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YVES RICHARD

LAURENT VERON

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Isotropic singularities of solutions of nonlinear elliptic inequalities

by

Yves RICHARD and Laurent VERON

Département de Mathématiques,
Faculté des Sciences, Parc de Grandmont 37200 Tours

ABSTRACT. — If g is nondecreasing function satisfying the weak singularities existence condition then all the positive solutions of $\Delta u \leq g(u) + f$ in $B_1(0) \setminus \{0\}$ where f is radial and integrable in $B_1(0)$ are isotropic in measure near 0. We apply this result to solutions of $\Delta u \pm g(u) = 0$ in particular when $g(r) \sim r|^{q-1}$, $g(r) \sim e^{\beta r}$, or $g(r) = r(L_n^+ r)^\alpha$.

Key words : Elliptic equations, fundamental solutions, singularities, convergence in measure.

RÉSUMÉ. — Si g est une fonction croissante sur $\mathbb{R}$ vérifiant la condition d'existence de singularités faibles et f une fonction intégrable radiale dans $B_1(0)$, alors toutes les solutions positives de $\Delta u \leq g(u) + f$ dans $B_1(0) \setminus \{0\}$ sont isotropes en mesure près de 0. Nous appliquons ce résultat aux solutions de $\Delta u \pm g(u) = 0$, en particulier quand $g(r) \sim r|^{q-1}$, $g(r) \sim e^{\beta r}$ ou $g(r) = r(L_n^+ r)^\alpha$.

Classification A.M.S. : 35J60.

0. INTRODUCTION

Let Ω be an open subset of $\mathbb{R}^N$ containing 0 and $\Omega' = \Omega \setminus \{0\}$. In the past few years many results about the behaviour near 0 of a positive function $u \in C^2(\Omega')$ satisfying

$$(0.1) \quad \Delta u = u^q$$

or

$$(0.2) \quad \Delta u = -u^q$$

($q > 1$) in Ω' have been published ([1], [2], [7], [8], [11], [23]). Although those equations are very different (existence or nonexistence of a comparison principle between their solutions), there exists a great similarity between them in the case $N \geq 3$ and $1 < q < N/(N-2)$ in the sense that there always exist solutions satisfying

$$(0.3) \quad \lim_{x \rightarrow 0} |x|^{N-2} u(x) = \gamma$$

with $\gamma > 0$, which implies that

$$(0.4) \quad \Delta u = u^q - C(N) \gamma \delta_0$$

or

$$(0.5) \quad \Delta u = -u^q - C(N) \gamma \delta_0$$

holds in $D'(\Omega)$ ([23], [11]) where δ_0 is the Dirac measure at 0 and $C(N) = (N-2) |S^{N-1}|$ if $N \geq 3$, $C(2) = 2\pi$, but the two proofs of this phenomenon run very differently. In fact the main point to notice is that for a u satisfying (0.3) u^q is integrable near 0 and this leads us to a new type of isotropy which is the key-stone for the study of isolated singularities of positive solutions of nonlinear elliptic inequalities of the following type

$$(0.6) \quad \Delta u \leq g(u) + f.$$

Assume $N \geq 3$, g is a continuous nondecreasing function defined on $[0, +\infty)$ satisfying the weak singularities existence condition

$$(0.7) \quad \int_0^1 g(r^{2-N}) r^{N-1} dr < +\infty,$$

$f \in L^1_{\text{loc}}(\Omega)$ is radial near 0 and $u \in C^2(\Omega')$ is a positive solution of (0.6) in Ω' . Then

(i) either there exists $\gamma \in [0, +\infty)$ such that $r^{N-2} u(r, \cdot)$ converges in measure on S^{N-1} to γ as r tends to 0,

(ii) or $\lim_{x \rightarrow 0} |x|^{N-2} u(x) = +\infty$.

In the case $N=2$ it is necessary to introduce the exponential order of growth of g [20]

$$(0.8) \quad a_g^+ = \inf \left\{ a > 0 : \int_0^{+\infty} e^{-ar} g(r) dr < +\infty \right\},$$

and we prove that *under the same conditions on f and u satisfying (0.6) in Ω' ; then*

– if $a_g^+ = 0$ we have either (i) or (ii) with $|x|^{2-N}$ replaced by $\text{Ln}(1/|x|)$

– if $a_g^+ > 0$ we have

(iii) *either there exists $\gamma \in [0, 2/a_g^+]$ such that $u(r, \cdot)/\text{Ln}(1/r)$ converges in measure to γ on S^1 as r tends to 0,*

(iv) *or $\lim_{x \rightarrow 0} u(x)/\text{Ln}(1/|x|) \geq 2/a_g^+$.*

Those results play an important role for the description of isolated singularities of nonnegative solutions of

$$(0.9) \quad \Delta u = g(u).$$

For example, when $N \geq 3$ we prove that *if g is nondecreasing and satisfies the weak singularities existence condition, then any $u \in C^2(\Omega')$ nonnegative and satisfying (0.9) in Ω' is such that $|x|^{N-2} u(x)$ converges to some $\gamma \in \mathbb{R}^+ \cup \{+\infty\}$ as x tends to 0.* This result extends to the case $N=2$ with some minor modifications. An other important tool for proving this type of result is Serrin and Ni's symmetry theorem [12].

When g has nonpositive values we prove that when $N \geq 3$ any nonnegative solution $u \in C^2(\Omega')$ of (0.9) is such that $r^{N-2} u(r, \cdot)$ converges in $L^1(S^{N-1})$ to some $\gamma \in [0, +\infty)$ as r tends to 0. Under a moderate growth assumption on g we prove that $\lim_{x \rightarrow 0} |x|^{N-2} u(x) = \gamma$. When $N=2$ the situation is quite

more complicated. Using a result due to John and Nirenberg we prove that *when g has nonpositive values and is of exponential or subexponential type any nonnegative solution u of (0.9) in Ω' satisfies*

$$(0.10) \quad \lim_{x \rightarrow 0} u(x)/\text{Ln}(1/|x|) = \gamma \in [0, 2/a_g^+].$$

The last section is devoted to the study of the behavior near 0 of positive solutions of

$$(0.11) \quad \Delta u = u(\text{Ln}^+ u)^a$$

in $\Omega' (\alpha > 0)$. This equation reduces to a Hamilton-Jacobi equation in setting $v = Ln^+ u$ and v satisfies

$$(0.12) \quad \Delta v + |Dv|^2 = v^\alpha$$

on $\{x \in \Omega' : u(x) \geq 1\}$. If we set $g(r) = r(Ln^+ r)^\alpha$, it is clear that (0.7) is always satisfied, hence for any $\gamma \geq 0$ there always exist solutions satisfying (0.3); however Vazquez *a priori* estimate condition

$$(0.13) \quad \int_{r_0}^{+\infty} \frac{ds}{\sqrt{s g(s)}} < +\infty$$

for some $r_0 > 0$ is satisfied if and only if $\alpha > 2$ and we prove the following:

Assume $N \geq 3$ and $u \in C^2(\Omega')$ is a nonnegative solution of (0.11) in Ω' ; then

– if $0 < \alpha \leq 2$

(i) either u can be extended to Ω as a C^2 solution of (0.11) in Ω

(ii) or there exists $\gamma > 0$ such that $\lim_{x \rightarrow 0} |x|^{N-2} u(x) = \gamma$.

– if $\alpha > 2$

(iii) either u behaves as in (i) or (ii)

(iv) or $u(x) = \gamma(\alpha, N) e^{\gamma(\alpha) |x|^{2/(2-\alpha)}} (1 + O(|x|^{2/(\alpha-2)}))$ near 0 with

$\gamma(\alpha) = \left(\frac{2}{\alpha-2}\right)^{2/(\alpha-2)}$ and $\gamma(\alpha, N) = e^{(\alpha-(N-1)(\alpha-2))/2\alpha}$. This result extends in dimension 2.

The contents of this article is the following:

1. Isotropic solutions of elliptic inequalities
2. Singular solutions of $\Delta u = \pm g(u)$
3. Singularities of $\Delta u = u(Ln^+ u)^\alpha$.

1. ISOTROPIC SOLUTIONS OF ELLIPTIC INEQUALITIES

Throughout this section Ω is an open subset of $\mathbb{R}^N$, $N \geq 2$ containing 0, $\Omega' = \Omega \setminus \{0\}$ and g is a nondecreasing function. For the sake of simplicity we shall assume that g is continuous. If $N \geq 3$ it is wellknown that the following condition

$$(1.1) \quad \int_0^1 g(r^{2-N}) r^{N-1} dr < +\infty,$$

is a necessary and sufficient condition for the existence for any $\gamma \geq 0$ of a solution ψ belonging to some appropriate Marcinkiewicz space of

$$(1.2) \quad -\Delta\psi + g(\psi) = C(N) \gamma \delta_0$$

in $D'(\Omega)$ [3], or equivalently of a solution of

$$(1.3) \quad -\Delta\psi + g(\psi) = 0$$

in Ω' with a weak singularity at 0, that is such that

$$(1.4) \quad \lim_{x \rightarrow 0} |x|^{N-2} u(x) = \gamma,$$

[22]. Moreover $g(\psi) \in L^1_{\text{loc}}(\Omega)$.

If $N=2$ the situation is more complicated and we define the exponential order of growth of g

$$(1.5) \quad a_g^+ = \inf \left\{ a > 0 : \int_0^{+\infty} e^{-ar} g(r) dr < +\infty \right\}$$

[20], and the condition $\gamma \in [0, 2/a_g^+]$ is a necessary and sufficient condition for the existence of a function $\psi \in C^2(\Omega')$ satisfying (1.3) in Ω' and

$$(1.6) \quad \lim_{x \rightarrow 0} \psi(x)/L_n(1/|x|) = \gamma.$$

Moreover for such a ψ , $g(\psi) \in L^1_{\text{loc}}(\Omega)$ and (1.2) holds in $D'(\Omega')$ [21]. Our first result is the following

PROPOSITION 1.1. — Assume $\bar{B}_R = \{x \in \mathbb{R}^N : |x| \leq R\} \subset \Omega$, $g(0) = 0$, $f \in L^1_{\text{loc}}(\Omega)$ is nonnegative and $u \in C^2(\Omega')$ is a nonnegative solution of

$$(1.7) \quad \Delta u \leq g(u) + f$$

in Ω' . If $v \in C^2(\bar{B}_R \setminus \{0\})$ is a radial nonnegative solution of

$$(1.8) \quad \Delta v = g(v)$$

in $B_R \setminus \{0\}$ such that $g(v + \bar{\delta}) \in L^1(B_R)$ for some $\bar{\delta} > 0$, then there exists $\alpha \geq 0$ such that for any $q \in [1, \infty)$

$$(1.9) \quad \lim_{x \rightarrow 0} |x|^{1-N} \int_{|y|=|x|} |\alpha - \omega(y)/\mu(y)|^q dS = 0$$

where $\omega = \inf(u, v)$, $\mu(x) = |x|^{2-N}$ if $N \geq 3$ and $\mu(x) = L_n(1/|x|)$ if $N=2$.

The main ingredient for proving this result is the following theorem due to Brezis and Lions [5].

LEMMA 1.1. — Assume $N \geq 2$, $\omega \in L^1_{\text{loc}}(\Omega')$ satisfies

$$\Delta \omega \in L^1_{\text{loc}}(\Omega') \text{ in the sense of distributions in } \Omega',$$

$$(1.10) \quad \omega \geq 0 \quad \text{a. e. in } \Omega',$$

$$\Delta \omega \leq a\omega + F \quad \text{a. e. in } \Omega',$$

where a is some nonnegative constant and $F \in L^1_{\text{loc}}(\Omega)$. Then $\omega \in L^1_{\text{loc}}(\Omega)$ and there exist $\alpha \geq 0$ and $\Phi \in L^1_{\text{loc}}(\Omega)$ such that

$$(1.11) \quad -\Delta \omega = \Phi + \alpha C(N) \delta_0$$

in $D'(\Omega)$.

