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Existence of a closed geodesic on p -convex sets

by

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ABSTRACT. — The existence of a non constant closed geodesic on some nonsmooth sets is proved.

Key words : Closed geodesics, Lusternik-Fet theorem, nonsmooth analysis, p -convex sets.

RÉSUMÉ. — On montre l'existence d'une géodésique fermée non constante sur certains ensembles non réguliers.

0. INTRODUCTION

A well-known result by Lusternik-Fet (*see*, for instance, [12]) establishes the existence of a non-constant closed geodesic in a compact regular Riemannian manifold without boundary.

In [15], this result is generalized to cover manifolds with boundary.

In both cases, the problem is reduced to a research of critical points for the energy functional $f(\gamma) = \frac{1}{2} \int_0^1 |\gamma'|^2 ds$ on the space of the admissible

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paths $X = \{ \gamma \in W^{1,2}(0,1; M); \gamma(0) = \gamma(1) \}$ where M is the manifold considered.

In this paper, we shall extend Lusternik-Fet result to cover a more general situation, namely p -convex sets. Such class of sets was introduced in [9] and in a less restrictive version in [2], where is also proved the existence of infinitely many geodesics on M orthogonal to M_0 and M_1 , under the hypothesis that M , M_0 and M_1 are p -convex subsets of $\mathbb{R}^n$.

Examples of p -convex sets are $C_{\text{loc}}^{1,1}$ -submanifolds (possibly with boundary) of a Hilbert space and images under a $C_{\text{loc}}^{1,1}$ -diffeomorphism of convex sets.

The motivation for considering Lusternik-Fet result in the context of p -convex sets comes from some remarks about regularity of f and X .

In the case handled by Lusternik-Fet, f is a regular functional and X is a regular Riemannian manifold, on the contrary, in [15], even if M is a regular manifold, X has not a natural structure of manifold and f is not regular. All that suggests that the more natural way to deal with this problem is to consider as starting-point irregular sets.

This consideration prompted the present work.

Other typical problems in differential geometry, concerning sets with a certain degree of irregularity, are treated in [17].

For proving our result, we use a variational technique adapted for non regular functionals. We characterize closed geodesics as “critical points” for the energy functional f on the space X of the admissible paths. Then, we prove that f is included in the class of ϕ -convex functions (see, for instance, [10]). For such functions, some adaptations of classical variational methods in critical point theory (such as deformation lemmas) are available (see, for instance, [4], [8], [13]).

The present work is divided in 4 sections.

In the first section, we recall the definition of p -convex sets and describe some properties of them. In the second one, we give a variational characterization for closed geodesics. The third section is a topological one. We deduce some homotopic properties of X . They together with a suitable deformation lemma are the basic tools for the proof of the existence of at least a non-constant closed geodesic on a p -convex subset of $\mathbb{R}^n$, in section four.

1. SOME RECALLS ON p -CONVEX SETS

In this section, we shall define p -convex sets and describe their properties.

Before, let us recall some notions of non-smooth analysis (cf. [3] to [7], [9], [10]).

From now on, H will be a real Hilbert space, $|\cdot|$ and $(\cdot, \cdot)$ its norm and scalar product, respectively.

DEFINITION 1.1 (see also [3] and [6]). — Let Ω be an open subset of H and $f: \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ a map.

We set

$$D(f) = \{u \in \Omega : f(u) < +\infty\}.$$

Let u belong to $D(f)$. The function f is said to be subdifferential at u if there exists $\alpha \in H$ such that

$$\liminf_{v \rightarrow u} \frac{f(v) - f(u) - (\alpha, v - u)}{|v - u|} \geq 0.$$

We denote by $\partial^- f(u)$ the (possibly empty) set of such α 's and we set

$$D(\partial^- f) = \{u \in D(f) : \partial^- f(u) \neq \emptyset\}.$$

It is easy to check that $\partial^- f(u)$ is convex and closed $\forall u \in D(f)$.

If $u \in D(\partial^- f)$, $\text{grad}^- f(u)$ will denote the element of minimal norm of $\partial^- f(u)$. Moreover, let M be a subset of H . We denote by I_M the function:

$$I_M(u) = \begin{cases} 0, & u \in M \\ +\infty, & u \in H \setminus M. \end{cases}$$

It is easy to check that $\partial^- I_M(u)$ is a cone $\forall u \in M$.

We will call normal cone to M at u the set $\partial^- I_M(u)$ and tangent cone to M at u its negative polar $(\partial^- I_M(u))^-$, i. e.,

$$(\partial^- I_M(u))^- = \{v \in H : (v, w) \leq 0, \forall w \in \partial^- I_M(u)\}.$$

DEFINITION 1.2. — A point $u \in D(f)$ is said to be critical from below for f if $0 \in \partial^- f(u)$; $c \in \mathbb{R}$ is said to be a critical value of f if there exists $u \in D(f)$ such that

$$0 \in \partial^- f(u) \quad \text{and} \quad f(u) = c.$$

DEFINITION 1.3 (see also [5], [10]). — Let Ω be an open subset of H . A function $f: \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to have a φ -monotone subdifferential if there exists a continuous function

$$\varphi : D(f) \times \mathbb{R}^2 \rightarrow \mathbb{R}^+$$

such that:

$$(\alpha - \beta, u - v) \geq -(\varphi(u, f(u), |\alpha|) + \varphi(v, f(v), |\beta|)) |u - v|^2$$

whenever

$$u, v \in D(\partial^- f), \quad \alpha \in \partial^- f(u) \quad \text{and} \quad \beta \in \partial^- f(v).$$

If $p \geq 1$, f is said to have a φ -monotone subdifferential of order p if there exists a continuous function

$$\chi : D(f)^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^+$$

such that:

$$(\alpha - \beta, u - v) \geq -\chi(u, v, f(u), f(v))(1 + |\alpha|^p + |\beta|^p)|u - v|^2$$

whenever

$$u, v \in D(\partial^- f), \quad \alpha \in \partial^- f(u) \quad \text{and} \quad \beta \in \partial^- f(v).$$

Now let us give the definition of p -convex sets (cf. [2]).

