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Periodic bounce trajectories with a low number of bounce points (*)

by

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ABSTRACT. — In this paper we study the existence of a periodic trajectory with prescribed period, which bounces against the boundary of an open subset of $\mathbb{R}^N$, in presence of a potential field. We prove the existence of periodic solutions with at most $N+1$ bounce points.

Key words : Periodic bounce trajectory, bounce point, nonregular point, Morse index.

RÉSUMÉ. — Dans ce papier on étudie l'existence d'une trajectoire périodique à période fixée, qui rejaillit sur le bord d'un sous ensemble ouvert de $\mathbb{R}^N$ dans un champ de potentiel. On démontre qu'il existe des solutions périodiques avec $N+1$ points de rejaillissement au plus.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with boundary $\partial\Omega$ of class C^2.

A bounce trajectory in $\bar{\Omega}$ is a piecewise linear path with corners at $\partial\Omega$, for which the usual low of reflection is satisfied, namely the segments make equal angles with the tangent plane. A bounce point is a corner point for our path.

The main result of this paper is the following:

(1.1) THEOREM. — *Let Ω be as above. Then there exists at least one periodic nonconstant trajectory in $\bar{\Omega}$ with at most* $N+1$ *bounce points.*

(1.2) *Remark.* — The conclusion of Theorem (1.1) is optimal in the sense that it is possible to construct a set Ω for wich there are not trajectories with only N bounce points. For $N=1$ this is obvious. For $N=2$ we refer to [6], [13] for such a controexample.

(1.3) *Remark.* — The result of Theorem (1.1) is somewhat surprising. In fact analougous problems exibit a more complicated fenomenology.

For example the Cauchy problem has a solution (in general non unique) provided that the concept of solution is generalized to include trajectories which spend some time lying on the boundary (*see* [7] to [10], [15] and Remark (2.14)).

The illumination problem (*i. e.* existence of bounce trajectories with prescribed extreme points) may not have any solution even in a generalized sense (*see* [16, 18] for controexamples and [11], [14] for some recent results).

We refer also to [12], [14] where the existence of periodic trajectories of special type has been proved in some particular cases.

Theorem (1.1) can be obtained as a consequence of a more general result. Perhaps now it is convenient to give a rigorous definition.

Let $V \in C^1(\bar{\Omega}, \mathbb{R})$, $\nabla V(x)$ the gradient of V at x and $\nu(x)$ the exterior unit normal to $\bar{\Omega}$ in $x \in \partial\Omega$.

(1.4) DEFINITION. — *A loop γ from S^1 to $\bar{\Omega}$ is called a periodic bounce trajectory with respect to the potential* V *if:*

(i) $\gamma \in C^2(S^1)$ *except for at most a finite number of instants* $t_1, \ldots, t_l$ *for wich* $\gamma(t_j) \in \partial\Omega$;

(ii) $\gamma''(t) + \nabla V(\gamma(t)) = 0$ *for every* $t_1, \ldots, t_l$;

(iii) *for every* $t \in \{t_1, \ldots, t_l\}$ *there exist the limits* $\lim_{s \to t} \gamma'_{\pm}(s) := \gamma'_{\pm}(t)$ *and*

$$(1.5) \quad \gamma'_+(t) - \langle \gamma'_+(t), \nu(\gamma(t)) \rangle \nu(\gamma(t)) = \gamma'_-(t) - \langle \gamma'_-(t), \nu(\gamma(t)) \rangle \nu(\gamma(t)),$$

$$(1.6) \qquad \langle \gamma'_+(t), \nu(\gamma(t)) \rangle = -\langle \gamma'_-(t), \nu(\gamma(t)) \rangle \neq 0;$$

(iv) *the set* $\{t_1, \ldots, t_l\}$ *is not empty.*

The instants $t_1, \ldots, t_l$ *for wich* (1.5) *and* (1.6) *hold are called bounce instants, while the points* $\gamma(t_j)$ *are called bounce points.*

Notice that $\gamma(t_j) \in \partial\Omega$ does not implies that $\gamma(t_j)$ is a bounce point according to our definition. In fact it may happen that $\langle \gamma'_+(t), \nu(\gamma(t)) \rangle = -\langle \gamma'_-(t), \nu(\gamma(t)) \rangle = 0$.

Using the above definition we can enunciate the following

(1.7) THEOREM. — *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set with boundary of class* C^2 *and* $V \in C^2(\bar{\Omega}, \mathbb{R})$. *Then there exists* $T_0 > 0$ (*depending of* Ω *and* V) *such that for every* $T \in (0, T_0)$ *there exists a* T*-periodic nonconstant bounce trajectory* (*with respect to the potential* V) *having at most* $N+1$ *bounce* instants.

In particular if $V = 0$ *then* $T_0 = +\infty$.

(1.8) COROLLARY. — *Under the assumptions of Theorem* (1.7), *for every* $T > 0$ *there exist infinitely many bounce trajectories* $\gamma_1, \ldots, \gamma_k, \ldots$ *having at most* $N+1$ *bounce* points. *Moreover if every* γ_k *is not contained in the set* $\{x \in \Omega : \nabla V(x) = 0\}$, *they are all geometrically distinct, i. e.* $\mathrm{Im}(\gamma_r) \neq \mathrm{Im}(\gamma_s)$ *for every* $r \neq s$.

(1.9) *Remark.* — If the set $\{x \in \Omega : \nabla V(x) = 0\}$ includes a bounce trajectory γ, it may happen that all the γ_k's have the following form:

$$\gamma_k(t) = \gamma_1(t/k).$$

i. e. they are not geometrically distinct.

The proof of Theorem (1.7) is based on an approximation scheme which uses the penalization method. The approximating problem can be solved as in [2]. A bounce trajectory is obtained as limit of regular solutions of a Lagrangian system constrained in a potential well. The approximating problem is studied with variational methods and the number of the bounce points is related to the Morse index of an approximating trajectory. However for technical reason it is convenient to use a generalization of the Conley index (*see* [3]) and a theorem related to it (*see* [4] or [5]).

2. THE APPROXIMATION SCHEME

In this section we show how the existence of a bounce trajectory (in a generalized sense) can be obtained as limit of regular solutions of a Lagrangian system.

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with boundary $\partial\Omega$ of class C^2 and ν the exterior unit normal to $\bar{\Omega}$. Let $h \in C^2(\bar{\Omega})$ be a function such that:

$$(2.1)\qquad \begin{array}{ll} \text{(i)} & h(x) = \operatorname{dist}(x, \partial\Omega) \text{ if } \operatorname{dist}(x, \partial\Omega) \leqq d_0; \\ \text{(ii)} & h(x) > d_0 \text{ if } \operatorname{dist}(x, \partial\Omega) > d_0; \\ \text{(iii)} & h(x) \leqq 1 \text{ for every } x \in \bar{\Omega}; \\ \text{(iv)} & |\nabla h(x)| \leqq 1 \text{ for every } x \in \Omega, h(x) = 1 \text{ far from } \partial\Omega; \end{array}$$

where d_0 is a costant small enough to assure the regularity of dist $(x, \partial\Omega)$.

