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On the local solution of the tangential Cauchy-Riemann equations (*)

by

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ABSTRACT. — We study the solution operators P and homotopy formula introduced by G. M. Henkin for the tangential Cauchy-Riemann complex of a suitable small domain D on a strictly pseudoconvex real hypersurface in complex n-space. The main difficulties stem from the fact that P is an integral operator with a rather complicated kernel. For $U \subset\subset D$, we derive a C^k-norm estimate of the form $\| P\varphi \|_{U,k} \leqq K \| \varphi \|_{D,k}$, where the constant K blows up as U increases to D. We obtain careful control of the rate of this blow-up and of the dependence of K on the derivatives of the function defining the real hypersurface. Our estimates are sufficient for application to the local CR embedding problem.

RÉSUMÉ. — Nous étudions les opérateurs intégraux P dans la formule d'homotopie de G. M. Henkin pour le complexe tangentiel de Cauchy-Riemann sur un petit domaine d'une hypersurface réelle strictement pseudoconvexe dans l'espace $\mathbb{C}^n$. Avec les C^k-normes pour les domaines $U \subset\subset D$ nous dérivons une borne, $| P\varphi |_{U,k} \leqq K | \varphi |_{D,k}$, dans laquelle le constant K tend vers $+\infty$ lorsque U tend vers D. Nous constatons cette croissance de K et la dépendance de K sur les dérivées de la fonction qui définit l'hypersurface.

Mots clés : Hypersurface réelle, complexe tangentiel, noyau intégral.

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INTRODUCTION

This paper is concerned with the $\bar{\partial}_b$, or tangential Cauchy-Riemann, complex on a small portion M_ρ of a strictly pseudo-convex real hypersurface M^{2n-1} in complex space $\mathbb{C}^n$. Under suitable restrictions on M_ρ, there exist solution operators P and Q satisfying the homotopy formula

$$\varphi = \bar{\partial}_b P\varphi + Q\bar{\partial}_b \varphi, \tag{0.1}$$

for $(0, s)$-forms φ, $1 \leqq s \leqq n-3$, restricted to M_ρ. We shall study the regularity properties of certain of these operators.

Various aspects of the equation (0.1) have been studied by a number of people since the early works of H. Lewy [6], Kohn-Rossi [5], and Andreotti-Hill [1]. We should mention the works of Henkin [4], Romanov [8], and Skoda [9], in particular. We shall work with the explicit operators constructed by Henkin in [4], in the formulation given by Harvey and Polking [3]. As shown in [4] (0.1) holds on the compact manifold-with-boundary

$$M_\rho = \{ z \in M : r^0(z) < \rho \}, \tag{0.2}$$

where r^0 is a (suitable) pluriharmonic function. In addition to the results of [4] and [3], the higher differentiability properties of similar such P and Q were studied by Boggess [2].

For M as above of differentiability class C^l, we take M_ρ as in (0.2) with r^0 a real function of one of the holomorphic coordinates, both suitably chosen. Our results yield estimates of the form

$$\| P\varphi \|_{\rho(1-\sigma), k} \leqq c_k \delta^{-s} \| \varphi \|_{\rho, k}. \tag{0.3}$$

Here, $0 < \sigma < 1$, $\delta = \operatorname{dist}(\partial M_{\rho(1-\sigma)}, \partial M_\rho)$, $0 \leqq k \leqq l-3$, $s = s(k) > 0$, and $\| \varphi \|_{\rho, k}$ is the usual sup norm, taken over M_ρ, of the derivatives up to order k of the coefficients of the form φ. The same estimate holds for Q. A much more precise result is stated in theorem (4.1) below.

The formula (0.1) and the estimates (0.3) for $s=1$ form a major element of our proof [11] of the local embedding theorem for formally integrable, strictly pseudoconvex CR structures of dimension $2n-1$. The restriction $1 \leqq s \leqq n-3$ limits it to $2n-1 \geqq 7$. To be sure there is a "weak" homotopy formula (0.1) for the case $1 \leqq s = n-2$, as we shall indicate below. However, in this degree the operator Q inherits an additional term for which we have no estimate. The argument of [11] is based on the methods of Nash and Moser, with (0.1), (0.3) being used in solving the "linearized problem".

Hopefully, our estimates in theorem (4.1) will eventually be improved. This would probably decrease the derivative loss in the main result of [11]. For $k=0$, Henkin [4] has obtained (0.3) with $s=0$. For $k>0$, it seems difficult to avoid $s>0$. Major difficulties stem from the boundary integrals occuring in P (and Q). Estimates similar to (0.3) for the $\bar\partial$-complex were used in [10] to give a proof of the sharp form of the Newlander-Nirenberg theorem. The paper [10] may serve as a useful introduction to the methods of the present work and of [11].

In section 1 we recall the construction of Henkin's $\bar\partial_b$-homotopy formula. We take the first derivatives of $P\varphi$ in sections 2 and 3, and estimate the higher derivatives in section 4.

1. THE HENKIN $\bar\partial_b$-HOMOTOPY FORMULA

We begin by sketching the particular results needed from [4], making use of the exterior calculus developed in [3]. Let $w \in \mathbb{C}^n$, $\xi \in \mathbb{C}^m$, and $w = g(\xi)$ be a sufficiently smooth map. Using a dot product notation, we define a (1,0)-form

$$\omega = \omega^g = \frac{g \,.\, dw}{g \,.\, w}, \qquad g \,.\, dw = \sum_{j=1}^{n} g_j \, dw_j, \qquad \text{etc.,} \tag{1.1}$$

on the set of $(\xi, w) \in \mathbb{C}^m \times \mathbb{C}^n$ for which $g \,.\, w \neq 0$. This may be considered a generalization of the Cauchy kernel, since

$$\omega^g = \frac{dw_n}{w_n}, \qquad \text{if} \quad g_\alpha = 0, \quad 1 \leqq \alpha \leqq n-1, \quad g_n \neq 0. \tag{1.2}$$

Given l such maps, $g^{(j)}$, $1 \leqq j \leqq l$, from $\mathbb{C}^m$ to $\mathbb{C}^n$ and the corresponding $(1,0)$-forms ω^j, we define, on the set where all denominators are non-zero, the $(n, n-l)$-form

$$\Omega^{1 \cdots l} = \omega^1 \wedge \ldots \wedge \omega^l \wedge \sum (\bar\partial \omega^1)^{\alpha_1} \wedge \ldots \wedge (\bar\partial \omega^l)^{\alpha_l}. \tag{1.3}$$

Here the sum is over all l-tuples of non-negative integers $(\alpha_1, \ldots, \alpha_l)$ for which $\alpha_1 + \ldots + \alpha_l = n - l$.