LEMMA 1.2. — Assume $N \geq 2$, $h \in L^1(B_R)$ is radial and φ is a nonnegative radial solution of

$$(1.12) \quad -\Delta \varphi = h$$

in $D'(B_R \setminus \{0\})$ [resp. in $D'(B_R)$]. Then there exists $v \in [0, +\infty)$ such that $\lim_{x \rightarrow 0} \varphi(x)/\mu(x) = v$ [resp. $\lim_{x \rightarrow 0} \varphi(x)/\mu(x) = 0$].

Proof. — From Lemma 1.1 there exists $v \geq 0$ such that

$$(1.13) \quad -\Delta \varphi = h + v C(N) \delta_0$$

in $D'(B_R)$ and $\tilde{\varphi} = \varphi - v\mu$ satisfies (1.12) in $D'(B_R)$. Without any loss of generality we can assume that h is nonnegative in $B(0, R)$, hence $r \mapsto r^{N-1} \tilde{\varphi}_r(r)$ is nonincreasing and then keeps a constant sign near 0.

Case 1. — $r^{N-1} \tilde{\varphi}_r(r) > 0$ on $(0, \varepsilon]$. For n large enough define

$$(1.14) \quad \eta_n(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{1}{n}, \\ \frac{1}{2} \left(1 + \cos \left(n\pi \left(r - \frac{1}{n} \right) \right) \right) & \text{if } \frac{1}{n} \leq r \leq \frac{2}{n}, \\ 0 & \text{if } \frac{2}{n} \leq r \leq \varepsilon. \end{cases}$$

$0 \leq \eta_n \leq 1$ on $[0, \varepsilon]$ and $\int_0^\varepsilon \eta_{nr}(r) dr = -1$. From (1.12) we get

$$\left| \int_0^\varepsilon \tilde{\varphi}_r(r) \eta_{nr}(r) r^{N-1} dr \right| = \int_0^\varepsilon h(r) \eta_n(r) r^{N-1} dr.$$

Using the monotonicity of $r^{N-1} \varphi_r(r)$ we deduce

$$(1.15) \quad 0 \leq \left(\frac{2}{n} \right)^{N-1} \tilde{\varphi}_r \left(\frac{2}{n} \right) \leq \left| \int_{1/n}^{2/n} \tilde{\varphi}_r(r) \eta_{nr}(r) r^{N-1} dr \right| \leq \int_0^{2/n} h(r) r^{N-1} dr$$

which implies $\lim_{n \rightarrow +\infty} \left(\frac{2}{n}\right)^{N-1} \tilde{\varphi}_r\left(\frac{2}{n}\right) = 0$ and

$$(1.16) \quad \lim_{r \rightarrow 0} r^{N-1} \tilde{\varphi}_r(r) = 0.$$

Case 2. — $r^{N-1} \tilde{\varphi}_r(r) \leq 0$ on $(0, \varepsilon]$. Using the same method as above we get

$$(1.17) \quad 0 \leq -\left(\frac{1}{n}\right)^{N-1} \tilde{\varphi}_r\left(\frac{1}{n}\right) \leq \int_0^{2/n} h(r) r^{N-1} dr$$

which again implies (1.16).

From (1.16) it is clear that $\lim_{x \rightarrow 0} \tilde{\varphi}(x)/\mu(x) = 0$.

Proof of Proposition 1.1. — Let p be the $C^{1,1}$ even convex function defined on $\mathbb{R}$ by

$$p(t) = \begin{cases} |t| - \delta/2 & \text{for } |t| \geq \delta > 0 \\ t^2/2\delta & \text{for } |t| \leq \delta \end{cases}$$

and let ω_δ be $\frac{1}{2}(u+v-p(u-v))$. Then

$$(1.18) \quad \Delta\omega_\delta = \frac{1}{2}\Delta(u+v) - \frac{1}{2}p'(u-v)\Delta(u-v) - \frac{1}{2}p''(u-v)|\nabla(u-v)|^2$$

It is clear that $\Delta\omega_\delta \in L^1_{\text{loc}}(\mathbf{B}_R \setminus \{0\})$ and $0 \leq \omega \leq \omega_\delta \leq \omega + \delta/4$. Moreover

$$(1.19) \quad \Delta\omega_\delta \leq \frac{1}{2}\Delta(u+v) - \frac{1}{2}p'(u-v)\Delta(u-v) = F.$$

We now set $\mathbf{B}_R \setminus \{0\} = G_1 \cup G_2 \cup G_3$ with

$$(1.20) \quad \begin{aligned} G_1 &= \{x \in \mathbf{B}_R \setminus \{0\} : (u-v)(x) > \delta\} \\ G_2 &= \{x \in \mathbf{B}_R \setminus \{0\} : (u-v)(x) < -\delta\} \\ G_3 &= \{x \in \mathbf{B}_R \setminus \{0\} : |(u-v)(x)| \leq \delta\}. \end{aligned}$$

On G_1 , $p'(u-v) = 1$ and $F = \Delta v = g(v) = g\left(\omega_\delta - \frac{\delta}{4}\right)$. On G_2 , $p'(u-v) = -1$ and

$$F = \Delta u \leq g(u) + f = g\left(\omega_\delta - \frac{\delta}{4}\right) + f \leq g(v) + f.$$

On G_3 , $p'(u-v) = (u-v)/\delta$, hence

$$(1.21) \quad F = \frac{1}{2} \left(1 - \frac{u-v}{\delta} \right) \Delta u + \frac{1}{2} \left(1 + \frac{u-v}{\delta} \right) \Delta v \\ \leq \frac{1}{2} \left(1 - \frac{u-v}{\delta} \right) g(u) + \frac{1}{2} \left(1 + \frac{u-v}{\delta} \right) g(v) + f$$

and by the continuity of g there exists $\theta = \theta(x) \in [0, 1]$ such that $F \leq g(\theta u + (1-\theta)v) + f$. If we assume for example that $v \leq u \leq v + \delta$, then $F \leq g(u) + f$ and $0 \leq u - \omega_\delta \leq \frac{3}{4} \delta$ which implies that

$$F \leq g\left(\omega_\delta + \frac{3}{4} \delta\right) + f \leq g(v + \delta) + f.$$

We do the same if $u \leq v \leq u + \delta$ and finally

$$(1.22) \quad \Delta \omega_\delta \leq g\left(\omega_\delta + \frac{3}{4} \delta\right) + f \leq g(v + \delta) + f$$

holds in $B_R \setminus \{0\}$. We take now $\delta \leq \bar{\delta}$, so the right-hand side of (1.22) is integrable in B_R and there exists $\alpha \geq 0$ such that

$$(1.23) \quad -\Delta \omega_\delta = \Phi + \alpha C(N) \delta_0$$

in $D'(B_R)$ with $\Phi \in L^1_{\text{loc}}(B_R)$.

From Lemma 1.2. $\omega_\delta(x)/\mu(x)$ remains bounded near 0 and it is the same with $\varphi_\delta = \omega_\delta - \alpha\mu$. Moreover φ_δ satisfies

$$(1.24) \quad -\Delta \varphi_\delta = \Phi$$

in $D'(B_R)$. Let

$$\bar{\varphi}_\delta(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} \varphi_\delta(r, \sigma) d\sigma$$

and

$$\bar{\Phi}(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} \Phi(r, \sigma) d\sigma$$

be the spherical averages of φ_δ and Φ respectively, (r, σ) being the spherical coordinates in $\mathbb{R}^N \setminus \{0\}$, then

$$(1.25) \quad -\Delta \bar{\varphi}_\delta = \bar{\Phi} \leq |\bar{\Phi}|.$$

Applying Lemma 1.2 we deduce that $\lim_{r \rightarrow 0} \bar{\varphi}(r)/\mu(r) = 0$. As a consequence

$$\lim_{r \rightarrow 0} \int_{S^{N-1}} |\omega_\delta(r, \cdot)/\mu(r) - \alpha| d\sigma = 0,$$

which implies (with the uniform boundedness)

$$(1.26) \quad \lim_{r \rightarrow 0} \int_{S^{N-1}} |\omega_\delta(r, \cdot)/\mu(r) - \alpha|^q d\sigma = 0$$

for any $q \in [1, +\infty)$. As $0 \leq \omega \leq \omega_\delta \leq \omega + \delta/4$ we deduce

$$(1.27) \quad \lim_{r \rightarrow 0} \int_{S^{N-1}} |\omega(r, \cdot)/\mu(r) - \alpha|^q d\sigma = 0,$$

which is (1.9).

Remark 1.1. — As $\{\Delta\omega_\delta\} = \Phi$ is integrable in B_R and $\Phi = \Delta\omega_\delta = F - \frac{1}{2}p''(u-v)|\nabla(u-v)|^2$ we get

$$(1.28) \quad \frac{1}{2}p''(u-v)|\nabla(u-v)|^2 \leq \Phi + g(v+\delta) + f$$

and then $p''(u-v)|\nabla(u-v)|^2 \in L^1(B_R)$.

DEFINITION 1.1. — Assume (E, Σ, μ) is an abstract measure space where Σ is a σ -algebra of subsets of E and μ a positive σ -additive and complete measure such that $\mu(E) < +\infty$, and $\{\psi_r\}_{r \in (0, R)}$ a subset of measurable functions (for the measure μ) with value in $\mathbb{R}$. We say that $\{\psi_r\}$ converges in measure to some measurable function ψ as r tends to 0 if for any $\varepsilon > 0$ we have

$$(1.29) \quad \lim_{r \rightarrow 0} \mu(\{x \in E : |\psi_r(x) - \psi(x)| > \varepsilon\}) = 0.$$

It is equivalent to say that from any sequence $\{r_n\}$ converging to 0 we can extract a subsequence $\{r_{n_k}\}$ such that $\{\psi_{r_{n_k}}\}$ converges to ψ μ -a. e. on E as n_k goes to $+\infty$.

The generic isotropy result is the following

THEOREM 1.1. — Assume $N \geq 3$, g satisfies (1.1), $f \in L^1_{\text{loc}}(\Omega')$ is radial near 0 and $u \in C^2(\Omega')$ is nonnegative and satisfies

$$(1.30) \quad \Delta u \leq g(u) + f$$

in Ω' . Then we have the following

(i) either $r^{N-2} u(r, \cdot)$ converges in measure on S^{N-1} to some nonnegative real number γ as r tends to 0,

(ii) or

$$(1.31) \quad \lim_{x \rightarrow 0} |x|^{N-2} u(x) = +\infty.$$

Proof. — We recall that $(r, \sigma) \in (0, +\infty) \times S^{N-1}$ are the spherical coordinates in $\mathbb{R}^N \setminus \{0\}$. For $\lambda > 0$ let v_λ be the solution of

$$(1.32) \quad \begin{aligned} \Delta v_\lambda &= g(v_\lambda) + |f| \quad \text{in } B_R \setminus \{0\} \subset \Omega' \\ v_\lambda &= 0 \quad \text{on } \partial B_R \\ \lim_{x \rightarrow 0} |x|^{N-2} v_\lambda(x) &= \lambda. \end{aligned}$$

Such a v_λ exists, is radial and positive near 0. As $|f|$ is radial it does not affect the behaviour of v_λ near 0 (see Lemma 1.2).

From Proposition 1.1 there exists $v(\lambda) \geq 0$ such that

$$(1.33) \quad \lim_{r \rightarrow 0} r^{N-2} \inf(u(r, \cdot), v_\lambda(r)) = v(\lambda)$$

in $L^q(S^{N-1})$, $1 \leq q < +\infty$, and $v(\lambda) \leq \lambda$ from convexity. Moreover the function $\lambda \mapsto v(\lambda)$ is nondecreasing.

Case 1. — Assume $\lim_{\lambda \rightarrow +\infty} v(\lambda) = \gamma < +\infty$. For $\lambda > \gamma$ we have (1.33).

