to be compact in any reasonable topology. In particular, there exists a minimizing sequence with no convergent subsequences: the direct method cannot possibly succeed.

The remainder of this paper concerns applications of Thm. 2.7.

3. A Regularity Result.

The main result of this section is a regularity theorem in the spirit of [7 , Thm. 2.1] (see also [8 , Cor. 1]). To prove it, we apply a local existence result based on Thm. 2.7. Thus, let a point $(t_0,x_0)\in\mathbf{R}\times\mathbf{R}^n$ be given, and suppose U is a neighbourhood of (t_0,x_0) on which a function $L:U\times\mathbf{R}^n\to\mathbf{R}$ is defined and obeys

$$L \text{ is locally Lipschitz on } U\times\mathbf{R}^n, \tag{3.1}$$

$$L(t,x,\cdot) \text{ is strictly convex at infinity for each } (t,x)\in U. \tag{3.2}$$

These conditions are simply the Basic Hypotheses restricted to $U\times\mathbf{R}^n$.

3.1 THEOREM [5]. If (3.1) and (3.2) hold, then there exist $\epsilon>0$ and $R>0$ for which $W=[t_0-\epsilon,t_0+\epsilon]\times(x_0+R\bar{B})$ is a subset of U on which the following holds. For every $M>0$ and every $\rho\in(0,R)$, there exists $\delta>0$ so small that for every pair of endpoints (a,x_a) and (b,x_b) in W obeying

$$0<b-a<\delta, \quad |x_b-x_a|\le M(b-a), \quad \min\{|x_b-x_0|,|x_a-x_0|\}<\rho, \tag{3.3}$$

the following problem has a Lipschitz solution:

$$\min\{\int_a^b L(t,x(t),\dot{x}(t))\,dt \;:\; x(a)=x_a,\; x(b)=x_b,\; |x(t)-x_0|<R \;\; \forall t\in[a,b]\}. \tag{3.4}$$

In fact, all its solutions are Lipschitz.

Proof (Sketch). Without loss of generality, we take $(a,x_a)=(0,0)$. Then it suffices to check (H1)-(H5) and apply Thm. 2.7. Now (H1) and (H2) follow from (3.1) and (3.2) in the region of interest. As for (H3), if it is violated by L then one can define a new Lagrangian

$$\tilde{L}(t,x,v) = L(t,x,v) - \langle\zeta,v\rangle - \gamma$$

for which (H2) implies (H3), while the solution set to the problem above remains unchanged. For a sufficiently small choice of δ, conditions (3.3) permit $\Lambda(\bar{x})$ to be made arbitrarily small, where $\bar{x}$ is the straight line between the endpoints: this gives (H6). And (H7) follows from Prop. 2.5.

3.2 THEOREM. In the presence of the Basic Hypotheses, let x be any strong local solution to problem (P). Let $\tau\in[a,b]$ be any point for which

$$\liminf_{\substack{s,t\to\tau \\ a\le s\le\tau\le t\le b \\ s\neq t}} \frac{|x(t)-x(s)|}{t-s} < +\infty. \tag{3.5}$$

Then τ lies in a relatively open subinterval I of $[a,b]$ on which x is Lipschitzian. In particular, there is a relatively open subset Ω of $[a,b]$ of full measure in which x is locally Lipschitzian.

Proof. Let τ be a point satisfying (3.5). Then there must be some number $M>0$ together with sequences a_i, b_i converging to τ while obeying

$$a_i<b_i, \quad \tau\in \text{rel int } [a_i,b_i], \quad \text{and} \quad \frac{|x(b_i)-x(a_i)|}{b_i-a_i}\le M \quad \forall i. \tag{3.6}$$

Note that the Basic Hypotheses imply that conditions (3.1) and (3.2) hold (with $U=[a,b]\times\mathbb{R}^n$) at the point $(t_0,x_0)=(\tau,x(\tau))$. Therefore Thm. 3.1 applies, and provides certain positive constants R and ϵ. We may reduce R and ϵ, if necessary, to assure that x solves problem (P) relative to all arcs y obeying $|y-x|_\infty\le 2R$, and that $|x(t)-x(\tau)|<R$ $\forall t\in[\tau-\epsilon,\tau+\epsilon]$. In terms of the constant M defined above and the choice $\rho=R/2$, Thm. 3.1 asserts the existence of some $\delta>0$ for which conditions (3.3) imply that all solutions of problem (3.4) are Lipschitzian.