DEFINITION 1.4. — Let M be a subset of H . M is said to be a p -convex set if there exists a continuous function $p : M \rightarrow \mathbb{R}^+$ such that

$$(\alpha, v - u) \leq p(u)|\alpha||v - u|^2$$

whenever $u, v \in M$ and $\alpha \in \partial^- I_M(u)$.

Examples of p -convex sets are the following ones:

- (1) the $C_{\text{loc}}^{1,1}$ -submanifolds (possibly with boundary) of H ;
- (2) the convex subsets of H ;
- (3) the images under a $C_{\text{loc}}^{1,1}$ -diffeomorphism of convex sets;
- (4) the subset of $\mathbb{R}^n : \{x : \max |x_i| \leq 1, \sum x_i^2 \geq 1\}$ [note that it is not included in the classes (1), (2), (3)].

Several properties of p -convex sets are proved in [2]. We recall some of them.

Let us define the following set relatively to a p -convex set M :

DEFINITION 1.5. — Let us denote by $\hat{A}$ the set of u 's $\in H$ with the two properties:

- (i) $\delta_p(u, M) < 1$ where $\delta_p(u, M) = \limsup_{\substack{|u-w| \rightarrow d(u, M) \\ w \in M}} 2p(w)|u-w|$.
- (ii) $\exists r \geq 0$ such that $M \cap \{v \in H : |v-u| \leq r\}$ is closed in H and not empty.

Obviously, $M \subset \hat{A}$ and:

PROPOSITION 1.6. — Let $M \subset H$ be p -convex and locally closed. Then $\hat{A}$ is open and $\forall u \in \hat{A}$ there exists one and only one $w \in M$ such that $|u-w| = d(u, M)$.

Moreover, if we set $\pi(u) = w$, then

- (i) $(u - \pi(u)) \in \partial^- I_M(\pi(u))$ and $2p(\pi(u))|u - \pi(u)| < 1, \forall u \in \hat{A}$.
- (ii) $|\pi(u_1) - \pi(u_2)| \leq (1 - p(\pi(u_1))|u_1 - \pi(u_1)| - p(\pi(u_2))|u_2 - \pi(u_2)|)^{-1}|u_1 - u_2|, \forall u_1, u_2 \in \hat{A}$.
- (iii) $(t\pi(u) + (1-t)u) \in \hat{A}, \forall u \in \hat{A}, \forall t \in [0, 1]$.

Remark 1.7. — Let us set $A = \{u \in \hat{A} : 4p(\pi(u))|u - \pi(u)| < 1\}$. Then A is an open set containing M and one can easily prove that $\pi : A \rightarrow M$ is Lipschitz continuous of constant two.

PROPOSITION 1.8. — *Let $M \subset H$ be locally closed and p -convex. Then*

$$\lim_{s \rightarrow 0^+} \frac{\pi(u+sv) - u}{s} = P_u(v)$$

$\forall u \in M$ and $\forall v \in H$, where P_u is the projection on the tangent cone to M at u , i. e. $(\partial^- I_M(u))^-$.

PROPOSITION 1.9. — *Let $M \subset H$ be locally closed and p -convex. Let us take $u \in M$ and $B(u, r) = \{v \in H : |v - u| < r\} \subset \hat{A}$. Then*

$$|su_1 + (1-s)u_0 - \pi(su_1 + (1-s)u_0)| \leq 2p(\pi(su_1 + (1-s)u_0))s(1-s)|u_0 - u_1|^2$$

$\forall s \in [0, 1]$ and $\forall u_0, u_1 \in B(u, r)$.

PROPOSITION 1.10. — *Let $M \subset H$ be locally closed and p -convex. Then M is an absolute neighbourhood retract (see [14] for the definition of absolute neighbourhood retract).*

Finally, let us point out that the two definitions of tangent cone given in [1] and in [3] coincide in the case of p -convex sets. Indeed:

PROPOSITION 1.11. — *Let $M \subset H$ be locally closed and p -convex. Then $\forall u \in M$*

$$C_M(u) = T_M(u) = (\partial^- I_M(u))^- ,$$

where $C_M(u)$ and $T_M(u)$ are respectively the tangent cone and the contingent cone to M at u .

2. VARIATIONAL CHARACTERIZATION OF CLOSED GEODESICS

In this section, H will indicate a real Hilbert space, $M \subset H$ a locally closed p -convex set and we will deal with closed geodesics on M , namely:

DEFINITION 2.1. — *A curve $\gamma : [0, 1] \rightarrow M$ is said to be a closed geodesic on M if*

- (a) $\gamma \in W^{2,1}(0, 1; H)$;
- (b) $\gamma''(s) \in \partial^- I_M(\gamma(s))$ a. e. in $]0, 1[$;
- (c) $\gamma(0) = \gamma(1)$ and $\gamma'_+(0) = \gamma'_-(1)$.

We want to characterize them as critical points for the energy functional

$$f : L^2(0, 1; H) \rightarrow \mathbb{R} \cup \{+\infty\}$$

defined in such a way:

$$f(\gamma) = \begin{cases} \frac{1}{2} \int_0^1 |\gamma'|^2 ds, & \gamma \in X \\ +\infty, & \gamma \in L^2(0,1; H) \setminus X \end{cases}$$

where

$$X = \{ \gamma \in W^{1,2}(0,1; H) : \gamma(s) \in M, \forall s, \gamma(0) = \gamma(1) \}$$

is the so called space of the admissible paths.