Assume $\{r_n\}$ is some sequence converging to 0, then there exists a subsequence $\{r_{n_k}\}$ such that

$$(1.34) \quad \lim_{n_k \rightarrow +\infty} r_{n_k}^{N-2} \inf(u(r_{n_k}, \sigma), v_\lambda(r_{n_k})) = v(\lambda) \quad \text{a.e. on } S^{N-1}.$$

As $v(\lambda) < \gamma$ and $\lim_{n_k \rightarrow +\infty} r_{n_k}^{N-2} v_\lambda(r_{n_k}) = \gamma$ we deduce that

$$\inf(u(r_{n_k}, \sigma), v_\lambda(r_{n_k})) = u(r_{n_k}, \sigma) \quad \text{a.e. on } S^{N-1}$$

for n_k large enough and

$$(1.35) \quad \lim_{n_k \rightarrow +\infty} r_{n_k}^{N-2} u(r_{n_k}, \sigma) = v(\lambda) \quad \text{a.e. on } S^{N-1}.$$

For $\lambda' > \lambda$ we repeat this operation with $\{r_n\}$ replaced by $\{r_{n_k}\}$ and there exists a subsequence $\{r_{n_{k_i}}\}$ such that

$$(1.36) \quad \lim_{n_{k_i} \rightarrow +\infty} r_{n_{k_i}}^{N-2} u(r_{n_{k_i}}, \sigma) = v(\lambda') \quad \text{a.e. on } S^{N-1}.$$

From (1.35) and (1.36) we deduce that $v(\lambda') = v(\lambda) = \gamma$ for $\lambda > \gamma$ which implies (i).

Case 2. — Assume $\lim_{\lambda \rightarrow +\infty} v(\lambda) = +\infty$. For $\delta > 0$ we call p the function introduced in the proof of Proposition 1.1 and for $\lambda > 0$, $\tilde{\omega}_\delta = \frac{1}{2}(u + v_\lambda - p(u - v_\lambda)) + \frac{3}{4}\delta$. From (1.22) we have

$$(1.37) \quad \Delta \tilde{\omega}_\delta \leq g(\tilde{\omega}_\delta) + |f|.$$

Moreover $r^{N-2} \tilde{\omega}_\delta(r, \cdot)$ converges to $v(\lambda)$ in $L^q(S^{N-1})$ ($1 \leq q < +\infty$) as r tends to 0. We consider now $w = v_{v(\lambda)}$ the solution of (1.32) and we set

$$s = \frac{r^{N-2}}{N-2},$$

$$w'(s) = r^{N-2} w(r), \quad \tilde{\omega}'_\delta(s, \sigma) = r^{N-2} \tilde{\omega}_\delta(r, \sigma), \quad \varphi(s) = f(r).$$

Then (1.32) and (1.37) become

$$(1.38) \quad s^2 (\omega'_\delta)_{ss} + \frac{1}{(N-2)^2} \Delta_{S^{N-1}} \tilde{\omega}'_\delta \leq k s^{N/(N-2)} \left(g\left(\frac{\tilde{\omega}'_\delta}{s(N-2)}\right) + \varphi \right),$$

$$s^2 w'_{ss} = k s^{N/(N-2)} \left(g\left(\frac{w'}{s(N-2)}\right) + |\varphi| \right),$$

where $k = k(N) = (N-2)^{(4-N)/(N-2)}$ and $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on S^{N-1} . Consider a C^∞ function ρ such that $\rho \in L^\infty(\mathbb{R})$, $\rho \equiv 0$ on $(-\infty, 0)$, $\rho' > 0$ on $(0, +\infty)$ and $j(r) = \int_0^r \rho(\tau) d\tau$. From convexity and monotonicity we have

$$(1.39) \quad s^2 \frac{d^2}{ds^2} \int_{S^{N-1}} j(w' - \omega'_\delta) d\sigma \geq 0.$$

As $\int_{S^{N-1}} j(w' - \omega'_\delta) d\sigma \leq C \int_{S^{N-1}} |w' - \omega'_\delta| d\sigma$ and as $w'(s)$ and $\tilde{\omega}'_\delta(s, \cdot)$ converges to $v(\lambda)$ in $L^1(S^{N-1})$ as s tends to 0 we deduce that $\int_{S^{N-1}} j(w' - \omega'_\delta) d\sigma = 0$ on $(0, R^{N-2}/(N-2)]$ and $w' \leq \tilde{\omega}'_\delta$ or

$$(1.40) \quad v_{v(\lambda)}(r) \leq \omega_\delta(r, \sigma) \leq \omega(r, \sigma) + \delta/4$$

which implies

$$(1.41) \quad v(\lambda) \leq \lim_{x \rightarrow 0} |x|^{N-2} \omega(x) \leq \lim_{x \rightarrow 0} |x|^{N-2} u(x)$$

and we get (1.31).

Remark 1.2. — If u satisfies (i) then $v_\gamma(x) \leq u(x)$ in $B_R \setminus \{0\}$.

Remark 1.3. — If u is a radial solution of (1.29), $u \geq 0$, in $B_R \setminus \{0\}$, then a simple adaptation of the proof of Theorem 1.1 shows that $|x|^{N-2} u(x)$ admits a limit in $[0, +\infty]$ as x tends to 0.

The 2-dimensional version of Theorem 1.1 is the following

THEOREM 1.2. — Assume $N=2$, $f \in L^1(\Omega)$ is radial near 0 and $u \in C^2(\Omega)$ is a nonnegative solution of (1.29) in Ω' . Then

— If $a_g^+ = 0$ the alternative of Theorem 1.1 holds with $|x|^{2-N}$ replaced by $\text{Ln}(1/|x|)$.

— If $a_g^+ > 0$, we have the following alternative

(i) either there exists a nonnegative real number $\gamma \in [0, 2/a_g^+)$ such that $u(r, \cdot)/\text{Ln}(1/r)$ converges in measure on S^1 to γ as r tends to 0,

(ii) or

$$(1.43) \quad \lim_{x \rightarrow 0} u(x)/\text{Ln}(1/|x|) \geq 2/a_g^+.$$

Proof. — Case 1. — Assume $a_g^+ = 0$. We define $v(\lambda)$ as

$$(1.44) \quad \lim_{r \rightarrow 0} (\text{Ln}(1/r))^{-1} \inf(u(r, \cdot), v_\lambda(r)) = v(\lambda).$$

As $v(\lambda)$ is nondecreasing and v_λ exists for every $\lambda > 0$ we can proceed as in the proof of Theorem 1.1 if $\lim_{\lambda \rightarrow +\infty} v(\lambda) = \gamma < +\infty$. If

$\lim_{\lambda \rightarrow +\infty} v(\lambda) = +\infty$ we introduce $\tilde{\omega}_\delta$ and $v_{v(\lambda)} = w$ as in Theorem 1.1 and

make the following change of variable

$$(1.45) \quad \begin{aligned} t &= \text{Ln}(1/r) \\ w'(t) &= w(r), \quad \tilde{\omega}'_\delta(t, \sigma) = \tilde{\omega}_\delta(r, \sigma), \quad f'(t) = f(r). \end{aligned}$$

Hence w' and $\tilde{\omega}'_\delta$ satisfies

$$(1.46) \quad \begin{aligned} (\tilde{\omega}'_\delta)_{tt} + (\tilde{\omega}'_\delta)_{\theta\theta} &\leq e^{-2t} (g(\omega'_\delta) + f') \\ w'_{tt} &= e^{-2t} (g(w') + |f'|) \end{aligned}$$

on $(T, +\infty) \times S^1$ and with the same function j as before

$$(1.47) \quad \frac{d^2}{dt^2} \int_{S^1} j(w' - \omega'_\delta) d\theta \geq 0.$$

As $t^{-1}(w' - \omega'_\delta)$ converges to 0 in $L^1(S^1)$ we deduce that $j(w' - \omega'_\delta) = 0$ and we get finally

$$(1.48) \quad \lim_{x \rightarrow 0} u(x)/\text{Ln}(1/|x|) = +\infty.$$

Case 2. — Assume $a_g^+ > 0$ and set $\gamma = \lim_{\lambda \uparrow 2/a_g^+} v(\lambda)$. Clearly $\gamma \leq 2/a_g^+$. If $\gamma < 2/a_g^+$ we can proceed as in Theorem 1.1. If $\gamma = 2/a_g^+$ we get as in Case 1

$$(1.49) \quad \inf(u(x), v_\lambda(x)) \geq v_{v(\lambda)}(x) - \frac{\delta}{4}$$

for any $\lambda \leq \frac{2}{a_g^+}$ and $x \in B_R \setminus \{0\}$. We can take in particular $\lambda = \frac{2}{a_g^+} = v(\lambda)$ and we get (ii).

2. SINGULAR SOLUTIONS OF $\Delta u = \pm g(u)$

The first application of Theorem 1.1 is the following

THEOREM 2.1. — Assume $N \geq 3$, g is a nondecreasing locally Lipschitz continuous function satisfying (1.1) and $u \in C^2(\Omega')$ is a nonnegative solution of

$$(2.1) \quad \Delta u = g(u)$$

in Ω' . Then $|x|^{N-2} u(x)$ admits a limit in $[0, +\infty]$ as x tends to 0.

Proof. — From Theorem 1.1 we can assume that there exist $\gamma \in [0, +\infty)$ and a sequence $\{r_n\}$ converging to 0 such that

$$(2.2) \quad \lim_{n \rightarrow +\infty} r_n^{N-2} u(r_n, \cdot) = \gamma \quad \text{a. e. in } S^{N-1}.$$

Case 1. — Assume $\gamma > 0$. For $\varepsilon > 0$ set w_ε the solution of

$$(2.3) \quad \begin{aligned} \Delta w_\varepsilon &= g(w_\varepsilon) & \text{in } \Gamma_{\varepsilon, R} &= \{x \in \mathbb{R}^N : \varepsilon < |x| < R\} \\ w_\varepsilon &= u & \text{on } \partial B_\varepsilon \\ w_\varepsilon &= \max_{x \in \partial B_R} u(x) & \text{on } \partial B_R \end{aligned}$$

(we may assume that $\bar{B}_R \subset \Omega$). From maximum principle $u \leq w_\varepsilon$ in $\Gamma_{\varepsilon, R}$. Let $u^\varepsilon = u + w_\varepsilon(R)$, then

$$(2.4) \quad -\Delta u^\varepsilon + g(u^\varepsilon) \geq 0$$

and finally $u \leq w_\varepsilon \leq u^\varepsilon$ in $\Gamma_{\varepsilon, R}$ and there exists a sequence $\{\varepsilon_n\}$ converging to 0 and a function $w \in C^2(\bar{B}_R \setminus \{0\})$ satisfying $-\Delta w + g(w) = 0$ in $B_R \setminus \{0\}$ such that $\{w_{\varepsilon_n}\}$ converges to w in the C_{loc}^1 -topology of $\bar{B}_R \setminus \{0\}$.

