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Nonlinear Evolution Equations and Analyticity. I

by

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ABSTRACT. — We prove a theorem on abstract nonlinear evolution equations $\partial_t u = F(t, u)$ in a Banach space, which aims at estimating certain families of Liapunov functions for the solutions. The theorem is useful in proving global analyticity (in space variables) of solutions of various partial differential equations, such as the equations of Korteweg de-Vries, Benjamin-Ono, Euler, Navier-Stokes, nonlinear Schrödinger, etc. In this paper it is shown that if the initial state of the KdV or the BO equation has an analytic continuation that is analytic and L^2 in a strip containing the real axis, then the solution (which is unique in $H^2(\mathbb{R})$) has the same property for all time, though the width of the strip might decrease with time.

Key-words: Evolution equation, Liapunov function (family), Equations of (generalized) Korteweg de-Vries, Benjamin-Ono, Euler, Navier-Stokes, Sobolev space, Sobolev norm, Taylor series, Radius of convergence.

RÉSUMÉ. — On démontre un théorème sur des équations d'évolution non-linéaires abstraites dans un espace de Banach, visant à estimer certaines fonctions de Liapounov pour les solutions. Ce théorème sert à démontrer l'analyticité globale (en espace) des solutions de diverses équations aux dérivées partielles. On montre en particulier que si la condition initiale de l'équation de Korteweg de-Vries ou de Benjamin-Ono a un prolongement analytique L^2 dans une bande autour de l'axe réel, alors la solution (unique dans H^2) a la même propriété à tout instant, bien que la largeur de la bande puisse décroître avec le temps.

1. LIAPUNOV FAMILIES

In what follows we consider nonlinear evolution equations of the form

$$\text{(E)} \qquad \partial_t u = \mathrm{F}(t, u), \qquad t \geqslant 0, \quad \partial_t = \partial/\partial t .$$

We shall prove a theorem which is useful in deducing the analyticity (in space variables) of solutions of some nonlinear partial differential equations. The main problem here is to *estimate* the solutions that are already known to exist and have regularity sufficient to justify various formal operations.

Actually it will be possible to construct a theory in which existence and regularity (including analyticity) can be established simultaneously, but it seems that such an attempt leads to undesirable complications.

In this section we prove the main theorem, which uses a class of Liapunov functions involving a parameter. In subsequent sections we apply the theorem to prove the analyticity of solutions of the (generalized) KdV equation and the BO (Benjamin-Ono) equation. Other typical nonlinear equations, such as the Euler equation, the Navier-Stokes equation, nonlinear Schrödinger equation, etc. will be dealt with in subsequent publications. It should be noted that our results do not prove analyticity in t; this is in general impossible. On the other hand, we do not assume that $\mathrm{F}(t, u)$ is analytic in t; only continuity in t suffices. In this respect our results resemble a theorem of Nagumo [7], but differ from the latter in being nonlocal.

We start with preliminary definitions and lemmas. All Banach spaces X, Z, ... considered below are assumed to be real. X* denotes the dual space of X.

Let Z be a Banach space. We say a real-valued function Φ on an open subset $\mathrm{O} \subset \mathrm{Z}$ is weak* C^1 if the Gâteaux derivative $\mathrm{D}\Phi$ exists with values in Z* and is *demi*continuous* from Z to Z* (i. e. continuous from strong Z to weak* Z*). We recall that the mean value theorem holds for Φ:

$$(1.1) \quad \Phi(v) - \Phi(u) = \int_0^1 \langle v - u, \mathrm{D}\Phi(u + \lambda(v - u)) \rangle \, d\lambda \qquad (u, v \in \mathrm{O}),$$

provided the segment $uv \subset \mathrm{O}$.

Next suppose that we have another Banach space $\mathrm{X} \supset \mathrm{Z}$ with the inclusion continuous and dense. Suppose further that $\mathrm{D}\Phi$ considered above is demi*continuous (not only from Z to Z* but also) from Z to X*. (This statement makes sense since $\mathrm{X}^* \subset \mathrm{Z}^*$ by canonical identification.) Then we say Φ is (Z, X)-weak* C^1.

Finally suppose we have a family $\{ \Phi_\sigma ; -\infty < \sigma < \bar{\sigma} \leqslant \infty \}$ of real-valued functions on O such that $\Phi_\cdot(.)$ is $(\tilde{\mathrm{Z}}, \tilde{\mathrm{X}})$-weak* C^1 on $\tilde{\mathrm{O}}$, where

$\tilde{Z} = \mathbb{R} \times Z$, $\tilde{X} = \mathbb{R} \times X$, and $\tilde{O} = (-\infty, \bar{\sigma}) \times O$. We denote by $D\Phi_\sigma$ the partial derivative in Z-variable only.