We introduce the vector field v,

$$v = w \,.\, \partial_w = \sum_{j=1}^{n} w_j \frac{\partial}{\partial w_j}, \tag{1.4}$$

and the interior product ι_v. One readily verifies

$$\iota_v \, \omega^g = 1, \qquad \iota_v (\bar\partial \omega^g) = 0. \tag{1.5}$$

If $\beta_1 + \ldots + \beta_l = n - l + 1$, $\beta_i \geqq 0$, then

$$0 = \omega^1 \wedge \ldots \wedge \omega^l \wedge (\bar\partial \omega^1)^{\beta_1} \wedge \ldots \wedge (\bar\partial \omega^l)^{\beta_l}, \tag{1.6}$$

since each term has a wedge product of $n+1$ of the differentials dw_j, $1 \leqq j \leqq n$. If we take the interior product of equation (1.6) with (1.4) and use (1.5), we get

$$0=\sum_{j=1}^{l}(-1)^{j+1}\omega^1\wedge\ldots\wedge\hat{\omega}^j\wedge\ldots\wedge\omega^l\wedge(\bar{\partial}\omega^1)^{\beta_1}\wedge\ldots\wedge(\bar{\partial}\omega^l)^{\beta_l}. \quad (1.7)$$

This formula is used to derive the generalized Koppelman lemma,

$$\bar{\partial}\Omega^{1\ldots l}=\sum_{j=1}^{l}(-1)^j\Omega^{1\ldots\hat{j}\ldots l}. \quad (1.8)$$

For this we write

$$\begin{aligned}\bar{\partial}\Omega^{1\ldots l}&=\sum_{j=1}^{l}(-1)^{j+1}\omega^1\wedge\ldots\wedge\bar{\partial}\omega^j\wedge\ldots\wedge\omega^l\wedge\Sigma\\&=\sum_{j=1}^{l}(-1)^{j+1}\omega^1\wedge\ldots\wedge\hat{\omega}^j\wedge\ldots\wedge\omega^l\wedge\{\Sigma_{(j,1)}+\Sigma_{(j,0)})\\&\quad+\sum_{j=1}^{l}(-1)^j\omega^1\wedge\ldots\wedge\hat{\omega}^j\wedge\ldots\wedge\omega^l\wedge\Sigma_{(j,0)}.\end{aligned}$$

Here $\sum$ denotes the sum in (1.3), $\Sigma_{(j,1)}$ denotes the similar sum with $\alpha_1+\ldots+\alpha_l=n-l+1$, $\alpha_j\geqq 1$, and $\Sigma_{(j,0)}$ the sum with $\alpha_1+\ldots+\alpha_l=n-l+1$, $\alpha_j=0$. The expression $\Sigma_{(j,1)}+\Sigma_{(j,0)}$ is independent of j, so the first alternating sum vanishes by (1.7). The second alternating sum is precisely the right hand side of (1.8). Only the cases

$$\bar{\partial}\Omega^1=0, \quad (1.9)$$

$$\bar{\partial}\Omega^{12}=\Omega^1-\Omega^2, \quad (1.10)$$

$$\bar{\partial}\Omega^{123}=-\Omega^{23}+\Omega^{13}-\Omega^{12}, \quad (1.11)$$

are used for our construction. For this we take $\xi=(\zeta,z)$, ζ, $z\in\mathbb{C}^n$, and make the substitution $w=\zeta-z$. Decomposition according to z-type gives

$$\Omega^{1\ldots l}(\zeta,z)=\sum_{i=0}^{n}\sum_{s=0}^{n-l}\Omega^{1\ldots l}_{i,s}(\zeta,z), \quad (1.12)$$

where the subscript (i,s) indicates that the "double" form is of type (i,s) in z and type $(n-i,\,n-l-s)$ in ζ.

We shall work with a real hypersurface M which is a graph over the (z_α,x_n)-coordinate hyperplane $y_n=0$, $z_n=x_n+iy_n$. We assume that the defining function r is at least three times continuously differentiable, and

$$\begin{aligned}&M: r=0, \qquad R=O(|z|^3),\\&r(z)=-y_n+\sum_{\alpha,\beta=1}^{n-1}g_{\alpha\bar{\beta}}z_\alpha\bar{z}_\beta+R(z_\alpha,x_n).\end{aligned} \quad (1.13)$$

Here, the hermitian matrix $g_{\alpha\bar{\beta}}$, the Levi form of M at 0, is assumed to be a small perturbation of the identity matrix $\delta_{\alpha\bar{\beta}}$. We define

$$M_\rho = \{ z \in M : r^0(z_n) < \rho \}, \tag{1.14}$$

where r^0 is a sufficiently smooth real valued function of the last holomorphic coordinate z_n only. This is a slight departure from [4], where r^0 is assumed to be pluri-harmonic. In either case a most natural choice would be $r^0 = \operatorname{Re} \log z_n$, for a suitable branch of log, so that $M_\rho = \{ z \in M : |z_n| < e^\rho \}$. (A different choice of r^0 turns out to be more appropriate in [11].) We further define

$$\begin{aligned} g^+(\zeta, z) &= g^+(\zeta) = r_\zeta \equiv (r_{z_1}(\zeta), \ldots, r_{z_n}(\zeta)), \\ g^-(\zeta, z) &= g^-(z) = r_z = (r_{z_1}(z), \ldots, r_{z_n}(z)), \\ g^0(\zeta, z) &= g^0(\zeta) = r^0_\zeta = (0, \ldots, 0, r^0_{z_n}(\zeta)), \end{aligned} \tag{1.15}$$

and denote the corresponding forms ω by ω^+, ω^-, and ω^0. For ζ, $z \in M_\rho$ and $w = \zeta - z$, one shows (*e. g. see* sec. 4 of [11]) that $g^+ . w$ and $g^- . w$ vanish only for $\zeta = z$, if ρ is sufficiently small. (We assume M_ρ shrinks to 0 as $\rho \to 0$.)