Notice that the function h verifies the following properties:

$$(2.1)\qquad \begin{array}{ll} \text{(v)} & \lim\limits_{x \to x_0} -\nabla h(x) = \nu(x_0) \text{ for every } x_0 \in \partial\Omega; \\ \text{(vi)} & h_0 := \sup\limits_{x \in \Omega,\, y \neq 0} \dfrac{\langle h''(x) y, y \rangle}{|y|^2} < \infty. \end{array}$$

Let $U \in C^2(\Omega, \mathbb{R}^+)$ be defined as follows:

$$(2.2)\qquad U(x) = \frac{1}{h^2(x)} - 1,$$

(the term -1 has been added so that $U(x) = 0$ for any x far from $\partial\Omega$: this will semplify the notation) and let $V \in C^2(\bar{\Omega}, \mathbb{R}^+)$.

The following proposition shows that a bounce solution can be obtained by a suitable approximation scheme. The proposition is somewhat more general of what we need. It uses a "concept" of generalized solution used in [7] to [11], and [15] which allows solutions which may spend some time lying on $\partial\Omega$.

(2.3) Proposition. — *Let* $T > 0$ *and* $\varepsilon > 0$. *Let* $\gamma_\varepsilon \in C^2([0, T], \Omega)$ *a* T-*periodic solution of the Lagrangian system*:

$$(2.4)\qquad \gamma_\varepsilon'' + \nabla V(\gamma_\varepsilon) + \varepsilon \nabla U(\gamma_\varepsilon) = 0$$

such that:

$$(2.5)\qquad E(\gamma_\varepsilon) := \frac{1}{2}|\gamma_\varepsilon'|^2 + V(\gamma_\varepsilon) + \varepsilon\, U(\gamma_\varepsilon) \leqq K\,(^1)$$

(1) Notice that $E(\gamma_\varepsilon)$ is a constant of the motion, *i.e.* the energy of γ_ε.

where K *is a real constant independent of* ε.

Then γ_ε *has a subsequence convergent in* $H^1(S^1, \bar{\Omega})$ ([2]) *to a curve* $\gamma \in H^1(S^1, \bar{\Omega})$ *satisfying the following properties:*

$$(2.6) \qquad \gamma \text{ is Lipschitz continuous;}$$

there is a positive finite real Borel measure μ *on* $[0, T]$ *with* $\operatorname{supt} \mu \subset C(\gamma) := \{ t \in [0, T] : \gamma(t) \in \partial\Omega \}$ *such that* $\gamma'' = -\nabla V(\gamma) - \nu(\gamma)\mu$ *in the distributions sense, i. e.*

$$(2.7) \qquad \int_0^T \langle \gamma', v' \rangle \, dt - \int_0^T \langle \nabla V(\gamma), v \rangle \, dt = \int_{C(\gamma)} \langle \nu(\gamma), v \rangle \, d\mu$$

for every $v \in C^\infty([0, T], \mathbb{R}^N)$ *such that* $v(0) = v(T)$:
γ *has left and right derivative in every* $t \in [0, T]$ *and*

$$(2.8) \qquad \frac{1}{2} |\gamma'_\pm(t_2)|^2 - \frac{1}{2} |\gamma'_\pm(t_1)|^2 = V(\gamma(t_1) - V(\gamma(t_2))$$

for every $t_1, t_2 \in [0, T]$;

$$(2.9) \quad \gamma'_+(t) - \langle \gamma'_+(t), \nu(\gamma(t)) \rangle \nu(\gamma(t)) = \gamma'_-(t) - \langle \gamma'_-(t), \nu(\gamma(t)) \rangle \nu(\gamma(t))$$

for every $t \in C(\gamma)$;

$$(2.10) \qquad \langle \gamma'_+(t), \nu(\gamma(t)) \rangle = -\langle \gamma'_-(t), \nu(\gamma(t)) \rangle$$

for every $t \in C(\gamma)$.

Proof. – By (2.4) we have

$$(2.11) \quad \int_0^T \langle \gamma'_\varepsilon, v' \rangle \, dt - \int_0^T \langle \nabla V(\gamma_\varepsilon), v \rangle \, dt - \varepsilon \int_0^T \langle \nabla U(\gamma_\varepsilon), v \rangle \, dt = 0$$

for every $v \in H^1(S^1; \mathbb{R}^N)$.

Let $v_\varepsilon = -\nabla h(\gamma_\varepsilon)$. By (2.5) γ'_ε is bounded in L^∞ because we have supposed $U(x) \geqq 0$, $V(x) \geqq 0$ for every $x \in \Omega$. Moreover by (2.1) (vi) $v'_\varepsilon = -h''(\gamma_\varepsilon)\gamma'_\varepsilon$ is bounded in L^∞. Since also $\langle \nabla V(\gamma_\varepsilon), v_\varepsilon \rangle$ is bounded in L^∞, by (2.11) we get that

$$\varepsilon \int_0^T \langle \nabla U(\gamma_\varepsilon), v_\varepsilon \rangle \, dt = 2\varepsilon \int_0^T \frac{|\nabla h(\gamma_\varepsilon)|^2}{h^3(\gamma_\varepsilon)} \, dt \; .$$

is bounded independently of ε. By (2.1) (v) $|\nabla h(x)| \geqq 1/2$ in a neighbourhood of $\partial\Omega$, therefore there exists M_0 independent of ε such that

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([2]) Here $H^1(S^1, \bar{\Omega}) = \{ q \in AC(0, T; \bar{\Omega}) : q' \in L^2(0, T; \mathbb{R}^N), \; q(0) = q(T) \}$.

(2.12)
$$\int_0^T \frac{2\varepsilon}{h^3(\gamma_\varepsilon)} dt \leqq M_0.$$

Then $\varepsilon < \nabla U(\gamma_\varepsilon) = \dfrac{-2\varepsilon \nabla h(\gamma_\varepsilon)}{h^3(\gamma_\varepsilon)}$ is bounded in L^1, hence, by (2.4), γ_ε'' is bounded in L^1.

Since for every $1 < p < +\infty$ $H^{1,1}([0, T]; \mathbb{R}^N)$ is compactly embedded in L^p, up to a subsequence, there exists $\gamma \in H^1(S^1; \mathbb{R}^N)$ such that $\gamma_\varepsilon \to \gamma$ in H^1 (and uniformly). Obviously $\gamma(t) \in \bar{\Omega}$, $\forall t \in [0, T]$, $\gamma(0) = \gamma(T)$ and γ is Lipchitz continuous.