For each i, consider the following problem:

$$\min\{ \int_{a_i}^{b_i} L(t,y(t),\dot y(t))\, dt \ :\ y(a_i)=x(a_i),\ y(b_i)=x(b_i),\ |y(t)-x(\tau)|<R\ \forall t\in[a_i,b_i] \}.$$

For some sufficiently large value of i, conditions (3.6) and the continuity of x imply (3.3), so each solution of (P_i) is Lipschitzian. But the strong local minimality of x in (P) implies that the restriction of x to $[a_i,b_i]$ solves (P_i). Therefore x is Lipschitzian on this interval, whose relative interior contains τ.

The existence of the set Ω follows because (3.5) holds at every point $\tau \in [a,b]$ where x is differentiable, and such points form a set of full measure in $[a,b]$.

4. Existence and Regularity in the Large.

Although the intermediate existence theorem concerns (P_R), which involves the state constraint $|x|_\infty < R$, it can often be used to treat the unconstrained problem (P) as well. This is most evident when the solutions of (P) can be assigned an *a priori* bound in the supremum norm, for then the solution sets for (P) and (P_R) coincide for sufficiently large values of R. In this section we describe a very mild growth condition which generates such an *a priori* bound, and use it to prove new global regularity results for certain slow-growth problems.

Suppose there are nonnegative constants β and γ and a positive-valued function $\bar{\alpha}:[0,+\infty)\to\mathbb{R}$, for which the following holds for every $R>0$:

$$L(t,x,v) \geq \bar{\alpha}(R)|v| - \beta|x| - \gamma \qquad \forall(t,x,v)\in[a,b]\times R\bar{B}\times\mathbb{R}^n. \tag{4.1}$$

We may then estimate the objective value of any admissible arc x as follows: if $R=|x|_\infty$,

$$\begin{aligned}\Lambda(x) &\geq \bar{\alpha}(R)\int_a^b |\dot{x}(t)|\,dt \; - \; (\beta R+\gamma)(b-a)\\ &\geq \bar{\alpha}(R)\,(R-\min\{|x_a|, |x_b|\}) - (\beta R+\gamma)(b-a).\end{aligned} \tag{4.2}$$

Suppose now that $\bar{x}$ is an admissible arc for problem (P). If we assume that for some $R_0>0$,

$$R\geq R_0 \quad\Rightarrow\quad \bar{\alpha}(R)\left[R - \min\{|x_a|, |x_b|\}\right] - (\beta R+\gamma)(b-a) > \Lambda(\bar{x}), \tag{4.3}$$

then (4.2) gives, for all admissible arcs x,

$$\Lambda(x) \leq \Lambda(\bar{x}) \quad\Rightarrow\quad |x|_\infty < R_0. \tag{4.4}$$

Consequently the solutions of (P), if any, are to be found among the solutions of (P_{R_0}). Now the solution set for (P_{R_0}) remains unchanged if L is replaced by the Lagrangian $\tilde{L}=L+\beta R_0+\gamma$. This new Lagrangian obeys (H3) in the set $[a,b]\times R_0\bar{B}$, with $\bar{\alpha}=\bar{\alpha}(R_0)$ from (4.1). And (4.3) can be rearranged to give

$$\tilde{\Lambda}(\bar{x})/\bar{\alpha}(R_0) + \min\{|x_a|, |x_b|\} < R_0 : \tag{4.5}$$

this strict inequality is preserved when $\bar{\alpha}(R_0)$ is replaced by a sufficiently large value of α in

$(0,\bar{\alpha}(R_0))$, which confirms (H4). Only (H5) remains to check.

