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The global Cauchy problem for the non linear Klein-Gordon equation-II

by

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ABSTRACT. — We study the Cauchy problem for a class of non linear Klein-Gordon equations of the type $\ddot{\varphi}-\Delta\varphi+f(\varphi)=0$ by a contraction method. We prove the existence and uniqueness of strongly continuous global solutions in the energy space $H^1(\mathbb{R}^n)\oplus L^2(\mathbb{R}^n)$ for arbitrary space dimension n under assumptions on f that cover the case of a sum of powers $\lambda|\varphi|^{p-1}\varphi$ with $1 \leqq p < 1+4/(n-2)$, $n \geqq 2$ and $\lambda > 0$ for the highest p. This provides an alternative proof of existence and uniqueness to that presented in a previous paper [7]. Some of the results can be extended to the critical case $p=1+4/(n-2)$.

Key words : Nonlinear Klein-Gordon, Cauchy problem, contraction method.

RÉSUMÉ. — On étudie le problème de Cauchy pour une classe d'équations de Klein-Gordon non linéaires du type $\ddot{\varphi}-\Delta\varphi+f(\varphi)=0$ par une méthode de contraction. On prouve l'existence et l'unicité de solutions

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méthode de contraction. On prouve l'existence et l'unicité de solutions globales fortement continues dans l'espace d'énergie $H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ pour une dimension d'espace n quelconque, avec des hypothèses sur f qui couvrent le cas d'une somme de puissances $\lambda|\varphi|^{p-1}\varphi$ avec $1 \leqq p < 1+4/(n-2)$, $n \geqq 2$ et $\lambda > 0$ pour la valeur de p la plus élevée. On obtient ainsi une démonstration de l'existence et de l'unicité différente de celle donnée dans un article précédent [7]. Une partie des résultats s'étend au cas critique $p = 1+4/(n-2)$.

1. INTRODUCTION

A large amount of work has been devoted to the study of the Cauchy problem for the non linear Klein-Gordon equation

$$\square \varphi \equiv \ddot{\varphi} - \Delta\varphi = -f(\varphi) \tag{1.1}$$

where φ is a complex valued function defined in space time $\mathbb{R}^{n+1}$, the upper dot denotes the time derivative, Δ is the Laplace operator in $\mathbb{R}^n$ and f is a non linear complex valued function, a typical form of which is the sum of two powers

$$f(\varphi) = \lambda_0 \varphi + \lambda\varphi|\varphi|^{p-1} \tag{1.2}$$

with $1 \leqq p < \infty$ and $\lambda \geqq 0$ ([3]-[10], [12], [14]-[18], [20]-[26], [30], [31]).

We refer to a previous paper with the same title [7] for a more detailed introduction and for a number of technical points which will not be repeated here. The available proofs of existence of global solutions make use at some stage of the conservation of the energy, thereby yielding solutions which are locally bounded functions of time with values in a space not larger than the energy space X_e [*see* (2.31) below]. It is therefore natural to study the Cauchy problem with initial data in that space. The solutions thereby obtained will be called finite energy solutions. All the available uniqueness proofs of solutions are of a perturbative nature. For finite energy solutions, an elementary power counting argument shows that they require assumptions on f which amount to

$$p - 1 < 4/(n-2) \tag{1.3}$$

in the special case (1.2). It is well known that uniqueness in smaller spaces can be obtained under weaker conditions: an example of that possibility will be given below (*see* Proposition 2.1 and the subsequent discussion). The available proofs of existence are of two types. Compactness proofs, non perturbative in character, yield the existence of (weak) global finite energy solutions under assumptions weaker than (1.3), but without uniqueness (*see* [12], [21], [25] and Proposition 2.1 of [7]). The main result of our previous paper [7] was the proof of uniqueness of those weak solutions under assumptions which reduce to (1.3) in the special case (1.2). Proofs of existence by contraction, which are perturbative, again require (1.3) for finite energy solutions, possibly weaker assumptions in smaller spaces. However, there is so far no general proof by contraction of the existence of finite energy solutions under the assumption (1.3) for general n. The available proofs either require stronger restrictions on the interaction or use smaller spaces than the energy space ([3], [4], [6], [9], [10], [14], [15], [16], [22], [24], [30]).

The purpose of the present paper is to provide a general proof by contraction of the existence and uniqueness of finite energy solutions. For that purpose, we first state a fairly abstract uniqueness result (*see* Proposition 2.1), which is one of the ingredients of the uniqueness proof given in [7] (*see* Proposition 3.1 of [7]). We next derive a local existence result by a contraction argument (*see* Proposition 2.2). That argument is slightly more elaborate than the most usual ones and is inspired by a recent paper on the non linear Schrödinger equation [11]. We show that the previous results cover the case of arbitrary initial data of finite energy for a suitable choice of the relevant spaces (*see* Proposition 2.3). We prove that the solutions thereby obtained have finite energy for all times (*see* Proposition 3.1) and conclude that they can be continued to all times (*see* Proposition 3.2). The local results are collected in Section 2. The results which use the conservation of energy are collected in Section 3. Most of the results of Section 2 and some of the results of Section 3 can be extended to the critical value $p=1+4/(n-2)$; the required modifications are described in Section 4. We restrict our attention to $n \geqq 2$ since the special case $n=1$ would require slightly modified statements. In addition that case, as well as the case $n=2$, can be treated by simpler (Hilbert space) methods.

We now comment briefly on the proof of uniqueness of finite energy solutions. This proof is basically the same as that given in [7] and is based on a partial contraction method on bounded sets of finite energy solutions. An essential use is made of the fact that such solutions possess space-time integrability properties, namely belong to $L^q_{loc}(\mathbb{R}, \dot{B}^\rho_r)$ (*see* definition below) for suitable values of ρ, r and q. In [7], those properties are derived in as much generality as possible for solutions in $L^\infty_{loc}(\mathbb{R}, H^1)$. This requires to consider values of r such that $\gamma(r) \equiv (n-1)(1/2-1/r) \approx 1$, and is achieved by estimating the corresponding norms sublinearly through the integral equation associated with (1.1) (*see* Lemma 3.3 of [7]). For the purpose of proving uniqueness, it would suffice to derive those properties for smaller values of r, namely for $\gamma(r) \leqq (n-1)/(n+1)$. This possibility has been considered in [5]. Here we follow the more economical point of view by solving the local problem in a space slightly smaller than a suitable $L^q(., \dot{B}^\rho_r)$, but not contained in $L^\infty(., H^1)$, which leads again to values of r such that $\gamma(r) \leqq (n-1)/(n+1)$.