By (2.12), the sequence of positive real functions $\dfrac{2\varepsilon}{h^3(\gamma_\varepsilon)}$ converges (up to a subsequence) in L^1-weak*. Since $[L^1(S^1; \mathbb{R})]^* \subset [C^0(S^1; \mathbb{R})]^*$ (where [ ]* denotes the dual space) we get that

$$\frac{2\varepsilon}{h^3(\gamma_\varepsilon)} \to \mu \in [C^0(S^1; \mathbb{R})]^* \text{ weakly.}$$

By well known theorems, μ is a positive finite Borel measure. Moreover if $\bar{t} \notin C(\gamma)$ we have that $\varepsilon U(\gamma_\varepsilon) \to 0$ uniformly in a neighbourhood of $\bar{t}$, therefore supt $\mu \subset C(\gamma)$.

Since (2.1) (v) holds, when ε tends to 0 by (2.11) we get (2.7).

By (2.7) $\gamma' \in BV(S^1; \mathbb{R}^N)$ (³) and (2.9) holds.

To prove (2.8) we shall need the following property:

(2.13)
$$\lim \varepsilon U(\gamma_\varepsilon(t)) = 0 \quad a.e. \text{ in } [0, T],$$

up to a subsequence.

Since $\gamma_\varepsilon' \to \gamma'$ in L^2, up to a subsequence, $\gamma_\varepsilon' \to \gamma'$ *a.e.* in $[0, T]$. Since $\varepsilon U(x) \geqq 0$, $\forall x \in \Omega$, the real number $E(\gamma_\varepsilon)$ defined at (2.5) is bounded indepently of ε, therefore there exists $w \in L^\infty([0, T]; \mathbb{R}^N)$ such that

$$\varepsilon U(\gamma_\varepsilon(t)) \to w(t) \quad a.e. \text{ in } [0, T].$$

We claim that $w(t) = 0$ *a.e.* Indeed

$$\varepsilon U(\gamma_\varepsilon(t)) = \frac{\varepsilon}{h^2(\gamma_\varepsilon(t))}$$

and

(³) Then γ has left and right derivative in every $t \in S^1$ which are left continuous and right continuous respectively.

$$\varepsilon \nabla U(\gamma_\varepsilon(t)) = \varepsilon \frac{-2\nabla h(\gamma_\varepsilon(t)}{h(\gamma_\varepsilon(t))} U(\gamma_\varepsilon(t)).$$

Therefore if $w(t) \neq 0$ on a set $E \subset [0, T]$ having positive Lebesgue measure, we have $|\varepsilon \nabla U(\gamma_\varepsilon(t))| \to +\infty, \forall t \in E$, hence, by Fatou Lemma,

$$\liminf_{\varepsilon \to 0} \int_E |\varepsilon \nabla U(\gamma_\varepsilon(t))| \, dt = +\infty$$

in contradiction with the boundness of $\varepsilon \nabla U(\gamma_\varepsilon(t))$ in L^1.

By (2.13) and (2.5)

$$\frac{1}{2}|\gamma'(t_2)|^2 - \frac{1}{2}|\gamma'(t_1)|^2 = V(\gamma(t_1) - V(\gamma(t_2))$$

for almost every $t_1, t_2 \in [0, T]$. Since the left derivative of γ is left continuous and the right derivative is right continuous we get (2.8).

By (2.8) with $t_1 = t_2$ we get $|\gamma'_+(t)| = |\gamma'(t)|$, $\forall t \in [0, T]$. Then, since (2.9) holds, it must be

$$|\langle \gamma'_+(t), \nu(\gamma(t)) \rangle| = |\langle \gamma'_-(t), \nu(\gamma(t)) \rangle|$$

for every $t \in C(\gamma)$. If $\langle \gamma'_+(t), \nu(\gamma(t)) \rangle \neq 0$ it must be

$$\langle \gamma'_+(t), \nu(\gamma(t)) \rangle = -\langle \gamma'_-(t), \nu(\gamma(t)) \rangle$$

because $\gamma(t) \in \bar{\Omega}\ \forall t$. Then (2.10) is proved. ∎

(2.14) *Remark.* — For every couple $(\gamma_0, p_0) \in \Omega \times \mathbb{R}^N$ the Cauchy problem has at least one solution, *i. e.* there exists a curve γ with initial conditions

$$\begin{cases} \gamma(t_0) = \gamma_0 \\ \gamma'(t_0) = p_0 \end{cases} \tag{2.15}$$

which satisfies (2.7)-(2.10).

Proof. — It is easy to check that the equation (2.4) has always a unique solution γ_ε satisfying (2.15) for every $t \in \mathbb{R}$ and its energy is

$$\frac{1}{2} p_0^2 + V(\gamma_0) + \varepsilon U(\gamma_0).$$

For any $T > 0$ by (2.4) we have

$$\int_{-T}^{T} \langle \gamma''_\varepsilon + \nabla V(\gamma_\varepsilon) + \varepsilon \nabla U(\gamma_\varepsilon), v \rangle = 0$$

for every $v \in H^1([-T, T]; \mathbb{R}^N)$. Therefore

$$\int_{-T}^{T} \langle \gamma'_\varepsilon, v' \rangle dt - \int_{-T}^{T} \langle \nabla V(\gamma_\varepsilon), v \rangle dt - \varepsilon \int_{-T}^{T} \langle \nabla U(\gamma_\varepsilon), v \rangle dt = \langle \gamma'_\varepsilon(T), v(T) \rangle - \langle \gamma'_\varepsilon(-T), v(-T) \rangle$$

for every $v \in H^1([-T, T]; \mathbb{R}^N)$.

At this point, since γ'_ε is bounded in L^∞ independently of ε, as in the proof of Proposition (2.3) we get the conclusion. ■

3. THE EXISTENCE OF A SOLUTION OF THE APPROXIMATING PROBLEM

To enunciate the abstract theorem which we use to study the approximating problem we recall the Palais-Smale condition and the definition of Morse index.

Let X be a real Hilbert space with norm $\| \ \|$ and scalar product $\langle \ , \ \rangle$ and let Λ be an open set in X. If $J \in C^1(\Lambda, \mathbb{R})$, J' will denote its Frechet derivative which can be identified, by virtue of $\langle \ , \ \rangle$ with a function from Λ to X.

(3.1) DEFINITION. — *We say that* J *satisfies the Palais-Smale condition (P.S.) on* Λ *if every sequence* γ_n *such that* $J(\gamma_n)$ *is bounded and* $J'(\gamma_n) \to 0$ *has a subsequence which converges to* $\bar{\gamma} \in \Lambda$.

(3.2) DEFINITION. — *Let* $J \in C^2(\Lambda, \mathbb{R})$ *and* $\gamma \in \Lambda$ *such that* $J'(\gamma) = 0$. *We call Morse index of* γ *the dimension of the space spanned by the eigenvectors of* $J''(\gamma)$ *corresponding to the strictly negative eigenvalues.*

We denote by $m(\gamma)$ *the morse index of* γ.