Moreover

$$(2.5) \quad u \leq w \leq u^1 = u + \max_{\partial B_R} u(x)$$

From Remark 1.2 $\lim_{x \rightarrow 0} |x|^{N-2} w(x) = \gamma$, hence we deduce from Serrin and

Ni's results [12] that w is radial and from (2.2) and (2.5)

$$(2.6) \quad \lim_{n \rightarrow +\infty} r_n^{N-2} w(r_n) = \gamma.$$

If $w'(s) = w'(r^{N-2}/(N-2)) = r^{N-2} w(r)$, then

$$(2.7) \quad s^2 w'_{ss} = k(N) s^{N/(N-2)} g(w'/s(N-2))$$

we deduce that $s \rightarrow w'(s) - k(N)(N-2)^2/(2N) s^{N/(N-2)} g(0)$ is convex and

$$(2.8) \quad \lim_{r \rightarrow 0} r^{N-2} w(r) = \gamma = \lim_{x \rightarrow 0} |x|^{N-2} u(x).$$

Case 2. — Assume $\gamma = 0$. For $\varepsilon > 0$ and $v > 0$ set $w_{\varepsilon, v}$ the solution of

$$(2.9) \quad \begin{aligned} \Delta w_{\varepsilon, v} &= g(w_{\varepsilon, v}) \quad \text{in } \Gamma_{\varepsilon, R} \\ w_{\varepsilon, v} &= u + v\varepsilon^{\frac{2-N}{2}} \quad \text{on } \partial B_\varepsilon \\ w_{\varepsilon, v} &= \max_{x \in \partial B_R} (u(x) + v|x|^{2-N}) \quad \text{on } \partial B_R. \end{aligned}$$

As in case 1 we have

$$(2.10) \quad u(x) \leq w_{\varepsilon, v}(x) \leq u(x) + v|x|^{2-N} + w_{\varepsilon, v}(R)$$

in $\Gamma_{\varepsilon, R}$. For $0 < v' < v$ let $v_{v'}$ be the radial solution of $-\Delta v_{v'} + g(v_{v'}) = C(N)v'\delta_0$ in $D'(B_R)$ such that $v_{v'} = 0$ on ∂B_R . As $\lim_{x \rightarrow 0} |x|^{N-2} v_{v'}(x) = v'$ we deduce that for ε small enough $v_{v'} < w_{\varepsilon, v}$ on ∂B_ε

and finally

$$(2.11) \quad w_{\varepsilon, v} \geq v_{v'}$$

in $\Gamma_{\varepsilon, R}$ and as in Case 1 there exists a subsequence $\{\varepsilon_n\}$ such that $\lim \varepsilon_n = 0$ and a function w^v satisfying $-\Delta w^v + g(w^v) = 0$ in B_R such that $w_{\varepsilon, v}$ converges to w^v in the C^1_{loc} topology of $\bar{B}_R \setminus \{0\}$ and we have

$$(2.12) \quad \max(u, v_{v'}) \leq w^v \leq u + v|x|^{2-N} + \max_{\partial B_R} u(x).$$

Applying again [12] we deduce that w^v is radial and as in Case 1 we get that

$$(2.13) \quad \lim_{x \rightarrow 0} |x|^{N-2} u(x) \leq \lim_{x \rightarrow 0} |x|^{N-2} w^v(x) = v.$$

As v is arbitrary $\lim_{x \rightarrow 0} |x|^{N-2} u(x) = 0$ and u can be extended to Ω as a C^2 solution of (2.1) in Ω .

In the same way we can prove the two dimensional case

THEOREM 2.2. — Assume $N=2$ and g is a nondecreasing locally Lipschitz continuous function defined on $\mathbb{R}^+$. If $u \in C^2(\Omega')$ is a nonnegative solution of (2.1) in Ω' , we have the following:

- if $a_g^+ = 0$ $u(x)/\text{Ln}(1/|x|)$ admits a limit in $[0, +\infty]$ as x tends to 0;
- if $a_g^+ > 0$ and g satisfies

$$(2.14) \quad \text{for any } a \geq 0 \quad \lim_{r \rightarrow +\infty} e^{-ar} g(r) \text{ exists in } [0, +\infty],$$

$u(x)/\text{Ln}(1/|x|)$ admits a limit in $[0, 2/a_g^+]$ as x tends to 0.

Proof. — If $a_g^+ = 0$ we proceed as in Theorem 2.1. If $a_g^+ = +\infty$ and g satisfies (2.14), u can be extended to Ω as a C^2 solution of (2.1) in Ω [21]. If $0 < a_g^+ < +\infty$ we have two cases

(i) either there exists $\gamma \in [0, 2/a_g^+)$ and a sequence $\{r_n\}$ converging to 0 such that

$$(2.15) \quad \lim_{n \rightarrow +\infty} u(r_n \cdot) / \text{Ln}(1/r_n) = \gamma \quad \text{a.e. in } S^1$$

(ii) or $\lim_{x \rightarrow 0} u(x)/\text{Ln}(1/|x|) \geq 2/a_g^+$.

In case (i) we have $\lim_{x \rightarrow 0} u(x)/\text{Ln}(1/|x|) = \gamma$ as in Theorem 2.1. In case (ii) we have an *a priori* estimate thanks to (2.14) [21]:

$$(2.16) \quad u(x) \leq \left(\frac{2}{a_g^+} + \varepsilon \right) \text{Ln}(1/|x|) + B(\varepsilon)$$

near 0 for any $\varepsilon > 0$. This clearly implies

$$(2.17) \quad \lim_{x \rightarrow 0} u(x)/\text{Ln}(1/|x|) = 2/a_g^+.$$

THEOREM 2.3. — Assume $N \geq 3$, g is a continuous function defined on $[0, +\infty)$ such that $\lim_{r \rightarrow +\infty} g(r)/r = K$ for some $K > -\infty$ and $u \in C^2(\Omega')$ is a nonnegative solution of

$$(2.18) \quad -\Delta u = g(u)$$

in Ω' . Then there exists $\gamma \in [0, +\infty)$ such that

$$(2.19) \quad \lim_{x \rightarrow 0} |x|^{1-N} \int_{|y|=|x|} |\gamma - |x|^{N-2} u(y)| dS = 0,$$

$g(u) \in L^1_{\text{loc}}(\Omega)$ and u solves

$$(2.20) \quad -\Delta u = g(u) + C(N) \gamma \delta_0$$

in $D'(\Omega)$. If we assume moreover that

$$(2.21) \quad \int_0^1 \inf(g(\alpha r^{2-N}), g(\beta r^{2-N})) r^{N-1} dr = +\infty$$

for any $\alpha, \beta > 0$, then $\gamma = 0$.

Proof. — The fact that $g(u) \in L^1_{\text{loc}}(\Omega)$ and u satisfies (2.20) for some $\gamma \geq 0$ is proved in [5]. If $\bar{u}(r)$ [res. $\overline{g(u)}(r)$] is the spherical average of u [resp. $g(u)$] then

$$(2.22) \quad \Delta \bar{u} = \overline{g(u)}$$

in $B_R \setminus \{0\} \subset \Omega'$ and we deduce from Lemma 1.2 that

$$(2.23) \quad \lim_{x \rightarrow 0} |x|^{1-N} \int_{|y|=|x|} |\gamma' - |x|^{N-2} u(y)| dS = 0$$

for some $\gamma' \geq 0$ and $\bar{u}$ solves

$$(2.24) \quad -\Delta \bar{u} = \overline{g(u)} + C(N) \gamma' \delta_0$$

in $D'(B_R)$. Whence $\gamma = \gamma'$. Let us assume now that $\gamma > 0$ and g satisfies (2.21) for any $\alpha, \beta > 0$. As $r^{N-2} u(r, \cdot)$ converges to γ in $L^1(S^{N-1})$ it converges in measure and for any $\eta \in (0, |S^{N-1}|)$ there exists $r_0 \in (0, R)$ such that for any $r \in (0, r_0)$ there exists a measurable subset $\omega(r) \subset S^{N-1}$ such that $|\omega(r)| \geq \eta$ and $|r^{N-2} u(r, \sigma) - \gamma| < \gamma/2$ for $\sigma \in \omega(r)$. As $g(r) \geq K'r - L$ and $u \in L^1_{\text{loc}}(B_R)$ there is no loss of generality to assume that $g(r) \geq 0$ on $(0, +\infty)$, hence

$$(2.25) \quad \int_{B_{r_0}} g(u) dx = \int_0^{r_0} \int_{S^{N-1}} g(u) r^{N-1} d\sigma dr \geq \int_0^{r_0} \int_{\omega(r)} g(u) r^{N-1} d\sigma dr.$$

For $\rho \in (0, r_0]$ and $\sigma \in \omega(\rho)$, $\frac{\gamma}{2} \rho^{2-N} \leq u(\rho, \sigma) < 2\gamma \rho^{2-N}$ and as g is continuous, $g(u(\rho, \sigma)) \geq \inf\left(g\left(\frac{\gamma}{2} \rho^{2-N}\right), g(2\gamma \rho^{2-N})\right)$. As g satisfies (2.21) we

get

$$(2.26) \quad \int_{B_{r_0}} g(u) dx \geq \eta \int_0^{r_0} \inf \left(g \left(\frac{\gamma}{2} r^{2-N} \right), g(2\gamma r^{2-N}) \right) r^{N-1} dr = +\infty,$$

contradiction. Hence $\gamma = 0$.

Under an assumption of monotonicity on g we get a much more accurate result:

PROPOSITION 2.1. — Assume $N \geq 3$, g is a nondecreasing locally Lipschitz continuous function defined on $[0, +\infty)$ and $u \in C^2(\Omega')$ is a nonnegative solution of (2.18) in Ω' . Assume also that $\bar{B}_R \subset \Omega$ and that there exists a radial continuous function Φ defined in $\bar{B}_R \setminus \{0\}$ and satisfying

$$(2.27) \quad \begin{aligned} -\Delta \Phi &\geq g(\Phi) \quad \text{in } D'(\bar{B}_R \setminus \{0\}), \\ \Phi &\geq u \quad \text{in } \bar{B}_R \setminus \{0\}. \end{aligned}$$

Then $|x|^{N-2}u(x)$ converges to some nonnegative real number γ when x tends to 0.

Proof. — From Remark 1.3 $|x|^{N-2}\Phi(x)$ converges to some $\gamma' \geq 0$ as x tends to 0. If $\gamma' = 0$ then $\lim_{x \rightarrow 0} |x|^{N-2}u(x) = 0$. Let us assume that $\gamma' > 0$.

From Brezis and Lions' result

$$-\Delta \Phi = -\{\Delta \Phi\} + C(N)\gamma'\delta_0$$

with $-\{\Delta \Phi\} \in L^1_{\text{loc}}(B_R)$ which implies that $g(\Phi) \in L^1(B_R)$ and g satisfies (1.1). From Theorem 2.3 there exists $\gamma \in [0, \gamma']$ such that $r^{N-2}u(r, \cdot)$ converges to γ in $L^1(S^{N-1})$ as r tends to 0. We consider now the sequence of functions $\{u^N\}$ defined by $u^0 = \Phi$ and for $N \geq 1$

$$(2.28) \quad \begin{aligned} -\Delta u^N &= g(u^{N-1}) + C(N)\gamma\delta_0 \quad \text{in } D'(B_R) \\ u^N &= \Phi \quad \text{on } \partial B_R. \end{aligned}$$

Then u^N is radial and $u \leq u^N \leq u^{N-1} < \Phi$. It is clear that $\{u^N\}$ converges in $C^1_{\text{loc}}(\bar{B}_R \setminus \{0\})$ to a radial function $\bar{u}$ which satisfies

$$(2.29) \quad -\Delta \bar{u} = g(\bar{u}) + C(N)\gamma\delta_0 \quad \text{in } D'(B_R)$$

and $\bar{u} \geq u$. As a consequence of Lemma 1.2 $\lim_{x \rightarrow 0} |x|^{N-2}\bar{u}(x) = \gamma$. From

Remark 1.2 $\lim_{x \rightarrow 0} |x|^{N-2}u(x) = \gamma$ which ends the proof.

Remark 2.1. — The hypothesis of radially of Φ which is rather restrictive can be withdrawn if we know that $\lim_{x \rightarrow 0} u(x) = +\infty$ and

$\Phi \geq \sup_{|x|=\mathbf{R}} u(x)$. In that case we can consider the following iterative scheme with $\Phi^0 = \Phi$ and

$$(2.30) \quad \begin{aligned} -\Delta \Phi^N &= g(\Phi^{N-1}) + C(N) \gamma' \delta_0 \quad \text{in } D'(\mathbf{B}_{\mathbf{R}}) \\ \Phi^N &= \sup_{|x|=\mathbf{R}} u(x) \quad \text{on } \partial \mathbf{B}_{\mathbf{R}}. \end{aligned}$$

Then $u \leq \Phi^N \leq \Phi^{N-1} \leq \Phi$ and $\{\Phi^N\}$ converges in $C_{\text{loc}}^1(\bar{\mathbf{B}}_{\mathbf{R}} \setminus \{0\})$ to some Φ^- satisfying

$$(2.31) \quad \begin{aligned} -\Delta \Phi^- &= g(\Phi^-) + C(N) \gamma' \delta_0 \quad \text{in } D'(\mathbf{B}_{\mathbf{R}}) \\ \Phi^- &= \sup_{|x|=\mathbf{R}} u(x) \quad \text{on } \partial \mathbf{B}_{\mathbf{R}} \end{aligned}$$

and $\Phi^- \geq u$. As $\lim_{x \rightarrow 0} \Phi^-(x) = +\infty$ we deduce from Serrin and Ni' results

[12] that Φ^- is radial and we can apply Lemma 1.2.

