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LIMIT PROBLEMS IN OPTIMAL CONTROL THEORY

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1. INTRODUCTION

In this paper we deal with sequences of optimal control problems of the form

$$(P_h) \quad \min \left\{ \int_0^1 f_h(t, y, u) dt : y' = g_h(t, y, u), y(0) = y_h^0 \right\}$$

where the state variable y belong to the Sobolev space $W^{1,1}(0,1;\mathbf{R}^n)$ and the control variable u is in $L^1(0,1;\mathbf{R}^m)$. We are interested in the asymptotic behaviour (as $h \rightarrow +\infty$) of the optimal pairs (u_h, y_h) of (P_h) ; more precisely, we shall construct a new problem (P_∞) such that

if (u_h, y_h) is an optimal pair of (P_h) and if (u_h, y_h) tends to (u_∞, y_∞) in the topology $wL^1(0,1;\mathbf{R}^m) \times L^\infty(0,1;\mathbf{R}^n)$, then (u_∞, y_∞) is an optimal pair for (P_∞) .

The basic tool for treating the asymptotic problem above is the Γ -convergence theory which has been already used very fruitfully for many variational problems (see for instance [1],[2],[4],[5],[8],[9])

Here we use a more sophisticated version of the usual Γ -limits, because we shall consider our problems (P_h) as minimization problems on the product space $U \times Y$ (U is the space of controls and Y the space of states) for the functionals

$$F_h(u, y) = \begin{cases} \int_0^1 f_h(t, y, u) dt & \text{if } y' = g_h(t, y, u), y(0) = y_h^0 \\ +\infty & \text{otherwise,} \end{cases}$$

and the spaces U and Y will play a different role with respect to Γ -convergence.

In Section 2 we develop the abstract theory we shall need in the following; in Section 3 we show the applications to problems (P_h) above, and we give an example showing that in some situations the domain of problem (P_∞) is not given by a state equation $y' = g_\infty(t, y, u)$ but coincides with the entire product space $U \times Y$.

2. THE ABSTRACT FRAMEWORK

Let us denote by U and Y two topological spaces and let $F_h: U \times Y \rightarrow \overline{\mathbf{R}}$ be a sequence of functions; by $Z(+)$ we shall denote the "sup" operator and by $Z(-)$ the "inf" operator. For every $u \in U$ and $y \in Y$ we define

$$\Gamma_{\text{seq}}(N^\alpha, U^\beta, Y^\gamma) \lim_h F_h(u, y) = \bigwedge_{(u_h) \in S(u)} \bigvee_{(y_h) \in S(y)} \bigwedge_{k \in N} \bigvee_{h \geq k} F_h(u_h, y_h)$$

where α, β, γ are the signs $+$ or $-$, and $S(u)$ and $S(y)$ respectively denote the set of all sequences $u_h \rightarrow u$ in U and $y_h \rightarrow y$ in Y . For example we have

$$\Gamma_{\text{seq}}(N^+, U^-, Y^+) \lim_h F_h(u, y) = \inf_{u_h \rightarrow u} \sup_{y_h \rightarrow y} \limsup_{h \rightarrow \infty} F_h(u_h, y_h).$$

When a Γ -limit is independent of the sign $+$ or $-$ associated to one of the spaces N, U, Y this sign will be omitted. For example, if

$$\Gamma_{\text{seq}}(N^+, U^-, Y^+) \lim_h F_h(u, y) = \Gamma_{\text{seq}}(N^+, U^+, Y^+) \lim_h F_h(u, y),$$

then their common value will be indicated by $\Gamma_{\text{seq}}(N^+, U, Y^+) \lim_h F_h(u, y)$.

The following propositions are proved in [4].

PROPOSITION 2.1. *Let (u_h, y_h) be a minimum point for F_h , or simply a pair such that*

$$\lim_h F_h(u_h, y_h) = \lim_h \left[\inf_{U \times Y} F_h \right].$$

Assume that (u_h, y_h) converges to (u_∞, y_∞) in $U \times Y$ and that there exist

$$F_\infty = \Gamma_{\text{seq}}(N, U^-, Y^-) \lim_h F_h.$$

Then we have

- (i) (u_∞, y_∞) is a minimum point for F_∞ on $U \times Y$;
- (ii) $\lim_h \left[\inf_{U \times Y} F_h \right] = \min_{U \times Y} F_\infty$.