From (1.10), (1.12), and the decomposition $\bar\partial = \bar\partial_\zeta + \bar\partial_z$, we get

$$\bar\partial_\zeta \Omega^{+-}_{0,s} + \bar\partial_z \Omega^{+-}_{0,s-1} = \Omega^+_{0,s} - \Omega^-_{0,s}.$$

Since ω^+ is holomorphic in z, Ω^+ contains no differentials $d\bar{z}_j$. Hence, $\Omega^+_{0,s} = 0$ for $s \geqq 1$. Since ω^- is holomorphic in ζ, $\Omega^-_{0,s} = 0$ for $n - 1 - s \geqq 1$, or $s \leqq n - 2$. Thus,

$$\bar\partial_\zeta \Omega^{+-}_{0,s} + \bar\partial_z \Omega^{+-}_{0,s-1} = 0, \qquad 1 \leqq s \leqq n-2. \tag{1.16}$$

For a form $\varphi(\zeta)$ of type $(0, s)$ in ζ, $1 \leqq s \leqq n-2$, $\varphi(\zeta) \wedge \Omega^{+-}_{0,s}(\zeta, z)$ is of type $(n, n-1)$ in ζ, and (1.16) gives

$$d_\zeta(\varphi \wedge \Omega^{+-}_{0,s}) = \bar\partial_\zeta(\varphi \wedge \Omega^{+-}_{0,s}) = \bar\partial_\zeta \varphi \wedge \Omega^{+-}_{0,s} - \bar\partial_z(\varphi \wedge \Omega^{+-}_{0,s-1}).$$

We apply Stokes' theorem on the manifold-with-boundary $\{ \zeta \in M_\rho : |\zeta - z| \geqq \varepsilon \}$, $\varepsilon > 0$, for a fixed z in M_ρ, and let ε tend to zero. The resulting residue at z is a non-zero constant multiple of $\varphi(z)$. Moving the exterior derivative $\bar\partial_z$ past the integral sign, we obtain formally

$$\varphi - B\varphi = \bar\partial P_0 \varphi + Q_0 \bar\partial \varphi. \tag{1.17}$$

Here,

$$P_0 \varphi(z) = c_1 \int_{M_\rho} \varphi(\zeta) \wedge \Omega^{+-}_{0,s-1}(\zeta, z), \tag{1.18}$$

$$Q_0 \psi^{(0,s+1)}(z) = c_2 \int_{M_\rho} \psi(\zeta) \wedge \Omega^{+-}_{0,s}(\zeta, z), \tag{1.19}$$

$$B\varphi(z) = c_3 \int_{\partial M_\rho} \varphi(\zeta) \wedge \Omega^{+-}_{0,s}(\zeta, z). \tag{1.20}$$

The preceding argument is rigorous if φ vanishes in a neighborhood of z. For the general case we may assume that φ has compact support in M_ρ and apply either theorem (3.2) of [4] or theorem (9.13) of [3]. In these theorems (1.17) is verified in the sense of currents of type $(n, n-1-q)$ along M, which results in equality only mod $\bar{\partial} r$. This is to be understood in (0.1) or (1.27) below. Only the tangential part of the homotopy formula, which gives equality, is used in [11].

To transform the boundary integral (1.20), we use (1.11), which gives

$$\bar{\partial}_\zeta \Omega^{0+-}_{0,s} + \bar{\partial}_z \Omega^{0+-}_{0,s-1} = -\Omega^{+-}_{0,s} + \Omega^{0-}_{0,s} - \Omega^{0+}_{0,s}. \tag{1.21}$$

We note that ω^0 is given by (1.2), and $w_n = \zeta_n - z_n$ does not vanish for $\zeta \in \partial M_\rho$, $z \in M_\rho$. Also, $\bar{\partial}\omega^0 = 0$, $\bar{\partial}\omega^+ = \bar{\partial}_\zeta \omega^+$, and $\bar{\partial}\omega^- = \bar{\partial}_z \omega^-$, so that

$$\left.\begin{aligned} \Omega^{0+}_{0,*} &= \Omega^{0+}_{0,0}, \\ \Omega^{0-}_{0,*} &= \Omega^{0-}_{0,n-2} = \omega^0 \wedge \omega^- \wedge (\bar{\partial}_z \omega^-)^{n-2}, \qquad (dw \to d\zeta). \end{aligned}\right\} \tag{1.22}$$

Thus,

$$\bar{\partial}_\zeta \Omega^{0+-}_{0,s} + \bar{\partial}_z \Omega^{0+-}_{0,s-1} = -\Omega^{+-}_{0,s}, \qquad 1 \leqq s \leqq n-3. \tag{1.23}$$

We insert (1.23) into (1.20), use Stokes' theorem over ∂M_ρ to throw $\bar{\partial}_\zeta$ onto $\varphi(\zeta)$ in the first integral and take $\bar{\partial}_z$ outside the second integral to get

$$B\varphi = \bar{\partial} P_1 \varphi + Q_1 \bar{\partial}\varphi, \tag{1.24}$$

where

$$P_1 \varphi(z) = c_4 \int_{\partial M_\rho} \varphi(\zeta) \wedge \Omega^{0+-}_{0,s-1}(\zeta, z), \tag{1.25}$$

$$Q_1 \psi^{(0,s+1)}(\zeta) = c_5 \int_{\partial M_\rho} \psi(\zeta) \wedge \Omega^{0+-}_{0,s}(\zeta, z). \tag{1.26}$$

Thus, since our forms are restricted to M, $\bar{\partial} = \bar{\partial}_b$, and we have

$$\left.\begin{aligned} \varphi &= \bar{\partial}_b P\varphi + Q\bar{\partial}_b \varphi, \\ P &= P_0 + P_1, \qquad Q = Q_0 + Q_1. \end{aligned}\right\} \tag{1.27}$$

We briefly consider the case $1 \leqq s = n-2$, in which we have (1.27) with the boundary integral of $\varphi \wedge \Omega^{0-}_{0,n-2}$ added to the right hand side. Following Henkin [4], we approximate $\omega^0 = (\zeta_n - z_n)^{-1} d\zeta_n$ by $p_j(\zeta_n, z_n)\, d\zeta_n$, where p_j is a sequence of polynomials converging uniformly for ζ_n on the arc $r^0(\zeta_n) = 0$, $\operatorname{Im}\zeta_n > 0$, and $r^0(z_n) < 0$, z_n fixed. We also approximate $r_z.(\zeta - z)$ by $r_z(z_\varepsilon).(\zeta - z_\varepsilon)$, $z_\varepsilon = (z_\alpha, z_n - \varepsilon i)$. Denote the resulting form by $\Omega_{(j,\varepsilon)}(\zeta, z)$.