LEMMA 1.1 (chain rule). — Let $u(\cdot) \in C(I; O) \cap C^1(I; X)$ and $\sigma(\cdot) \in C^1(I; \mathbb{R})$ with $\sigma(t) < \bar{\sigma}$, where $I \subset \mathbb{R}$ is an interval. Then $\Phi_{\sigma(.)}(u(\cdot)) \in C^1(I; \mathbb{R})$ with

$$(1.2) \qquad \partial_t [\Phi_{\sigma(t)}(u(t))] = \langle \partial_t u(t), D\Phi_{\sigma(t)}(u(t)) \rangle + \sigma'(t) \partial_\sigma \Phi_{\sigma(t)}(u(t)).$$

Proof. — We may assume that $0 \in I$. We apply (1.1) to the function $\Phi_.(\cdot)$ on $(-\infty, \bar{\sigma}) \times O$ into $\mathbb{R}$, with u replaced by $(\sigma(0), u(0))$ and v by $(\sigma(t), u(t))$, obtaining

$$(1.3) \quad \Phi_{\sigma(t)}(u(t)) - \Phi_{\sigma(0)}(u(0)) = \int_0^1 [\langle u(t) - u(0), D\Phi_{\tilde{\sigma}}(\tilde{u}) \rangle + (\sigma(t) - \sigma(0)) \partial_\sigma \Phi_{\tilde{\sigma}}(\tilde{u})] d\lambda,$$

where $\tilde{u} = u(0) + \lambda(u(t) - u(0))$ and $\tilde{\sigma} = \sigma(0) + \lambda(\sigma(t) - \sigma(0))$; note that $\tilde{u} \in O$ and $\tilde{\sigma} < \bar{\sigma}$ if $|t|$ is sufficiently small. If we divide (1.3) by t and let $t \to 0$, then $t^{-1}(u(t) - u(0)) \to \partial_t u(0)$ in X and $t^{-1}(\sigma(t) - \sigma(0)) \to \sigma'(0)$, while $D\Phi_{\tilde{\sigma}}(\tilde{u}) \rightharpoonup D\Phi_{\sigma(0)}(u(0))$ (weak* convergence) in X^* for each fixed λ, and similarly $\partial_\sigma \Phi_{\tilde{\sigma}}(\tilde{u}) \to \partial_\sigma \Phi_{\sigma(0)}(u(0))$. If $|t|$ is sufficiently small, however, $\tilde{u}$ is arbitrarily close to $u(0)$ in Z, uniformly in $\lambda \in [0, 1]$, and similarly for $\tilde{\sigma}$. Therefore $D\Phi_{\tilde{\sigma}}(\tilde{u})$ and $\partial_\sigma \Phi_{\tilde{\sigma}}(\tilde{u})$ are uniformly bounded in X^* by the demi*continuity. Application of the bounded convergence theorem thus shows that (1.2) is true for $t = 0$. The same is true for each $t \in I$. Since the right member of (1.2) is continuous in t, the lemma is proved.

REMARK 1.2. — As is seen from the proof, Lemma 1.1 remains true if we replace all the t-derivatives involved by the right (or left) derivatives. Thus $u(\cdot) \in C(I; O) \cap C^{1+}(I; X)$ and $\sigma(\cdot) \in C^{1+}(I; \mathbb{R})$ imply $\Phi_{\sigma(t)}(u(t)) \in C^{1+}(I; \mathbb{R})$, where C^{1+} indicates that the right derivative exists and is right-continuous.

REMARK 1.3. — There are some problems in which we are not able to construct the family Φ_σ with the required continuity properties but in which it is still possible to find a substitute function Φ_σ that satisfies the inequality obtained from (1.2) by replacing $=$ by $\leqslant$. Such a function will serve the same purpose.