We conclude this introduction by giving the main notation used in this paper. For any r, $1 \leqq r \leqq \infty$, we denote by $\| . \|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^n)$. With each r it is convenient to associate the variables $\beta(r)$, $\gamma(r)$ and $\delta(r)$ defined by

$$2\beta(r)/(n+1) = \gamma(r)/(n-1) = \delta(r)/n = 1/2 - 1/r.$$

For any integer k, we denote by $H^k \equiv H^k(\mathbb{R}^n)$ the usual Sobolev spaces. We shall use the homogeneous Besov spaces of arbitrary order and the associated Sobolev inequalities, for which we refer to [1], [29] and to the Appendix of [7]. We use the notation $\dot{B}^\rho_r \equiv \dot{B}^\rho_{r,2}(\mathbb{R}^n)$ for those spaces. For any interval I, for any Banach space B, we denote by $\mathscr{C}(I, B)$ [resp. by $\mathscr{C}_w(I, B)$] the space of strongly (resp. weakly) continuous functions from I to B and by $\mathscr{C}^1(I, B)$ the space of strongly continously differentiable functions from I to B. For any q, $1 \leqq q \leqq \infty$, we denote by $L^q(I, B)$ [resp. $L^q_{loc}(I, B)$] the space of measurable functions φ from I to B such that $\| \varphi(.); B \| \in L^q(I)$ [resp. $\| \varphi(.); B \| \in L^q_{loc}(I)$]. If I is open, we denote by $\mathscr{D}'(I, B)$ the space of vector valued distributions from I to B [12].

We shall need the operators $\omega = (-\Delta)^{1/2}$, $K(t) = \omega^{-1} \sin \omega t$ and $\dot{K}(t) = \cos \omega t$. The operators $K(t)$ and $\dot{K}(t)$ are bounded and strongly continuous with respect to t in H^k for any k.

2. THE LOCAL CAUCHY PROBLEM

In this section, we study the Cauchy problem for the equation (1.1) and we derive the local existence and uniqueness of solutions with initial data in a space which can be taken as large as the energy space.

In all the paper, we assume the interaction f in (1.1) to satisfy the following assumption

(A 1) $f\in\mathscr{C}^1(\mathbb{C}, \mathbb{C})$, $f(0)=0$ and for some p, $1\leqq p<\infty$ and all $z\in\mathbb{C}$

$$|f'(z)|\equiv \operatorname{Max}\{|\partial f/\partial z|, |\partial f/\partial \bar{z}|\}\leqq C(1+|z|^{p-1}). \tag{2.1}$$

We shall work mainly with the integral equation associated with (1.1), and for that purpose we define formally

$$G(t_1, t_2; \varphi)(t)=-\int_{t_1}^{t_2} d\tau\, K(t-\tau) f(\varphi(\tau)), \tag{2.2}$$

$$F(t_0; \varphi)(t)=G(t_0, t; \varphi)(t), \tag{2.3}$$

where φ is a suitably regular function of space time, so that

$$F(t_2; \varphi)-F(t_1; \varphi)=G(t_1, t_2; \varphi). \tag{2.4}$$

The integral equation associated with (1.1) is then

$$\varphi=\varphi^{(0)}+F(t_0; \varphi)\equiv A(t_0, \varphi^{(0)}; \varphi) \tag{2.5}$$

where $\varphi^{(0)}$ is a given function of space time. For any interval $I\subset\mathbb{R}$ and suitable values of l, q, r and q_1, we shall use the notation

$$\mathscr{X}_0(I)=L^q(I, L^l),\ \mathscr{X}_1(I)=L^{q_1}(I, L^r)$$

and a similar notation for local spaces. The next lemma gives a meaning to the quantities (2.2) and (2.3) and describes some of their properties.

LEMMA 2.1. — *Let $n\geqq 2$, let f satisfy* (A 1). *Let l, r, q, q_1 satisfy*

$$\begin{cases} 1\leqq l, r, q, q_1\leqq\infty \\ 1<r<\infty \quad \text{if } n=2; \qquad |\gamma(r)|\leqq 1 \quad \text{if } n>3 \end{cases} \tag{2.6}$$

$$(p-1)n/l\leqq \operatorname{Min}\{1+\gamma(r), n(1-\gamma(r))\} \tag{2.7}$$

$$(p-1)/q+1/q_1\leqq 1 \tag{2.8}$$

$$\eta_1\equiv 2-(p-1)(n/l+1/q)>0. \tag{2.9}$$

Let I be a bounded interval and let $\varphi\in\mathscr{X}_1(I)\cap\mathscr{X}_0(I)$. Then

(1) *For any $t_0\in I$, $F(t_0; \varphi)\in\mathscr{X}_1(I)$ and $F(t_0; \varphi)$ is a continuous function of t_0 with values in that space. For any $t_0\in I$ and any $\varphi_1, \varphi_2\in\mathscr{X}_1(I)\cap\mathscr{X}_0(I)$,*

the following estimate holds

$$\| F(t_0;\ \varphi_1)-F(t_0;\ \varphi_2);\ \mathscr{X}_1(I) \| \leqq C_1 \| \varphi_1-\varphi_2;\ \mathscr{X}_1(I) \| \times \{ |I|^2+|I|^{n_1} (\sum_{i=1,\ 2} \| \varphi_i;\ \mathscr{X}_0(I) \|^{p-1}) \} \quad (2.10)$$

(2) *For any* $t_1,\ t_2 \in I$, $G(t_1,\ t_2;\ \varphi) \in \mathscr{X}_{1\,\mathrm{loc}}(\mathbb{R})$ *and for any bounded interval* J, $G(t_1,\ t_2;\ \varphi)$ *is a continuous function of* t_1, t_2 *with values in* $\mathscr{X}_1(J)$. *For any* $t_1,\ t_2 \in I$, *any* $\varphi_1,\ \varphi_2 \in \mathscr{X}_1(I) \cap \mathscr{X}_0(I)$ *and any bounded interval* $J \supset I$, *the following estimate holds*

$$\| G(t_1,\ t_2;\ \varphi_1)-G(t_1,\ t_2;\ \varphi_2);\ \mathscr{X}_1(J) \| \leqq C_1 \| \varphi_1-\varphi_2;\ \mathscr{X}_1([t_1,\ t_2]) \| \times \{ |J|^2+|J|^{n_1} (\sum_{i=1,\ 2} \| \varphi_i;\ \mathscr{X}_0([t_1,\ t_2]) \|^{p-1}) \}. \quad (2.11)$$

(3) *For any* $t_0 \in I$, $F(t_0;\ \varphi)$ *satisfies* $\square F(t_0;\ \varphi)=f(\varphi)$ *in* $\mathscr{D}'(I \times \mathbb{R}^n)$ *and for any* $t_1,\ t_2 \in I$, $G(t_1,\ t_2;\ \varphi)$ *satisfies* $\square G(t_1,\ t_2;\ \varphi)=0$ *in* $\mathscr{D}'(\mathbb{R}^{n+1})$.