Then

$$\int_{\partial M_\rho} \varphi(\zeta) \wedge \Omega^{0-}_{0,n-2}(\zeta, z) = \lim_{\substack{j \to \infty \\ \varepsilon \to 0}} \int_{\partial M_\rho} \varphi(\zeta) \wedge \Omega_{(j,\varepsilon)}(\zeta, z)$$

$$= \lim_{\substack{j \to \infty \\ \varepsilon \to 0}} \int_{M_\rho} \bar{\partial}\varphi(\zeta) \wedge \Omega_{(j,\varepsilon)}(\zeta, z) \equiv Q_2(\bar{\partial}_b \varphi)(z), \quad (1.28)$$

since $\Omega_{(j,\varepsilon)}$ is holomorphic in ζ and without singularity on M_ρ. Thus, (1.27) holds with $Q = Q_0 + Q_1 + Q_2$. However, we are not able to obtain any useful bounds for the operator Q_2.

Returning to (1.27) with $1 \leqq s \leqq n-3$, we introduce the notation

$$p(\zeta, z) = r_\zeta . (\zeta - z), \qquad q(\zeta, z) = r_\zeta . (\zeta - z), \qquad w_n = \zeta_n - z_n, \quad (1.29)$$

in order to write out the kernels more explicitly. Then,

$$\Omega^{+-}_{0,s} = \frac{\partial_\zeta r \wedge (r_z . d\zeta) \wedge (\bar{\partial}_\zeta \partial_\zeta r)^{n-2-s} \wedge (\bar{\partial}_z r_z \wedge d\zeta)^s}{p(\zeta, z)^{n-1-s} q(\zeta, z)^{s+1}}, \quad (1.30)$$

and

$$\Omega^{0+-}_{0,s} = \frac{d\zeta_n \wedge \partial_\zeta r \wedge (r_z . d\zeta) \wedge (\bar{\partial}_\zeta \partial_\zeta r)^{n-3-s} \wedge (\bar{\partial}_z r_z \wedge d\zeta)^s}{w_n p(\zeta, z)^{n-2-s} q(\zeta, z)^{s+1}}. \quad (1.31)$$

These expressions make evident the following property of the four operators P_0, Q_0, P_1, Q_1. Each annihilates the ideal of forms generated by $\bar{\partial}_r$. This is because each integrand contains the factor $\partial_\zeta r$, and restricting to $r=0$ *i. e.* to M, $d_\zeta r = \partial_\zeta r + \bar{\partial}_\zeta r = 0$. Thus any term in $\varphi(\zeta)$ or $\psi(\zeta)$ containing $\bar{\partial}_\zeta r$ is annihilated by the wedge product.

We need to determine the nature of the operators P and Q as acting on the coefficients of the form $\varphi^{(0,s)}(\zeta)$ or $\psi^{(0,s+1)}(\zeta)$ relative to the differentials $d\bar{\zeta}$. For this let D_ρ be the projection of M_ρ onto $y_n = 0$, so that by (1.13) M_ρ is a graph over D_ρ. If $f(\zeta)$ is a typical such coefficient, then P_0 and Q_0 are (sums of) operators of the form

$$Kf(z) = \int_{D_\rho} f(\zeta)\, k(\zeta, z)\, dV(\zeta), \quad (1.32)$$

while P_1 and Q_1 are operators of the form

$$Lf(z) = \int_{\partial D_\rho} f(\zeta)\, l(\zeta, z)\, dS(\zeta). \quad (1.33)$$

Here, dV and dS are the Euclidean volume and surface measures in $\mathbb{R}^{2n-1}$. Occuring in each numerator in (1.30), (1.31) is

$$\partial_\zeta r \wedge (r_z . d\zeta) = r_\zeta . d\zeta \wedge (r_z - r_\zeta) . d\zeta.$$

We use $\zeta(t)=z+t(\zeta-z)$ to write

$$r_z(\zeta)-r_z(z)=\int_{t=0}^{1}\{r_{zz}(\zeta(t)).(\zeta-z)+r_{z\bar z}(\zeta(t)).(\bar\zeta-\bar z)\}. \tag{1.34}$$

We then have an expression of each $d\bar z$ component of the $(n, n-1)_\zeta$-form $\varphi^{(0,s)}(\zeta)\wedge\Omega^{+}_{0,\bar s-1}(\zeta,z)$ as a linear combination of the differentials

$$d\zeta_1\wedge\ldots\wedge d\zeta_n\wedge d\bar\zeta_1\wedge\ldots\wedge \widehat{d\bar\zeta_i}\wedge\ldots\wedge d\bar\zeta_n,\qquad 1\leqq i\leqq n, \tag{1.35}$$

having coefficients which are rational combinations of p, q, the first and second derivatives $\partial^1 r$, $\partial^2 r$ evaluated at ζ or z or integrated as in (1.34), and of ζ_i-z_i, $\bar\zeta_i-\bar z_i$. Further, we express (1.35) as $a_j(\zeta)\,dV(\zeta)$, where each $a_j(\zeta)$ is an easily computed expression in $\partial^1 r(\zeta)$. It follows from (1.30) that $k(\zeta,z)$ can be put into the form

$$\left.\begin{aligned} k(\zeta,z)&=A(\zeta,z)\,B^{IJ}p^{-\alpha}q^{-\beta},\\ B^{IJ}&\equiv(\zeta-z)^{I}(\bar\zeta-\bar z)^{J},\end{aligned}\right\} \tag{1.36}$$

where $\alpha\geqq 1$, $\beta\geqq 1$, and I, J are non-negative multi-indices. (Initially $|I|+|J|=1$.) A is constructed from ζ, z and up to a certain number (initially 2) of derivatives $\partial^j r$ of the defining function as described. Similarly, the kernel $l(\zeta,z)$ has the form

$$l(\zeta,z)=A(\zeta,z)\,B^{IJ}p^{-\alpha}a^{-\beta}w_n^{-\gamma}. \tag{1.37}$$

where in addition $\gamma\geqq 0$. As we shall see only such kernels will arise in taking derivatives of (1.32) and (1.33). We shall only have to consider z-derivatives of $l(\zeta,z)$.