We now define a Liapunov family for (E), assuming that F is continuous on $IT \times O$ to X, where $IT = [0, T]$, $T > 0$. A *Liapunov family* $\{ \Phi_\sigma; -\infty < \sigma < \bar{\sigma} \leqslant \infty \}$ for (E) on O is a family of real-valued functions on O satisfying the following conditions. $\Phi_.(\cdot)$ is $(\tilde{Z}, \tilde{X})$-weak* C^1 on $(-\infty, \bar{\sigma}) \times O$ into $\mathbb{R}$ and

$$(1.4) \qquad \langle F(t, v), D\Phi_\sigma(v) \rangle \leqslant \beta(\Phi_\sigma(v)) + \alpha(\Phi_\sigma(v)) \partial_\sigma \Phi_\sigma(v)$$

whenever $v \in O$ with $\Phi_\sigma(v) < \bar{r}$, where $\alpha(r)$, $\beta(r)$ are real-valued continuous functions defined for $-\infty < r < \bar{r} \leqslant \infty$. It is assumed (for simplicity) that $\alpha(r) \geqslant 0$. Note that the left member of (1.4) makes sense since $F(t, v) \in X$ and $D\Phi_\sigma(v) \in X^*$.

Given a Liapunov family $\{ \Phi_\sigma \}$ for (E) on O, we say $\phi \in O$ is $\{ \Phi_\sigma \}$-*admissible* if there is $b < \sigma$ such that $\Phi_b(\phi) < r$. If ϕ is $\{ \Phi_\sigma \}$-admissible, we can solve the ordinary differential equation

$$(1.5) \qquad \partial_t \rho = \beta(\rho), \qquad t \geqslant 0, \qquad \rho(0) = \Phi_b(\phi).$$

We denote by $\rho(t)$ the *maximal* solution of (1.5) (in the sense of values, not of the interval of existence), which exists on a certain interval IT_1, $T_1 > 0$. Then we define another function

$$(1.6) \qquad \sigma(t) = b - \int_0^t \alpha(\rho(\tau))d\tau.$$

σ also exists on IT_1 and satisfies $\sigma(t) < \bar{\sigma}$ (recall that $\alpha \geqslant 0$ and $b < \bar{\sigma}$).

With these definitions, we can state the main theorem.

THEOREM 1. — Let $O \subset Z \subset X$ be as above. Let F be continuous on $IT \times O$ into X. Let $\{ \Phi_\sigma; -\infty < \sigma < \bar{\sigma} \}$ be a Liapunov family for (E) on O, so that we have inequality (1.4) with certain continuous functions α, β defined on $(-\infty, \bar{r})$. Let u be a solution of (E) such that $u \in C(IT; O) \cap C^1(IT; X)$, where $u(0) = \phi$ is $\{ \Phi_\sigma \}$-admissible, so that we can construct the functions $\rho(\cdot)$, $\sigma(\cdot)$ as above. Under these conditions, we have

$$(1.7) \qquad \Phi_{\sigma(t)}(u(t)) \leqslant \rho(t) \qquad \text{for} \quad t \in IT \cap IT_1.$$

Proof. — The proof is a simple application of a comparison theorem for ordinary differential equations. Writing

$$\omega(t) = \Phi_{\sigma(t)}(u(t)), \qquad \kappa(t) = \partial_\sigma \Phi_{\sigma(t)}(u(t))$$

for simplicity, we obtain from (E), (1.2), (1.4), and (1.6) the inequality

$$\partial_t \omega(t) \leqslant \beta(\omega(t)) + \kappa(t)(\alpha(\omega(t)) - \alpha(\rho(t))).$$

Subtracting (1.5) from this inequality gives

$$\partial_t(\omega - \rho) \leqslant \beta(\omega) - \beta(\rho) + \kappa(t)(\alpha(\omega) - \alpha(\rho)),$$

where the variable t is suppressed in most places. It follows from the comparison theorem that $\omega(t) \leqslant \rho(t)$, since this is true for $t = 0$; note that $\omega(0) = \Phi_b(\phi) = \rho(0)$. (The last argument is rather formal and sketchy. To be more precise, one may replace ρ by the solution of a modified equation $\partial_t \rho = \beta(\rho) + \varepsilon$ with $\rho(0) = \Phi_b(\phi) + \varepsilon$, obtain the desired inequality, and then let $\varepsilon \searrow 0$. Moreover, since it is not known *a priori* that $\omega(t) < \bar{r}$, one has to work within the interval IT_2 where this is true and show eventually that T_2 can be made equal to min $\{ T, T_1 \}$. These procedures are more or less standard.)