For this purpose, let us state:

THEOREM 2.2. — *Let us take $\gamma \in X$. Then $\partial^- f(\gamma) \neq \emptyset$ if and only if*

$$\gamma \in W^{2,2}(0,1; H) \quad \text{and} \quad \gamma'_+(0) = \gamma'_-(1);$$

in such a case

$$\| \text{grad}^- f(\gamma) \|_{L^2} \leq \| \gamma'' \|_{L^2} \leq \theta(\bar{p}, f(\gamma)) (1 + \| \text{grad}^- f(\gamma) \|_{L^2})$$

where $\bar{p} = \max_{[0,1]} (p \circ \gamma)$ and $\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ is a continuous function.

Moreover, if $0 \in \partial^- f(\gamma)$ then $\gamma \in W^{2,\infty}(0,1; H)$.

Before the proof, we give some lemmas which are essentially contained in [2].

If $\gamma \in X$ and $\delta \in L^2(0,1; H)$, we set:

$$(P_\gamma \delta)(s) = P_{\gamma(s)} \delta(s)$$

where $P_{\gamma(s)}$ is the projection on the tangent cone to M at $\gamma(s)$.

By Proposition 1.8, $P_\gamma \delta \in L^2(0,1; H)$.

LEMMA 2.3 (see [2], Lemma 3.3). — *Let us take $\delta \in W^{1,2}(0,1; H)$ and $\gamma \in W^{1,2}(0,1; H)$ such that $\gamma(s) \in M, \forall s \in [0,1]$. Then*

$$\liminf_{t \rightarrow 0^+} \frac{\frac{1}{2} \int_0^1 |(\gamma + t\delta)'|^2 ds - \frac{1}{2} \int_0^1 |\pi(\gamma + t\delta)'|^2 ds}{t} \geq -2 \int_0^1 p(\gamma) |\delta - P_\gamma \delta| \cdot |\gamma'|^2 ds.$$

LEMMA 2.4. — *Let us take $\gamma \in X$ and $\alpha \in \partial^- f(\gamma)$. Then*

$$\int_0^1 (\gamma', \delta') ds \geq \int_0^1 (\alpha, P_\gamma \delta) ds - 2 \int_0^1 p(\gamma) |\delta - P_\gamma \delta| \cdot |\gamma'|^2 ds$$

$\forall \delta \in W^{1,2}(0,1; H)$ with $\delta(0) = \delta(1)$.

Proof. — Let us take $\delta \in W^{1,2}(0,1; H)$ with $\delta(0) = \delta(1)$.

We observe that, if $t > 0$ is sufficiently small, we can define $\pi(\gamma + t\delta)$ and:

$$\pi(\gamma + t\delta)(s) \in M, \quad \pi[(\gamma + t\delta)(0)] = \pi[(\gamma + t\delta)(1)].$$

Then

$$\frac{1}{2} \int_0^1 |\pi(\gamma + t\delta)'|^2 ds = f(\pi(\gamma + t\delta)).$$

Now, let us consider $\alpha \in \partial^- f(\gamma)$. By Proposition 1.8, we have:

$$\begin{aligned} \int_0^1 (\gamma', \delta') ds - \int_0^1 (\alpha, P_\gamma \delta) ds \\ = \lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^1 \left\{ \frac{1}{2} |(\gamma + t\delta)'|^2 - \frac{1}{2} |\gamma'|^2 - \alpha(\pi(\gamma + t\delta) - \gamma) \right\} ds \\ \geq \liminf_{t \rightarrow 0^+} \frac{1}{t} \int_0^1 \left\{ \frac{1}{2} |\pi(\gamma + t\delta)'|^2 - \frac{1}{2} |\gamma'|^2 - \alpha(\pi(\gamma + t\delta) - \gamma) \right\} \\ + \liminf_{t \rightarrow 0^+} \frac{1}{2t} \int_0^1 \{ |(\gamma + t\delta)'|^2 - |\pi(\gamma + t\delta)'|^2 \} ds. \end{aligned}$$

Recalling that $\left(\frac{\pi(\gamma + t\delta) - \gamma}{t} \right)$ is bounded in $L^2(0,1; H)$, the thesis is a consequence of Definition 1.1 and Lemma 2.3. ■

LEMMA 2.5 (see [2], Lemma 3.5). — Let $\alpha \in L^2(0,1; H)$ and $\gamma \in W^{1,2}(0,1; H)$ be such that $\gamma(s) \in M$, $\forall s \in [0,1]$.

Let us suppose that:

$$\begin{aligned} \int_0^1 (\gamma', \delta') ds \geq \int_0^1 (\alpha, P_\gamma \delta) ds - 2 \int_0^1 p(\gamma) |\delta - P_\gamma \delta| \cdot |\gamma'|^2 ds \\ \forall \delta \in W_0^{1,2}(0,1; H). \end{aligned}$$

Then

$$\gamma \in W^{2,2}(0,1; H), \quad \gamma''(s) + \alpha(s) \in \partial^- I_M(\gamma(s)) \quad \text{a. e.},$$

and

$$\|\gamma''\|_{L^2} \leq \left[1 + 2\bar{p} \left(\int_0^1 |\gamma'|^2 ds \right)^{1/2} \right] \left(2\bar{p} \int_0^1 |\gamma'|^2 ds + \|\alpha\|_{L^2} \right)$$

where $\bar{p} = \max_{[0,1]} p \circ \gamma$.