We denote by μ, $\mu(k)$, or $\mu(l)$ an upper bound.

$$\mu\geqq 2(\alpha+\beta)+\gamma-|I|-|J|. \tag{1.38}$$

For integrals like (1.32) over M_ρ we shall always have $\gamma=0$ and $\mu=2n-1$. As shown in [4], [3], $k(\zeta,z)$ is then absolutely integrable in ζ, uniformly in z. Thus Kf as well as Lf are continuous over the interior of M.

We denote by δ_z a vector field in $\mathbb{C}^n$ tangent to M,

$$\delta_z=v(z).\partial_z+\bar v(z).\partial_{\bar z}, \tag{1.39}$$

and by δ_ζ the corresponding operator in ζ-coordinates. As a fixed basis of such fields δ_z, we shall take the real and imaginary parts of

$$\begin{gathered} r_n(z)\,\partial_{z^\alpha}-r_\alpha(z)\,\partial_{z^n},\qquad 1\leqq\alpha\leqq n-1,\\ ir_{\bar n}(z)\,\partial_{z^n}-ir_n(z)\,\partial_{\bar z^n}.\end{gathered}$$

In particular, the coefficients v are constructed from the first derivatives of r. Any of the $(z_\alpha=x_\alpha+iy_\alpha, x_n)$-coordinate partial derivatives is a linear combination, with function coefficients, of the fields of this basis, and conversely. Thus, to measure the C^j-norm of Kf or Lf, it suffices to apply up to j of these vector fields δ_z.

2. FIRST DERIVATIVES OF THE KERNEL k

We proceed to differentiate (1.32) using (1.39) and prepare to throw the derivative onto f via integration by parts. The nature of the kernel (1.36) complicates the process. For similar arguments involving the $\bar{\partial}$ complex, one may consult [3] or [7], for example. We shall make use of the operator

$$T_\zeta = i(r_{\bar{\zeta}}.\partial_\zeta - r_\zeta.\partial_{\bar{\zeta}}), \tag{2.1}$$

which is tangent to M but transverse to the holomorphic tangent planes to M. It has been used in [7] and by a number of other people. From (1.29) and $|r_\zeta| \geqq \frac{1}{4}$, we may assume that

$$|T_\zeta p| \geqq c, \qquad |T_\zeta q| \geqq c, \tag{2.2}$$

for all ζ, $z \in M_\rho$ and a constant $c > 0$, by taking ρ sufficiently small.

To compute

$$\begin{aligned}(\delta_z + \delta_\zeta) k = [(\delta_z + \delta_\zeta) A . B^{IJ} \\ + A(\delta_z + \delta_\zeta) B^{IJ}] p^{-\alpha} q^{-\beta} + AB^{IJ}(\delta_z + \delta_\zeta)[p^{-\alpha} q^{-\beta}],\end{aligned} \tag{2.3}$$

we note that

$$(\delta_z + \delta_\zeta)(\zeta_i - z_i) = v_i(\zeta) - v_i(z) = \sum v_{iIJ} B^{IJ}, \quad (|I| + |J| = 1),$$

where by (1.34) the coefficients v_{iIJ} involve $\partial^j r$, $j \leqq 2$. Hence,

$$(\delta_z + \delta_\zeta) B^{IJ} = \sum A^{IJ}_{KL} B^{KL} \qquad (|K| + |L| = |I| + |J|), \tag{2.4}$$

the coefficients A^{IJ}_{KL} depending through v on $\partial^j r$, $j \leqq 2$. Also, we have

$$\begin{aligned}(\delta_z + \delta_\zeta)[p^{-\alpha}] &= F T_\zeta [p^{-\alpha}], \qquad F = \frac{(\delta_z + \delta_\zeta) p}{T_\zeta p}, \\ (\delta_z + \delta_\zeta)[q^{-\beta}] &= E T_\zeta [q^{-\beta}], \qquad E = \frac{(\delta_z + \delta_\zeta) q}{T_\zeta q}.\end{aligned} \tag{2.5}$$

Hence,

$$\begin{aligned}(\delta_z + \delta_\zeta)[p^{-\alpha} q^{-\beta}] &= E p^{-\alpha} T_\zeta [q^{-\beta}] + F T_\zeta [p^{-\alpha}] q^{-\beta} \\ &= T_\zeta [E p^{-\alpha} q^{-\beta}] - (T_\zeta E) p^{-\alpha} q^{-\beta} - \alpha G (T_\zeta p) p^{-\alpha-1} q^{-\beta}, \\ G &\equiv F - E.\end{aligned} \tag{2.6}$$

From (2.5)

$$E(\zeta, z)=(T_\zeta q)^{-1}\{(\delta_z r_z).(\zeta-z)+r_z.(v(\zeta)-v(z))\}$$
$$=\sum E_{IJ} B^{IJ} \qquad (|I|+|J|=1), \quad (2.7)$$

where E_{IJ} has denominator $T_\zeta q$ and numerator involving $\partial^j r$, $j\leqq 2$. From (2.6)

$$\begin{aligned} G(\zeta, z) &= \left(\frac{1}{T_\zeta p}-\frac{1}{T_\zeta q}\right)(\delta_z+\delta_\zeta) p+\frac{1}{T_\zeta q}\{(\delta_z+\delta_\zeta) p-(\delta_z+\delta_\zeta) q\}, \\ \frac{1}{T_\zeta p}-\frac{1}{T_\zeta q} &= (T_\zeta p\, T_\zeta q)^{-1}\{i(r_z-r_\zeta).r_{\bar\zeta}-(T_\zeta r_\zeta).(\zeta-z)\}, \\ (\delta_z+\delta_\zeta) p &= (\delta_\zeta r_\zeta).(\zeta-z)+r_\zeta.(v(\zeta)-v(z)), \\ (\delta_z+\delta_\zeta) p-(\delta_z+\delta_\zeta) q &= (\delta_\zeta r_\zeta-\delta_z r_z).(\zeta-z)+(r_\zeta-r_z).(v(\zeta)-v(z)). \end{aligned} \quad (2.8)$$