REMARK 1.4. — *a*) Theorem 1 and its proof remain valid if $u \in C(IT; O) \cap C^{1+}(IT; X)$ satisfies $\partial_t^+ u = F(t, u)$; see Remark 1.2.

b) Suppose there are several equations (E_j), $j = 1, \ldots, N$, of the form (E), where the right members F_j have the properties assumed for F with I and $O \subset Z \subset X$ in common. If $\{ \Phi_\sigma \}$ is a Liapunov family on O for each of the (E_j), it is easy to see that $\{ \Phi_\sigma \}$ is also a Liapunov family for (E) with $F = F_1 + \ldots + F_N$. This applies, for example, to the KdV equation and the BO equation to be discussed in the following sections. For the KdV, we may take $F_1(t, u) = - \partial^3 u$, $F_2(t, u) = - a(u)\partial u$ and for the BO, $F_1 = - H\partial^2 u$ with the same F_2.

c) A family analogous to Φ_σ was used by Fritz and Dobrushin [*3*] in the study of certain dynamical systems. The authors thank Professors Dell'Antonio and Doplicher for this information.

2. ANALYTICITY OF SOLUTIONS OF THE KdV AND BO EQUATIONS

In this section we apply Theorem 1 to prove the analyticity (in the space variable) in a certain global sense for solutions of the (generalized) KdV equation

$$\text{(G)} \qquad \partial_t u = F(u) \equiv - \partial^3 u - a(u)\partial u, \qquad x \in \mathbb{R}, \quad t \geqslant 0,$$

where $\partial = \partial/\partial x$. It is assumed that $a(\lambda)$ is real-analytic in $\lambda \in \mathbb{R}$; no growth rate is assumed for $a(\lambda)$. A similar result can be proved for the BO equation, for which see Remark 2.1, *c*), below.

First we summarize known results for the solutions of (G) (see [*6*]). If $\phi \in H^s = H^s(\mathbb{R})$ with $s \geqslant 2$, (G) has a unique solution $u \in C(IT; H^s) \cap C^1(IT; H^{s-3})$ with $u(0) = \phi$ on an interval $IT \equiv [0, T]$, where $T > 0$ is determined by $| \phi |_1$ only. (We use $| \quad |_s$ to denote the H^s-norm.) T may be chosen arbitrarily large if either $| \phi |_1$ is sufficiently small or $a(\lambda)$ has a limited growth rate (say, $a(\lambda) \leqslant o(| \lambda |^4)$). It follows that $u \in C(IT; H^\infty)$ if $\phi \in H^\infty \equiv \cap H^s$. In what follows we shall restrict $\phi \in H^\infty$ to a certain class of analytic functions.

For each $r > 0$, denote by $S(r)$ the strip $\{ -\infty < \operatorname{Re} x < \infty, \ -r < \operatorname{Im} x < r \}$ in the complex x-plane. Let $A(r)$ be the set of all analytic functions f on $S(r)$ such that $f \in L^2(S(r'))$ for each $r' < r$, $r' > 0$ and that $f(x) \in \mathbb{R}$ for $x \in \mathbb{R}$. We shall identify $f \in A(r)$ with its trace on the real axis if there is no possibility of confusion.

$A(r)$ is a Fréchet space with these $L_2(S(r'))$-norms as the generating system of seminorms. The analyticity for (G) can now be stated by the following theorem.

THEOREM 2. — Let $u \in C(IT; H^\infty)$ be a solution of (G). If $u(0) = \phi \in A(r_0)$ for some $r_0 > 0$, there is $r_1 > 0$ such that $u \in C(IT; A(r_1))$.

REMARK 2.1. — *a*) In terms of the inductive limit $A = \cup \{ A(r); r > 0 \}$, Theorem 2 says that $u(0) \in A$ implies $u(t) \in A$, i. e. the property $u(t) \in A$ is *persistent* for (G).

b) For the proper KdV equation, which corresponds to $a(\lambda) = \lambda$ in (G), it is known [6] that $u \in C([0, \infty); H^\infty)$ if $\phi \in H^\infty$. Thus $u(t) \in A$ for all $t \geqslant 0$ if $\phi \in A$. This result is consistent with a statement in Deift-Trubowitz [2].

c) The BO equation is obtained from the KdV equation simply by replacing ∂^3 with $H\partial^2$, where H is the Hilbert transform. For the BO equation, the H^∞-persistence property stated above has been established recently by Iorio [4]. It follows that Theorem 2 is also true for the BO equation, with T arbitrarily large. Indeed, replacing ∂^3 with $H\partial^2$ has no effect at all in the estimates given in section 3.

d) For these equations one can prove the analyticity of the time derivatives $\partial_t^k u(t)$, but we shall not go into the proof. On the other hand, Theorem 2 may be generalized to include the case in which $a(u)$ in (G) is replaced with $a(t, u)$ involving t explicitly, provided a is continuous jointly in t, u and analytic in u. A further generalization to the case $a = a(t, x, u)$ is possible, if a depends on x analytically in an appropriate global sense.