LEMMA 2.6. — Let us take $\gamma \in X \cap W^{2,1}(0,1; H)$ with $\gamma'_+(0) = \gamma'_-(1)$ and $\alpha \in L^1(0,1; H)$ such that $\alpha + \gamma'' \in \partial^- I_M(\gamma)$ a. e. Then $\forall \eta \in X$,

$$f(\eta) \geq f(\gamma) + \int_0^1 (\alpha, \eta - s) ds - \theta_1(\bar{p})(1 + \|\gamma''\|_{L^1}^2 + \|\alpha\|_{L^1}^2) \|\eta - \gamma\|_{L^2}^2$$

where $\bar{p} = \max_{[0,1]} p \circ \gamma$ and $\theta_1 : \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous function.

Proof. — If $\eta \in X$, then:

$$\begin{aligned} f(\eta) - f(\gamma) - \int_0^1 (\alpha, \eta - \gamma) ds \\ = \frac{1}{2} \int_0^1 |\eta' - \gamma'|^2 ds + \int_0^1 (\gamma', \eta' - \gamma') ds - \int_0^1 (\alpha, \eta - \gamma) ds \\ = \frac{1}{2} \int_0^1 |\eta' - \gamma'|^2 ds - \int_0^1 (\alpha + \gamma'', \eta - \gamma) ds. \end{aligned}$$

By p -convexity of M , we have:

$$\begin{aligned} \frac{1}{2} \int_0^1 |\eta' - \gamma'|^2 ds - \int_0^1 (\alpha + \gamma'', \eta - \gamma) ds \\ \geq \frac{1}{2} \int_0^1 |\eta' - \gamma'|^2 ds - \int_0^1 p(\gamma) |\alpha + \gamma''| \cdot |\eta - \gamma|^2 ds \\ \geq \frac{1}{2} \int_0^1 |\eta' - \gamma'|^2 ds - \bar{p} \|\alpha + \gamma''\|_{L^1} \|\eta - \gamma\|_{L^\infty}^2. \quad (2.6.1) \end{aligned}$$

Using in (2.6.1) the following estimate:

$$\|\eta - \gamma\|_{L^\infty}^2 \leq \|\eta - \gamma\|_{L^2}^2 + 2 \|\eta - \gamma\|_{L^2} \|\eta' - \gamma'\|_{L^2}$$

and then applying Young's inequality to the factor

$$2 \|\eta - \gamma\|_{L^2} \|\eta' - \gamma'\|_{L^2},$$

we obtain:

$$\begin{aligned} \frac{1}{2} \int_0^1 |\eta'|^2 ds - \frac{1}{2} \int_0^1 |\gamma'|^2 ds - \int_0^1 (\alpha, \eta - \gamma) ds \\ \geq \frac{1}{2} \int_0^1 |\eta' - \gamma'|^2 ds - \bar{p} \|\alpha + \gamma''\|_{L^1} (\|\eta - \gamma\|_{L^2}^2 + 2 \|\eta - \gamma\|_{L^2} \|\eta' - \gamma'\|_{L^2}) \\ \geq \frac{1}{2} \int_0^1 |\eta' - \gamma'|^2 ds - 2\bar{p}^2 \|\alpha + \gamma''\|_{L^1}^2 \|\eta - \gamma\|_{L^2}^2 \\ - \bar{p} \|\alpha + \gamma''\|_{L^1} \|\eta - \gamma\|_{L^2}^2 - \frac{1}{2} \int_0^1 |\eta' - \gamma'|^2 ds \end{aligned}$$

which gives the thesis. ■

Now we come back to the

Proof of theorem 2.2. — If $\partial^- f(\gamma) \neq \emptyset$, as a consequence of Definition 1.1 and Lemmas 2.4, 2.5, we get:

$$\gamma \in W^{2,2}(0,1; H)$$

and

$$\|\gamma''\|_{L^2} \leq (1 + 2\bar{p} \sqrt{2f(\gamma)})(4\bar{p} f(\gamma) + \|\alpha\|_{L^2}).$$

If $0 \in \partial^- f(\gamma)$, from Lemma 2.4, we obtain $\forall \delta \in W_0^{1,2}(0,1; H)$:

$$\int_0^1 (\gamma', \delta') ds \geq -2 \int_0^1 p(\gamma) |\delta - P_\gamma \delta| \cdot |\gamma'|^2 ds \quad (2.2.1)$$

Since

$$\begin{aligned} \gamma' &\in L^\infty(0,1; H), \\ \left| \int_0^1 (\gamma', \delta') ds \right| &\leq 2\bar{p} \|\gamma'\|_{L^\infty}^2 \|\delta\|_{L^1}, \quad \forall \delta \in W_0^{1,2}(0,1; H) \end{aligned}$$

and by duality:

$$\gamma'' \in L^\infty(0,1; H).$$

Now, let us prove that $\gamma'_-(1) = \gamma'_+(0)$.

Let us consider $v \in H$ and $\forall n \in \mathbb{N}$, $\rho_n \in W^{1,2}(0,1)$ such that

$$\begin{aligned} 0 &\leq \rho_n \leq 1, & \rho_n(0) &= \rho_n(1) = 1, \\ \rho_n &= 0 & \text{in } &\left[\frac{1}{2n}, 1 - \frac{1}{2n} \right]. \end{aligned}$$

Then, let us define the following functions:

$$\delta_n = \rho_n v, \quad \forall n \in \mathbb{N}.$$

Again, from Lemma 2.4, we have:

$$\int_0^1 (\gamma', \delta'_n) ds \geq \int_0^1 (\alpha, P_\gamma \delta_n) ds - 2 \int_0^1 p(\gamma) |\delta_n - P_\gamma \delta_n| \cdot |\gamma'|^2 ds \quad (2.2.2)$$