It follows that

$$G(\zeta, z)=\sum G_{IJ} B^{IJ} \qquad (|I|+|J|=2), \quad (2.9)$$

where G_{IJ} has denominator $T_\zeta p T_\zeta q$ and numerator involving $\partial^j r$, $j\leqq 3$. Since one has (*see e.g.* section 4 of [11])

$$|p(\zeta, z)|\geqq c|\zeta-z|^2, \qquad |q(\zeta, z)|\geqq c|\zeta-z|^2, \qquad c>0, \quad (2.10)$$

the additional factor p^{-1} in (2.6) is nullified by the factor G. If we write

$$AB^{IJ} T_\zeta[E p^{-\alpha} q^{-\beta}]$$
$$=T_\zeta[E k]-(T_\zeta A) EB^{IJ} p^{-\alpha} q^{-\beta}-AET_\zeta[B^{IJ}] p^{-\alpha} q^{-\beta}, \quad (2.11)$$

then from (2.3), (2.6) we get

$$\begin{aligned} (\delta_z+\delta_\zeta) k &= T_\zeta k^1+k^0, \\ \mu(k^0)=\mu(k), &\qquad \mu(k^1)=\mu(k)-1. \end{aligned} \quad (2.12)$$

More explicitly,

$$\begin{aligned} k^0 &= \sum A^0_{KL\,\rho\sigma} B^{KL} p^{-\rho} q^{-\sigma}, \qquad 2(\rho+\sigma)-|K|-|L|\leqq\mu, \\ k^1 &= \sum A^1_{KL\,\rho\sigma} B^{KL} p^{-\rho} q^{-\sigma}, \qquad 2(\rho+\sigma)-|K|-|L|\leqq\mu-1, \end{aligned} \quad (2.13)$$

where

$$\begin{aligned} A^0_{KL\,\rho\sigma} &= w_1[A], \qquad A^1_{KL\,\rho\sigma}=w_0[A], \\ w_0 A=S_0 A, &\qquad w_1 A=S_1 \partial^1 A+S_2 A. \end{aligned} \quad (2.14)$$

This means that w_0 is the zeroeth order operator multiplication by $E=S_0(\partial^j r, j\leqq 2)$, and w_1 is a first order operator with coefficients $S_1=S_1(\partial^j r, j\leqq 1)$, $S_2(\partial^j r, j\leqq 3)$. In each case S_0, S_1, S_2 have denominators in $T_\zeta p$, $T_\zeta q$ and numerators involving the indicated number of derivatives of r.

3. FIRST DERIVATIVES OF Kf AND Lf

We compute $\delta_z K f$ in the sense of distributions. Let g be a smooth function with compact support in D_ρ, then

$$\int_{D_\rho} Kf(z)\,\delta_z g(z)\,dV(z) = \lim_{\varepsilon \to 0} I_\varepsilon,$$

$$I_\varepsilon = \iint_{|\zeta - z| > \varepsilon} f(\zeta)\,k(\zeta, z)\,\delta_z g(z)\,dV(\zeta)\,dV(z).$$

Using (2.12) we rewrite the integrand as

$$\begin{aligned} f k\,\delta_z g &= f\,\delta_z[kg] - f\,\delta_z[k]g \\ &= f\,\delta_z[kg] + f(\delta_\zeta k - T_\zeta k^1 - k^0)g \\ &= f\,\delta_z[kg] + \delta_\zeta[f k]g - (\delta_\zeta f)kg \\ &\quad - T_\zeta[f k^1]g + (T_\zeta f)k^1 g - f k^0 g. \end{aligned}$$

Hence, $I_\varepsilon = I_1 + I_2 + I_3 + I_4 + I_5$,

$$I_1 = -\int_z \int_{|\zeta - z| > \varepsilon} \{\delta_\zeta f\,k - T_\zeta f\,k^1\}\,dV(\zeta)\,g(z)\,dV(z),$$

$$I_2 = -\int_z \int_{|\zeta - z| > \varepsilon} f\,k^0\,dV(\zeta)\,g(z)\,dV(z),$$

$$I_3 = \int_\zeta f(\zeta) \int_{|\zeta - z| > \varepsilon} \delta_z[kg]\,dV(z)\,dV(\zeta),$$

$$I_4 = \int_z \int_{|\zeta - z| > \varepsilon} \delta_\zeta[f k]\,dV(\zeta)\,g(z)\,dV(z),$$

$$I_5 = -\int_z \int_{|\zeta - z| > \varepsilon} T_\zeta[f k^1]\,dV(\zeta)\,g(z)\,dV(z).$$

We shall transform I_3, I_4 and I_5 using Stokes' theorem on the inner integrals. For this we consider the integrals as over $M \subset \mathbb{C}^n$, and denote by $N(\zeta)$, $N_z(\zeta)$, and $N_\zeta(z)$, respectively, the outward unit normals tangent to M for the domains M_ρ, $\{\zeta : |\zeta - z| < \varepsilon\}$, and $\{z : |\zeta - z| < \varepsilon\}$ which lie on M. Since M is of class at least C^3, we have

$$N_\zeta(z) + N_z(\zeta) = O(|\zeta - z|). \tag{3.1}$$

We denote by $\langle\ ,\ \rangle$ and div, the real Euclidean inner product and divergence relative to M. The resulting integrals over the interior of M_ρ

for the three terms give

$$\begin{gathered}\mathrm{I}_6=\iint_{|\zeta-z|>\varepsilon} f(\zeta)\,k^2(\zeta, z)\,g(z)\,d\mathrm{V}(\zeta)\,d\mathrm{V}(z),\\ k^2\equiv(\operatorname{div}\delta_z+\operatorname{div}\delta_\zeta)\,k-\operatorname{div}\mathrm{T}_\zeta)\,k^1.\end{gathered}\tag{3.2}$$