Proof. — The proof of parts (1) and (2) follows from estimates which we now derive (*see* also Proposition 3.1 of [7]). We decompose f as $f=f_1+f_2$ with $|f_1'(z)| \leqq C$ and $|f_2'(z)| \leqq C|z|^{p-1}$ and we estimate separately the contributions of f_1 and f_2 to F and G. We use the estimate ([13], [19], [27])

$$\| K(t) u \|_r \leqq C |t|^{1-\delta(r)+\delta(s)} \| u \|_s \quad (2.12)$$

which holds for

$$\begin{cases} 0 \leqq \delta(r)-\delta(s) \leqq \mathrm{Min}\{ 1+\gamma(r),\ n(1-\gamma(r)) \} \\ \qquad 1<s,\ r<\infty \qquad \text{if} \quad n \neq 2. \end{cases} \quad (2.13)$$

The contribution of f_1 is estimated with $s=r$ by

$$\| K(t-\tau)(f_1(\varphi_1(\tau))-f_1(\varphi_2(\tau))) \|_r \leqq C |t-\tau|\, \| \varphi_1(\tau)-\varphi_2(\tau) \|_r. \quad (2.14)$$

The contribution of f_2 is estimated by

$$\| K(t-\tau)(f_2(\varphi_1(\tau))-f_2(\varphi_2(\tau))) \|_r \leqq C |t-\tau|^{1-\delta(r)+\delta(s)} \times \| f_2(\varphi_1(\tau))-f_2(\varphi_2(\tau)) \|_s \leqq C |t-\tau|^{1-\delta(r)+\delta(s)} \times \| \varphi_1(\tau)-\varphi_2(\tau) \|_r \{ \| \varphi_1(\tau) \|_l^{p-1}+\| \varphi_2(\tau) \|_l^{p-1} \} \quad (2.15)$$

with

$$(p-1)\,n/l=\delta(r)-\delta(s) \quad (2.16)$$

which determines s in the range (2.13) under the assumption (2.7). Substituting (2.14) and (2.15) into the definitions of F and G and using the Young inequality for the time integral, which is allowed by (2.8) and (2.9), we obtain (2.10) and (2.11). The continuity of G follows from (2.11) and that of F from (2.4) and from the continuity of G.

Part (3) follows from a duality argument and an easy computation.

Q.E.D.

The previous lemma ensures some form of equivalence between the equation (1.1) and the integral equation (2.5) and allows to relate the integral equations corresponding to different initial times.

COROLLARY 2.1. — *Let $n \geqq 2$, let f satisfy* (A 1) *and let l, r, q and q_1 satisfy* (2.6)-(2.9). *Let* I *be a bounded open interval and let $\varphi \in \mathscr{X}_1(\mathrm{I}) \cap \mathscr{X}_0(\mathrm{I})$. Then*

(1) φ *satisfies* (1.1) *in $\mathscr{D}'(\mathrm{I}, \mathbb{R}^n)$ if and only if for some (all) $t_0 \in \mathrm{I}$, $\varphi^{(0)} \equiv \varphi - \mathrm{F}(t_0; \varphi)$ satisfies $\square\, \varphi^{(0)} = 0$ [in $\mathscr{D}'(\mathrm{I} \times \mathbb{R}^n)$].*

(2) *If φ satisfies* (2.5), *then for all $t_1 \in \mathrm{I}$, φ also satisfies the equation $\varphi = \mathrm{A}(t_1, \varphi^{(1)}; \varphi)$ with $\varphi^{(1)} = \varphi^{(0)} + \mathrm{G}(t_0, t_1; \varphi)$, and $\varphi^{(1)}$ is a continuous function of t_1 with values in $\mathscr{X}_1(\mathrm{I})$.*

We have stated Lemma 2.1 in terms of bounded time intervals. We could as well have used instead unbounded intervals at the expense of using $\mathscr{X}_{i\ \mathrm{loc}}$ instead of $\mathscr{X}_i$ at suitable places.

We are now in a position to state a preliminary uniqueness result.

PROPOSITION 2.1. — *Let $n \geqq 2$, let f satisfy* (A 1) *and let l, r, q* and q_1 satisfy (2.6)-(2.9). *Let I be an open interval, let $t_0 \in \mathrm{I}$, let $\varphi^{(0)} \in \mathscr{X}_{1\ \mathrm{loc}}(\mathrm{I})$. Then the equation* (2.5) *has at most one solution in $\mathscr{X}_{1\ \mathrm{loc}}(\mathrm{I}) \cap \mathscr{X}_{0\ \mathrm{loc}}(\mathrm{I})$.*

Proof. — Let φ_1 and φ_2 be two solutions of (2.5) with the same $\varphi^{(0)}$. Then

$$\varphi_1 - \varphi_2 = \mathrm{F}(t_0; \varphi_1) - \mathrm{F}(t_0; \varphi_2).$$

Using the estimate (2.10) with I replaced by a small interval J containing t_0 and taking that interval sufficiently small, so that

$$\mathrm{C}_1 \{ |\mathrm{J}|^2 + |\mathrm{J}|^{\eta_1} (\sum_{i=1,\ 2} \| \varphi_i; \mathscr{X}_0(\mathrm{J}) \|^{p-1}) \} \leqq 1/2,$$

we obtain a linear inequality which implies that $\varphi_1 = \varphi_2$ in that interval. Iterating the process yields $\varphi_1 = \varphi_2$ everywhere in I.

Q.E.D.

We now discuss briefly the assumptions of Proposition 2.1. The uniqueness result of that proposition can accomodate arbitrarily large values of p. In fact, with the most favourable choice of r and q_1, namely $\gamma(r)=(n-1)/(n+1)$ and $q_1=\infty$, (2.7) and (2.8) become

$$(p-1)\,n/l \leqq 2\,n/(n+1)$$
$$(p-1)/q \leqq 1$$

which together with (2.9) can be satisfied for arbitrarily large p by taking l and q sufficiently large. However, whereas finite energy solutions belong to $L^{\infty}_{\text{loc}}(\mathbb{R}, L^r)$ for that choice of r, they do not in general belong to $\mathscr{X}_{0\ \text{loc}}(\mathbb{R})$ if l and/or q are too large. Actually, the condition of finite energy will allow only for $n/l+1/q \geqq n/2-1$, and (2.9) will then result in the condition (1.3) to ensure the uniqueness of finite energy solutions. That point will be discussed in more detail in Proposition 2.3.

The study of the local existence problem will require estimates for the operator K acting in homogeneous Besov spaces.

LEMMA 2.2. — *Let $n \geqq 2$ and $0 \leqq \gamma(r) \leqq 1$. Then* K *satisfies the estimate*

$$\| K(t)\,u;\ \dot{B}^{\rho}_{r} \| \leqq C\,|t|^{-\mu}\,\| u;\ \dot{B}^{\rho'}_{r'} \| \tag{2.17}$$

for all ρ, ρ', r', μ *such that*

$$0 \leqq 1+\mu=\rho+\delta(r)-\rho'-\delta(r')$$
$$\leqq \frac{1}{2}(\gamma(r)-\gamma(r'))(1+1/\gamma(r)) \leqq 1+\gamma(r) \tag{2.18}$$

[*which implies in particular* $|\gamma(r')| \leqq \gamma(r)$].