For the proof of Theorem 2, it is convenient to use an equivalent set of norms in $A(r)$ involving only $f(x)$ for real x. Such norms are given by

$$\| f \|^2_{\sigma,s} = \sum_{j=0}^{\infty} (j!)^{-2} e^{2j\sigma} | \partial^j f |^2_s \qquad (\sigma < r). \tag{2.1}$$

Here the real parameter s is not essential, since it is easy to see that

$$\| f \|_{\sigma,s} \leqslant c \| f \|_{\sigma',s'} \quad \text{if} \quad \sigma < \sigma', \qquad \| f \|_{\sigma,s} \leqslant \| f \|_{\sigma,s'} \quad \text{if} \quad s < s',$$

with c depending on σ, σ', s, and s'.

The equivalence of the set of norms (2.1) to the previous ones is shown by the following lemma, to be proved in section 4.

LEMMA 2.2. — $A(r) \subset H^\infty$ for each $r > 0$. Moreover, $f \in A(r)$ implies that $\| f \|_{\sigma,s} < \infty$ for any σ, s with $e^\sigma < r$. Conversely, if $f \in H^\infty(\mathbb{R})$ satisfies $\| f \|_{\sigma,s} < \infty$ for some $s \in \mathbb{R}$ for each $e^\sigma < r$, then f (has analytic continuation) $\in A(r)$.

In view of Lemma 2.2, to prove Theorem 2 it suffices to apply Theorem 1 to the Liapunov family $\{ \| u \|_{\sigma,2} \}$ with $e^{\sigma(0)} < r_0$. For practical computation, however, it is convenient to work with a finite sum in (2.1). Thus we set

$$\| f \|^2_{\sigma,2;m} = \sum_{j=0}^{m} (j!)^{-2} e^{2j\sigma} | \partial^j f |^2_2 . \tag{2.1'}$$

We apply Theorem 1 to estimate $\| u(t) \|_{\sigma(t),2;m}$ with an appropriate function $\sigma(t)$, uniformly in m, and then let $m \to \infty$ to obtain an estimate for $\| u(t) \|_{\sigma(t),2}$. Since it is sufficient to assume $u(t) \in H^{m+5}$ to perform various formal computations required in the estimate, we apply Theorem 1 with $Z = H^{m+5}$, $X = H^{m+2}$, and the Liapunov family

$$\Phi_{\sigma;m}(v) = (1/2) \| v \|^2_{\sigma,2;m} \tag{2.2}$$

defined on an appropriate open set $O \subset Z$. The property of the family (2.2) to the Liapunov for (G) depends on the following lemma.

LEMMA 2.3. — For $v \in H^{m+5}$, we have

$$\langle F(v), D\Phi_{\sigma;m}(v) \rangle \leqslant \overline{K}(| v |_2)\Phi_{\sigma;m}(v) + \overline{\alpha}(\sigma, | v |_2, \Phi_{\sigma;m}(v))\partial_\sigma\Phi_{\sigma;m}(v) \tag{2.3}$$

for sufficiently small σ (algebraically!), where $\overline{K}$ and $\overline{\alpha}$ are real-valued, nonnegative functions with the following properties. $\overline{K}$ is continuous and monotone nondecreasing on $\mathbb{R}_+$. $\overline{\alpha}(\sigma, \mu, v)$ is continuous and monotone nondecreasing on a subset of $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ determined by an inequality of the form $\sigma \leqslant \overline{\sigma}(\mu, v)$, with $\overline{\sigma}$ monotone nonincreasing on $\mathbb{R}_+ \times \mathbb{R}_+$. Thus $\overline{\alpha}(\sigma, | v |_2, \Phi_{\sigma;m}(v))$ is well-defined for any $v \in H^{m+2}$ if σ is small enough, and (2.3) makes sense (and is claimed to be true) for such σ.

The proof of Lemma 2.3 will be given in next section. Here we prove Theorem 2 using Lemma 2.3. We start with the proof that (2.2) is indeed a Liapunov family for (G) on a certain open set $O \subset Z$. For this we have to exhibit the set O and the functions α, β that appear in the basic estimate (1.4). Actually O, α, and β will depend on the solution u under consideration.