PROPOSITION 2.2. — Assume $N \geq 3$, g is a nondecreasing locally Lipschitz continuous function defined on $[0, +\infty)$ satisfying for some $q > N/2$.

$$(2.32) \quad \sup(g'(\varphi), g'(\psi)) \in L_{\text{loc}}^q(\Omega)$$

for any φ and ψ continuous and nonnegative in Ω' such that $g(\varphi)$ and $g(\psi) \in L_{\text{loc}}^1(\Omega)$. If $u \in C^2(\Omega')$ is a nonnegative solution of (2.18) in Ω' , then $|x|^{N-2} u(x)$ converges to some nonnegative real number γ as x tends to 0.

Proof. — From Theorem 2.3 we have (2.20) for some $\gamma \geq 0$ and $g(u) \in L_{\text{loc}}^1(\Omega)$.

Case 1. — $\gamma = 0$. Without any restriction we can assume that $u > \varepsilon$ in $\bar{\mathbf{B}}_{\mathbf{R}} \setminus \{0\} \subset \Omega'$ and we write (2.20) as

$$(2.33) \quad \Delta u + du + g(0) = 0$$

in $\mathbf{B}_{\mathbf{R}} \setminus \{0\}$ where $d(x) = (g(u) - g(0))/u$. As $g(u) \in L^1(\mathbf{B}_{\mathbf{R}})$ (2.32) implies that $d \in L^q(\mathbf{B}_{\mathbf{R}})$ and we deduce from [18] that either u has a removable singularity at 0 or

$$(2.34) \quad 0 < \lim_{x \rightarrow 0} |x|^{N-2} u(x) < \overline{\lim}_{x \rightarrow 0} |x|^{N-2} u(x) < +\infty,$$

which is impossible as $\gamma = 0$.

Case 2. — $\gamma > 0$. Let v_γ be the solution of

$$(2.35) \quad \begin{aligned} -\Delta v_\gamma &= g(v_\gamma) + C(N) \gamma \delta_0 \quad \text{in } D'(\mathbf{B}_{\mathbf{R}}), \\ v_\gamma &= 0 \quad \text{on } \partial \mathbf{B}_{\mathbf{R}}, \end{aligned}$$

v_γ is constructed using an increasing sequence of approximate solutions as in [11], $0 \leq v_\gamma \leq u$ in $B_R \setminus \{0\}$ and v_γ is radial. Let w be $u - v_\gamma$, then

$$(2.36) \quad \Delta w + dw = 0$$

in $B_R \setminus \{0\}$ with $d = (g(u) - g(v_\gamma))/(u - v_\gamma) \in L^q(B_R)$. Then we deduce from [18] that either w has a removable singularity at 0 or

$$(2.37) \quad 0 < \lim_{x \rightarrow 0} |x|^{N-2} w(x) \leq \overline{\lim}_{x \rightarrow 0} |x|^{N-2} w(x)$$

which is impossible as

$$(2.38) \quad \gamma = \lim_{x \rightarrow 0} |x|^{N-2} v_\gamma(x) = \lim_{x \rightarrow 0} |x|^{N-2} u(x).$$

Remark 2.2. — Under the hypotheses of Proposition 2.2 two nonnegative solutions u_i ($i=1, 2$) of

$$(2.39) \quad -\Delta u = g(u) + C(N) \gamma \delta_0$$

in $D'(\Omega)$ are such that $u_1 - u_2 \in L_{loc}^\infty(\Omega)$. As for the solvability of (2.39) we have

PROPOSITION 2.3. — Assume $N \geq 3$, Ω is bounded with a C^1 boundary $\partial\Omega$ and g is a nondecreasing function defined on $[0, +\infty)$, satisfying (1.1) and $g(r) = o(r)$ near 0. Then there exists $\gamma^* \in (0, +\infty]$ with the following properties:

(i) for any $\gamma \in [0, \gamma^*)$ there exists at least one nonnegative function $u \in C^1(\bar{\Omega} \setminus \{0\})$ vanishing on $\partial\Omega$ solution of (2.39) in $D'(\Omega)$,

(ii) for $\gamma > \gamma^*$ no such u exists.

Proof. — Step 1. Assume $\Omega = B_R$. — A function u vanishing on ∂B_R is a radial solution of (2.40) in $D'(B_R)$ if and only if the function $v(t) = u(r)$, with $t = r^{2-N}$, satisfies

$$(2.40) \quad \begin{aligned} v_{tt} + \frac{1}{(N-2)^2} t^{-2(N-1)/(N-2)} g(v) &= 0 \quad \text{on } (R^{2-N}, +\infty), \\ v(R^{2-N}) &= 0, \\ \lim_{t \rightarrow +\infty} v(t)/t &= \gamma. \end{aligned}$$

As v is concave the last condition is equivalent to

$$(2.41) \quad \lim_{t \rightarrow +\infty} v_t(t) = \gamma.$$

For $\alpha > 0$, let v^α be the solution of the initial value problem defined on a maximal interval $[R^{2-N}, T^*)$

$$(2.42) \quad \begin{aligned} v_{tt}^\alpha + \frac{1}{(N-2)^2} t^{-2(N-1)/(N-2)} g(v^\alpha) &= 0 \quad \text{on } (R^{2-N}, T^*), \\ v^\alpha(R^{2-N}) &= 0, \\ v_t^\alpha(R^{2-N}) &= \alpha. \end{aligned}$$

If $T^* < +\infty$ then $\lim_{t \uparrow T^*} v^\alpha(t) = 0$ as a consequence of concavity and there exists $T \in (R^{2-N}, T^*)$ such that $v_t(T) = 0$. If $T^* = +\infty$ and $\lim_{t \rightarrow +\infty} v_t(t) = 0$ then the same relation holds with $T = +\infty$. As a consequence if no solution v^α of (2.42) satisfies (2.41) with $\gamma > 0$ we have

$$(2.43) \quad (N-2)^2 \alpha = \int_{R^{2-N}}^T t^{-2(N-1)/(N-2)} g(v^\alpha(t)) dt$$

and the right-hand side of (2.43) is majorized by $\int_{R^{2-N}}^{+\infty} t^{-2(N-1)/(N-2)} g(\alpha(t - R^{2-N})) dt$, which implies

$$(2.44) \quad (N-2)^2 \alpha R^{-N} < \int_0^{+\infty} (t+1)^{-2(N-1)/(N-2)} g(\alpha R^{2-N} t) dt,$$

or

$$(2.45) \quad (N-2)^2 R^{-2} < \int_0^{+\infty} t(t+1)^{-2(N-1)/(N-2)} \frac{g(\alpha R^{2-N} t)}{\alpha R^{2-N} t} dt.$$

For $\varepsilon > 0$ there exists $\eta > 0$ such that $\alpha R^{2-N} t < \eta$ implies $g(\alpha R^{2-N} t) < \varepsilon \alpha R^{2-N} t$. Hence the right-hand side of (2.45) is majorized by

$$\begin{aligned} \frac{R^{N-2}}{\alpha} \int_{R^{N-2}\eta/\alpha}^{+\infty} (t+1)^{-2(N-1)/(N-2)} g(\alpha R^{2-N} t) dt \\ + \varepsilon \int_0^{R^{N-2}\eta/\alpha} t(t+1)^{-2(N-1)/(N-2)} dt \end{aligned}$$

or

$$\begin{aligned} \alpha^{2(N-1)/(N-2)} \int_\eta^{+\infty} (R^{N-2}s + \alpha)^{-2(N-1)/(N-2)} g(s) ds \\ + \varepsilon \int_0^{+\infty} t(t+1)^{-2(N-1)/(N-2)} dt. \end{aligned}$$

Consequently

$$(2.46) \quad \lim_{\alpha \rightarrow 0} \int_0^{+\infty} t(t+1)^{-2(N-1)/(N-2)} \frac{g(\alpha R^{2-N}t)}{\alpha R^{2-N}t} dt = 0$$

contradicting (2.45). As a consequence there exists $\alpha^* > 0$ such that for any $\alpha \in (0, \alpha^*)$ the solution v^α of (2.42) is defined on $[R^{2-N}, +\infty)$ and satisfies (2.41) for some $\gamma > 0$.

Step 2. The general case. — There exists $R > 0$ such that $\Omega \subset B_R$. If $\tilde{\gamma} > 0$ is such that there exists a solution v to (2.40), then for any $\gamma \in [0, \tilde{\gamma}]$ the sequence $\{u_n\}$ defined by $u_0 = 0$ and for $n \geq 1$

$$(2.47) \quad \begin{aligned} -\Delta u^n &= g(u^{n-1}) + C(N) \gamma \delta_0 \quad \text{in } D'(\Omega), \\ u^n &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

increases, is majorized by v in Ω and converges to some u which vanishes on $\partial\Omega$ and satisfies (2.39) in $D'(\Omega)$. For the same reasons, the set of $\gamma > 0$ such that there exists a nonnegative solution of (2.39) vanishing on $\partial\Omega$ is an interval.

Remark 2.3. — If $\lim_{r \rightarrow +\infty} g(r)/r > 0$ it is proved in [11] that $\gamma^* < +\infty$. If we no longer assume that $\lim_{r \rightarrow 0} g(r)/r = 0$ it can be proved that for any $v_0 > 0$ there exists $R_0 > 0$ such that for any $\Omega \subset B_{R_0}$ and any $\gamma \in [0, v_0)$ there exists a solution u of (2.39) in $D'(\Omega)$.

The two-dimensional version of Theorem 2.3 is the following

THEOREM 2.4. — Assume $N=2$, g is a continuous function defined on $[0, +\infty)$ such that $\lim_{r \rightarrow +\infty} g(r)/r > -\infty$ and $u \in C^2(\Omega')$ is a nonnegative solution of (2.18) in Ω' . Then there exists $\gamma \in [0, +\infty)$ such that

$$(2.48) \quad \lim_{x \rightarrow 0} |x|^{-1} \int_{|y|=|x|} |\gamma - u(y)/\text{Ln}(1/|x|)| dS = 0,$$

$g(u) \in L^1_{\text{loc}}(\Omega)$ and u solves

$$(2.49) \quad -\Delta u = g(u) + 2\pi\gamma\delta_0$$

in $D'(\Omega)$. If we assume moreover that

$$(2.50) \quad \int_0^1 \inf(g(\alpha \text{Ln}(1/r)), g(\beta \text{Ln}(1/r))) r dr = +\infty$$

for any $\alpha, \beta > 0$, then $\gamma = 0$.

Remark 2.4. — When $a_g^+ = 0$, Proposition 2.2 which holds in the case $N=2$ with $|x|^{2-N}$ replaced by $\text{Ln}(1/|x|)$ provides an interesting criterion for proving that

$$(2.51) \quad \lim_{x \rightarrow 0} u(x)/\text{Ln}(1/|x|) = \gamma$$

for some $\gamma \geq 0$. Proposition 2.1 is also valid in the case $N=2$ (with the same modifications).

We introduce now a class new of g 's defined on $[0, +\infty)$ which are those satisfying

$$(2.52) \quad \forall \sigma > 0, \quad \lim_{r \rightarrow +\infty} e^{-\sigma r} g(r) = l(\sigma) \text{ exists in } [0, +\infty],$$

and we have [20]

$$(2.53) \quad a_g^+ = \sup \{ \sigma > 0 : l(\sigma) = +\infty \} = \inf \{ \sigma > 0 : l(\sigma) = 0 \}.$$

THEOREM 2.5. — Assume $N=2$, g is a continuous function defined on $[0, +\infty)$ satisfying $\lim_{r \rightarrow +\infty} g(r)/r > -\infty$ and (2.52) with $a_g^+ < +\infty$ and

$u \in C^2(\Omega')$ is a nonnegative solution of (2.18) in Ω' and assume also

(i) either $a_g^+ = 0$,

(ii) or $a_g^+ > 0$ and $\int_0^1 g\left(\frac{2}{a_g^+} \text{Ln}(1/r)\right) r dr = +\infty$.