Proof. — For $2 \leqq r \leqq \infty$, K satisfies the estimate ([2], [15])

$$\| K(t)\,u;\ \dot{B}^{\rho}_{r} \| \leqq C\,|t|^{-\gamma(r)}\,\| u;\ \dot{B}^{\rho+2\beta(r)-1}_{\bar{r}} \| \tag{2.19}$$

with $1/r+1/\bar{r}=1$. On the other hand, from (2.12) with $r=s$, we obtain for any j

$$\| K(t)\,u * \varphi_j \|_r \leqq C\,|t|\,.\,\| u * \varphi_j \|_r$$

where $\{\varphi_j\}$ is the dyadic decomposition used in the definition of the Besov spaces (*see* the Appendix of [7] for the notation), so that

$$\| K(t)\,u;\ \dot{B}^{\rho}_{r} \| \leqq C\,|t|\,.\,\| u;\ \dot{B}^{\rho}_{r} \|. \tag{2.20}$$

Finally, we estimate directly, by using the fact that $|\sin y| \leqq |y|$,

$$\| K u * \varphi_j \|_2 \leqq |t| . \| u * \varphi_j \|_2,$$

$$\| K u * \varphi_j \|_\infty \leqq \sum_{|k-j| \leqq 1} \| K u * \varphi_j * \varphi_k \|_\infty \leqq \sum_{|k-j| \leqq 1} \| K \varphi_k \|_\infty \| u * \varphi_j \|_1,$$

$$\| K \varphi_k \|_\infty = \operatorname{Sup} (2\pi)^{-n/2} \left| \int d\xi \, |\xi|^{-1} \sin t |\xi| \, \hat{\varphi}_k(\xi) \, e^{ix\xi} \right|$$

$$\leqq (2\pi)^{-n/2} |t| \, 2^{nk} \| \hat{\varphi}_0 \|_1$$

so that by the definition of $\dot{B}^\rho_r$ and by interpolation in the r variable,

$$\| K(t) u; \dot{B}^\rho_r \| \leqq C |t| . \| u; \dot{B}^{\rho + 2\delta(r)}_r \|. \tag{2.21}$$

The estimate (2.17) then follows by interpolation between (2.19), (2.20) and (2.21).

Q.E.D

For any interval $I \subset \mathbb{R}$ and for suitable values of ρ, r and q, we shall use the notation $\mathscr{X}_2(I) = L^q(I, \dot{B}^\rho_r)$. We shall denote by $B_i(I, R)$ the closed ball of radius R in $\mathscr{X}_i(I)$, $i = 1, 2$. The main ingredient of the local existence proof is the following estimate.

LEMMA 2.3. — *Let $n \geqq 2$, let f satisfy* (A 1), *let ρ, r and q satisfy $0 \leqq \rho < 1$ and*

$$0 \leqq \gamma(r) \leqq (n-1)/(n+1) \tag{2.22}$$

$$(p-1)(n/r - \rho) \leqq 1 + \gamma(r) \tag{2.23}$$

$$p \leqq q \tag{2.24}$$

$$\eta_2 \equiv 2 - (p-1)(n/r - \rho + 1/q) > 0. \tag{2.25}$$

Let I *be a bounded open interval, let $t_0 \in I$ and let $\varphi \in \mathscr{X}_2(I)$. Then*

(1) $F(t_0; \varphi) \in \mathscr{X}_2(I)$ *and* $F(t_0; \varphi)$ *satisfies the estimate*

$$\| F(t_0; \varphi); \mathscr{X}_2(I) \| \leqq C_2 \{ |I|^2 \| \varphi; \mathscr{X}_2(I) \| + |I|^{\eta_2} \| \varphi; \mathscr{X}_2(I) \|^p \}. \tag{2.26}$$

(2) *For any bounded interval $J \supset I$ and for any $t_1, t_2 \in I$, $G(t_1, t_2; \varphi)$ is a continuous function of t_1, t_2 with values in $\mathscr{X}_2(J)$, and satisfies the following estimate*

$$\| G(t_1, t_2, \varphi); \mathscr{X}_2(J) \|$$
$$\leqq C_2 \{ |J|^2 \| \varphi; \mathscr{X}_2([t_1, t_2]) \| + |J|^{\eta_2} \| \varphi; \mathscr{X}_2([t_1, t_2]) \|^p \}. \tag{2.27}$$

Proof. — We decompose $f=f_1+f_2$ as in the proof of Lemma 2.1 and we estimate the contributuons of f_1 and f_2 separately. We consider only the contribution of f_2; that of f_1 is obtained from it by replacing p by 1. We apply (2.17) with $u=f_2(\varphi)$ and estimate the right hand side by Lemma 3.2 of [7] and the Sobolev inequalities, thereby obtaining

$$\| K(t-\tau) f_2(\varphi);\, \dot{B}_{r'}^{\rho'} \| \leqq C\,|t-\tau|^{-\mu}\,\| \varphi;\, \dot{B}_r^{\rho} \|^p \tag{2.28}$$

provided $\rho' \leqq \rho$ and $p(n/r-\rho)=n/r'-\rho'$, or equivalently

$$(p-1)(n/r-\rho)=\rho+\delta(r)-\rho'-\delta(r')=1+\mu.$$

This relation determines μ and allows to apply Lemma 2.2 with $\rho' \leqq \rho$ provided one can find r' such that

$$0 \leqq \delta(r)-\delta(r') \leqq (p-1)(n/r-\rho)$$
$$\leqq \frac{1}{2}(\gamma(r)-\gamma(r'))(1+1/\gamma(r)) \leqq 1+\gamma(r). \tag{2.29}$$

For given r in the range (2.22), one can find r' satisfying (2.29) provided (2.23) holds. We then substitute (2.28) into the definition of F and G and estimate the time integral by the Young inequality, which is possible provided (2.24) and (2.25) hold, and yields the second term in the right hand side of (2.26) and (2.27). The term with f_1 is estimated as indicated above and yields the first term in the right hand side of (2.26) and (2.27).

Q.E.D

We are now in a position to prove the local existence and uniqueness of solutions of the equation (2.5).

PROPOSITION 2.2. — *Let* $n \geqq 2$, *let* f *satisfy* (A 1), *let* ρ, r, q *and* q_1 *satisfy* $0 \leqq \rho < 1$, $1 \leqq q \leqq q_1 \leqq \infty$ *and* (2.22)-(2.25). *Then for any* $R>0$, *there exists* $T(R)>0$ *such that for any* $t_0 \in \mathbb{R}$ *and for any* $\varphi^{(0)} \in B_2(I, R) \cap \mathscr{X}_1(I)$, *where* $I=[t_0-T(R), t_0+T(R)]$, *the equation* (2.5) *has a solution in* $B_2(I, 2R) \cap \mathscr{X}_1(I)$, *satisfying* $\| \varphi;\, \mathscr{X}_1(I) \| \leqq 2 \| \varphi^{(0)};\, \mathscr{X}_1(I) \|$. *That solution is unique in* $\mathscr{X}_2(I) \cap \mathscr{X}_1(I)$.