PROPOSITION 2.2. *Let $\{F_h\}$ and $\{G_h\}$ be two sequences of function from $U \times Y$ into $[0, +\infty]$,*

and let $(u, y) \in U \times Y$. Assume there exist

$$\Gamma_{\text{seq}}(N, U^-, Y) \lim_h F_h(u, y) \quad \text{and} \quad \Gamma_{\text{seq}}(N, U, Y^-) \lim_h G_h(u, y).$$

Then we have

$$\Gamma_{\text{seq}}(N, U^-, Y^-) \lim_h [F_h + G_h](u, y) = \Gamma_{\text{seq}}(N, U^-, Y) \lim_h F_h(u, y) + \Gamma_{\text{seq}}(N, U, Y^-) \lim_h G_h(u, y).$$

In many applications, the introduction of a new auxiliary variable can be helpful; the following proposition shows the behaviour of Γ -limits with respect to this operation.

PROPOSITION 2.3. *Let $F_h: U \times Y \rightarrow \bar{\mathbb{R}}$ be a sequence of functions, let V be another topological space, and let $\Xi_h: U \times Y \rightarrow \wp(V)$ be a sequence of multimappings. Assume that the following compactness condition is satisfied:*

for every converging sequence (u_h, y_h) with $F_h(u_h, y_h)$ bounded, there exist a sequence $v_h \in \Xi_h(u_h, y_h)$ relatively compact in V .

Then setting

$$\Phi_h(u,v,y) = \begin{cases} F_h(u,y) & \text{if } v \in \Xi_h(u,y) \\ +\infty & \text{otherwise,} \end{cases}$$

we have for every $(u,y) \in U \times Y$

$$\begin{aligned} & \inf \left\{ \Gamma_{\text{seq}}(N^-, (U \times V)^-, Y^-) \lim_h \Phi_h(u,v,y) : v \in V \right\} \leq \\ & \leq \Gamma_{\text{seq}}(N^-, U^-, Y^-) \lim_h F_h(u,y) \leq \Gamma_{\text{seq}}(N^+, U^-, Y^-) \lim_h F_h(u,y) \leq \\ & \leq \inf \left\{ \Gamma_{\text{seq}}(N^+, (U \times V)^-, Y^-) \lim_h \Phi_h(u,v,y) : v \in V \right\} . \end{aligned}$$

Therefore, if for every $(u,v,y) \in U \times V \times Y$ there exists

$$\Gamma_{\text{seq}}(N, (U \times V)^-, Y^-) \lim_h \Phi_h(u,v,y) ,$$

we have

$$\Gamma_{\text{seq}}(N, U^-, Y^-) \lim_h F_h(u,y) = \inf \left\{ \Gamma_{\text{seq}}(N, (U \times V)^-, Y^-) \lim_h \Phi_h(u,v,y) : v \in V \right\} .$$

Proof. It is enough to repeat, with just some slight modifications, the proof of Proposition 2.4 of [3]. ■

In the following, if A is a set we denote by χ_A the function

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise.} \end{cases}$$

3. APPLICATIONS TO CONTROL PROBLEMS

Let k, m, n be positive integers and let $p > 1$. The space Y of states we consider is the Sobolev space $W^{1,1}(0,1; \mathbb{R}^n)$ endowed with the $L^\infty(0,1; \mathbb{R}^n)$ topology, and the space U of

controls is the space $L^p(0,1;\mathbf{R}^n)$ endowed with its weak topology (weak* if $p=+\infty$). The cost functions are of the form

$$(3.1) \quad J_h(u,y) = \int_0^1 f_h(t,y,u) dt$$

where $f_h: [0,1] \times \mathbf{R}^n \times \mathbf{R}^m \rightarrow [0,+\infty]$ are Borel functions. Finally, the state equations are

$$(3.2) \quad \begin{cases} y' \in a_h(t,y) + B_h(t,y) b_h(t,u) \\ y(0) = y_h^0 \end{cases}$$

where $a_h: [0,1] \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ and $B_h: [0,1] \times \mathbf{R}^n \rightarrow \mathbf{R}^{nk}$ are Borel functions, and the multimappings $b_h: [0,1] \times \mathbf{R}^m \rightarrow \wp(\mathbf{R}^{nk})$ are Borel measurable (i.e. the sets $\{(t,u,v) \in [0,1] \times \mathbf{R}^n \times \mathbf{R}^k : v \in b_h(t,u)\}$ are Borel sets).