(3.3) LEMMA. — *Let* Λ *be an open subset of the real Hilbert space* X. *Let* $J \in C^2(A, \mathbb{R})$, $0 \in \Lambda$, $J(0) \leqq 0$. *Assume that:*

(J_1) *if* $\gamma_n \to \gamma_0 \in \partial\Lambda$ *then* $J(\gamma_n) \to -\infty$;

(J_2) J *satisfies* (*P.S.*) *on* Λ;

(J_3) there exists an N-*dimensional space* E_N ($N \geqq 1$) *such that:*

(i) $$J_{|E_N \cap \Lambda} \leqq 0$$

(ii) there exists $\rho > 0$, $\alpha > 0$ such that $B_\rho := \{\gamma \in X : \|\gamma\| \leqq \rho\} \subset \Lambda$ and $\inf_S J > \alpha$, where $S = \partial B_\rho \cap E_N^\perp$ and $E_N^\perp = \{v \in X : \langle v, w \rangle = 0 \ \forall w \in E_N\}$;

(iii) there exists $e \in E_N^{\perp} \setminus \{0\}$ such that the set

$$Q_\Lambda = \{ y + re : y \in E_N, r \geqq 0 \} \cap \Lambda$$

is bounded.

Then if $\beta < +\infty$ *is such that*

$$\sup_{Q_\Lambda} J < \beta,$$

J *has a critical point* γ (4) *such that:*

$$a < J(\gamma) < \beta$$

and

$$m(\gamma) \leqq N + 1.$$

The existence of a critical point γ such that $\alpha < J(\gamma) < \beta$ can be obtained by a slight variant of the linking theorems (*see e.g.* [1, 17] and its proof can be carried out in a similar way.

Indeed if we put $J(\gamma) = -\infty$ $\forall \gamma \in X \setminus \Lambda$, because of (J_1), (J_3) (i) and (J_3) (iii), there exists $R > 0$ such that

$$Q_\Lambda \subset = y + re : y \in E_N, \| y \| \leqq R, 0 \leqq r \leqq R \}.$$

$$\sup_{\partial Q} J \leqq 0 \text{ and } \sup_{Q} J < \beta.$$

Moreover S and ∂Q link (*see* Proposition (2.2) of [1]), so using (J_1) and (J_2) we are able to prove the existence of a critical point $\gamma \in \Lambda$ such that $\alpha < J(\gamma) < \beta$.

To get the estimate on the Morse index of the critical point γ, we use a generalization of the Morse-Conley index (*see* [3]). In fact Lemma (3.3) can be obtained as a variant of Corollary (3.19) of [4] (*see* also [5]).

We refer to the appendix where an idea of the proof is given.

Now we are able to prove the existence of a solution for the approximating problem using a technique introduced in [2]. Actually here the situation is simpler because J satisfies (P.S.) on Λ and $J(\gamma_n)$ tends to $-\infty$ when γ_n approaches $\partial\Lambda$. By Lemma (3.3) we get also an estimate of the Morse index of the approximating solution. This estimate will be used to give the estimate of the bounce points of the solution.

(3.4) Proposition. — *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set with boundary* $\partial\Omega$ *of class* C^2, $V \in C^2(\bar{\Omega}, \mathbb{R}^+)$ *and* $U \in C^2(\Omega, \mathbb{R}^+)$ *be the function defined at* (2.2).

(4) *i.e.* $J'(\gamma) = 0$.

Then there exist $T_0>0$, *depending of* Ω *and* V, *such that for every* $T\in(0,T_0)$ *and* $\varepsilon>0$ *there exists* $\gamma_\varepsilon\in C^2(\mathbb{R},\Omega)$, T-*periodic solution of the Lagrangian system* (2.4), *verifying the following properties*:

(i) $$0<E^-\leqq E(\gamma_\varepsilon)\leqq E^+,$$

where $E^-, E^+\in\mathbb{R}^+\setminus\{0\}$ *do not depend on* ε *and the energy* $E(\gamma_\varepsilon)$ *is defined at* (2.5);

(ii) $$0<\alpha<J_\varepsilon(\gamma_\varepsilon)<\beta$$

where α, $\beta\in\mathbb{R}^+\setminus\{0\}$ *do not depend on* ε, $J_\varepsilon\in C^2(\Lambda,\mathbb{R})$ *is the functional*

$$J_\varepsilon(\gamma)=\frac{1}{2}\int_0^T|\gamma'|^2\,dt-\int_0^T V(\gamma)\,dt-\varepsilon\int_0^T U(\gamma)\,dt, \tag{3.5}$$

and

$$\Lambda=\{\gamma\in H^1(0,T;\mathbb{R}^N):\gamma(0)=\gamma(T),\gamma(t)\in\Omega,\forall\, t\in[0,T]\};$$

(iii) $$\frac{1}{2}\int_0^T|\gamma_\varepsilon'|^2\,dt>\alpha\geqq\int_0^T\langle\nabla V(\gamma_\varepsilon),\gamma_\varepsilon\rangle\,dt;$$

(iv) $$m(\gamma_\varepsilon)\leqq N+1.$$

In order to prove Proposition (3.4) applying Lemma (3.3), we need some preliminary notations and results. Let

$$X=\{\gamma\in H^1(0,T;\mathbb{R}^N):\gamma(0)=\gamma(T)\}$$

with inner product

$$\langle v,w\rangle_X=\int_0^T\langle v',w'\rangle\,dt+\left\langle\int_0^T v\,dt,\int_0^T w\,dt\right\rangle$$

where $\langle\ ,\ \rangle$ is the standard inner product in $\mathbb{R}^N$.

Let Λ be as in the statement of Proposition (3.4), that is

$$\Lambda=\{\gamma\in X:\gamma(t)\in\Omega,\forall\, t\in[0,T]\}.$$

It is easy to check that

$$J_\varepsilon'(\gamma)\,v=\int_0^T\langle\gamma',v'\rangle\,dt-\int_0^T\langle\nabla V(\gamma),v\rangle\,dt-\varepsilon\int_0^T\langle\nabla U(\gamma),v\rangle\,dt$$

for every $\gamma\in\Lambda$, for every $v\in X$.

If γ_ε is a critical point for J (that is $J'(\gamma_\varepsilon)\,v=0$ $\forall\, v\in X$) then γ_ε is the restriction to the interval $[0,T]$ of a T-periodic solution of (2.4).