Integrating by parts and passing to the limit as $n \rightarrow \infty$, we obtain:

$$(\gamma'_-(1) - \gamma'_+(0), v) \geq 0, \quad \forall v \in H$$

and then

$$\gamma'_-(1) = \gamma'_+(0).$$

Now suppose that $\gamma \in W^{2,2}(0,1; H)$ and $\gamma'_+(0) = \gamma'_-(1)$. By applying Lemma 2.6 with $\alpha = -\gamma''$, we get $-\gamma'' \in \partial^- f(\gamma)$, so that

$$\|\text{grad}^- f(\gamma)\|_{L^2} \leq \|\gamma''\|_{L^2}. \quad \blacksquare$$

THEOREM 2.7. — *Let us consider $\gamma \in X \cap W^{2,2}(0,1; H)$ with $\gamma'_+(0) = \gamma'_-(1)$ and $\alpha \in L^2(0,1; H)$.*

Then $\alpha \in \partial^- f(\gamma)$ if and only if $\alpha(s) + \gamma''(s) \in \partial^- I_M(\gamma(s))$ a. e.

Moreover $\text{grad}^- f(\gamma) = -P_\gamma(\gamma'')$.

Proof. — If $\alpha \in \partial^- f(\gamma)$, by Lemmas 2.4 and 2.5 we get

$$\alpha(s) + \gamma''(s) \in \partial^- I_M(\gamma(s)) \quad \text{a. e.}$$

Viceversa, if $\alpha(s) + \gamma''(s) \in \partial^- I_M(\gamma(s))$ a. e., we apply Lemma 2.6 obtaining $\alpha \in \partial^- f(\gamma)$.

Now, since $-P_\gamma \gamma'' \in L^2$ and $-P_\gamma \gamma'' \in \partial^- f(\gamma)$, if $\alpha \in \partial^- f(\gamma)$ then

$$\int_0^1 (\alpha + \gamma'', P_\gamma \gamma'') ds \leq 0.$$

This means:

$$\int_0^1 (P_\gamma \gamma'', \gamma'') ds \leq - \int_0^1 (\alpha, P_\gamma \gamma'') ds.$$

So that,

$$\|P_\gamma \gamma''\|_{L^2}^2 \leq \|\alpha\|_{L^2} \|P_\gamma \gamma''\|_{L^2}. \quad \blacksquare$$

Now, we are ready to state the desired characterization:

THEOREM 2.8. — *Let us consider $\gamma \in X$. Then: $0 \in \partial^- f(\gamma)$ if and only if γ is a closed geodesic on M ; in this case $\gamma \in W^{2,\infty}(0,1; H)$ and the function $s \rightarrow |\gamma'(s)|$ is constant.*

Proof. — If γ is a closed geodesic on M , we can apply Lemma 2.6 with $\alpha=0$ obtaining $0 \in \partial^- f(\gamma)$.

Vice versa, if $0 \in \partial^- f(\gamma)$, from Theorem 2.2 we get:

$$\gamma \in W^{2,\infty}(0,1; H) \quad \text{and} \quad \gamma'_+(0) = \gamma'_-(1).$$

Moreover, by Theorem 2.7 we get

$$\gamma''(s) \in \partial^- I_M(\gamma(s)) \quad \text{a. e.}$$

so that, γ is a closed geodesic on M .

Finally, since $|\gamma'|^2$ is Lipschitz continuous, in order to prove that the function $s \rightarrow |\gamma'(s)|$ is constant, we will show that

$$(|\gamma'|^2)' = 0 \quad \text{a. e.}$$

Let us consider

$$\alpha \in \partial^- I_M(\gamma(s)).$$

From Definition 1.1, we have:

$$(\alpha, \gamma(t) - \gamma(s)) \leq |\gamma(t) - \gamma(s)| \varepsilon(\gamma(t) - \gamma(s)) \quad (2.8.1)$$

where

$$\lim_{\substack{v \rightarrow 0 \\ v \in L^2}} \varepsilon(v) = 0.$$

Dividing by $(t-s)$ and passing to the limit as $t \rightarrow s^+$ and $t \rightarrow s^-$ in (2.8.1), we get:

$$(\alpha, \gamma'(s)) = 0, \quad \forall \alpha \in \partial^- I_M(\gamma(s)), \quad \forall s \in]0, 1[$$

which gives the thesis recalling that

$$(|\gamma'(s)|^2)' = 2(\gamma'(s), \gamma''(s)) \quad \text{and} \quad \gamma''(s) \in \partial^- I_M(\gamma(s)) \quad \text{a. e.} \quad \blacksquare$$

At this point, the proof of the existence of closed geodesics on M is reduced to the research of critical points for f .

The method we want to use for this aim is based on the evolution theory, as developed in [5], [6], [7], [9] and [10]. Therefore we need to prove that f has a φ -monotone subdifferential of order two:

THEOREM 2.9. — *Let M be closed in H . Then f is l. s. c. and there exists a continuous function*

$$\varphi_0: L^2 \times \mathbb{R} \rightarrow \mathbb{R}^+$$

such that:

$$f(\eta) \geq f(\gamma) + \int_0^1 (\alpha, \eta - \gamma) ds - \varphi_0(\gamma, f(\gamma)) (1 + \|\alpha\|_{L^2}^2) \|\eta - \gamma\|_{L^2}^2$$

whenever $\eta, \gamma \in X$ and $\alpha \in \partial^- f(\gamma)$.

In particular, f has a φ -monotone subdifferential of order two.

Proof. — First we will prove that f is l. s. c.