Then there exists $\gamma \in \left[0, \frac{2}{a_g^+}\right)$ such that $u - \gamma \text{Ln} \frac{1}{r}$ is locally bounded in Ω .

Proof. — The main ingredient for proving this is a theorem due to John and Nirenberg ([9], Th. 7.21) that we recall

«Let $u \in W^{1,1}(G)$ where $G \subset \Omega$ is convex and suppose that there exists a constant K such that

$$(2.54) \quad \int_{G \cap B_r} |\nabla u| dx \leq K r \quad \text{for any ball } B_r$$

then there exist positive constant μ_0 and C such that

$$(2.55) \quad \int_G \exp\left(\frac{\mu}{K} |u - u_G|\right) dx \leq C (\text{diam}(G))^2$$

where $\mu = \mu_0 |G| (\text{diam}(G))^{-2}$ and $u_G = \frac{1}{|G|} \int_G u dx$.

From Theorem 2.4 there exists $\gamma \geq 0$ such that $u(r, \cdot)/\text{Ln}(1/r)$ converges to γ in $L^1(S^1)$ as r tends to 0 and $g(u) \in L_{\text{loc}}^1(\Omega)$. Set $w = u - \gamma \text{Ln}(1/|x|)$.

then

$$(2.56) \quad -\Delta w = g(u)$$

in $D'(\Omega)$. It is now classical that $\nabla w \in M_{loc}^2(\Omega)$ where $M^2(G)$ is the usual Marcinkiewicz space over G . If we take $G = \bar{B}_R \subset \Omega$ then ∇w satisfies (2.54) for some $K > 0$, which implies

$$(2.57) \quad \int_{B_\rho} e^{xw} dx \leq C(\rho)$$

for some $\alpha > 0$ and $0 < \rho \leq R$.

Case 1. — Assume $a_g^+ = 0$. Then for any $\varepsilon > 0$ we have

$$(2.58) \quad |g(r)| \leq K_\varepsilon e^{\varepsilon r}$$

for some $K_\varepsilon > 0$ and any $r \geq 0$. From (2.57) we have

$$(2.59) \quad \int_{B_\rho} e^{xw} |x|^{\alpha\gamma} dx \leq C(\rho).$$

If $\gamma > 0$ we have for $p, \sigma > 1$ and $\lambda > 0$

$$(2.60) \quad \int_{B_\rho} e^{p\varepsilon u} dx \leq \left(\int_{B_\rho} e^{\sigma p\varepsilon u} |x|^{\sigma\lambda} dx \right)^{1/\sigma} \left(\int_{B_\rho} |x|^{-\sigma'\lambda} dx \right)^{1/\sigma'}$$

($\sigma' = \sigma/(\sigma-1)$). We set $\sigma p\varepsilon = \alpha$, $\sigma\lambda = \alpha\gamma$, hence $\lambda = \gamma p\varepsilon$, $\sigma = \frac{\alpha}{p\varepsilon}$ and $\sigma'\lambda = \alpha\gamma p\varepsilon/(\alpha - p\varepsilon)$.

Hence for any $p > 1$ we can take ε small enough so that $\sigma'\lambda < 2$ and $\sigma > 1$. As a consequence $g(u) \in L^p(B_\rho)$ and $w \in L^\infty(B_\rho)$. If $\gamma = 0$, (2.59) implies that $g(u) \in L^p(B_\rho)$ for any $p \in [1, \infty)$ and $u \in L^\infty(B_\rho)$.

Case 2. — Assume $a_g^+ > 0$ and $\int_0^1 g\left(\frac{2}{a_g^+} \ln(1/r)\right) r dr = +\infty$.

Step 1. — $0 \leq \gamma < \frac{2}{a_g^+}$. Assume the contrary that is $\gamma \geq \frac{2}{a_g^+}$. As $a_g^+ > 0$ we have $\lim_{r \rightarrow +\infty} g(r) = +\infty$ and from Remark 1.2

$$(2.61) \quad u(x) > v_\gamma(x),$$

where v_γ satisfies

$$(2.62) \quad -\Delta v_\gamma + g(v_\gamma) = 2\pi\gamma\delta_0$$

in $D'(\mathbf{B}_R)$, $v_\gamma = 0$ on $\partial\mathbf{B}_R$. As a consequence [21] $\lim_{x \rightarrow 0} u(x) = +\infty$ and for $|x| < R'$ small enough

$$(2.63) \quad -\Delta u \geq 2\pi\gamma\delta_0$$

in $D'(\mathbf{B}_R)$. As a consequence $u(x) \geq \gamma L n\left(\frac{1}{|x|}\right) - l$, which implies

$$\int_{\mathbf{B}_R} g(u) dx = +\infty, \text{ contradiction.}$$

Step 2. — We claim that for any $\alpha > 0$ there exist $\rho \in (0, R]$ such that (2.57) holds. We fix $0 < R' < R$ and write $w = w_1 + w_2$ where w_1 is harmonic in $\mathbf{B}_R$, and take the value w on $\partial\mathbf{B}_R$ and w_2 satisfies

$$(2.64) \quad -\Delta w_2 = g(u)$$

in $\mathbf{B}_R$ and $w_2 = 0$ on $\partial\mathbf{B}_R$. As $\nabla w_1 \in L^2(\mathbf{B}_R)$ we deduce

$$(2.65) \quad \|\nabla w_1\|_{M^2(\mathbf{B}_\rho)} \xrightarrow{\rho \rightarrow 0} 0$$

and for w_2 we have

$$(2.66) \quad \|\nabla w_2\|_{M^2(\mathbf{B}_R)} \leq C \|g(u)\|_{L^1(\mathbf{B}_R)}$$

where C is independent of R' . As a consequence we get

$$(2.67) \quad \lim_{\rho \rightarrow 0} \|\nabla w\|_{M^2(\mathbf{B}_\rho)} = 0$$

and the constant K in (2.55) can be taken as small as we want provided $G = \mathbf{B}_\rho$ and u is replaced by w . This implies that for any $\alpha > 0$ we can find $\rho \in (0, R)$ such that (2.57) holds.

Step 3: End of the proof. — From the definition of a_g^+ , for any $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that

$$(2.68) \quad |g(r)| \leq K_\varepsilon e^{(a_g^+ + \varepsilon)r}$$

for $r \geq 0$, and we have from (2.59)

$$(2.69) \quad \int_{\mathbf{B}_\rho} e^{p(a_g^+ + \varepsilon)u} dx \leq \left(\int_{\mathbf{B}_\rho} e^{\sigma p(a_g^+ + \varepsilon)u} |x|^{\sigma\lambda} dx \right)^{1/\sigma} \left(\int_{\mathbf{B}_\rho} |x|^{-\sigma'\lambda} dx \right)^{1/\sigma'}.$$

We take $\sigma p(a_g^+ + \varepsilon) = \alpha$, $\sigma\lambda = \alpha\gamma$ [we assume $\gamma > 0$ other-wise $g(u) \in L_{\text{loc}}^p(\Omega)$ for any $p > 1$ and $w \in L_{\text{loc}}^\infty(\Omega)$] and $\lambda = \gamma p(a_g^+ + \varepsilon)$, $\sigma = \alpha/p(a_g^+ + \varepsilon)$ and $\lambda\sigma' = \alpha\gamma p(a_g^+ + \varepsilon)/(\alpha - p(a_g^+ + \varepsilon))$. As $\gamma a_g^+ < 2$ there exist $p > 1$, $\varepsilon > 0$, $\alpha > 0$ such that $\sigma'\lambda < 2$ which implies $g(u) \in L_{\text{loc}}^p(\Omega)$ and we end the proof as in Case 1.

Remark 2.5. — If $a_g^+ = +\infty$ then $\gamma=0$ from Theorem 2.4. In that case it is unlikely that Theorem 2.5 still holds. However we conjecture that $\lim_{x \rightarrow 0} u(x)/\ln(1/|x|) = 0$.

Concerning the existence of solutions of (2.49) the following result can be proved as in Proposition 2.3.

PROPOSITION 2.4. — Assume $N=2$, Ω is bounded with a C^1 boundary $\partial\Omega$ and g is a nondecreasing function defined on $[0, +\infty)$ such that $a_g^+ \in (0, +\infty]$ and $g(r)=o(r)$ near 0. Then there exists $\gamma^* \in (0, 2/a_g^+]$ with the following properties:

(i) for any $\gamma \in [0, \gamma^*)$ there exists at least one nonnegative function $u \in C^1(\bar{\Omega} \setminus \{0\})$ vanishing on $\partial\Omega$ solution of (2.49) in $D'(\Omega)$,

(ii) for $\gamma > \gamma^*$ no such u exists.

Remark 2.6. — If $g(r)=e^{ar}$ it is easy to see that γ^* exists only if $\text{diam.}(\Omega)$ is small enough. Moreover in that case $\gamma^* < \frac{2}{a_g^+} = \frac{2}{a}$.

3. SINGULARITIES OF $\Delta u = u(Ln^+ u)^\alpha$

Our first result deals with the one-dimensional case

THEOREM 3.1. — Assume $u \in C^2(0, R)$ is a nonnegative solution of

$$(3.1) \quad u_{rr} = u(Ln^+ u)^\alpha \quad \text{in } (0, R).$$

Then:

— if $0 < \alpha < 2$,

$u(r)$ admits a finite limit as r tends to 0;

— if $\alpha > 2$,

(i) either $u(r)$ admits a finite limit as r tends to 0,

(ii) or

$$(3.2) \quad \begin{cases} u(r) = \sqrt{e} e^{\gamma(\alpha)} r^{2/(2-\alpha)} (1 + O(r^{2/(\alpha-2)})), \\ u_r(r) = -\sqrt{e} (\gamma(\alpha))^{\alpha/2} r^{\alpha/(2-\alpha)} e^{\gamma(\alpha)} r^{2/(2-\alpha)} (1 + O(r^{2/(\alpha-2)})), \end{cases}$$

near 0 where

$$(3.3) \quad \gamma(\alpha) = \left(\frac{2}{\alpha-2} \right)^{2/(\alpha-2)}.$$

From (3.1) u is convex and $u(r)$ admits a limit in $\mathbb{R}^+ \cup \{+\infty\}$ as r tends to 0. If this limit is larger than 1, (3.1) is equivalent to

$$(3.4) \quad v_{rr} + v_r^2 = v^\alpha$$

on some interval $(0, R')$ with the transformation $u = e^v$. Theorem 3.1 is an immediate consequence of the following result

LEMMA 3.1. — Assume $v \in C^2(0, R')$ is a nonnegative solution of (3.4) in $(0, R')$. Then

— if $0 < \alpha \leq 2$, v remains bounded near 0;

— if $\alpha > 2$

(i) either v remains bounded near 0,

(ii) or

$$(3.5) \quad \begin{cases} r^{2/(\alpha-2)} v(r) = \gamma(\alpha) + \frac{1}{2} r^{2/(\alpha-2)} + O(r^{4/(\alpha-2)}) \\ r^{\alpha/(\alpha-2)} v_r(r) = -(\gamma(\alpha))^{\alpha/2} + O(r^{4/(\alpha-2)}). \end{cases}$$

Proof. — Assuming that u is unbounded near 0, then $\lim_{r \rightarrow 0} u(r) = +\infty = \lim_{r \rightarrow 0} v(r)$ and v is decreasing near 0. So we can define

$$(3.6) \quad \begin{cases} \rho = v \in [\sigma, +\infty), \\ h(\rho) = v_r^2, \end{cases}$$

and (3.5) become

$$(3.7) \quad \frac{1}{2} h_\rho + h = \rho^\alpha \quad \text{in } [\sigma, +\infty).$$

$$\text{Hence } h(\rho) = h(\sigma) e^{2(\sigma-\rho)} + 2 e^{-2\rho} \int_\sigma^\rho s^\alpha e^{2s} ds.$$