(3.6) LEMMA. — *Let* $(\gamma_n)\subset\Lambda$ *be a sequence converging to* γ *weakly in* H^1. *Assume that* $\gamma\in\partial\Lambda$. *Then*

$$\lim_{n \to +\infty} \int_0^T \frac{1}{h^2(\gamma_n(t))} dt = +\infty.$$

Proof. — Since $\gamma \in \partial\Lambda$, there exists $t_0 \in [0, T]$ such that $\gamma(t_0) \in \partial\Omega$. Obviously we can suppose $t_0 = 0$. We have

$$|\gamma_n(t) - \gamma_n(0)| \leqq \int_0^t |\gamma_n'| \, ds \leqq t^{1/2} \left(\int_0^T |\gamma_n'|^2 \, ds \right)^{1/2} \leqq t^{1/2} \| \gamma_n \|_X.$$

Since (2.1) (iv) holds and $\| \gamma_n \|_X \leqq C$ for some $C > 0$, we have

$$|h(\gamma_n(t)) - h(\gamma_n(0))| \leqq |\gamma_n(t) - \gamma_n(0)| \leqq t^{1/2} \| \gamma_n \|_X \leqq t^{1/2} C.$$

Since γ_n converges to γ weakly in H^1, γ_n converges to γ also in L^∞. In particular $\gamma_n(0) \to \gamma(0) \in \partial\Omega$. Then $h(\gamma_n(0)) \to 0$. Let $b_n = h(\gamma_n(0))$. We have

$$h(\gamma_n(t)) \leqq b_n + t^{1/2} C.$$

Then

$$\frac{1}{h^2(\gamma_n(t))} \geqq \frac{1}{(b_n + t^{1/2} C)^2} \geqq \frac{1}{2} \left(\frac{1}{b_n^2 + C^2 t} \right)$$

hence

$$\int_0^T \frac{1}{h^2(\gamma_n(t))} dt \geqq \frac{1}{2} \int_0^T \frac{1}{b_n^2 + C^2 t} dt = \left(\frac{1}{2C^2} \right) \log \left(1 + \frac{C^2 T}{b_n^2} \right).$$

Since $b_n \to 0$ we get the thesis. ∎

(3.7) Lemma. — *Let $(\gamma_n) \subset \Lambda$ be a sequence such that $J_\varepsilon(\gamma_n)$ is bounded from above and $J_\varepsilon'(\gamma_n) \to 0$.*

Then there exists a subsequence $\gamma_{n_k} \to \bar{\gamma} \in \Lambda$. In particular J_ε satisfies (P.S.) on Λ.

Proof. — Since for every $x_0 \in \partial\Omega$

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} \frac{\langle \nabla U(x), -\nabla h(x) \rangle}{U(x)} = +\infty,$$

and Ω is bounded, for every $\delta > 0$ there exists $a_\delta \in \mathbb{R}^+$ such that

$$U(x) \leqq \delta < \nabla U(x), -\nabla h(x) > + a_\delta \tag{3.8}$$

for every $x \in \Omega$.

Since $J'(\gamma_n) \to 0$ we have

$$\int_0^T \langle \gamma_n', v' \rangle dt - \int_0^T \langle \nabla V(\gamma_n), v \rangle dt - \varepsilon \int_0^T \langle \nabla U(\gamma_n), v \rangle dt = a_n \| v \|_X \tag{3.9}$$

for every $v \in X$, where $a_n \to 0$.

Because of (2.1) (vi) $-\nabla h(\gamma_n) \in X$, then by (3.8), (3.9), (2.1) (vi) and (2.1) (iv) we get

$$\begin{aligned}\varepsilon \int_0^T U(\gamma_n)\,dt \leqq \delta\varepsilon \int_0^T &< \nabla U(\gamma_n), -\nabla h(\gamma_n)\,dt + T a_\delta \\ &\leqq \delta\left[h_0 \int_0^T |\gamma_n'|^2\,dt + T \sup_{\Omega\bar\Omega} |\nabla V| \right.\\ &\left. + |a_n| \left(h_0 \left(\int_0^T |\gamma_n'|^2\,dt\right)^{1/2} + T\right)\right] + T a_\delta.\end{aligned}$$

Then there exists M_1 independent of n such that

$$\begin{aligned}\varepsilon \int_0^T U(\gamma_n)\,dt &\leqq \delta\left[2h_0 \int_0^T |\gamma_n'|^2\,dt + M_1\right] + T a_\delta \\ &= \delta\left[4h_0 J(\gamma_n) + 4h_0 \int_0^T V(\gamma_n)\,dt + 4h_0\varepsilon \int_0^T U(\gamma_n)\,dt + M_1\right] + T a_\delta.\end{aligned}$$

Since $J(\gamma_n)$ is bounded from above there exists M_2 independent of n such that

$$\varepsilon \int_0^T U(\gamma_n)\,dt \leqq \delta\left[4h_0\varepsilon \int_0^T U(\gamma_n)\,dt + M_2\right] + T a_\delta.$$

Then if $4h_0\delta = 1/2$ we have

$$\frac{1}{2}\int_0^T U(\gamma_n)\,dt \leqq M \tag{3.10}$$

where M is a constant independent of n.

Now $J(\gamma_n)$ is bounded from above, therefore by (3.10) $\int_0^T |\gamma_n'|^2\,dt$ is bounded. Then, up to a subsequence, γ_n is weakly convergent in H^1 (and strongly in L^∞) to $\bar\gamma \in X$ such that $\bar\gamma(t) \in \bar\Omega$ for every $t \in [0, T]$.

By (3.10) and Lemma (3.6) $\bar\gamma(t) \in \Omega$, $\forall t \in [0, T]$.

At this point by standard argument we can easily prove that the subsequence γ_n is strongly convergent in H^1 to $\bar\gamma \in \Lambda$. ■

Proof of Proposition (3.4). – By Lemma (3.6) and Lemma (3.7) J_ε satisfies (J_1) and (J_2).

Obviously we can suppose $O \in \Omega$. Let us pose

$$E_N = \{\gamma \in X : \gamma \text{ is constant}\},$$
$$E_N^\perp = \left\{\gamma \in X : \int_0^T \gamma\,dt = 0\right\},$$

and

$$S_\rho=\{\gamma\in E_N^\perp : \|\gamma\|_X=\rho\}$$

where $\rho>0$.