Let us take $\{\gamma_n\}_n \in X$ such that:

$$\lim_n \gamma_n = \gamma \quad \text{in } L^2(0,1; H) \quad \text{and} \quad f(\gamma_n) \leq c.$$

By definition of f , $\{\gamma_n\}_n$ converges weakly to γ in $W^{1,2}(0,1; H)$ and

$$\frac{1}{2} \int_0^1 |\gamma'|^2 \leq c.$$

So, we have only to prove that $\gamma \in X$.

But, since $\{\gamma_n\}_n$ converges uniformly to γ in $[0,1]$ and M is closed, we deduce that

$$\gamma(s) \in M, \quad \forall s \in [0,1]$$

and from $\gamma_n(1) = \gamma_n(0)$, $\forall n \in \mathbb{N}$, we have: $\gamma(0) = \gamma(1)$.

So, $\gamma \in X$.

Now, using Theorem 2.2, Theorem 2.7 and Lemma 2.6, we obtain the existence of a continuous function $\theta_2: \mathbb{R}^2 \rightarrow \mathbb{R}^+$ such that

$$f(\eta) \geq f(\gamma) + \int_0^1 (\alpha, \eta - \gamma) ds - \theta_2(\bar{p}, f(\gamma)) (1 + \|\alpha\|_{L^2}^2) \|\eta - \gamma\|_{L^2}^2$$

whenever $\eta, \gamma \in X$, $\alpha \in \partial^- f(\gamma)$ and were $\bar{p} = \max_{[0,1]} p \circ \gamma$.

By paracompactness and partition of unity, we obtain the existence of φ_0 . ■

3. HOMOTOPICAL PROPERTIES OF THE SPACE OF THE ADMISSIBLE PATHS

In this section, we want to deduce some “homotopical” properties of the space of the admissible paths X endowed with the $W^{1,2}$ -topology. To this aim, let us recall the following result contained in [16] (see Theorem 8.14, page 189).

THEOREM 3.1. — *Let $p: X \rightarrow B$ be a fibration. Let $x_0 \in X$, $b_0 = p(x_0)$, $F = p^{-1}(b_0)$. If p has a cross section, then*

$$\pi_q(X, x_0) \approx \pi_q(F, x_0) \oplus \pi_q(B, b_0), \quad \forall q \geq 2$$

while $\pi(X, x_0)$ is a semi-direct product of $\pi_1(F, x_0)$ by $\pi_1(B, b_0)$.

From now on, if M is a metric space and $u_0 \in M$, we will denote by $\Omega(M, u_0)$ its loop space with base point u_0 and we will set:

$$X^* = \{ \gamma \in C([0, 1]; M) \text{ such that } \gamma(0) = \gamma(1) \}$$

endowed with the topology of the uniform convergence.

Remark 3.2. — *The map $p: X^* \rightarrow M$ defined by $p(\gamma) = \gamma(0)$ is a fibration and*

$$\text{if } u_0 \in M, \text{ then } p^{-1}(u_0) = \Omega(M, u_0).$$

Moreover, the map $\lambda: M \rightarrow X^*$ defined by

$$\lambda(u_0)(s) = u_0, \quad \forall s \in [0, 1]$$

is a cross section.

As a consequence of Theorem 3.1, let us prove:

THEOREM 3.3 (see, also, Lemmas 2.11 and 2.12 in [11]) *Let $M \subset \mathbb{R}^n$ be compact, p -convex, connected and non-contractible in itself. Then, there exists $k \in \mathbb{N}$ such that:*

(i) *There exists a continuous map $g: S^k \rightarrow X^*$ which is not homotopic to a constant.*

(ii) *Every continuous map $\tilde{g}: S^k \rightarrow M$ is homotopic to a constant.*

Proof. — First of all, let us observe that, by Proposition 1.10, M is also arcwise connected. If M is not simply connected, then X^* is not arcwise connected, so that there exists a continuous map $g: S^0 \rightarrow X^*$ which is not homotopic to a constant. On the other hand, M is arcwise connected, then every continuous map $\tilde{g}: S^0 \rightarrow M$ is homotopic to a constant.

If M is simply connected, then X^* and $\Omega(M)$ are arcwise connected. Since by Proposition 1.10, M is an A.N.R., $\pi_h(M)$ is not trivial for some h (cf. [14]). Let $k+1$ be the first integer such that $\pi_{k+1}(M)$ is not trivial ($k \geq 1$). Applying Theorem 3.1, we have:

$$\pi_k(X^*) \approx \pi_k(\Omega(M)) \approx \pi_{k+1}(M).$$

Then $\pi_k(X^*)$ is not trivial, on the contrary $\pi_k(M)$ is trivial, so that the theorem is proved.

THEOREM 3.4. — *Let $M \subset \mathbb{R}^n$ be compact and p -convex. If there exists $k \geq 0$ and a continuous map $g: S^k \rightarrow X^*$ which is not homotopic to a constant, then there exists a continuous map $\tilde{g}: S^k \rightarrow X$ which is not homotopic to a constant.*

For the proof of this theorem, we need the following result contained in [8] (see Theorem 3.17).

THEOREM 3.5. — *Let W be an open subset of a real Hilbert space V and $g: W \rightarrow \mathbb{R} \cup \{+\infty\}$ be a l. s. c. function with a ϕ -monotone subdifferential of order 2. Then there exists a map $j: D(g) \rightarrow D(g)$ such that:*

- (i) $j(g^b) \subset g^b, \forall b \in \mathbb{R}$, where $g^b = \{u \in \Omega : g(u) \leq b\}$;
- (ii) $j: (g^b, |\cdot|_V) \rightarrow (g^b, d^*)$ where

$$d^*(u, v) = |u - v| + |g(u) - g(v)|, \quad \forall u, v \in D(g)$$

is continuous and it is a homotopy inverse of the identity function: $\text{Id}: (g^b, d^*) \rightarrow (g^b, |\cdot|_V)$.