Then the control problems we are concerned are

$$(P_h) \quad \inf \{ J_h(u,y) : (u,y) \in \Lambda_h \}$$

or equivalently

$$(P_h) \quad \inf \{ F_h(u,y) : (u,y) \in U \times Y \}$$

where

$$(3.3) \quad \Lambda_h = \{ (u,y) \in U \times Y : y' \in a_h(t,y) + B_h(t,y) b_h(t,u), y(0) = y_h^0 \}$$

$$(3.4) \quad F_h = J_h + \chi_{\Lambda_h}.$$

We introduce now an auxiliary variable $v \in L^q(0,1;\mathbf{R}^k)$ with $q>1$ and define a new sequence of functionals by setting

$$(3.5) \quad \Phi_h(u,v,y) = \begin{cases} F_h(u,y) & \text{if } v \in b_h(t,u) \\ +\infty & \text{otherwise.} \end{cases}$$

In this way the problems (P_h) take the form

$$\inf \left\{ \int_0^1 \left[f_h(t,y,u) + \chi_{\{v \in b_h(t,u)\}} \right] dt : y' = a_h(t,y) + B_h(t,y) v, y(0) = y_h^0 \right\}.$$

In order to apply the abstract theory presented in Section 2 (more precisely Proposition 2.1), we have to calculate the $\Gamma_{\text{seq}}(N, U^-, Y^-)$ limit of the sequence F_h . To do this, we make

some hypotheses on f_h, a_h, B_h, b_h .

(3.6) For every $t \in [0,1], r \geq 0, y \in \mathbb{R}^n$ with $|y| \leq r$ we have (if $1/q + 1/q' = 1$)

$$\begin{aligned} |a_h(t, y)| &\leq M_h(t, r) & \text{with} & \quad \|M_h(\cdot, r)\|_{L^1(0,1)} \leq M(r) < +\infty \\ |B_h(t, y)| &\leq N_h(t, r) & \text{with} & \quad \|N_h(\cdot, r)\|_{L^{q'}(0,1)} \leq N(r) < +\infty. \end{aligned}$$

(3.7) For every $t \in [0,1], r \geq 0, y_1, y_2 \in \mathbb{R}^n$ with $|y_1|, |y_2| \leq r$ we have

$$\begin{aligned} |a_h(t, y_1) - a_h(t, y_2)| &\leq \alpha_h(t, r) |y_1 - y_2| & \text{with} & \quad \|\alpha_h(\cdot, r)\|_{L^1(0,1)} \leq \alpha(r) < +\infty \\ |B_h(t, y_1) - B_h(t, y_2)| &\leq \beta_h(t, r) |y_1 - y_2| & \text{with} & \quad \|\beta_h(\cdot, r)\|_{L^{q'}(0,1)} \leq \beta(r) < +\infty. \end{aligned}$$

(3.8) There exist $\lambda > 0$ and $a \in L^1(0,1)$ such that

$$\lambda(|u|^p + |v|^q) - a(t) \leq f_h(t, 0, u) + \chi_{\{v \in b_h(t, u)\}}$$

for every $t \in [0,1], u \in \mathbb{R}^m, v \in \mathbb{R}^k$. When $p = +\infty$ or $q = +\infty$ the quantities $|u|^p$ and $|v|^q$ in the left-hand side have to be substituted by $\chi_{\{u \in H\}}$ and $\chi_{\{v \in K\}}$ respectively, where $H \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^k$ are bounded sets.

(3.9) For every $t \in [0,1], r \geq 0, u \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}^n$ with $|y_1|, |y_2| \leq r$ we have

$$f_h(t, y_1, u) \leq f_h(t, y_2, u) + \rho_r(t, |y_1 - y_2|) + \sigma_r(t, |y_1 - y_2|) |f_h(t, y_2, u)|^{(\alpha-1)/\alpha}$$

for a suitable $\alpha \in [1, +\infty]$ and functions $\rho_r(t, s), \sigma_r(t, s)$ from $[0,1] \times [0, +\infty[$ into $[0, +\infty[$ measurable in t , increasing and continuous in s , with $\rho_r(t, 0) = \sigma_r(t, 0) = 0$, and such that $z \rightarrow \rho_r(t, |z(t)|), z \rightarrow \sigma_r(t, |z(t)|)$ are continuous operators from Y into $L^1(0,1), L^\alpha(0,1)$ respectively.