Since $O\in\Omega$ we can suppose that there exists $\rho_0>0$ such that the function h defined at (2.1) is equal to 1 for every x such that $|x|\leqq\rho_0$. Then we have

$$U(x)=0, \forall x : |x|\leqq\rho_0. \tag{3.11}$$

Moreover, since $V(x)\geqq 0$ for every $x\in\bar\Omega$,

$$J_\varepsilon(0)\leqq 0 \qquad \text{and} \qquad J_{|E_N\cap\Lambda}\leqq 0.$$

Now we choose

$$T_0=\min\left\{\rho_0^2, \frac{1}{4(\sup_{x\in\bar\Omega} V(x))}, \frac{1}{4(\sup_{x\in\bar\Omega}|\nabla V(x)|)(\sup_{x\in\bar\Omega}|x|)}\right\}$$

If $\gamma\in E_N^\perp$ we have

$$|\gamma(t)|\leqq\int_0^T|\gamma'|\,ds$$

for every $t\in[0, T]$, therefore

$$\|\gamma\|_{L^\infty}\leqq(T)^{1/2}\left(\int_0^T|\gamma'|^2\,ds\right)^{1/2}, \qquad \forall\gamma\in E_N^\perp$$

and

$$\|\gamma\|_{L^\infty}\leqq(T)^{1/2}\rho, \qquad \forall\gamma\in S_\rho. \tag{3.12}$$

Let $\rho=1$ and $S=S_1$. Since $T<T_0\leqq\rho_0^2$, by (3.11) and (3.12) we have

$$U(\gamma(t))=0, \qquad \forall t\in[0, T], \quad \forall\gamma\in S.$$

Then for every $\gamma\in S$

$$J_\varepsilon(\gamma)=\frac{1}{2}\int_0^T|\gamma'|^2\,dt-\int_0^T V(\gamma)\,dt\geqq\frac{1}{2}-T\sup_{\Omega\bar\Omega}V.$$

Since $T<T_0\leqq\dfrac{1}{4(\sup_{x\in\bar\Omega}V(x))}$, we have $\dfrac{1}{2}-T\sup_{\bar\Omega}V>\dfrac{1}{4}$, hence

$$J_\varepsilon(\gamma)>\frac{1}{4}:=\alpha, \qquad \forall\gamma\in S. \tag{3.13}$$

Moreover since $T < T_0 \leqq \dfrac{1}{4(\sup_{x\in\bar\Omega} |\nabla V(x)|)(\sup_{x\in\bar\Omega} |x|)}$ we have

$$\alpha = \frac{1}{4} \geqq T(\sup_{x\in\bar\Omega} |\nabla V(x)|)(\sup_{x\in\bar\Omega} |x|)$$

so we get

$$\alpha \geqq \int_0^T \langle \nabla V(\gamma), \gamma \rangle \, dt, \qquad \forall \gamma \in \Lambda. \tag{3.14}$$

Let $e \in \mathbb{R}^N$ with $\|e\| = 1$ and

$$Q_\Lambda = \left\{ E_N^\perp + r \sin\left(\frac{2\pi}{T} t\right) e : r \geqq 0 \right\} \cap \Lambda.$$

If $\gamma \in Q_\Lambda$

$$\gamma = y + r \sin\left(\frac{2\pi}{T} t\right) e \in \Omega, \forall t \in [0, T].$$

where $y \in \mathbb{R}^N$. Therefore

$$|y| < d, \qquad r < 2d, \qquad \text{where} \quad d = \sup_{x\in\bar\Omega} |x|.$$

Then Q_Λ is bounded in X and

$$J_\varepsilon(\gamma) \leqq \frac{1}{2} \int_0^T |\gamma'|^2 \, dt < \frac{4 d^2 \pi^2}{T} := \beta \tag{3.15}$$

for every $\gamma \in Q_\Lambda$.

Then by Lemma (3.3) J_ε has a critical point γ_ε (5) such that

$$0 < \alpha < J_\varepsilon(\gamma_\varepsilon) < \beta \tag{3.16}$$

and

$$m(\gamma_\varepsilon) \leqq N + 1. \tag{3.17}$$

Since $V(x) \geqq 0$, $U(x) \geqq 0$, $\forall x \in \Omega$, by (3.16) we have

$$\frac{1}{2} \int_0^T |\gamma_\varepsilon'|^2 \, dt > \alpha,$$

hence by (3.14), (iii) of Proposition (3.4) follows.

---

(5) Which is the restriction to [0, T] of a T-periodic solution of class C^2 of (2.4).

It remains to prove the estimate for $E(\gamma_\varepsilon)$. Since $E(\gamma_\varepsilon)$ is a constant of the motion

$$(3.18)\qquad TE(\gamma_\varepsilon)=\frac{1}{2}\int_0^T|\gamma_\varepsilon'|^2\,dt+\int_0^T V(\gamma_\varepsilon)\,dt+\varepsilon\int_0^T U(\gamma_\varepsilon)\,dt.$$

Since $V(x)\geqq 0$ and $U(x)\geqq 0$ $\forall x\in\Omega$, by (3.16) and (3.18) we get

$$\alpha\leqq TE(\gamma_\varepsilon)\leqq\beta+2T\sup_{x\in\bar\Omega}V(x)+2\varepsilon\int_0^T U(\gamma_\varepsilon)\,dt.$$

Moreover, as in the proof of Lemma (3.7) we get that $\varepsilon\int_0^T U(\gamma_\varepsilon)\,dt$ is bounded from above by a constant M independent of ε. Then Proposition (3.4) holds with $E^-=\frac{\alpha}{T}$ and $E^+=\frac{\beta}{T}+2\sup_{x\in\bar\Omega}V(x)+\frac{2M}{T}$. ■

4. PROOF OF THE MAIN RESULT

Now we want to find a bounce trajectory with at most N+1 bounce points (where N is the dimension of the space), using the approximation scheme introduced in section 2 and Lemma (3.3).

To prove Theorem (1.7) obviously we can suppose $V(x)\geqq 0$ $\forall x\in\bar\Omega$.

For every $\varepsilon>0$ let γ_ε be the curve found in Proposition (3.4). By Proposition (2.3), up to a subsequence, γ_ε is convergent in $H^1(S^1,\bar\Omega)$ to a curve γ: $[0,T]\to\bar\Omega$ which verifies (2.6), (2.7), (2.8), (2.9) and (2.10) and which is the restriction to $[0,T]$ of a T-periodic curve.

By (ii) of Proposition (3.4) γ is not constant (because V and U are positive on Ω).

To prove that γ has at most N+1 bounce points it is useful to introduce the following notions of "nonregular point for γ".

(4.1) Definition. — *Let γ as above. We say that $\bar t\in[0,T]$ is a "nonregular instant for γ" if there exists $\bar\delta>0$ such that for every $\delta\in(0,\bar\delta)$ the weak equation*

$$(4.2)\qquad \int_{\bar t-\delta}^{\bar t+\delta}\langle\gamma',v'\rangle\,dt-\int_{\bar t-\delta}^{\bar t+\delta}\langle\nabla V(\gamma),v\rangle\,dt=0,\qquad \forall v\in H_0^1(\bar t-\delta,\bar t+\delta;\mathbb{R}^N)$$

is not verified.

We call "nonregular points for γ*" the points* $\gamma(\bar{t})\in\partial\Omega$ *such that* $\bar{t}$ *is a nonregular instants for* γ.

(4.3) *Remark.* — Notice that if we prove that γ has at most $N+1$ nonregular instants, by Proposition (2.3) we get that they are bounce instants *i. e.* γ verifies (i), (ii) and (iii) of Definition (1.4), with $l1\leqq N+1$.

To prove Theorem (1.7) we need also the following Lemmas.