As

$$\int_\sigma^\rho s^\alpha e^{2s} ds = \frac{1}{2} [s^\alpha e^{2s}]_\sigma^\rho - \frac{\alpha}{4} [s^{\alpha-1} e^{2s}]_\sigma^\rho + \frac{\alpha(\alpha-1)}{4} \int_\sigma^\rho s^{\alpha-2} e^{2s} ds$$

and

$$\frac{e^{-2\rho}}{\rho^\alpha} \int_\sigma^\rho s^{\alpha-2} e^{2s} ds = O\left(\frac{1}{\rho^2} + \frac{1}{\rho^\alpha}\right)$$

we get

$$(3.8) \quad \frac{h(\rho)}{\rho^\alpha} = 1 - \frac{\alpha}{2\rho} + O\left(\frac{1}{\rho^2} + \frac{1}{\rho^\alpha}\right)$$

as p goes to $+\infty$, which implies

$$(3.9) \quad \lim_{r \rightarrow 0} \frac{v_r(r)}{v^{\alpha/2}(r)} = -1$$

Integrating (3.9) implies that $v^{(2-\alpha)/2}(r)$ (if $0 < \alpha < 2$) or $\ln v(r)$ (if $\alpha = 2$) remains bounded near 0 which is a contradiction. So we are left with the case $\alpha > 2$, $\lim_{r \rightarrow 0} v(r) = +\infty$. From (3.8) we have

$$(3.10) \quad \frac{v_r}{v^{\alpha/2}} = -1 + \frac{\alpha}{4v} + O\left(\frac{1}{v^2}\right),$$

near 0, which implies $\lim_{r \rightarrow 0} r^{2/(\alpha-2)} v(r) = \left(\frac{2}{\alpha-2}\right)^{2/(\alpha-2)} = \gamma(\alpha)$. As a consequence $\frac{1}{v(r)} = \frac{1+o(1)}{\gamma(\alpha)} r^{2/(\alpha-2)}$ and (3.10) becomes

$$(3.11) \quad \frac{v_r}{v^{\alpha/2}} = -1 + \frac{1+o(1)}{\gamma(\alpha)} \frac{\alpha}{4} r^{2/(\alpha-2)}$$

Integrating (3.11) on $(0, r)$ for r small yields

$$(3.12) \quad v(r) = \gamma(\alpha) r^{2/(2-\alpha)} \left(1 + \frac{1+o(1)}{2\gamma(\alpha)} r^{2/(\alpha-2)}\right),$$

which implies, with (3.10),

$$(3.13) \quad \frac{v_r}{v^{\alpha/2}} = -1 + \frac{\alpha}{4\gamma(\alpha)} r^{2/(\alpha-2)} + O(r^{4/(\alpha-2)}).$$

Reasoning as before we get

$$(3.14) \quad v(r) = \gamma(\alpha) r^{2/(2-\alpha)} + \frac{1}{2} + O(r^{2/(\alpha-2)})$$

near 0 and

$$(3.15) \quad r^{\alpha/(\alpha-2)} v_r(r) = -(\gamma(\alpha))^{\alpha/2} + O(r^{4/(\alpha-2)}).$$

We assume now that Ω is an open subset of $\mathbb{R}^N$, $N \geq 2$, containing 0, $\Omega' = \Omega \setminus \{0\}$ and we consider the following equation in Ω'

$$(3.16) \quad \Delta u = u(Ln^+ u)^\alpha$$

where $u \in C^2(\Omega')$ is nonnegative.

LEMMA 3.2. — If $\alpha > 2$ and $\bar{B}_R \subset \Omega$; then there exists a constant $C = C(\alpha, N, R, \text{dist}(\partial B_R, \partial \Omega))$ such that

$$(3.17) \quad u(x) \leq e^{C|x|^{2/(2-\alpha)}} \quad \text{in } \bar{B}_R \setminus \{0\}.$$

Proof. — We define $\beta(t) = t(Ln^+ t)^\alpha$, $j(t) = \int_0^t \beta(s) ds$ and $\tau(t) = \int_t^{+\infty} \frac{dt}{\sqrt{j(s)}}$. As $\tau(2) < +\infty$ we deduce from Vazquez's result that the equation (3.16) satisfies the a priori interior estimate property [19]: if $x_0 \in \Omega'$ and if the cube $Q_\rho(x_0) = \{x \in \mathbb{R}^N : \sup_{1 \leq i \leq N} |x^i - x_0^i| < \rho\}$ is included in Ω' , then for any $a \in (0, 1)$ there exists a constant $\mu = \mu(a) > 0$ such that

$$(3.18) \quad u(x_0) \leq \frac{N}{a} \tau^{-1}(\mu\rho).$$

So the main point is to get a precise estimate on τ^{-1} . If $s_0 > e^{a/2}$ and $C(s_0) = \frac{1}{2} - \frac{\alpha}{4Ln s_0}$ it is easy to check that

$$j(t) > C(s_0) t^2 (Ln t)^\alpha \quad \text{for } t > s_0.$$

If $C_0 = \frac{2}{(\alpha-2)\sqrt{C(s_0)}}$, then $\tau(s) < C_0 (Ln s)^{(2-\alpha)/2}$ for $s > s_0$ and

$$(3.19) \quad \tau^{-1}(y) \leq e^{C_0^2/(\alpha-2) y^{2/(2-\alpha)}}.$$

for $0 < y < \tau(s_0)$. For $|x| < \frac{\sqrt{N}}{2} R$, $Q_{\frac{2|x|}{\sqrt{N}}}(x) \subset B_R$. We set

$$R_0 = \min\left(\frac{1}{2}R, \frac{1}{2} \frac{\tau(s_0)}{\mu}\right)$$

and for $|x| \leq R_0$ we can apply (3.18), (3.19) which gives

$$(3.20) \quad u(x) \leq \frac{N}{a} e^{((C_0 \sqrt{N})/2)^{2/(\alpha-2)} |x|^{2/(2-\alpha)}}.$$

The estimate in $B_R \setminus B_{R_0}$ is obtained from (3.18) with a simple compactness argument and we get (3.17).

LEMMA 3.3. — Assume $N \geq 2$, $\alpha > 0$ and $v \in C^2(\bar{B}_R \setminus \{0\})$ is a nonnegative solution of

$$(3.21) \quad v_{rr} + \frac{N-1}{r} v_r + v_r^2 = v^\alpha \quad \text{in } (0, R)$$

such that $\lim_{r \rightarrow 0} v(r) = +\infty$. Then for any $\varepsilon > 0$ there exists $r(\varepsilon) \in (0, R)$ such that

$$(3.22) \quad -\frac{N-1}{rv^{\alpha/2}} - 1 < \frac{v_r}{v^{\alpha/2}} \leq -1 + \varepsilon \quad \text{in } (0, r(\varepsilon)).$$

Proof. — From (3.21) it is clear that $v_r < 0$ on some $(0, r_0) \subset (0, R)$ and we get

$$(3.23) \quad v_{rr} + v_r^2 \geq v^\alpha \quad \text{in } (0, r_0).$$

Taking $v = \rho$ as a new variable and $h(\rho) = v_r^2$ as a new unknown we get as in Lemma 3.1

$$\frac{1}{2} h_\rho + h \geq \rho^\alpha \quad \text{for } \rho \geq \rho_0,$$

which implies $(e^{2\rho} h)_\rho \geq 2e^{2\rho} \rho^\alpha$ and by integration we get $\frac{h(\rho)}{\rho^\alpha} \geq 1 - \varepsilon$ for any $\varepsilon > 0$ and $\rho > \rho(\varepsilon)$, that is

$$(3.24) \quad \frac{v_r}{v^{\alpha/2}} \leq -1 + \varepsilon \quad \text{in } (0, r(\varepsilon)),$$

where $r(\varepsilon)$ is small enough. As a consequence $\lim_{r \rightarrow 0} v_r(r) = -\infty$. If we set $\omega = v_r$ we get from (3.21)

$$(3.25) \quad \omega_{rr} + \frac{N-1}{r} \omega_r + 2\omega\omega_r - \frac{N-1}{r^2} \omega = \alpha\omega v^{\alpha-1}.$$

As $\omega < 0$ on $(0, r_0)$, (3.25) implies

$$(3.26) \quad \omega_{rr} + \left(\frac{N-1}{r} + 2\omega \right) \omega_r < 0 \quad \text{in } (0, r_0).$$

Hence if $\omega_r(r_1) \leq 0$ for some $r_1 \in (0, r_0)$ we would have $\omega_r(r) < 0$ for $r \in (0, r_1)$ contradicting $\lim_{r \rightarrow 0} \omega(r) = -\infty$. As a consequence $\omega_r > 0$ and

$$(3.27) \quad v_r^2 + \frac{N-1}{r} v_r - v^\alpha \leq 0 \quad \text{in } (0, r_0).$$

A simple algebraic computation implies

$$(3.28) \quad -\frac{N-1}{2r} - \sqrt{\left(\frac{N-1}{2r} \right)^2 + v^\alpha} \leq v_r \leq 0$$

and

$$(3.29) \quad \frac{v_r}{v^{\alpha/2}} \geq -\frac{N-1}{rv^{\alpha/2}} - 1,$$

which ends the proof.

LEMMA 3.4. — Assume $N \geq 2$, $\alpha > 1$ and $u \in C^2(\bar{B}_R \setminus \{0\})$ is a nonnegative solution of

$$(3.30) \quad u_{rr} + \frac{N-1}{r} u_r = u(Ln^+ u)^\alpha \quad \text{in } (0, R).$$

Then $\lim_{r \rightarrow 0} u(r)/\mu(r) = +\infty$ if and only if $\lim_{r \rightarrow 0} r^{2/\alpha} Ln u(r) = +\infty$.

Proof. — Case 1 : $N \geq 3$. — We consider the following change of variable

$$(3.31) \quad s = r^{2-N}, \quad \tilde{u}(s) = u(r);$$

$\tilde{u}$ satisfies

$$(3.32) \quad \tilde{u}_{ss} = \frac{1}{(N-2)^2} s^{-2((N-1)/(N-2))} \tilde{u}(Ln^+ \tilde{u})^\alpha \quad \text{in } (S, +\infty),$$

with $S = R^{2-N}$, and if $\lim_{r \rightarrow 0} r^{N-2} u(r) = +\infty$ we have

$$(3.33) \quad \lim_{r \rightarrow +\infty} \tilde{u}(s)/s = \lim_{s \rightarrow +\infty} \tilde{u}_s(s) = +\infty.$$

From convexity $\tilde{u}(s) \leq s \tilde{u}_s(s) (1 + o(1))$ and

$$(Ln \tilde{u})^\alpha < (Ln s + Ln \tilde{u}_s + O(1))^\alpha \leq (N-2)^2 (Ln s)^\alpha (Ln \tilde{u}_s)^\alpha$$

for s large enough; so (3.32) becomes

$$(3.34) \quad \tilde{u}_{ss} \leq s^{-N/(N-2)} \tilde{u}_s (Ln \tilde{u}_s)^\alpha (Ln s)^\alpha.$$

As $\alpha > 1$

$$\int_\sigma^{+\infty} \frac{\tilde{u}_{ss}}{\tilde{u}_s (Ln \tilde{u}_s)^\alpha} ds = \frac{1}{\alpha-1} (Ln \tilde{u}_s(\sigma))^{1-\alpha}$$

and

$$\int_\sigma^{+\infty} s^{-N/(N-2)} (Ln s)^\alpha ds < A \sigma^{-2/(N-2)} (Ln \sigma)^\alpha$$

for some constant A and σ large enough. As a consequence $Ln \tilde{u}_s(\sigma) \geq B \sigma^{2/(N-2)(\alpha-1)} (Ln \sigma)^{\alpha/(1-\alpha)}$. A straightforward computation implies that for

any $\varepsilon > 0$ and for s large enough

$$\tilde{u}(s) \geq e^{s^{\varepsilon + 2/(1-\alpha)/(N-2)}},$$

which means

$$(3.35) \quad \text{Ln } u(r) \geq r^{\varepsilon + 2/(1-\alpha)},$$

for r small enough and $\lim_{r \rightarrow 0} r^{2/\alpha} \text{Ln } u(r) = +\infty$. Conversely

$$\lim_{r \rightarrow 0} r^{2/\alpha} \text{Ln } u(r) = +\infty \text{ implies } \lim_{r \rightarrow 0} u(r)/\mu(r) = +\infty \quad (N \geq 2).$$

Case 2: $N=2$. — We make the following change of variable

$$(3.36) \quad r = e^{-t}, \quad \tilde{u}(t) = u(r),$$

and we get (with $T = \text{Ln}(1/R)$)

$$(3.37) \quad \tilde{u}_{tt} = e^{-2t} \tilde{u} (\text{Ln } \tilde{u})^\alpha \quad \text{in } (T, +\infty).$$

If we assume $\lim_{r \rightarrow 0} u(r)/\text{Ln}(1/r) = +\infty$ then

$$\lim_{t \rightarrow +\infty} \tilde{u}(t)/t = \lim_{t \rightarrow +\infty} \tilde{u}_t(t) = +\infty$$

(by convexity) and we get

$$\frac{\tilde{u}_{tt}}{\tilde{u}_t (\text{Ln } \tilde{u}_t)} \leq e^{-2t} t (\text{Ln } t)^\alpha (1 + o(1)) \quad \text{for } t \gg T$$

and

$$(3.38) \quad \text{Ln } \tilde{u}_t(t) \geq B t^{1/(1-\alpha)} (\text{Ln } t)^{\alpha/(1-\alpha)} e^{-2t/(1-\alpha)}$$

for some $B > 0$ and t large enough, which implies

$$(3.39) \quad \tilde{u}(t) \geq e^{(2/(\alpha-1)-\varepsilon)t},$$

for any $\varepsilon > 0$ and t large. From (3.39) we get the result.