Proof of theorem 3.4. — Let k be a natural number and $g: S^k \rightarrow X^*$ a continuous map which is not homotopic to a constant.

Let us set

$$X_A^* = \{\gamma \in C([0, 1]; A); \gamma(0) = \gamma(1)\}$$

endowed with the topology of the uniform convergence, where A is the set defined in Remark 1.7.

By Proposition 1.6, X^* is a deformation retract of X_A^* . Then the map $g: S^k \rightarrow X_A^*$ is not homotopic to a constant.

Moreover, since X_A^* is an open subset of the Banach space:

$$X_{\mathbb{R}^n}^* = \{\gamma \in C([0, 1]; \mathbb{R}^n); \gamma(0) = \gamma(1)\},$$

by [14], we deduce that X_A^* is homotopically equivalent to

$$X_A = \{\gamma \in W^{1,2}(0, 1; \mathbb{R}^n); \gamma(0) = \gamma(1); \gamma(s) \in A\}$$

endowed with $W^{1,2}$ -topology.

Therefore, there exists a continuous map $f_1: S^k \rightarrow X_A$ which is not homotopic to a constant.

Now, let a be a real number such that

$$\frac{1}{2} \int_0^1 |\gamma'|^2 ds \leq a, \quad \forall \gamma \in f_1(S^k).$$

Then, setting

$$X_A^b = \left\{ \gamma \in X_A \text{ such that } \frac{1}{2} \int_0^1 |\gamma'|^2 ds \leq b \right\},$$

we have that $f_1: S^k \rightarrow X_A^b$ is not homotopic to a constant $\forall b \geq a$.

At this point, let us remark the following: $\forall \gamma \in X_A^b$ there exists $r(\gamma) > 0$ such that if

$$\eta \in W^{1,2}(0,1; \mathbb{R}^n),$$

$$\frac{1}{2} \int_0^1 |\eta'|^2 ds \leq b \quad \text{and} \quad \|\eta - \gamma\|_{L^2} < r(\gamma)$$

then $\eta(s) \in A, \forall s \in [0, 1]$.

Now, let us set

$$V = L^2(0,1; \mathbb{R}^n); \quad W = \bigcup_{\gamma \in X_A^b} B(\gamma, r(\gamma))$$

where $B(\gamma, r(\gamma))$ is the open ball in L^2 of center γ and radius $r(\gamma)$ and let us define a function $g: W \rightarrow \mathbb{R} \cup \{+\infty\}$ in such a way:

$$g(\gamma) = \begin{cases} \frac{1}{2} \int_0^1 |\gamma'|^2 ds & \text{if } \gamma \in X_A^b \\ +\infty & \text{if } \gamma \in W \setminus X_A^b \end{cases}$$

Obviously, g is the restriction to W of a convex and l. s. c. function on $L^2(0,1; \mathbb{R}^n)$.

Since $X_A^b = g^b$, by Theorem 3.5 we deduce that

$$i: X_A^b \rightarrow \tilde{X}_A^b,$$

where $\tilde{X}_A^b$ is defined as the space X_A^b endowed with the L^2 -topology, is a homotopy equivalence $\forall b \geq a$.

Therefore, $f_1: S^k \rightarrow \tilde{X}_A^b$ is not homotopic to a constant $\forall b \geq a$.

Now, let us consider the following homotopy H defined on $f_1(S^k) \times [0, 1]$, in such a way:

$$H(\gamma, t)(s) = t \pi(\gamma(s)) + (1-t) \gamma(s).$$

By Remark 1.7, we have:

$$|H(\gamma, t)'(s)| \leq 2t |\gamma'(s)| + (1-t) |\gamma'(s)| \leq 2 |\gamma'(s)|.$$

So that $H: f_1(S^k) \times [0, 1] \rightarrow \tilde{X}_A^b$ where $b \geq 4a$.

Let us take $f_2 = H(\cdot, 1) \circ f_1$.

The map $f_2: S^k \rightarrow \tilde{X}_A^b$ is not homotopic to a constant, moreover $f_2(S^k) \subset \tilde{X}^b$ where

$$\tilde{X}^b = \left\{ \gamma \in X: \frac{1}{2} \int_0^1 |\gamma'|^2 ds \leq b \right\}$$

endowed with the L^2 -topology.

Then, $f_2: S^k \rightarrow \tilde{X}^b$ is not homotopic to a constant $\forall b \geq 4a$. Now, applying Theorem 3.5 to

$$V = W = L^2(0,1; \mathbb{R}^n) \quad \text{and} \quad g \equiv f$$

where f is the energy functional defined in section 2, we deduce the existence of a map $j: \tilde{X} \rightarrow X$ where $\tilde{X}$ denotes the space X endowed with the L^2 -topology such that $\forall b, j(\tilde{X}^b) \subset X^b$. Moreover j is continuous and it is a homotopy inverse of the identity function.

Finally, let us consider the continuous map $f_3: S^k \rightarrow X^b$ defined by $f_3 = j \circ f_2$. It is not homotopic to a constant $\forall b \geq 4a$ and then $f_3: S^k \rightarrow X$ is not homotopic to a constant. ■

THEOREM 3.6. — *Let $M \subset \mathbb{R}^n$ be compact and p -convex and f the functional defined in section 2. Then there exists $a > 0$ such that*

$$f^a = \{ \gamma : \gamma \in X \quad \text{and} \quad f(\gamma) \leq a \}$$

endowed with the $W^{1,2}$ -topology is homotopically equivalent to M .