Proof. — We use the estimates of Lemmas 2.1 and 2.3 with $n/l=n/r-\rho$ so that (2.7), (2.8) and (2.9) are implied by (2.22)-(2.25) with $\eta_1=\eta_2 \equiv \eta$. Furthermore, by the Sobolev inequalities, $\dot{B}_r^{\rho} \subset L^l$ with

$$\| u \|_l \leqq C_3 \| u;\, \dot{B}_r^{\rho} \|$$

and $\mathscr{X}_2(.) \subset \mathscr{X}_0(.)$ with a similar inequality between the corresponding norms. We choose $T = T(R)$ such that

$$C_1\{(2T)^2 + (2T)^n 2(2C_3R)^{p-1}\} \leqq 1/2 \tag{2.30}$$

$$C_2\{(2T)^2 + (2T)^n(2R)^{p-1}\} \leqq 1/2. \tag{2.30'}$$

It follows from (2.30), (2.30′) that the operator $A(t_0, \varphi^{(0)})$ leaves the set $S = B_2(I, 2R) \cap \mathscr{X}_1(I)$ invariant and is contracting on S (by a factor 1/2) in the norm of $\mathscr{X}_1(I)$. Now for any $R_1 > 0$, $B_1(I, R_1) \cap B_2(I, 2R)$ is w^*-compact in $\mathscr{X}_1(I) \cap \mathscr{X}_2(I)$, therefore compact in the w^*-topology of $\mathscr{X}_1(I)$, therefore w^*-closed in $\mathscr{X}_1(I)$ and therefore strongly closed in $\mathscr{X}_1(I)$, so that also S is strongly closed in $\mathscr{X}_1(I)$. The existence and uniqueness of a solution in S then follows from the contraction mapping theorem. The more general uniqueness result follows from Proposition 2.1.

Q.E.D.

We now discuss briefly the assumptions of Proposition 2.2. The conditions (2.23)-(2.25) give upper limits on p, which in the most favourable case $\rho \lesssim 1$, $\gamma(r) = (n-1)/(n+1)$, $q = q_1 = \infty$, reduce to

$$(p-1)(n/2 - 1 - n/(n+1)) < 2n/(n+1)$$

or equivalently to

$$(p-1)(n/2 - 3/2 - 1/n) < 2.$$

That conditions is weaker than the condition (1.3) expected for finite energy solutions. In particular it does not restrict p for $n \leqq 3$. However, in the same way as for Proposition 2.1, finite energy solutions in general belong to $\mathscr{X}_2$ only if ρ, r and q are not too large, as will be seen below.

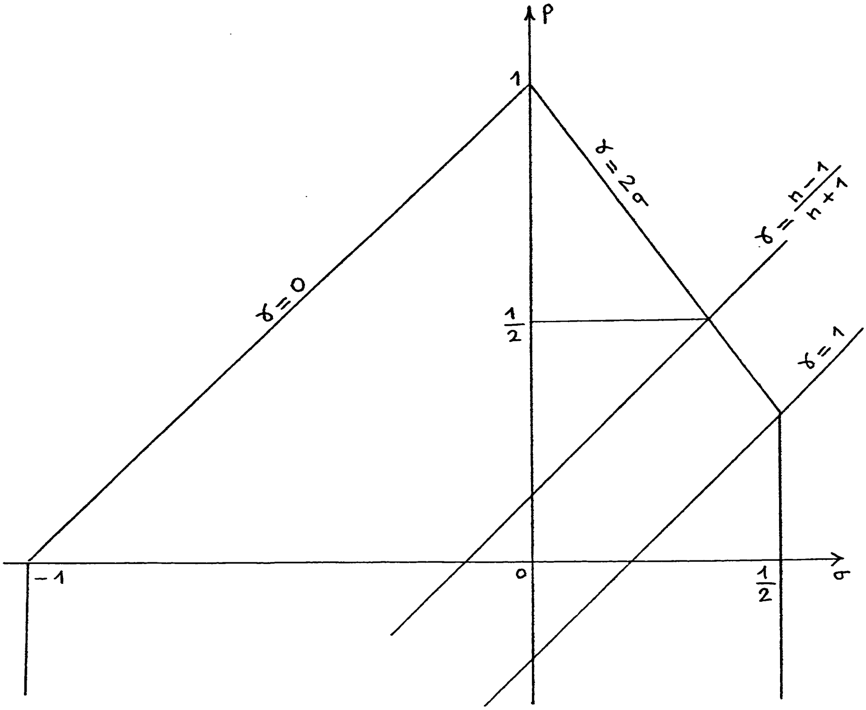
We now derive the implications of the previous results for solutions corresponding to initial data with finite energy. For that purpose, we first introduce the energy space

$$X_e = \{(\varphi_0, \psi_0) : \varphi_0 \in H^1, \psi_0 \in L^2\} = H^1 \oplus L^2. \tag{2.31}$$

With each $(\varphi_0, \psi_0) \in X_e$ is associated a solution of the free equation

$$\varphi^{(0)}(t) = \dot{K}(t - t_0)\varphi_0 + K(t - t_0)\psi_0 \tag{2.32}$$

in $\mathscr{C}(\mathbb{R}, H^1)$. Such solutions satisfy the following space time integrability properties (*see* [17], [23], [28], Lemma 3.1 of [7] and Figure).



Space time integrability properties of the solutions of the KG equation.
The case shown is $n=6$.

LEMMA 2.4. — *Let $n \geqq 2$. Let ρ, r and q satisfy*

$$\left.\begin{array}{c} 0 \leqq \delta(r) \leqq n/2 \\ -1 \leqq \sigma \equiv \rho+\delta(r)-1 < 1/2 \\ \sigma \leqq \gamma(r)/2. \end{array}\right\} \tag{2.33}$$

$$1/q=\operatorname{Max}(0, \sigma). \tag{2.34}$$

Then for any $(\varphi_0, \psi_0) \in X_e$, $\varphi^{(0)}$ as defined by (2.32) *belongs to $\mathscr{X}_2(\mathbb{R})$ and $\varphi^{(0)}$ satisfies the estimate*

$$\| \varphi^{(0)}; \mathscr{X}_2(\mathbb{R}) \| \leqq C(\| \psi_0 \|_2 + \| \nabla \varphi_0 \|_2). \tag{2.35}$$

We can now collect the information obtained thus far for solutions of (2.5) with finite energy initial data.

PROPOSITION 2.3. — *Let* $n \geqq 2$, *let* f *satisfy* (A1) *and* (1.3). *Then*

(1) *There exist* ρ, r *and* q *satisfying* $0 \leqq \rho < 1$, (2.22)-(2.25) *and* (2.33), (2.34).