An extremely versatile replacement for (H5) is the **extremal growth condition (EGC)**, defined as follows:

$$\forall\ R>0,\ \ \forall\ r>0,\ \lim_{s\to\infty} \Delta_R(r,s) = +\infty. \tag{EGC}$$

It is evident that (EGC) implies (H5), from which we deduce the following result.

4.1 THEOREM. Let L satisfy the growth condition (4.1). Suppose there is an arc $\bar{x}$ admissible for (P) such that

$$\liminf_{R\to\infty} \Big[\ \bar{\alpha}(R)[R - \min\{|x_a|,|x_b|\}] - (\beta R+\gamma)(b-a)\Big] > \Lambda(\bar{x}). \tag{4.6}$$

If (EGC) holds, then the solution set for (P) is a nonempty collection of Lipschitz arcs.

Note that the inequality condition (4.6) certainly holds if $\lim_{R\to\infty} R[\bar{\alpha}(R)-\beta(b-a)] = +\infty$. This condition is much weaker than the coercivity hypothesis typically invoked in existence theory, which we will discuss in Section 5. Conditions (H1), (H2), and (4.1) are also very mild. Thus the applicability of Thm. 4.1 depends primarily on the number of situations in which (EGC) can be shown to hold. This number is rather large, as illustrated by the applications of Thm. 4.1 which constitute the remainder of the paper. The first of these concerns a class of slow-growth Lagrangians familiar from the theory of parametric problems.

4.2 PROPOSITION. Let $L(t,x,v) = \varphi(x)(1+|v|^2)^{1/2}$, where φ is a positive-valued locally Lipschitz function. Then (EGC) holds. In particular, if one has the inequality

$$\liminf_{R\to\infty} \bar{\alpha}(R)[R - \min\{|x_a|,|x_b|\}] > \Lambda(\bar{x}) \tag{4.7}$$

for some admissible arc $\bar{x}$, where

$$\bar{\alpha}(R) = \min\{\varphi(x) : |x|\leq R\}, \tag{4.8}$$

then the solution set of (P) is a nonempty subset of $C^1[a,b]$.

Proof. Note that (4.1) holds with $\bar{\alpha}$ as in (4.8) and $\beta=\gamma=0$. Also, any Lipschitzian solution of (P) will automatically be smooth because the Lagrangian is strictly convex in v. Therefore the desired conclusions will follow from Thm. 4.1 once we verify (EGC).

To do this, fix $R, r>0$ and let a Lipschitz arc x on $[t_0,t_1]$ satisfy the conditions of Def. 2.4 for some $s>r$. Conditions 2.4(c)(e) imply that there is a constant c for which

$$\varphi(x(t))^2 = c^2(1+|\dot{x}(t)|^2) \qquad \forall t\in[t_0,t_1]. \tag{4.9}$$

(This is the second Erdmann condition—see [1].) Now 2.4(a) implies

$$c^2 \geq \bar{\alpha}(R)/(1+r^2),$$

whence (4.9) implies

$$|\dot{x}(t)|^2 \leq \bar{A}(R)(1+r^2)/\bar{\alpha}(R)^2 - 1 \qquad \forall t\in[t_0,t_1],$$

where $\bar{A}(R) = \max\{\varphi(x) : |x|\leq R\}$. In particular, the existence of such an arc x forces $s^2=\rho(s)^2$ to be less than the right side of this inequality. In other words, for all sufficiently large values of s, no such arc x exists. That is, $\Delta_R(r,s)=+\infty$, which verifies (EGC) and completes the proof. ////

Proposition 4.2 is a generalization of [5, Prop. 7.3], in which the function φ was required to be uniformly bounded away from zero on $\mathbf{R}^n$. Theorem 4.1, on the other hand, is a special case of [5, Thm. 7.1].