(It may be noted, in passing, that the proof of Theorem 2 is greatly simplified if $a(\lambda)$ is *entire* in λ, as in the proper KdV equation. In this case we may take $O = Z$, and the structure of the function α becomes much simpler.)

Given a solution $u \in C(\mathrm{IT}; H^\infty)$ of (G) with $u(0) = \phi \in A(r_0)$, choose σ_0 such that $e^{\sigma_0} < r_0$ and let $\mu = 1 + \max \{ | u(t) |_2, t \in \mathrm{IT} \}$. Let O be the set of all $v \in Z$ with $| v |_2 < \mu$; obviously O is open in Z, and $u(t) \in O$ for $t \in \mathrm{IT}$.

Using the functions $\overline{K}$, $\overline{\alpha}$, $\overline{\sigma}$ given in Lemma 2.3, we set

$$K = \overline{K}(\mu), \qquad \beta(r) = Kr \qquad (r \geqslant 0), \tag{2.4}$$

$$\rho(t) = (1/2) \| \phi \|^2_{b,2} e^{Kt} \qquad (t \geqslant 0), \qquad b < \sigma_0, \tag{2.5}$$

$$\bar{r} = 1 + \max \{ \rho(t); t \in \mathrm{IT} \}, \tag{2.6}$$

$$\overline{\sigma} = \overline{\sigma}(\mu, \bar{r}), \tag{2.7}$$

$$\alpha(r) = \overline{\alpha}(\overline{\sigma}, \mu, r) \qquad (0 \leqslant r < \bar{r}). \tag{2.8}$$

We note that $\alpha(r)$ is well-defined for $r < \bar{r}$. The same is of course true of $\beta(r)$. In view of Lemma 2.3, it is now easy to see that (1.4) is satisfied with these functions α, β and the constants $\overline{\sigma}$, $\bar{r}$. (Note that $v \in O$ implies $| v |_2 < \mu$.) This proves that $\{ \Phi_{\sigma;m} \}$ is a Liapunov family for (G) on O.

Moreover, ϕ is $\{\Phi_{\sigma;m}\}$-admissible, since

$$\Phi_{b;m}(\phi) \leqslant (1/2)\|\phi\|^2_{b,2} = \rho(0) \leqslant \bar{r} - 1 .$$

If we denote by ρ_m the solution of (1.5) with $\rho_m(0) = \Phi_{b;m}(\phi) \leqslant \rho(0)$, we have obviously $\rho_m(t) \leqslant \rho(t)$, $t \geqslant 0$, and we obtain by Theorem 1

$$(2.9) \qquad (1/2)\|u(t)\|^2_{\sigma_m(t),2;m} = \Phi_{\sigma_m(t);m}(u(t)) \leqslant \rho_m(t) \leqslant \rho(t),$$

where

$$(2.10) \qquad \sigma_m(t) = b - \int_0^t \alpha(\rho_m(\tau))d\tau \geqslant b - \int_0^t \alpha(\rho(\tau))d\tau \equiv \sigma(t).$$

Note that (2.9) holds as long as $\rho_m(t) < \bar{r}$. But since $\rho_m(t) \leqslant \rho(t)$ as long as it exists and since $\rho(t) \leqslant \bar{r} - 1$, (2.9) holds for all $t \in \mathrm{IT}$.

On letting $m \to \infty$, we thus obtain from (2.9)

$$(2.11) \qquad \|u(t)\|^2_{\sigma(t),2} \leqslant \|\phi\|^2_{\sigma_0,2} e^{\mathrm{K}t} .$$

This shows that $u(t) \in \mathrm{A}(r_1)$ if $r_1 = e^{\sigma(\mathrm{T})}$.

It remains to prove the continuity of $u(t)$ in $\| \quad \|_{\sigma,2}$ for $e^\sigma < r_1$. For this we need the following lemma, the proof of which is easy and will be omitted.

LEMMA 2.4. — Let $\phi_n \in \mathrm{A}(r)$, $n = 1, 2, \ldots$, be a sequence with $\|\phi_n\|_{\sigma,2}$ bounded, where $e^\sigma < r$. If $\phi_n \to 0$ in H^∞ as $n \to \infty$, then $\|\phi_n\|_{\sigma',2} \to 0$ for each $\sigma' < \sigma$.