(4.4) LEMMA. — *Let* $\bar{t}$ *be a nonregular instant for* γ *and* $I_\delta=[\bar{t}-\delta,\bar{t}+\delta]$ *with* $\delta\in(0,T/2)$. *Then*

$$\liminf_{\varepsilon\to 0}\varepsilon\int_{I_\delta}\frac{1}{h^3(\gamma_\varepsilon)}dt>0.$$

Proof. — Since γ_ε satisfies (2.4) and $U(x)$ is defined by (2.2) we have

$$J'_\varepsilon(\gamma_\varepsilon)v=\int_0^T\langle\gamma'_\varepsilon,v'\rangle dt-\int_0^T\langle\nabla V(\gamma_\varepsilon),v\rangle dt+2\varepsilon\int_0^T\frac{\langle\nabla h(\gamma_\varepsilon),v\rangle}{h^3(\gamma_\varepsilon)}dt=0$$

for every $v\in H^1(S^1,\mathbb{R}^N)$.

If, up to a subsequence, $\lim_{\varepsilon\to 0}\varepsilon\int_{I_\delta}\frac{1}{h^3(\gamma_\varepsilon)}dt=0$, going to the limit in ε we get

$$\int_{I_\delta}\langle\gamma',v'\rangle dt-\int_{I_\delta}\langle\nabla V(\gamma),v\rangle dt=0 \tag{4.4}$$

for every $v\in H_0^1(I_\delta,\mathbb{R}^N)$, which contradicts the hypothesis. ■

(4.5) LEMMA. — *Let* $B=\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)<r_0\}$ *where* r_0 *is such that*

$$\mathrm{dist}(x,\partial\Omega)<r_0 \text{ implies } |\nabla h(x)|\geqq\frac{1}{2} \quad (^6).$$

If $\gamma(t_0)\in\partial\Omega$ *then there exist* $\varepsilon_0>0$ *and* $\delta_0>0$ *such that:*

$$\forall\delta<\delta_0,\qquad\forall\varepsilon<\varepsilon_0,\qquad\forall t\in(t_0-\delta,t_0+\delta),\qquad\gamma_\varepsilon(t)\in B$$

Proof. — Let ε_0 be such that

$$\mathrm{dist}(\gamma_\varepsilon(t_0),\gamma(t_0))\leqq r_0/2,\qquad\forall\varepsilon<\varepsilon_0.$$

---

[6] Notice that r_0 exists because of (2.1) (iv).

By (i) of Proposition (3.4) $\frac{1}{2}|\gamma_\varepsilon'|^2 \leqq E^+$, $\forall t \in [0, T], \forall \varepsilon > 0$, then it suffices to choose $\delta_0 = \left(\frac{r_0}{4E^+}\right)$. ■

Proof of Theorem (1.7). — Assume, by contradiction, that there exists $N+2$ bounce instants for γ, $t_1 < t_2 < \ldots < t_{N+2} \in [0, T]$.

For every j let δ_j be as in Lemma (4.5) and such that (4.2) is not verified for every $\delta < \delta_j$ with $\bar{t} = t_j$.

Let $\delta_0 \leqq \min\{\delta_1, \ldots, \delta_{N+2}\}$ such that $\forall \delta < \delta_0$, we have $t_{j+1} - t_j > 2\delta$ for every $j = 1, \ldots, N+1$, and $T + t_1 - t_{N+2} > 2\delta$.

Let $I_j = [t_j - \delta, t_j + \delta]$ and $I_j' = \left[t_j - \frac{\delta}{2}, t_j + \frac{\delta}{2}\right]$ with $\delta \in (0, \delta_0)$.

Moreover for every j let ε_j be as in Lemma (4.5), $\varepsilon_0 \leqq \min\{\varepsilon_1, \ldots, \varepsilon_{N+2}\}$ and $\varepsilon < \varepsilon_0$.

For every $j = 1, \ldots, N+2$ let $\varphi_j \in C^1([0, T], [0, 1])$ such that

$$\varphi_j(t) = 0, \qquad \forall t \in [0, T] \setminus I_j$$
$$\varphi_j(t) = 1, \qquad \forall t \in I_j'.$$

Let $v_{\varepsilon j}(t) = -\varphi_j(t) \nabla h(\gamma_\varepsilon(t))$. We have

$$\langle J_\varepsilon''(\gamma_\varepsilon) v_{\varepsilon j}, v_{\varepsilon j} \rangle = \int_0^T |v_{\varepsilon j}'|^2 dt - \int_0^T \langle V''(\gamma_\varepsilon) v_{\varepsilon j}, v_{\varepsilon j} \rangle dt$$
$$+ 2\varepsilon \int_0^T \frac{\langle h''(\gamma_\varepsilon) v_{\varepsilon j}, v_{\varepsilon j} \rangle}{h^3(\gamma_\varepsilon)} dt - 6\varepsilon \int_0^T \frac{\langle \nabla h(\gamma_\varepsilon), v_{\varepsilon j} \rangle^2}{h^4(\gamma_\varepsilon)} dt.$$

Since $\int_0^T |\gamma_\varepsilon'|^2 dt$ is bounded from above by a constant independent of ε, by (2.1) (vi) also $\int_0^T |v_{\varepsilon j}'|^2 dt$ is. Under our hypotheses $V \in C^2(\bar{\Omega}, \mathbb{R})$, therefore $\int_0^T \langle V''(\gamma_\varepsilon) v_{\varepsilon j}, v_{\varepsilon j} \rangle dt$ is bounded independently of ε. By (2.1) (vi) and (2.12) also $2\varepsilon \int_0^T \frac{\langle h''(\gamma_\varepsilon) v_{\varepsilon j}, v_{\varepsilon j} \rangle}{h^3(\gamma_\varepsilon)} dt$ is bounded by a constant independent of ε. Moreover we have

$$\varepsilon \int_0^T \frac{\langle \nabla h(\gamma_\varepsilon), v_{\varepsilon j} \rangle^2}{h^4(\gamma_\varepsilon)} dt \geqq \varepsilon \int_{I_j} \frac{\varphi_j |\nabla h(\gamma_\varepsilon)|^4}{h^4(\gamma_\varepsilon)} dt \geqq \varepsilon \int_{I_j'} \frac{\varphi_j^2 |\nabla h(\gamma_\varepsilon)|^4}{h^4(\gamma_\varepsilon)} dt$$
$$\geqq [\text{by Lemma (4.5)}] \frac{1}{16} \varepsilon \int_{I_j'} \frac{1}{h^4(\gamma_\varepsilon)} dt$$

$$\geqq (\text{by Hölder inequality}) \; \frac{1}{16}\varepsilon\left(\frac{1}{\delta}\right)^{1/3}\left(\int_{I'_j}\frac{1}{h^3(\gamma_\varepsilon)}dt\right)^{4/3}$$
$$=\frac{1}{16}\left(\frac{1}{\delta}\right)^{1/3}\left(\varepsilon\int_{I'_j}\frac{1}{h^3(\gamma_\varepsilon)}dt\right)\left(\int_{I'_j}\frac{1}{h^3(\gamma_\varepsilon)}dt\right)^{1/3}.$$

Now by Lemma (3.6) and Hölder inequality

$$\lim_{\varepsilon\to 0}\int_{I'_j}\frac{1}{h^3(\gamma_\varepsilon)}dt=+\infty,$$

therefore, since $\gamma(t_j)$ is a nonregular point for γ, by Lemma (4.4)

$$\lim_{\varepsilon\to 0}\langle J''_\varepsilon(\gamma_\varepsilon)\,v_{\varepsilon j}, v_\varepsilon\rangle=-\infty.$$

Let $\bar{\varepsilon}$ be such that $\langle J''_\varepsilon(\gamma_\varepsilon)\,v_{\varepsilon j}, v_{\varepsilon j}\rangle\leqq -1$ for every $\varepsilon\leqq\bar{\varepsilon}$ and for every $j=1,\ldots,N+2$.