Growth conditions on the Bernstein function can also be used to confirm (EGC) for problems with slow growth. Let us now suppose that L is C^2 and L_{vv} is everywhere positive definite. Then the *Bernstein function* is given by

$$F = L_{vv}^{-1}\left[L_x - L_{vt} - L_{vx}\,v\right]$$

(with all evaluations at (t,x,v)): it arises when the Euler equation for a C^2 extremal x is written in the form $\ddot{x}=F(t,x,\dot{x})$. Our next result concerns a new growth condition on F which implies global existence and regularity.

4.3 THEOREM. Let $L\in C^2$ obey growth condition (4.1) and $L_{vv}(t,x,v)>0$ $\forall(t,x,v)\in[a,b]\times\mathbf{R}^n\times\mathbf{R}^n$. Furthermore, assume that for every $R>0$, there exist a nonnegative function $m\in L^1[a,b]$ and a constant $c\geq 0$ for which

$$|F(t,x,v)| \leq (m(t) + c|L(t,x,v)|)(1+|v|) \qquad \forall(t,x,v)\in[a,b]\times R\bar{B}\times\mathbf{R}^n. \tag{4.10}$$

Then (EGC) holds. In particular, if there is an arc $\bar{x}$ admissible for (P) such that (4.5) holds, then the solution set for (P) is a nonempty subset of $C^1[a,b]$.

Proof. Let us first replace (4.10) by a growth restriction which appears more restrictive. For every $R>0$, (4.1) implies that

$$|L(t,x,v)| \leq L(t,x,v) + 2(\beta R+\gamma) \qquad \forall(t,x,v)\in[a,b]\times R\bar{B}\times\mathbf{R}^n.$$

Consequently (4.10) implies that

$$|F(t,x,v)| \leq (m(t) + 2c(\beta R+\gamma) + c\,L(t,x,v))(1+|v|) \qquad \forall(t,x,v)\in[a,b]\times R\bar{B}\times\mathbf{R}^n.$$

This has the same form as (4.10), except that L appears directly instead of in modulus. We are therefore free to assume the following instead of (4.10):

$$|F(t,x,v)| \leq (m(t) + c\,L(t,x,v))(1+|v|) \qquad \forall(t,x,v)\in[a,b]\times R\bar{B}\times\mathbf{R}^n. \tag{4.11}$$

(We may also add a constant to $m(t)$, if necessary, to ensure that the coefficient of $1+|v|$ is nonnegative throughout the region of interest.)

We now turn to (EGC). Fix any positive values of r and R. Recalling the constant M of Prop. 2.6, we define

$$k = \int_a^b m(t)\,dt + cM(b-a),$$

$$s = (r+k)e^k.$$

It suffices to prove that $\Delta_R(r,s)=+\infty$, since $\rho_R(s)=s>r$ and Prop. 2.5 applies. On the contrary, suppose $\Delta_R(r,s)<+\infty$. Then there must be some interval $[t_0,t_1]$ carrying an arc x satisfying

$$\dot{p}(t) = L_x(t,x(t),\dot{x}(t)) \quad \text{a.e. } [t_0,t_1],$$

$$p(t) = L_v(t,x(t),\dot{x}(t)) \quad \text{a.e. } [t_0,t_1].$$

The second of these relations, together with the strict convexity in v of our Lagrangian, implies that $\dot{x}$ is continuous on $[t_0,t_1]$. From the regularity theorem of Weierstrass [1, 2.6.iii, p. 60], it follows that x is in fact C^2, and the extremality conditions above can be rewritten as

$$\ddot{x}(t) = F(t,x(t),\dot{x}(t)) \quad \forall t\in[t_0,t_1].$$

According to (4.11), we have

$$|\ddot{x}(t)| \le g(t)\,(1+|\dot{x}(t)|) \quad \forall t\in[t_0,t_1],$$

where $g(t) = m(t) + c\,L(t,x(t),\dot{x}(t))$. Gronwall's inequality, together with the initial condition $|\dot{x}(\tau)|\le r$ from 2.4(a), implies that

$$|\dot{x}(t)| \le (r+\|g\|_1)\, e^{\|g\|_1} \quad \forall t\in[t_0,t_1]. \tag{4.12}$$

But we may compute

$$\|g\|_1 = \int_{t_0}^{t_1} \left[m(t) + c\,L(t,x(t),\dot{x}(t))\right] dt$$

$$\leq \int_a^b m(t)\,dt + c\,V_R(r,s) \;\leq\; k$$

by Prop. 2.6, whereupon (4.12) gives $|\dot{x}(t)| \leq s \;\; \forall t \in [t_0, t_1]$. This contradicts condition 2.4(b) and completes the proof.