Let $\mathscr{X}_1$ *and* $\mathscr{X}_2$ *correspond to values of* ρ, r, q *provided by part* (1) *and to* $q_1 \geqq q$. *Then*

(2) *For any* $(\varphi_0, \psi_0) \in X_e$, *there exists* $T > 0$ *depending only on* $\|(\varphi_0, \psi_0); X_e\|$ *such that for any* $t_0 \in \mathbb{R}$, *the equation* (2.5) *with* $\varphi^{(0)}$ *defined by* (2.32) *has a unique solution in* $\mathscr{X}_1(I) \cap \mathscr{X}_2(I)$, *where* $I = [t_0 - T, t_0 + T]$.

(3) *For any* $(\varphi_0, \psi_0) \in X_e$, *for any interval* I, *for any* $t_0 \in I$, *the equation* (2.5) *with* $\varphi^{(0)}$ *defined by* (2.32) *has at most one solution in* $\mathscr{X}_{1\,\mathrm{loc}}(I) \cap \mathscr{X}_{2\,\mathrm{loc}}(I)$.

Proof. — *Part* (1): The condition (2.23) can be rewritten as

$$(p-1)(n/2-1-\sigma) \leqq 1+\gamma(r). \tag{2.36}$$

Under the condition (2.34), the conditions (2.24) and (2.25) reduce to

$$p\sigma \leqq 1 \tag{2.37}$$

and to (1.3) respectively. We now show that for any p satisfying (1.3), one can choose r and σ satisfying the remaining conditions, namely $0 \leqq \rho < 1$, (2.22), (2.33), (2.36) and (2.37). In fact one can take $\rho = 0$ if $p-1 \leqq 4/(n-1)$ and $\gamma(r) = (n-1)/(n+1)$ if $p-1 \geqq 4/(n-1)$. In the former case, σ is negative, and the relevant conditions are immediately seen to hold. In the latter case, (2.36) can be regarded as a lower bound on σ which is an increasing function of p, namely $\sigma \geqq n/2-1-2n/[(n+1)(p-1)]$. For the upper limit in the inequality (1.3) on p, that lower bound becomes

$$\sigma \geqq n/2-1-n(n-2)/[2(n+1)] = (n-2)/[2(n+1)],$$

which is compatible both with the upper bound $\sigma \leqq (n-1)/[2(n+1)]$ coming from (2.33) and with the upper bound coming from (2.37), which is a decreasing function of p and reduces to $\sigma \leqq (n-2)/(n+2)$ in the limiting case of (1.3). The condition $0 \leqq \rho < 1$ is trivially satisfied. This completes the proof of part (1).

Part (2): follows from part (1), from Lemma 2.4 and from Proposition 2.2 after noticing that $\varphi^{(0)} \in \mathscr{X}_{1\,\mathrm{loc}}(\mathbb{R})$ for all r satisfying (2.22) and for all q_1.

Part (3): follows from part (1), from Lemma 2.4 and from Proposition 2.1 with the same choice $n/l = n/r - \rho$ as in the proof of Proposition 2.2.

Q.E.D.

3. THE CONSERVATION OF ENERGY AND THE GLOBAL CAUCHY PROBLEM

In this section, we prove that under suitable assumptions on f, the solutions obtained in Proposition 2.3 satisfy the conservation of energy and can be continued for all times. In almost all this section, we assume the interaction f in (1.1) to satisfy the following assumption

(A2) There exists a function $V \in \mathscr{C}^1(\mathbb{C}, \mathbb{R})$ such that $V(0)=0$, $V(z)=V(|z|)$ for all $z \in \mathbb{C}$ and $f(z)=\partial V/\partial \bar{z}$. For all $R > 0$, V satisfies the estimate

$$V(R) \geqq -a^2 R^2 \tag{3.1}$$

for some $a \geqq 0$.

For $(\varphi, \psi) \in X_e$ and such that $V(\varphi) \in L^1$ [this is the case if (1.3) holds], the energy is defined by

$$E(\varphi, \psi) = \| \psi \|_2^2 + \| \nabla \varphi \|_2^2 + \int dx\, V(\varphi(x)). \tag{3.2}$$

The assumption (A2) formally implies the conservation of energy for the equation (1.1). In order to give an actual proof, we need to approximate the solutions constructed in Proposition 2.3 by smooth solutions of a regularized equation. For that purpose we choose an even non negative function $h_1 \in \mathscr{C}^\infty(\mathbb{R}^n)$ with compact support and such that $\| h_1 \|_1 = 1$. For any positive interger j, we define $h_j(x) = j^n h_1(jx)$,

$$f_j(\varphi) = h_j * f(h_j * \varphi)$$

and correspondingly, if (A2) holds,

$$E_j(\varphi, \psi) = \| \psi \|_2^2 + \| \nabla \varphi \|_2^2 + \int dx\, V(h_j * \varphi). \tag{3.3}$$

We consider the regularized equation

$$\varphi = h_j * \varphi^{(0)} + F_j(t_0; \varphi) \equiv A_j(t_0, \varphi^{(0)}; \varphi) \tag{3.4}$$

where F_j is defined by (2.2), (2.3) with f replaced by f_j. The approximation result can be stated as follows.

LEMMA 3.1. — *Let $n \geqq 2$, let f satisfy* (A1) *and* (1.3). *Let ρ, r, q and q_1 satisfy the conditions of Proposition* 2.3, *part* (1). *Then*

(1) *Proposition* 2.3, *part* (2) *holds for the equation* (3.4) *with the same* T, *independently of* j.

(2) *The solution φ_j of* (3.4) *obtained in part* (1) *converges to the solution φ of* (2.5) *in $\mathscr{X}_1(I)$ when $j \to \infty$.*

Proof. — Part (1) follows from the fact that the convolution with h_j is a contraction in the spaces L^r and $\dot{B}^\rho_r$, so that all the estimates of Section 2 hold with f replaced by f_j.

Part (2) is proved in the same way as the corresponding result in the proof of Proposition 3.2 of [7] (*see* especially (3.60) of [7]), the only difference being that the estimates from Proposition 3.1 of [7] are replaced by the similar estimates of Lemma 2.1 above.

Q.E.D.

We now state without proof the relevant properties of the solutions of the regularized equation (*see* [7], especially Propositions 2.1 and 3.2).

LEMMA 3.2. — *The solutions φ_j of* (3.4) *obtained in Lemma* 3.1 *satisfy the following properties.*

(1) *For any non negative integer k, $(\varphi_j, \dot{\varphi}_j) \in \mathscr{C}^1(I, H^{k+1} \oplus H^k)$ and φ_j satisfies the equation*

$$\square\, \varphi_j + f_j(\varphi_j) = 0.$$

(2) *Let in addition f satisfy* (A2). *Then φ_j satisfies the conservation of energy*

$$E_j(\varphi_j(t), \dot{\varphi}_j(t)) = E_j(h_j * \varphi_0, h_j * \psi_0) \equiv E_j \tag{3.5}$$

and the estimates

$$\| \varphi_j(t) \|_2 \leqq e(E_j, t-t_0) \leqq e(\bar{E}, t-t_0) \tag{3.6}$$

$$\| \dot{\varphi}_j(t) \|_2^2 + \| \nabla \varphi_j(t) \|_2^2 \leqq \dot{e}(E_j, t-t_0)^2 \leqq \dot{e}(\bar{E}, t-t_0)^2 \tag{3.7}$$

where

$$\bar{E} = \operatorname{Sup} E_j < \infty \tag{3.8}$$

and

$$e(E, \tau) = \| \varphi_0 \|_2 \cosh a |\tau| + (E + a^2 \| \varphi_0 \|_2^2)^{1/2} a^{-1} \sinh a |\tau|. \tag{3.9}$$

In particular $(\varphi_j, \dot{\varphi}_j)$ *is locally bounded in* $H^1 \oplus L^2$ *uniformly in* j.