With lemmas 3.2-3.4 we can describe the behaviour of nonnegative radial solutions of (3.16) with a strong singularity at 0, when $\alpha > 2$.

LEMMA 3.5. — Assume $N \geq 2$, $\alpha > 2$ and $u \in C^2(\bar{B}_R \setminus \{0\})$ is a nonnegative solution of (3.30) in $(0, R)$ such that $\lim_{r \rightarrow 0} u(r)/\mu(r) = +\infty$. Then the following

holds near 0

$$(3.40) \quad r^{2/(\alpha-2)} \text{Ln } u(r) = \gamma(\alpha) + \frac{\alpha - (N-1)(\alpha-2)}{2\alpha} r^{2/(\alpha-2)} + O(r^{4/(\alpha-2)}),$$

$$r^{\alpha/(\alpha-2)} (\text{Ln } u(r))_r = -(\gamma(\alpha))^{\alpha/2} + O(r^{4/(\alpha-2)}).$$

Proof. — From the preceeding lemmas $\lim_{r \rightarrow 0} v_r(r)/v^{\alpha/2}(r) = -1$ where $v = \text{Ln } u$. As a consequence

$$(3.41) \quad \begin{aligned} \lim_{r \rightarrow 0} r^{2/(\alpha-2)} v(r) &= \gamma(\alpha) \\ \lim_{r \rightarrow 0} r^{\alpha/(\alpha-2)} v_r(r) &= -(\gamma(\alpha))^{\alpha/2} \end{aligned}$$

and $\frac{N-1}{r} v_r(r) = (-1 + o(1)) \frac{(N-1)(\alpha-2)}{2} v^{\alpha-1}(r)$ near 0. Plugging this estimate into equation (3.21) yields

$$(3.42) \quad v_{rr} + v_r^2 = v^\alpha + C(1 + o(1)) v^{\alpha-1}$$

with $C = (N-1)(\alpha-2)/2$. Taking again $\rho = v$ as the variable and $h(\rho) = v_r^2$ as the unknown implies

$$\frac{1}{2} (e^{2\rho} h(\rho))_\rho = \rho^\alpha e^{2\rho} + C(1 + o(1)) \rho^{\alpha-1} e^{2\rho}$$

and

$$(3.43) \quad \frac{h(\rho)}{\rho^\alpha} = 1 + (1 + o(1)) \left(C - \frac{\alpha}{2} \right) \frac{1}{\rho} \quad \text{as } \rho \rightarrow +\infty.$$

If we set $A = \frac{\alpha}{4} - \frac{C}{2} = \frac{\alpha - (N-1)(\alpha-2)}{4}$ we have $\frac{v_r}{v^{\alpha/2}} = -1 + \frac{1+o(1)}{v} A$, which implies $v(r) = \gamma(\alpha) (1 + o(1)) r^{2/(2-\alpha)}$ and finally

$$(3.44) \quad \frac{v_r}{v^{\alpha/2}} = -1 + \frac{1+o(1)}{\gamma(\alpha)} A r^{2/(\alpha-2)}.$$

Integrating (3.44) on $(0, r]$ for some small r implies

$$v(r) - \gamma(\alpha) r^{2/(2-\alpha)} = (1 + o(1)) (2A/\alpha).$$

As $v_r = -v^{\alpha/2} \left(1 + O\left(\frac{1}{v}\right) \right)$, we have $\frac{N-1}{r} v_r = -C v^{\alpha-1} \left(1 + O\left(\frac{1}{v}\right) \right)$ and v satisfies

$$(3.45) \quad v_{rr} + v_r^2 = v^\alpha + C v^{\alpha-1} + O(v^{\alpha-2});$$

using ρ and $h(\rho)$ yields

$$(3.46) \quad \frac{h(\rho)}{\rho^\alpha} = 1 + \frac{2C-\alpha}{2} \frac{1}{\rho} + O\left(\frac{1}{\rho^2}\right),$$

$$(3.47) \quad \frac{v_r}{v^{\alpha/2}} = -1 + \frac{A}{v} + O\left(\frac{1}{v^2}\right),$$

and, as $v = \gamma r^{2/(2-\alpha)} (1 + O(r^{2/(\alpha-2)}))$,

$$(3.48) \quad \frac{v_r}{v^{\alpha/2}} = -1 + \frac{A}{\gamma(\alpha)} r^{2/(\alpha-2)} + O(r^{4/(\alpha-2)}).$$

Integrating (3.48) gives $v(r) = \gamma(\alpha) r^{2/(2-\alpha)} + \frac{2A}{\alpha} + O(r^{2/(\alpha-2)})$ which implies (3.40).

Remark 3.1. — If $N \geq 3$ and $\alpha = 2 \frac{N-1}{N-2}$, $\psi(r) = \gamma(\alpha) r^{2/(2-\alpha)}$ is a solution of (3.30) in $(0, +\infty)$.

We are now able to prove the main theorem of this section

THEOREM 3.2. — Assume $N \geq 2$, $\alpha > 0$ and $u \in C^2(\Omega')$ is a nonnegative solution of (3.16) in Ω' . Then

if $0 < \alpha \leq 2$:

- (i) either u can be extended to Ω as a C^2 solution of (3.16) in Ω ,
- (ii) or there exists $\gamma > 0$ such that $\lim_{x \rightarrow 0} u(x)/\mu(x) = \gamma$ and u satisfies

$$(3.49) \quad \Delta u = u(Ln^+ u)^\alpha - C(N) \gamma \delta_0$$

in $D'(\Omega)$;

if $\alpha > 2$:

- (iii) either u behaves as in (i) or (ii) above
- (iv) or $u(x) = \gamma(\alpha, N) e^{\gamma(\alpha) |x|^{2/(2-\alpha)}} (1 + O(|x|^{2/(\alpha-2)}))$

near 0 with $\gamma(\alpha) = \left(\frac{2}{\alpha-2} \right)^{2/(\alpha-2)}$ and $\gamma(\alpha, N) = e^{(\alpha-(N-1)(\alpha-2))/2\alpha}$.

Proof. — From Theorems 1.1, 1.2 we know that $u(x)/\mu(x)$ admits a limit in $(0, +\infty]$ as x tends to 0. If the limit is finite we get (i) or (ii) [(iii) if $\alpha > 2$] and (3.49) from Theorems 1.1, 1.2 and Remark 1.1 (if the limit is 0 then u is regular as in Proposition 2.5). So let us assume that

$$(3.50) \quad \lim_{x \rightarrow 0} u(x)/\mu(x) = +\infty.$$

For any $c > 0$ let φ_c be the solution of

$$(3.51) \quad \begin{aligned} (\varphi_c)_{rr} + \frac{N-1}{r} (\varphi_c)_r &= \varphi_c (Ln^+ \varphi_c)^\alpha \quad \text{in } (0, R), \\ \lim_{r \rightarrow 0} \varphi_c(r)/\mu(r) &= c, \quad \varphi_c(R) = \min_{|x|=R} u(x), \end{aligned}$$

(we assume $B_R \subset \Omega$). It is clear that $0 \leq \varphi_c \leq u$ for $0 < |x| < R$, $c \mapsto \varphi_c$ is increasing and $\lim_{c \rightarrow +\infty} \varphi_c = \varphi$ where φ satisfies

$$(3.52) \quad \begin{aligned} \varphi_{rr} + \frac{N-1}{r} \varphi_r &= \varphi (Ln^+ \varphi)^\alpha \quad \text{in } (0, R), \\ \lim_{r \rightarrow 0} \varphi(r)/\mu(r) &= +\infty, \quad \varphi(R) = \min_{|x|=R} u(x). \end{aligned}$$

Moreover $0 \leq \varphi \leq u$ in $B_R \setminus \{0\}$.

If $0 < \alpha \leq 2$ we can take R small enough such that $\varphi(R) > e$ and we construct in the same way as φ a function $\tilde{\varphi}$ such that $0 \leq \tilde{\varphi} \leq \varphi$ and

$$(3.53) \quad \begin{aligned} \tilde{\varphi}_{rr} + \frac{N-1}{r} \tilde{\varphi}_r &= \tilde{\varphi} (Ln^+ \tilde{\varphi})^2 \quad \text{in } (0, R), \\ \lim_{r \rightarrow 0} \tilde{\varphi}(r)/\mu(r) &= +\infty, \quad \tilde{\varphi}(R) = \varphi(R). \end{aligned}$$

From Lemma 3.4 $\lim_{r \rightarrow 0} r^{2/\alpha} Ln^+ \tilde{\varphi}(r) = +\infty$. If we set $\zeta = Ln^+ \tilde{\varphi}$, then Lemma

3.3 implies that $\lim_{r \rightarrow 0} \frac{\zeta_r}{\zeta}(r) = -1$ which implies by integration that ζ remains

bounded near 0 and so does $\tilde{\varphi}$, a contradiction.

We assume now $\alpha > 2$. We define ψ_n as the solution of

$$(3.54) \quad \begin{aligned} (\psi_n)_{rr} + \frac{N-1}{r} (\psi_n)_r &= \psi_n (Ln^+ \psi_n)^\alpha \quad \text{in } \left(\frac{1}{n}, R\right), \\ \psi_n\left(\frac{1}{n}\right) &= \max_{|x|=1/n} u(x), \quad \psi_n(R) = \max_{|x|=R} u(x). \end{aligned}$$

Using Lemma 3.2 and the same device as in the proof of Proposition 2.5 we deduce that for some subsequence $\{\psi_{n_k}\}$ we have $\lim_{n_k \rightarrow \infty} \psi_{n_k} = \psi$ in

the $C^1((0, R])$ -topology and ψ satisfies

$$(3.55) \quad \psi_{rr} + \frac{N-1}{r} \psi_r = \psi (Ln^+ \psi)^\alpha \quad \text{in } (0, R)$$

Moreover $0 \leq u \leq \psi$ in $B_R \setminus \{0\}$. Applying Lemma 3.5 to φ and ψ we get (iv).

Remark 3.2. — It is interesting to notice that if u is a positive solution of (3.16) with a strong singularity at 0, then $v = Ln u$ behaves like the explicit radial singular solution of the following *first order* equation in $\mathbb{R}^N \setminus \{0\}$ ($\alpha > 2$)

$$(3.56) \quad |DU|^2 = U^\alpha$$

that is $U(x) = \gamma(\alpha) |x|^{2/(2-\alpha)}$.

Remark 3.3. — There is an alternative way to prove Theorem 3.2 in the case $\alpha > 2$, it is to obtain Harnack type inequalities as in [23] and to use Lemmas 3.3-3.5 (see [16] for details). Unfortunately such inequalities are out of reach in the case $0 < \alpha \leq 2$ as Lemma 3.2 no longer holds.

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