A special case of (4.10) involves the following pair of growth conditions: there exist constants $k>0$ and $p \geq 1$ such that for every $R>0$, there are nonnegative quantities $c(R)$, $g(R)$, and $m \in L^1[a,b]$ obeying

$$L(t,x,v) \geq k|v|^p - g(R) \qquad \forall (t,x,v) \in [a,b] \times R\bar{B} \times \mathbb{R}^n, \tag{4.13}$$

$$|F(t,x,v)| \leq c(R)\,(m(t) + |v|^{1+p}) \qquad \forall (t,x,v) \in [a,b] \times R\bar{B} \times \mathbb{R}^n. \tag{4.14}$$

Clarke and Vinter [7] introduced these conditions (with $m(t)=1$) to generalize the original work of Bernstein. They showed that when L is coercive (see Section 5), (4.13) and (4.14) imply that all solutions to (P) are Lipschitz. To see that these conditions imply (4.10), fix $R>0$ and write

$$|F(t,x,v)| \leq c(R)\,(1+|v|)\,(m(t)+|v|^p)$$
$$\leq (c(R)/k)\,(1+|v|)\,(k\,m(t) + L(t,x,v) + g(R)).$$

The right-hand side has the form prescribed by (4.10), as required. Since it does not require coercivity and relies upon a weaker growth condition, Thm. 4.3 generalizes [7, Cor 3.4].

Here is a simple non-coercive example to which Thm. 4.3 applies. Let $\varphi: \mathbb{R}^n \to \mathbb{R}$ be positive-valued and of class C^2. Then (4.10) holds for $L(t,x,v) = \phi(x)(1+|v|^2)^{1/2}$, while (4.1) was confirmed in Prop. 4.2. (The Bernstein function in this case is $F(t,x,v) = \nabla\phi(x)(1+|v|^2)/\phi(x)$.)

5. Global Regularity for Coercive Lagrangians.

The Lagrangian $L:[a,b]\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ is called *coercive* if there are a non-negative, nondecreasing convex function $\theta:[0,+\infty)\to\mathbf{R}$ and non-negative constants β, γ for which

$$L(t,x,v) \geq \theta(|v|) - \beta|x| - \gamma \qquad \forall(t,x,v)\in[a,b]\times\mathbf{R}^n\times\mathbf{R}^n, \tag{5.1}$$

$$\lim_{r\to\infty} \theta(r)/r = +\infty. \tag{5.2}$$

Note that when L is coercive, (H2) reduces to the assumption—standard in existence theory—that L is convex in v. Also, given any convex function θ_0 satisfying (5.1) and (5.2), one can create a nondecreasing, non-negative convex function θ for which these conditions remain valid, at the possible expense of an increase in γ.

A coercive Lagrangian L certainly satisfies growth condition (4.1). Indeed, (5.2) implies that for any given $\bar{\alpha}>0$, there exists $\gamma_1\geq 0$ such that

$$\theta(r) + \gamma_1 \geq \bar{\alpha}r \qquad \forall r\geq 0.$$

Thus (5.1) implies that

$$L(t,x,v) \geq \bar{\alpha}|v| - \beta|x| - (\gamma+\gamma_1) \qquad \forall(t,x,v)\in[a,b]\times\mathbf{R}^n\times\mathbf{R}^n.$$

As we noted in Section 4, any choice of $\bar{\alpha}>\beta(b-a)$ in this relation will confirm both (4.1) and (4.5). Thus Thm. 4.1. takes the following form in the coercive case.