(3.10) There exist $u_h \in L^p(0,1; \mathbb{R}^m)$ and $v_h \in L^q(0,1; \mathbb{R}^k)$ such that $v_h(t) \in b_h(t, u_h(t))$ for a.e. $t \in (0,1)$, and the sequence $f_h(t, 0, u_h(t))$ is weakly compact in $L^1(0,1)$.

LEMMA 3.1. *Under the previous assumptions, the following compactness condition is satisfied: for every converging sequence (u_h, y_h) with $F_h(u_h, y_h)$ bounded, there exists a sequence $\{v_h\}$ relatively compact in V such that for a.e. $t \in [0,1]$*

$$v_h(t) \in b_h(t, u_h(t)) \quad \text{and} \quad y'_h(t) = a_h(t, y_h(t)) + B_h(t, y_h(t)) v_h(t).$$

Proof. Let (u_h, y_h) be converging in $U \times Y$ with $F_h(u_h, y_h)$ bounded; then we have $(u_h, y_h) \in \Lambda_h$, so that we can find measurable functions $v_h(t)$ with

$$v_h(t) \in b_h(t, u_h(t)) \quad \text{and} \quad y'_h(t) = a_h(t, y_h(t)) + B_h(t, y_h(t)) v_h(t)$$

for a.e. $t \in [0, 1]$. It remains to prove that the sequence v_h is bounded in $L^q(0, 1; \mathbb{R}^k)$. Since y_h is uniformly bounded, by (3.8) and (3.9) we have for a suitable $r > 0$

$$\begin{aligned} \lambda(|u_h|^p + |v_h|^q) - a(t) &\leq f_h(t, 0, u_h) \leq \\ &\leq f_h(t, y_h, u_h) + \rho_r(t, r) + \sigma_r(t, r) \|f_h(t, y_h, u_h)\|^{(\alpha-1)/\alpha} \leq \\ &\leq c f_h(t, y_h, u_h) + \gamma(t) \end{aligned}$$

where $c > 0$ is a constant and $\gamma \in L^1(0, 1)$. Then, from the boundedness of $J_h(u_h, y_h)$ we get that v_h is bounded in $L^q(0, 1; \mathbb{R}^k)$. ■

By Lemma 3.1, Proposition 2.3 applies, so that we have reduced our problem to the characterization of the $\Gamma_{\text{seq}}(\mathbf{N}, (U \times Y)^-, Y^-)$ limit of the sequence $\Phi_h(u, v, y)$ defined in (3.5). Set now

$$\begin{aligned} \bar{f}_h(t, y, u, v) &= f_h(t, y, u) + \chi_{\{v \in b_h(t, u)\}} \\ \bar{J}_h(u, v, y) &= \int_0^1 \bar{f}_h(t, y, u, v) dt \\ \bar{\Lambda}_h &= \left\{ (u, v, y) \in U \times V \times Y : y' = a_h(t, y) + B_h(t, y) v \text{ a.e. on } [0, 1], y(0) = y_h^0 \right\}. \end{aligned}$$

Therefore

$$\Phi_h = \bar{J}_h + \chi_{\bar{\Lambda}_h}^-$$

and, by Proposition 2.2 we may split the $\Gamma_{\text{seq}}(\mathbf{N}, (U \times Y)^-, Y^-)$ limit of Φ_h into the sum

$$(3.11) \quad \Gamma_{\text{seq}}(\mathbf{N}, (U \times V)^-, Y) \lim_h \bar{J}_h + \Gamma_{\text{seq}}(\mathbf{N}, U \times V, Y^-) \lim_h \chi_{\bar{\Lambda}_h}^-.$$

The two terms in the sum above can be computed by using Lemma 3.1 and Theorem 3.4 of [4].

More precisely we have

PROPOSITION 3.2. Assume that (3.8),(3.9),(3.10) hold and that for every $y \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^k$

$$(3.12) \quad \bar{f}_h^*(\cdot, y, \xi, \eta) \rightarrow \varphi(\cdot, y, \xi, \eta) \quad \text{weakly in } L^1(0,1)$$

where $\bar{f}_h^*$ are the polar functions of $\bar{f}_h$ defined by

$$\bar{f}_h^*(t, y, \xi, \eta) = \sup \left\{ \xi u + \eta v - \bar{f}_h(t, y, u, v) : u \in \mathbb{R}^m, v \in \mathbb{R}^k \right\}.$$

Then, for every $(u, v, y) \in U \times V \times Y$ we have

$$\Gamma_{\text{seq}}(N, (U \times V)^-, Y) \lim_h \bar{J}_h(u, v, y) = \int_0^1 \varphi^*(t, y, u, v) dt$$

where φ^* is the polar function of φ .