The result obtained above shows that $\|u(t)\|_{\sigma(\mathrm{T}),2}$ is bounded for $t \in \mathrm{IT}$. Since $u(t)$ is continuous in H^∞, it follows from the lemma that $u(t)$ is continuous in $\| \quad \|_{\sigma,2}$ if $\sigma < \sigma(\mathrm{T})$.

3. PROOF OF LEMMA 2.3

We have to estimate $\langle \mathrm{F}(v), \mathrm{D}\Phi_{\sigma;m}(v) \rangle$ for $v \in \mathrm{Z} = \mathrm{H}^{m+5}$, where F is given by (G) and $\Phi_{\sigma;m}$ by (2.2). To this end we first compute $\langle w, \mathrm{D}\Psi_j(v) \rangle$ where $\Psi_j(v) = (1/2)\,|\,\partial^j v\,|^2_2$. If we write $\Lambda = (1 - \partial^2)^{1/2}$, then $\Psi_j(v) = (1/2)\,|\,\Lambda^2 \partial^j v\,|_0^2$, hence $\mathrm{D}\Psi_j(v) = (-\partial)^j \Lambda^4 \partial^j v$, which is in $\mathrm{H}^{m-2j+1} \subset \mathrm{H}^{-m+1} \subset \mathrm{H}^{-m-2} = \mathrm{X}^*$ if $v \in \mathrm{Z} = \mathrm{H}^{m+5}$ and $j \leqslant m$. Therefore

$$(3.0) \quad \langle w, \mathrm{D}\Psi_j(v) \rangle = (\Lambda^2 \partial^j w \mid \Lambda^2 \partial^j v)_0 = (\partial^j w \mid \partial^j v)_2 \qquad (w \in \mathrm{H}^{m+2}).$$

where $(\ \mid\ )_s$ denotes the H^s-inner product on $\mathbb{R}$. Since $\Phi_{\sigma;m}$ is a linear combination of the Ψ_j, we thus obtain

$$(3.1) \qquad \langle \mathrm{F}(v), \mathrm{D}\Phi_{\sigma;m}(v) \rangle = \sum_{j=0}^{m} (j!)^{-2} e^{2j\sigma} (\partial^j v \mid \partial^j \mathrm{F}(v))_2 .$$

We first compute

$$(3.2)\qquad (\partial^j v \mid \partial^j \mathrm{F}(v))_2 = -(\partial^j v \mid \partial^j(\partial^3 v + a(v)\partial v))_2 .$$

Here the contribution of $\partial^3 v$ vanishes on integration by parts. Hence

$$(3.3)\qquad (\partial^j v \mid \partial^j \mathrm{F}(v))_2 = -(\partial^j v \mid a(v)\partial^{j+1} v)_2 - \mathrm{Q}_j(v),$$

$$(3.4)\qquad \mathrm{Q}_j(v) = \sum_{k=1}^{j} \binom{j}{k} (\partial^j v \mid (\partial^k a(v))(\partial^{j-k+1} v))_2 .$$

Using a standard integration by parts, it is easily seen that the first term in (3.3) is majorized by $(1/2)\overline{\mathrm{K}}(|v|_2)|\partial^j v|^2{}_2$, where $\overline{\mathrm{K}}$ is a certain continuous, monotone nondecreasing function, depending on a but independent of j. Hence

$$(3.5)\qquad \langle \mathrm{F}(v), \mathrm{D}\Phi_{\sigma;m}(v) \rangle \leqslant \overline{\mathrm{K}}(|v|_2)\Phi_{\sigma;m}(v) - \sum_{j=0}^{m} (j!)^{-2} e^{2j\sigma} \mathrm{Q}_j(v) .$$

The first term on the right of (3.5) has a form required in (2.3). Thus it remains to show that the second term has the form of the second term in (2.3). This requires some preparations.

First we note that the analytic function $a(\lambda)$ satisfies the estimates

$$(3.6)\qquad |a^{(p)}(\lambda)| \leqslant p!\mathrm{M}^p \qquad (p = 0, 1, 2, \dots),$$

where $a^{(p)} = d^p a/d\lambda^p$ and M can be chosen independent of λ if λ varies over a bounded set. From this it is easy to deduce the estimates

$$(3.7)\qquad |a^{(p)}(v)|_{2,ul} \leqslant p!\mathrm{M}^p \qquad (p = 0, 1, 2, \dots),$$

where $|\ \ |_{2,ul}$ denotes the uniformly local H^2-norm (see Kato [5]), and where $\mathrm{M} = \mathrm{M}(|v|_2)$ depends only on $|v|_2$.