5.1 THEOREM. Let L be coercive. If (EGC) holds, then the solution set of (P) is a nonempty collection of Lipschitz arcs.

Theorem 5.1 is similar in spirit to the classical existence theorem of Tonelli. It relies upon convexity in v and coercivity, as does Tonelli's theorem, but contains the extra conclusion that all solutions are Lipschitzian. The example in [6] shows that this desirable assertion is available only under additional hypotheses, among which Thm. 5.1 establishes

(EGC). The verification of (EGC), in turn, can be based on the following result.

5.2 LEMMA [5]. Let L be coercive. Then for every $R>0$ and $K>0$, there exists a constant $M>0$ such that

$$|p|\leq K \quad\Rightarrow\quad \partial_p H(t,x,p) \subseteq M\bar{B} \qquad \forall (t,x)\in[a,b]\times R\bar{B}.$$

Lemma 5.2 is used in [5] to prove that any coercive Lagrangian L which is independent of t necessarily satisfies (EGC), and hence that any autonomous fast-growth problem has only Lipschitzian solutions.

We now consider a time-dependent version of (P) involving a coercive Lagrangian L. Thus we assume that the endpoints (a,x_a) and (b,x_b), together with the quantities β, γ, and θ of (5.1)-(5.2), are fixed. We say that L satisfies the *extended Morrey condition* if, for every $R>0$, there are nonnegative constants c_0, c_1, c_2, and a nonnegative function $m\in L^1[a,b]$ such that

$$|q| \leq (1+|p|)\Big[\, m(t) + c_0|L(t,x,v)| + c_1\max\{\theta(|v|), c_2|v|\}\,\Big] \tag{5.3}$$

whenever $q\in\partial_x L(t,x,v)$ and $p\in\partial_v L(t,x,v)$ for some point $(t,x,v)\in[a,b]\times R\bar{B}\times\mathbb{R}^n$.

5.3 THEOREM. If a coercive Lagrangian L satisfies the extended Morrey condition, then the set of solutions to (P) is nonempty, and consists entirely of Lipschitz arcs.

Proof. Observe that the extended Morrey condition (5.3) implies the following growth condition, which appears more restrictive. For every $R>0$, there exists a nonnegative $m\in L^1[a,b]$ and a constant $c\geq 0$ such that

$$|q| \leq (1+|p|)(m(t) + c\,L(t,x,v)) \tag{5.4}$$

whenever $q\in\partial_x L(t,x,v)$ and $p\in\partial_v L(t,x,v)$ for some $(t,x,v)\in[a,b]\times R\bar{B}\times\mathbb{R}^n$. Indeed, let $m_0(t)$, c_0, c_1, and c_2 be quantities for which (5.3) holds. Then choose $\gamma_1\geq 0$ so large that

$$\theta(r) + \gamma_1 \geq c_2 r \qquad \forall r\geq 0.$$

In the region $[a,b]\times R\bar{B}\times\mathbb{R}^n$, one has

$$|L(t,x,v)| \leq L(t,x,v) + 2(\beta R+\gamma),$$

$$\max\{\theta(|v|), c_2|v|\} \leq \theta(|v|)+\gamma_1 \leq L(t,x,v) + \beta R + \gamma + \gamma_1.$$

Using these estimates in (5.3) gives (5.4), with $c=c_0+c_1$ and $m(t)=m_0(t)+2c_0(\beta R+\gamma)+c_1(\beta R+\gamma+\gamma_1)$. We therefore proceed under condition (5.4). (We may add a constant to $m(t)$, if necessary, to ensure that the second factor in (5.4) is nonnegative throughout the region of interest.)