Remark 3.1. — From the fact that $\varphi_j \in \mathscr{C}(I, L^r)$ for all j and from Lemma 3.1, part (2), where one can take $q_1 = \infty$, it follows that $\varphi \in \mathscr{C}(I, L^r)$ and that φ_j converges to φ in $\mathscr{C}(I, L^r)$ when $j \to \infty$.

We are now in a position to prove the conservation of energy for solutions of (2.5) with finite energy initial data.

PROPOSITION 3.1. — *Let* $n \geqq 2$, *let* f *satisfy* (A1), (1.3) *and* (A2). *Let* $(\varphi_0, \psi_0) \in X_e$, *let* I *be an open interval and let* $t_0 \in I$. *Let* ρ, r *and* q *satisfy* $0 \leqq \rho < 1$, (2.22)-(2.24) *and* (2.33), (2.34), *and let* $q_1 = \infty$. *Let* $\varphi^{(0)}$ *be defined by* (2.32) *and let* φ *be a solution of* (2.5) *in* $\mathscr{X}_1(I) \cap \mathscr{X}_2(I)$. *Then* $(\varphi, \dot{\varphi}) \in \mathscr{C}(I, H^1 \oplus L^2)$ *and* φ *satisfies the conservation of energy*

$$E(\varphi(t), \dot{\varphi}(t)) = E(\varphi_0, \psi_0) \equiv E \tag{3.10}$$

and the estimates

$$\| \varphi(t) \|_2 \leqq e(E, t - t_0) \tag{3.11}$$

$$\| \dot{\varphi}(t) \|_2^2 + \| \nabla \varphi(t) \|_2^2 \leqq \dot{e}(E, t - t_0)^2 \tag{3.12}$$

for all $t \in I$.

Proof. — It is sufficient to prove the result in any bounded subinterval $I' \subset\subset I$ containing t_0. Let

$$R = \operatorname*{Sup}_{s \in I'} \| \varphi^{(0)} + G(t_0, s; \varphi); \mathscr{X}_2(I') \|. \tag{3.13}$$

R is finite by (2.27). Let $T = T(R)$ be defined as in the proof of Proposition 2.2 [*see* especially (2.30), (2.30')]. It follows from Proposition 2.2 and Corollary 2.1 that for any $t \in I'$, the solution φ can be recovered in the interval $I' \cap [t - T, t + T]$ by solving the equation (2.5) with initial time t by contraction. We can cover I' by a finite number of intervals I_k of length $2T$ and centers at $t_k = t_0 + (1 - \varepsilon) k T$ for some $\varepsilon > 0$. The result in I' will follow from the corresponding one in I_k for successive values of $k = 0, \pm 1, \pm 2, \ldots$ It is therefore sufficient to prove

Proposition 3.1 in the special case where I is a small interval containing t_0 where the equation (2.5) can be solved by the contraction method of Proposition 2.2, and from now on we restrict our attention to that case.

The proof is now very similar to the proof of energy conservation given in the proof of Proposition 3.2 of [7]. We approximate φ in I by the regularized solutions φ_j of the equation (3.4) as described in Lemma 3.1. From (3.6), (3.7), Remark 3.1 and standard compactness arguments, it follows that φ_j converges to φ in the w^*-sense in $L^\infty(I, H^1)$. Furthermore, by a duality argument, $\dot\varphi_j$ converges to $\dot\varphi$ in the w^*-sense in $L^\infty(I, L^2)$, where $\dot\varphi$ is the derivative of φ in $\mathscr{D}'(I, L^2)$. In addition $(\varphi, \dot\varphi)$ satisfy (3.11), (3.12) for almost all t in I. Now $\varphi \in \mathscr{C}(I, L^r) \cap L^\infty(I, H^1)$ and $\dot\varphi \in L^\infty(I, L^2)$ so that $\varphi \in \mathscr{C}(I, L^s) \cap \mathscr{C}_w(I, H^1)$ for all s, $2 \leqq s < 2n/(n-2)$. Moreover, by Corollary 2.1, part (1), φ satisfies the differential equation (1.1) in $\mathscr{D}'(I \times \mathbb{R}^n)$ so that $\ddot\varphi \in L^\infty(I, H^{-1})$, $\dot\varphi \in \mathscr{C}(I, H^{-1})$ and therefore $\dot\varphi \in \mathscr{C}_w(I, L^2)$. The continuity properties of $(\varphi, \dot\varphi)$ then imply that (3.11), (3.12) hold for all $t \in I$.

We next obtain some additional convergence properties of φ_j to φ. From uniform boundedness in H^1 and convergence in $\mathscr{C}(I, L^r)$, it follows by interpolation that φ_j converges to φ in $\mathscr{C}(I, L^s)$ for all s, $2 < s < 2n/(n-2)$. Furthermore, by a direct estimation, one can show that φ_j converges to φ in $\mathscr{C}(I, L^2)$ (*see* especially (3.63) of [7]). On the other hand $\dot\varphi_j(t)$ converges to $\dot\varphi(t)$ weakly in L^2 for each $t \in I$ (*see* especially (3.64) of [7]) and $\varphi_j(t)$ converges to $\varphi(t)$ weakly in H^1 for each $t \in I$.

We now take the limit $j \to \infty$ in (3.5). The right hand side obviously converges to $E(\varphi_0, \psi_0)$ while the term containing V in the left hand side converges to $\int dx\, V(\varphi)$ by the convergence of φ_j to φ in $\mathscr{C}(I, L^s)$ for $2 \leqq s \leqq p+1$. Weak convergence of $(\varphi_j(t), \dot\varphi_j(t))$ to $(\varphi(t), \dot\varphi(t))$ in $H^1 \oplus L^2$ then implies the inequality

$$E(\varphi(t), \dot\varphi(t)) \leqq E(\varphi_0, \psi_0) \tag{3.14}$$

for all $t \in I$. Time reversal invariance of (1.1) together with Proposition 2.2 and Corollary 2.1 then imply the equality (3.10) for all $t \in I$. That equality, together with the fact that $\varphi \in \mathscr{C}(I, L^s)$, $2 \leqq s \leqq p+1$ implies the continuity of the norm of $(\varphi, \dot\varphi)$ in $H^1 \oplus L^2$ as a function of time, which together with weak continuity implies strong continuity of $(\varphi, \dot\varphi)$ in $H^1 \oplus L^2$ as a function of time.