For the proof of this theorem we will need the following lemma:

LEMMA 3.7. — *Let f^0 be the set of the constant curves. Then there exists $a > 0$ such that f^0 is a strong deformation retract of f^a endowed with the L^2 -topology.*

Proof. — Since M is compact, we can suppose that M is p -convex with $p \equiv \text{Const}$. Let us take $\gamma \in f^a$ and let us consider

$$t\gamma(0) + (1-t)\gamma(s) \quad \text{with} \quad t \in [0, 1].$$

We remark that:

$$\begin{aligned} d(t\gamma(0) + (1-t)\gamma(s), M) &\leq |t\gamma(0) + (1-t)\gamma(s) - \gamma(0)| \\ &= (1-t)|\gamma(s) - \gamma(0)| \leq \left(\int_0^1 |\gamma'|^2 ds \right)^{1/2} \leq \sqrt{2a}. \end{aligned} \quad (3.7.1)$$

Therefore, taking a such that $4p\sqrt{2a} < 1$, by (3.7.1), we have that

$$t\gamma(0) + (1-t)\gamma(s) \in A$$

where A is defined in Remark 1.7.

Now we can consider the map H defined on $f^a \times [0, 1]$ in this way:

$$H(\gamma, t)(s) = \pi(t\gamma(0) + (1-t)\gamma(s)).$$

Let us observe that by Proposition 1.9:

$$\begin{aligned} d(t\gamma(0) + (1-t)\gamma(s), M) &= |t\gamma(0) + (1-t)\gamma(s) - \pi(t\gamma(0) + (1-t)\gamma(s))| \\ &\leq 2pt(1-t)|\gamma(0) - \gamma(s)|^2 \leq 4pat(1-t). \end{aligned} \quad (3.7.2)$$

By (3.7.2) and (ii) of Proposition 1.6, we have:

$$\left| \frac{d}{ds} H(\gamma, t)(s) \right| \leq (1 - 8p^2 at(1-t))^{-1} (1-t) |\gamma'(s)| \leq |\gamma'(s)|$$

so that we deduce:

$$\int_0^1 \left| \frac{d}{ds} H(\gamma, t)(s) \right|^2 ds \leq 2a.$$

Therefore,

$$H(\gamma, t)(s): f^a \times [0, 1] \rightarrow f^a.$$

Moreover,

$$H(\gamma, 0)(s) = \gamma(s) \quad \text{and} \quad H(\gamma, 1)(s) = \gamma(0), \quad \forall s \in [0, 1]$$

To conclude the proof it is enough to point out that if we endow f^a with the L^2 -topology, H is a continuous map. ■

Proof of Theorem 3.6. — By applying Theorem 3.5 to

$$W = L^2(0, 1; \mathbb{R}^n) \quad \text{and} \quad g \equiv f$$

where f is the functional defined in section 2, we obtain that f^a endowed with the $W^{1,2}$ -topology is homotopically equivalent to f^a with the L^2 -topology.

On the other hand, M is homeomorphic to f^0 with the L^2 -topology. Using lemma 3.7 we get the thesis. ■

THEOREM 3.8. — *There exists $a > 0$ such that f^a and X (both endowed with the $W^{1,2}$ -topology) are not homotopically equivalent.*

Proof. — Obvious from Theorems 3.3, 3.4 and 3.6. ■

4. THE MAIN RESULT

After Theorem 2.8, the problem to establish the existence of a non-constant closed geodesic on M , compact, connected and p -convex subset of $\mathbb{R}^n$, is reduced to find critical points for the energy functional f on the space of the admissible paths X (see section 2 for the Definition of f and X).

To this aim, we need a deformation lemma like the one contained in [13]. We shall use a version included in [8] (see Lemma 4.4).

LEMMA 4.1. — *Let V be a real Hilbert space and $g: V \rightarrow \mathbb{R} \cup \{+\infty\}$ a l. s. c. function with a φ -monotone subdifferential of order 2. We set*

$$d^*(u, v) = |u - v| + |g(u) - g(v)|, \quad \forall u, v \in D(g).$$

Let $-\infty < a < b \leq +\infty$ be such that:

- (i) $0 \notin \partial^- g(u)$ whenever $u \in D(g)$ and $a \leq g(u) \leq b$;
- (ii) $\forall c \in [a, b[$ and $\forall \{u_n\}_n \subset D(\partial^- g)$ with $\lim_n g(u_n) = c$ and

$\lim_n \text{grad}^- g(u_n) = 0$, $\{u_n\}_n$ has a converging subsequence in V .

Then g^a is a strong deformation retract of g^b in g^b , where g^a and g^b are endowed with the metric d^* .

Combining this lemma with the topological results in section 3, we can state the desired result:

THEOREM 4.2. — *Let $M \subset \mathbb{R}^n$ be compact, p -convex, connected and non-contractible in itself.*

Then, there exists at least a non-constant closed geodesic on M .

Proof. — Let us consider the energy functional f defined in section 2. By Theorem 2.9, f is l. s. c. and it has a ϕ -monotone subdifferential of order 2.

Moreover, by Theorem 2.8, the thesis is equivalent to state that there exists $\gamma \in X$ such that $0 \in \partial^- f(\gamma)$, and $f(\gamma) > 0$. So, if, by contradiction, the thesis is not true, we can apply Lemma 4.1 with

$$V = L^2(0, 1; \mathbb{R}^n), \quad g \equiv f, \quad b = +\infty$$

and a given by Theorem 3.8.

We recall that condition (ii) is satisfied because M is compact and the metric d^* induces the $W^{1,2}$ -topology on $X = f^b$.

Then, by Lemma 4.1 we deduce that X and f^a are homotopically equivalent, which is impossible by Theorem 3.8. ■

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