In view of Thm. 5.1, it suffices to verify (EGC). Thus, fix $r, R>0$. We will show that $\Delta_R(r,s)=+\infty$ whenever $s\geq S$ and $\rho_R(s)\geq M$, where S is given by Prop. 2.5 and M is defined as follows. Let M_0 be the constant given by Prop. 2.6; define

$$\sigma_0 = \max\{\,|p| : \ p\in\partial_v L(t,x,v), \ (t,x,v)\in[a,b]\times R\bar{B}\times r\bar{B}\,\},$$

and set

$$K_0 = \|m\|_1 + c\,M_0(b-a), \qquad K = \sigma_0(K_0+1)e^{K_0}.$$

The constant M appearing in our claim is the one for which (cf. Lemma 5.2)

$$|p|\leq K \quad\Rightarrow\quad \partial_p H(t,x,p)\subseteq M\bar{B} \qquad \forall (t,x)\in[a,b]\times R\bar{B}.$$

To verify that $s \geq S$ and $\rho_R(s) > M$ force $\Delta_R(r,s) = +\infty$, suppose the contrary. Then some $s \geq S$ with $\rho_R(s) > M$ must admit a Lipschitz arc x on some interval $[t_0,t_1]$ where conditions 2.4(a)–(e) hold. In particular, conditions 2.4(c)(e) imply that x solves a certain optimal control problem, for which the conclusions of the Maximum Principle [3, Thm. 5.2.1] imply the following. There exists an arc $p \in AC[t_0,t_1]$ for which

$$\dot{p}(t) \in \partial_x L(t,x(t),\dot{x}(t)) \quad \text{a.e. } [t_0,t_1], \tag{5.5}$$

$$p(t) \in \partial_v L(t,x(t),\dot{x}(t)) \quad \text{a.e. } [t_0,t_1]. \tag{5.6}$$

According to (5.4), it follows that

$$|\dot{p}(t)| \leq (1+|p(t)|)\,h(t) \qquad \text{a.e. } [t_0,t_1], \tag{5.7}$$

where $h(t) = m(t) + c\,L(t,x(t),\dot{x}(t))$. Now h is nonnegative and integrable, so

$$|h|_1 = \int_{t_0}^{t_1} h(t)\,dt \leq |m|_1 + c\,V_R(r,s) \leq K_0,$$

where K_0 was chosen above. Now by Gronwall's inequality and condition 2.4(a), inequality (5.7) gives

$$|p(t)| \leq |p(\tau)|(|h|_1+1)e^{|h|_1} \leq \sigma_0(K_0+1)e^{K_0} = K, \qquad t \in [t_0,t_1].$$

According to (5.6) and Lemma 5.2,

$$\dot{x}(t) \in \partial_p H(t,x(t),p(t)) \subseteq M\bar{B} \qquad \text{a.e. } [t_0,t_1],$$

and this contradicts condition 2.4(b). This contradiction arises from our assumption that $\Delta_R(r,s) < +\infty$, which must therefore be false. The proof is complete.

Theorem 5.3 compares favourably with other regularity results based on conditions of Morrey type. Although it requires Lipschitzian t-dependence in place of the measurable t-dependence of [7], [4], the growth condition (5.3) is more general than that appearing in either of these references. In the smooth case, inequality (5.3) reduces to

$$|L_x| \leq (1+|L_v|)\Big[m(t) + c_0|L| + c_1\max\{\theta(|v|), c_2|v|\}\Big] \qquad \forall (t,x,v) \in [a,b]\times R\bar{B}\times \mathbf{R}^n,$$

whereas [7, Cor. 3.3] requires

$$|L_x| \le m(t) + c_0|L| + c_1|L_v|.$$

Condition (5.3) is clearly the more widely applicable of these two.

In [4], Clarke supposes that L is measurable in t, smooth in (x,v), and strictly convex in v, while satisfying

$$|L_x| \le (1+|L_v|)(m(t) + d_1|v|) + d_2|L|, \tag{5.8}$$

$$e_1|v|^{1+r} + e_2 \le L(t,x,v) \le e_3|v|^{1+s} + e_4, \tag{5.9}$$

where $m \in L^1$ and all constants are positive. Now (5.8) certainly implies (5.3), while only the first inequality in (5.9) is needed to make L coercive. Thus Thm. 5.3 generalizes the regularity result of [4, p. 399]. Besides allowing less restrictive growth conditions than (5.8)–(5.9), Thm. 5.3 allows L to be locally Lipschitz instead of smooth in (x,v), and makes no assumption of strict convexity in v.

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