We remark, although this is not needed here, that a similar argument shows that $(\varphi_j(t), \dot\varphi_j(t))$ converges to $(\varphi(t), \dot\varphi(t))$ strongly in $H^1 \oplus L^2$ for each $t \in \mathrm{I}$.

Q.E.D.

We are now in a position to prove the global existence and uniqueness result for finite energy solutions.

PROPOSITION 3.2. — *Let* $n \geqq 2$, *let* f *satisfy* (A1), (1.3) *and* (A2). *Let* $(\varphi_0, \psi_0) \in \mathrm{X}_e$ *and* $t_0 \in \mathbb{R}$. *Then the equation* (2.5) *with* $\varphi^{(0)}$ *given by* (2.32) *has a unique solution* φ *such that* $(\varphi, \dot{\varphi}) \in \mathscr{C}(\mathbb{R}, \mathrm{X}_e)$, *and that solution satisfies the conservation of energy* (3.10) *and the estimates* (3.11), (3.12).

Let in addition ρ, r, q *and* q_1 *satisfy* $0 \leqq \rho < 1$, (2.22)-(2.24), (2.33), (2.34) *and* $q_1 \geqq q$. *Then the solution is unique in* $\mathscr{X}_{1\,\mathrm{loc}}(\mathbb{R}) \cap \mathscr{X}_{2\,\mathrm{loc}}(\mathbb{R})$.

Proof. — Let ρ, r, q and q_1 be as in the proposition. Then, by Proposition 2.3 part (2), the equation (2.5) with initial data in X_e can be solved locally in time in $\mathscr{X}_1(\mathrm{I}) \cap \mathscr{X}_2(\mathrm{I})$, where the length of the interval I depends only on the norm of (φ_0, ψ_0) in X_e. By Proposition 3.1 the local solutions thereby obtained belong to $\mathscr{C}(\mathrm{I}, \mathrm{X}_e)$ and satisfy the *a priori* estimates (3.11) and (3.12). By standard arguments those two facts imply the existence of global solutions in $\mathscr{X}_{1\,\mathrm{loc}}(\mathbb{R}) \cap \mathscr{X}_{2\,\mathrm{loc}}(\mathbb{R}) \cap \mathscr{C}(\mathbb{R}, \mathrm{X}_e)$. Uniqueness in $\mathscr{X}_{1\,\mathrm{loc}}(\mathbb{R}) \cap \mathscr{X}_{2\,\mathrm{loc}}(\mathbb{R})$ follows from Proposition 2.3 part (3). Uniqueness in $\mathscr{C}(\mathbb{R}, \mathrm{X}_e)$, actually in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}, \mathrm{X}_e)$, follows from the fact that any solution in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}, \mathrm{X}_e)$ belongs to $\mathscr{X}_{1\,\mathrm{loc}}(\mathbb{R}) \cap \mathscr{X}_{2\,\mathrm{loc}}(\mathbb{R})$ by Lemma 3.3 of [7].

Q.E.D.

4. THE CAUCHY PROBLEM IN THE CRITICAL CASE

In this section we describe briefly that part of the results of the previous sections that still hold in the critical case $p = 1 + 4/(n-2)$ and the modifications required to derive them. This case has been already considered in [17] in space dimension $3 \leqq n \leqq 5$ with similar conclusions.

The main result of Section 2, namely Proposition 2.3, still holds with the only difference that the condition (2.34) forces $\eta_2 = 0$ in (2.25). The time integrals in the equation (2.5) can still be estimated by using the Hardy-Littlewood-Sobolev inequality instead of the Young inequality. This requires to avoid the forbidden limiting case associated with the former inequality, which is achieved by assuming

$$\begin{cases} 1 < q \leqq q_1 < \infty \\ p < q. \end{cases} \tag{4.1}$$

The results of Section 3 become more delicate in the critical case. Lemmas 3.1 and 3.2 hold with no other changes than (4.1) while Remark 3.1 clearly does not apply. Proposition 3.1 still holds under the additional assumption (4.1) and the fact that V can be decomposed as $V = V_1 + V_2$ where V_1 satisfies the estimate

$$|V_1(R)| < C(R^2 + R^{p'+1}) \tag{4.2}$$

for some p', $1 \leqq p' < 1 + 4/(n-2)$ and for all $R \in \mathbb{R}^+$ and where the map $\varphi \to \int dx\, V_2(\varphi)$ is weakly lower semicontinuous from H^1 to $\mathbb{R}$ on the bounded sets of H^1. The proof requires three modifications:

The first one regards the possibility of reconstructing the given solution φ in a bounded interval $I' \subset\subset I$ by a finite number of successive steps of local resolution. In fact, with $\eta_2 = 0$, the local Cauchy problem for the equation (2.5) with initial data $(\varphi(s), \dot{\varphi}(s))$ at time s can be solved in the interval $[s, s+T]$ provided

$$h_T(s) \equiv \| \varphi^{(0)} + G(t_0, s; \varphi); \mathscr{X}_2((s, s+T)) \| \leqq \varepsilon \tag{4.3}$$

for a suitable $\varepsilon > 0$. In order to cover I' with (a finite number of) intervals $[s_i, s_i + T]$ where (4.3) holds, it suffices to prove that $h_T(s)$ tends to zero uniformly for $s \in I'$ when T tends to zero. Now $h_T(s)$ is easily seen to be a continuous function of s by the estimates of Section 2, is obviously decreasing in T, and tends to zero for fixed s when T tends to zero. The result then follows from Dini's theorem.

The second modification regards the limit $j \to \infty$ in (3.5). In fact the contribution of V_1 in the left hand side of (3.5) converges to $\int dx\, V_1(\varphi)$ by the convergence of φ_j to φ in $\mathscr{C}(I, L^s)$ for $2 \leqq s < 2n/(n-2)$ while, by lower semicontinuity

$$\liminf_{j \to \infty} \int dx\, V_2(\varphi_j) \geqq \int dx\, V_2(\varphi),$$

thereby yielding (3.14).

The third modification concerns the proof of strong continuity in time of $(\varphi, \dot{\varphi})$ in $H^1 \oplus L^2$ from weak continuity and energy conservation, using again the lower semi-continuity of $\int dx\, V_2(\varphi)$. In fact all three terms of the energy [*see* (3.2)] are lower semi-continuous in t and therefore continuous by energy conservation, so that also the norm of $(\varphi, \dot{\varphi})$ in $H^1 \oplus L^2$ is continuous.

Actually the lower semi-continuity of V_2 can be dispensed with in the proof of energy conservation [but not in the proof of strong continuity of $(\varphi, \dot{\varphi})$ in $H^1 \oplus L^2$ with respect to time] at the expense of using Fatou's lemma.

Finally we are not able to extend Proposition 3.2 to the limiting case.

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