For the boundary terms we get $\mathrm{I}_7+\mathrm{I}_8$,

$$\mathrm{I}_7=\int_z\int_{\zeta\in\partial\mathrm{M}_\rho} f(\zeta)\,l^1(\zeta, z)\,d\mathrm{S}(\zeta)\,g(z)\,d\mathrm{V}(z),$$

$$\mathrm{I}_8=\iint_{|\zeta-z|=\varepsilon} f(\zeta)\,l^2(\zeta, z)\,d\mathrm{S}(\zeta)\,g(z)\,d\mathrm{V}(z),$$

where

$$\left.\begin{aligned} l^1&=\langle\mathrm{N}(\zeta), \delta_\zeta\rangle k-\langle\mathrm{N}(\zeta), \mathrm{T}_\zeta\rangle k^1\\ l^2&=\{\langle\mathrm{N}_\zeta(z), \delta_z\rangle+\langle\mathrm{N}_z(\zeta), \delta_\zeta\rangle\}\,k+\langle\mathrm{N}_z(\zeta), \mathrm{T}_\zeta\rangle k^1.\end{aligned}\right\}\tag{3.3}$$

Using (3.1) we see that the coefficient of k in l^2 is $O(|\delta-z|)$ so that $\mu(l^2)=\mu-1=2n-2$. Therefore, $\mathrm{I}_8\to 0$ as $\varepsilon\to 0$ by essentially the same argument as for formula (3.18) in lemma 3.3 of [4]. Since all the integrals over the interior are convergent, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathrm{I}_\varepsilon=-\int_z\int_\zeta\{\delta_\zeta f\,k-\mathrm{T}_\zeta f\,k^1\}\,d\mathrm{V}(\zeta)\,g(z)\,d\mathrm{V}(z)\\ -\int_z\int_\zeta f\,(k^0+k^2)\,d\mathrm{V}(\zeta)\,g(z)\,d\mathrm{V}(z)\\ +\int_z\int_{\zeta\in\partial\mathrm{M}_\rho} f\,l^1\,d\mathrm{S}(\zeta)\,g(z)\,d\mathrm{V}(z).\end{aligned}$$

This gives $\delta_z^*\mathrm{K}f=-(\delta_z+\operatorname{div}\delta_z)\,\mathrm{K}f$ in the sense of distributions. Since it and $\mathrm{K}f$ are both continuous, it is a derivative in the ordinary sense, and we have

LEMMA (3.1). — *If* $f\in\mathrm{C}^1(\mathrm{M})$ *and* $\mu=\mu(k)=2n-1$, *then*

$$\delta\mathrm{K}f=\mathrm{K}(\delta f)+\mathrm{K}_\delta^1(\mathrm{T}_\zeta f)+\mathrm{K}_\delta^3(f)+\mathrm{L}_\delta^1(f),\tag{3.4}$$

where the new operators have kernels $-k^1$, $k^3\equiv k^0+k^2-(\operatorname{div}\delta_z)\,k$, $-l^1$, *with* $\mu(k^1)=\mu-1$, $\mu(k^3)=\mu(l^1)=\mu$.

In order to state lemma (3.1) a little more precisely, we must introduce some new notation. For an integral over M_ρ (or D_ρ) of the form (1.32), (1.36) we use round brackets,

$$\mathrm{K}f=(f, \mathrm{A})_{\mathrm{II}\alpha\beta}\equiv(f, \mathrm{A})_\mu,\tag{3.5}$$

with the latter notation usually sufficing. Similarly, for a boundary integral of the form (1.33), (1.37) we get

$$Lf = \langle f, A \rangle_{IJ\alpha\beta\gamma} \equiv \langle f, A \rangle_{\mu}. \tag{3.6}$$

We also use the same notation for a finite sum of such integrals with the same value for μ (1.38). We also let ∂ represent any, and all, our first order operators δ. Then, with the notation (2.14) and $\mu = 2n-1$, we have

$$\partial (f, A)_{\mu} = (\partial f, w_0 A)_{\mu} + (f, w_1 A)_{\mu} + \langle f, w_0 A \rangle_{\mu}. \tag{3.7}$$

Our analysis of the derivatives of Lf is much simpler. We simply let the operator δ_z fall on the kernel $l(\zeta, z)$, worsening the singularity at the boundary. From (1.37) we have

$$\delta_z l = (\delta_z A) B^{IJ} p^{-\alpha} q^{-\beta} w_n^{-\gamma} + A \delta_z [B^{IJ} p^{-\alpha} q^{-\beta} w_n^{-\gamma}],$$

which leads to

$$\partial \langle f, A \rangle = \langle f, w_1 A \rangle_{\mu} + \langle f, w_0 A \rangle_{\mu+2} \equiv \langle f, w_1 A \rangle_{\mu+2}, \tag{3.8}$$

by regarding w_0 as a first order operator w_1 and replacing μ by $\mu+2$ in the first term.

4. HIGHER ORDER DERIVATIVES

From (1.30) and (1.31) we see that P and Q have the character

$$Pf = (f, A)_{\mu} + \langle f, A \rangle_{\mu-1}, \qquad \mu = 2n-1, \tag{4.1}$$

as operators on the coefficients f of the differential form φ. The two A's are constructed from ζ, z, and $\partial^j r$, $j \leqq 2$. From (3.8) we have, taking b derivatives

$$\partial^b \langle f, A \rangle_{\mu-1} = \langle f, w_1^b A \rangle_{\mu-1+2b}. \tag{4.2}$$

Taking a second derivative of $(f, A)_{\mu}$, using (3.7) and (3.8), and combining several terms gives

$$\partial^2 (f, A)_{\mu} = (\partial^2 f, w_0^2 A)_{\mu} + (\partial f, w_0 w_1 A)_{\mu} + (f, w_1^2 A)_{\mu} + \langle \partial f, w_0^2 A \rangle_{\mu} + \langle f, w_0 w_1 A \rangle_{\mu+2}.$$

The notation $w_0^{\alpha} w_1^{\beta} A$ indicates that α zeroeth order and β first order operators of the form (2.14) are applied to A in some order. After taking

b derivatives δ_z we get

$$\partial^b (f, \mathrm{A})_\mu = \sum_{j=0}^{b} (\partial^j f, w_0^j w_1^{b-j} \mathrm{A})_\mu + \sum_{j=0}^{b-1} \langle \partial^j f, w_0^{j+1} w_1^{b-1-j} \mathrm{A} \rangle_{\mu+2(b-1-j)}. \tag{4.3}$$