Since the curves $v_{\varepsilon j}$ have mutually disjoint supports the bilinear form $\langle J''_\varepsilon(\gamma_\varepsilon)\,v, v\rangle$ is negative in the linear subspace of X generated by them, which has dimension at least $N+2$. Consequently $J''_\varepsilon(\gamma_\varepsilon)$ has at least $N+2$ strictly negative eigenvalues, hence

$$m(\gamma_\varepsilon)\geqq N+2,\qquad \forall\,\varepsilon\leqq\bar{\varepsilon},$$

and this contradicts (iv) of Proposition (3.4). Then γ has at most $N+1$ nonregular points.

Because of Remark (4.3) it remains to prove that γ has at least a bounce point. By contradiction if γ has not bounce points, $\gamma\in C^2(S^1,\bar{\Omega})$ and

$$\gamma''+\nabla V(\gamma)=0,\qquad \forall\, t\in S^1.$$

Then $\langle\gamma''+\nabla V(\gamma),\gamma\rangle=0$, $\forall\, t\in[0,T]$ and since γ is T-periodic

$$\frac{1}{2}\int_0^T|\gamma'|^2\,dt=\int_0^T\langle\nabla V(\gamma),\gamma\rangle\,dt$$

and this constradicts (iii) of Proposition (3.4).

Theorem (1.7) is so completely proved. ■

Proof of Corollary (1.8). — Let $T>0$. By Theorem (1.7) there exists $m_1>0$ such that there exists a T/m_1-periodic nonconstant bounce trajectory γ_1 with at most $N+1$ bounce instants. Obviously γ_1 is T-periodic and has at most $N+1$ bounce points. Let T/k_1 ($k_1\geqq m_1$) its minimal period.

Always by Theorem (1.7) there exists $m_2>k_1$ such that there exists a T/m_2-periodic nonconstant bounce trajectory γ_2 with at most $N+1$ bounce instants. Let T/k_2 its minimal period. Since $k_2 \geqq m_2 > k_1$ we have $T/k_2 \neq T/k_1$, *i.e.* the minimal periods of γ_1 and γ_2 are different.

In such a way we can found a sequence of nonconstant T-periodic bounce trajectories with at most $N+1$ bounce points having different minimal periods.

In order to prove the last statement notice that if for instance γ_r and γ_s are not geometrically different there exist $k_1, k_2 \in \mathbb{N}$, $k_1 \neq k_2$ such that $\gamma_r(t/k_1) = \gamma_s(t/k_2)$. Then for every t different from the bounce instants we have

$$-\nabla V(\gamma_r(t)) = \gamma_r''(t) = (k_1/k_2)^2 \gamma_s(k_1 t/k_2))$$
$$= -(k_1/k_2)^2 \nabla V(\gamma_s(k_1 t/k_2)) = -(k_1/k_2)^2 \nabla V(\gamma_r(t)),$$

therefore if $\{x \in \Omega : \nabla V(x) = 0\}$ does not includes linear paths γ_1 and γ_2 must be geometrically different. ■

APPENDIX

Sketch of the proof of Lemma (3.3)

To give an idea of the proof of Lemma (3.3) we can suppose, as in [5], Th. 7.1, that α and β are not critical level for J. We put

$$J^c = \{u \in \Lambda : J(u) < c\}$$

and

$$J_a^b = \{u \in \Lambda : a < J(u) < b\}.$$

Essentially we must prove that

$$i(J_\alpha^\beta) = \sum_{n \geq 0} \dim H_n(J^\beta, J^\alpha, \mathbb{R})\, t^n = t^{N+1} + \text{other possibly therms},$$

(*see* [3, 4, 5]).

Then it suffices to prove that $H_{N+1}(J^\beta, J^\alpha, \mathbb{R}) \neq 0$.

Now we put

$$\Delta^c = (X \setminus \Lambda) \cup J^c,$$

Since $\overline{X \setminus \Lambda} \subset \operatorname{int}(\Delta^\alpha)$, by the excision property we have

$$H_{N+1}(J^\beta, J^\alpha, \mathbb{R}) = H_{N+1}(\Delta^\beta, \Delta^\alpha, \mathbb{R}).$$

Let

$$Q=\{y+re : y\in E_N, \|y\|\leqq R, 0\leqq r\leqq R\}$$

where R is so large that

$$\partial Q \setminus E_N \subset X\setminus \Lambda.$$

It is known (*see e.g.* [4] or [5]) that $H_N(X\setminus S, \mathbb{R})\neq 0$ and $[\partial Q]$ is a generator, hence the map

$$i_{1,N}: \quad H_N(\partial Q, \mathbb{R}) \to H_N(X\setminus S, \mathbb{R})$$

is different from 0.

Since the diagram

$$\begin{array}{ccc} H_N(\partial Q, \mathbb{R}) & \xrightarrow{i_{1,N}} & H_N(X\setminus S, \mathbb{R}) \\ {\scriptstyle i_{2,N}}\downarrow & & i_{3,N} \\ H_N(\Delta^\alpha, \mathbb{R}) & & \end{array}$$

is commutative, $[\partial Q]$ is a generator in $H_N(\Delta^\alpha, \mathbb{R})$.

Let us consider the exact sequence

$$\to H_{N+1}(\Delta^\beta, \Delta^\alpha, \mathbb{R}) \xrightarrow{\partial_{N+1}} H_N(\Delta^\alpha, \mathbb{R}) \xrightarrow{i_N} H_N(\Delta^\beta, \mathbb{R}) \to.$$

Now ∂Q is homotopic to a point in Q and therefore also in Δ^β. Then we have

$$i_N([\partial Q])=0.$$

Since Im $\partial_{N+1}=\operatorname{Ker} i_N \ni [\partial Q]$ it must be $H_{N+1}(\Delta^\beta, \Delta^\alpha, \mathbb{R})\neq 0$. ■

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