PROPOSITION 3.3. Assume that (3.6),(3.7) hold and that

$$(3.13) \quad \text{for every } y \in \mathbb{R}^n \quad a_h(\cdot, y) \rightarrow a(\cdot, y) \quad \text{weakly in } L^1(0,1; \mathbb{R}^n);$$

$$(3.14) \quad \text{for every } y \in \mathbb{R}^n \quad B_h(\cdot, y) \rightarrow B(\cdot, y) \quad \text{strongly in } L^q(0,1; \mathbb{R}^{nk});$$

$$(3.15) \quad y_h^0 \rightarrow y^0 \quad \text{in } \mathbb{R}^n.$$

Then we have

$$\Gamma_{\text{seq}}(N, U \times V, Y^-) \lim_h \chi_{\Lambda_h}^- = \chi_{\bar{\Lambda}}^-$$

where

$$\bar{\Lambda} = \left\{ (u, v, y) \in U \times V \times Y : y' = a(t, y) + B(t, y) v, y(0) = y^0 \right\}.$$

Finally, we are in a position to compute the $\Gamma_{\text{seq}}(N, U^-, Y^-)$ limit of F_h . In fact, by Propositions 2.3, 3.2, and 3.3, and by (3.11) we get for every $(u, y) \in U \times Y$

$$\begin{aligned} \Gamma_{\text{seq}}(N, U^-, Y^-) \lim_h F_h(u, y) &= \\ &= \inf \left\{ \int_0^1 \varphi^*(t, y, u, v) dt : y' = a(t, y) + B(t, y) v, y(0) = y^0 \right\} = \end{aligned}$$

$$= \int_0^1 f(t, y, u, y') dt + \chi_{\{y(0)=y^0\}}$$

where the function f is defined by

$$f(t, y, u, w) = \inf \{j(t, y, u, v) : w = a(t, y) + B(t, y) v\}.$$

We conclude with an example showing that in general the domain of the limit functional

$$F(u, y) = \int_0^1 f(t, y, u, y') dt + \chi_{\{y(0)=y^0\}}$$

is not given by a differential equation of the form $y'=g(t, y, u)$ but may be the whole space $U \times Y$.

EXAMPLE 3.4. Consider the sequence of optimal control problems

$$(P_h) \quad \min \left\{ \int_0^1 [u^2 + |y - y_0(t)|^2] dt : y' = a_h(t) y + b_h(t) u, y(0) = \xi \right\}$$

where u varies in $U = L^2(0, 1)$, y varies in $Y = W^{1,1}(0, 1)$, and $y_0 \in L^2(0, 1)$, $\xi \in \mathbf{R}$ are given.

About the functions a_h and b_h we assume that

$$\begin{cases} a_h \rightarrow a & \text{weakly in } L^1(0, 1) \\ b_h \rightarrow b & \text{weakly}^* \text{ in } L^\infty(0, 1) \\ b_h^2 \rightarrow \beta^2 & \text{weakly}^* \text{ in } L^\infty(0, 1). \end{cases}$$

It is not difficult to check that all hypotheses (3.6),..., (3.10) and (3.12),..., (3.15) are satisfied,

and after some standard calculations we find that the limit problem (P_∞) has the form

$$(P_\infty) \quad \min \left\{ \int_0^1 \left[u^2 + |y - y_0(t)|^2 + \frac{|y' - a(t)y - b(t)u|^2}{\beta^2(t) - b^2(t)} \right] dt : y(0) = \xi \right\}.$$

Note that it is $\beta^2(t) \geq b^2(t)$ for a.e. $t \in [0, 1]$, and

$$\beta^2 = b^2 \text{ a.e. on } [0, 1] \Leftrightarrow b_h \rightarrow b \text{ a.e. on } [0, 1].$$

In this last situation, problem (P_∞) takes the usual form

$$(P_\infty) \quad \min \left\{ \int_0^1 [u^2 + |y - y_0(t)|^2] dt : y' = a(t)y + b(t)u, y(0) = \xi \right\},$$

but this does not arrive in the general case. Take for instance

$$b_h(t) = \sin(ht)$$

and we get $b \equiv 0$ and $\beta^2 \equiv 1/2$, so that the limit problem is

$$\min \left\{ \int_0^1 [u^2 + |y - y_0(t)|^2 + 2|y' - a(t)y|^2] dt : y(0) = \xi \right\}.$$

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