We define norms according to

$$\|f\|_\rho = \sup\{|f(\zeta)| : \zeta \in \mathrm{M}_\rho\},$$
$$\|\mathrm{A}\|_\rho = \sup\{|\mathrm{A}(\zeta, z)| : \zeta, z \in \mathrm{M}_\rho\},$$
$$\|\partial^b f\|_\rho = \sup(\|\partial^{\mathrm{K}} f\|_\rho : |\mathrm{K}| = b\},$$
$$\|f\|_{\rho, b} = \max_{0 \leqq j \leqq b} \|\partial^j f\|_\rho.$$

For $\mu \leqq 2n-1$ and $\rho \leqq \rho_0$, ρ_0 fixed, we have

$$|(f, \mathrm{A})_\mu (z)| \leqq \|f\|_\rho \|\mathrm{A}\|_\rho \int_{\mathrm{M}_\rho} |\mathrm{B}^{\mathrm{IJ}} p^{-\alpha} q^{-\beta}| \, d\mathrm{V}(\zeta),$$

so that

$$\|(f, \mathrm{A})_\mu\|_\rho \leqq c \|f\|_\rho \|\mathrm{A}\|_\rho. \tag{4.4}$$

Also,

$$|\langle f, \mathrm{A} \rangle_\mu (z)| \leqq \|f\|_\rho \|\mathrm{A}\|_\rho \int_{\partial \mathrm{M}_\rho} |\mathrm{B}^{\mathrm{IJ}} p^{-\alpha} q^{-\beta} w^{-\gamma}| \, d\mathrm{S}(\zeta).$$

For $\zeta \in \partial \mathrm{M}_\rho$, $z \in \mathrm{M}_{\rho(1-\sigma)}$, we have

$$\begin{gathered} |\zeta - z| \geqq \delta, \qquad |w_n| \geqq c\delta, \\ |p(\zeta, z)| \geqq c\,\delta^2, \qquad |q(\zeta, z)| \geqq c\,\delta^2, \\ |\mathrm{B}^{\mathrm{IJ}} p^{-\alpha} q^{-\beta} w_n^{-\gamma}| \leqq c\delta^{-\mu}, \end{gathered} \tag{4.5}$$

where, as in (0.3),

$$\delta \leqq \operatorname{dist}(\mathrm{M}_{\rho(1-\sigma)}, \partial \mathrm{M}_\rho).$$

Thus

$$\|\langle f, \mathrm{A} \rangle_\mu\|_{\rho(1-\sigma)} \leqq c\delta^{-\mu} \|\mathrm{A}\|_\rho \|f\|_\rho. \tag{4.6}$$

Applied to (4.2), these remarks give

$$\begin{aligned} \|\partial^b \langle f, \mathrm{A} \rangle_{\mu-1}\|_{\rho(1-\sigma)} &\leqq c_b \|f\|_\rho \|w_1^b \mathrm{A}\|_\rho \delta^{-\mu-2b+1} \\ &\leqq c_b \|f\|_\rho \|w_1^b \mathrm{A}\|_\rho \delta^{-2(n+b-1)}, \end{aligned} \tag{4.7}$$

since $\mu = 2n-1$.

In (4.3) we may replace $w_0^{j+1} w_1^{b-1-j}$ in the second sum by $w_0^j w_1^{b-j}$, again by regarding w_0 as a w_1 (2.14). Then

$$\| \partial^b (f, A)_\mu \|_{\rho(1-\sigma)} \leqq c_b \Big\{ \| \partial^b f \|_\rho \| w_0^b A \|_\rho + \sum_{j=0}^{b-1} \| \partial^j f \|_\rho \| w_0^j w_1^{b-j} A \|_\rho \delta^{-\mu-2(b-1-j)} \Big\}$$

$$\leqq c_b \delta^{-2(n+b-1)+1} \sum_{j=0}^{b} \| \partial^j f \|_\rho \| w_0^j w_1^{b-j} A \|_\rho, \quad (4.8)$$

where we have used $\mu = 2n-1$ and assumed $\delta \leqq 1$. Combining (4.1), (4.7), and (4.8) gives

$$\| \partial^b Pf \|_{\rho(1-\sigma)} \leqq c \delta^{-2(n+b-1)} \sum_{j=0}^{b} \| \partial^j f \|_\rho \| w_0^j w_1^{b-j} A \|_\rho. \quad (4.9)$$

In a term $w_0^j w_1^{b-j} A$, A and the coefficients of w_0 and w_1 involve $\partial^i r$, $1 \leqq i \leqq 3$. Also there are $b-j$ further differentiations. Such a term is therefore a sum of terms of the form

$$F(\partial^i r) \partial^{\alpha_1} (\partial^i r) \dots \partial^{\alpha_s} (\partial^i r), \qquad \alpha_1 + \dots + \alpha_s = b-j, \qquad 1 \leqq i \leqq 3. \quad (4.10)$$

Here F is a certain rational function (modulo the operation (1.34)) in the derivatives $\partial^i r$. By (2.2) its denominators are bounded away from 0 by a positive constant depending on b. The construction of $A(\zeta, z)$ and related expressions involves r and its derivatives on the line segment in $\mathbb{C}^n$ from ζ to z (1.34) for all points ζ, $z \in M_\rho$. Therefore we denote

$$M_{\hat\rho} = \text{convex hull of } M_\rho, \qquad \| f \|_{\hat\rho} = \sup \{ |f(z)| : z \in M_{\hat\rho} \}, \qquad \text{etc.}$$

It follows that

$$\| w_0^j w_1^{b-j} A \|_\rho \leqq c_b (1 + \| r \|_{\hat\rho, b-j+3})^{\gamma(b)} \quad (4.11)$$

for some positive constants c_b, $\gamma(b)$ depending on b.

Combining the above gives the following more precise form of (0.3).

THEOREM (4.1). — *Let the real hypersurface* M *in* (1.13) *be of class* C^l *and the* $(0, s)$ *form* φ, $1 \leqq s \leqq n-3$, *be of class* C^k *on the closure of* M_ρ,