Second, we make frequent use of the formulas

$$(3.8)\qquad |fg|_2 \leqslant \gamma(|f|_2|g|_1 + |f|_1|g|_2),$$

$$(3.8')\qquad |fg|_2 \leqslant \gamma|f|_{2,ul}|g|_2,$$

where γ is a numerical constant. (For (3.8′) see [5].)

Third, we use the following formula for higher derivatives of composed functions (given e. g. in Bourbaki [*1*, Chap. I, 3, Exercise 7] in a slightly different form):

$$(3.9)\qquad \partial^k a(v) = \Sigma \frac{k!}{p!k_1! \dots k_p!} a^{(p)}(v)(\partial^{k_1} v) \dots (\partial^{k_p} v),$$

where summation is taken over all *positive* integers $p, k_1, \dots, k_p$ such that

$$(3.10)\qquad 1 \leqslant p \leqslant k, \qquad k_1 + \dots + k_p = k .$$

If we apply (3.8′) to (3.9) multiplied with $\partial^{j-k+1}v$, we obtain

$$(3.11) \quad |(\partial^k a(v))(\partial^{j-k+1}v)|_2 \leqslant 2\gamma\Sigma \frac{k!}{p!k_1!\dots k_p!} |a^{(p)}(v)|_{2,ul} |(\partial^{k_1}v)\dots(\partial^{k_p}v)(\partial^{j-k+1}v)|_2 .$$

Here we use (3.7) to estimate $|a^{(p)}(v)|_{2,ul}$. The remaining factor is estimated by repeated application of (3.8) followed by the simple estimate $|\partial^k v|_1 \leqslant |\partial^{k-1}v|_2$ for $k \geqslant 1$. Thus

$$(3.12) \quad |(\partial^{k_1}v)\dots(\partial^{k_p}v)(\partial^{j-k+1}v)|_2 \leqslant \gamma^p[|\partial^{k_1-1}v|_2\dots|\partial^{k_p-1}v|_2|\partial^{j-k+1}v|_2 + |\partial^{k_1}v|_2|\partial^{k_2-1}v|_2\dots|\partial^{k_p-1}v|_2|\partial^{j-k}v|_2 + (\text{cycl})]$$

where (cycl) denotes terms obtained by cyclic change of $k_1, \dots, k_p$ in the preceding expression.

It is convenient at this point to introduce the following short-hand notation.

$$(3.13) \qquad b_j = (j!)^{-1}e^{j\sigma}|\partial^j v|_2, \qquad j = 0, 1, 2, \dots, m .$$

We estimate $Q_j(v)$ (see (3.4)) by applying the Schwarz inequality to the inner product $(\ |\ )_2$ and using the estimates obtained above. On multiplying $Q_j(v)$ with $(j!)^{-2}e^{2j\sigma}$, we thus obtain, after some arrangement of the factorials,

$$(3.14) \quad -(j!)^{-2}e^{2j\sigma}Q_j(v) \leqslant \Sigma 2\gamma^{p+1}M^p e^{(p-1)\sigma} b_j[(b_{k_1-1}/k_1)\dots(b_{k_p-1}/k_p)(j-k+1)b_{j-k+1} + b_{k_1}(b_{k_2-1}/k_2)\dots(b_{k_p-1}/k_p)b_{j-k} + (\text{cycl})] .$$

We have to sum (3.14) over $0 \leqslant j \leqslant m$. For this we need the following technical lemma, the proof of which will be given at the end of the section.

LEMMA 3.1. — For nonnegative real numbers $b_0, b_1, \dots, b_m$, we have

$$(3.15) \qquad \Sigma b_j b_{k_1}(b_{k_2-1}/k_2)\dots(b_{k_p-1}/k_p)b_{j-k} \leqslant \gamma'^p B^p \tilde{B}^2 ,$$

$$(3.16) \quad \Sigma b_j(b_{k_1-1}/k_1)\dots(b_{k_p-1}/k_p)(j-k+1)b_{j-k+1} \leqslant (\text{same}),$$

where summation is to be taken over all positive $k_1, \dots, k_p$ and j such that $k_1 + \dots + k_p = k \leqslant j \leqslant m$, with $1 \leqslant p \leqslant m$ fixed. Here γ' is a numerical constant, and

$$(3.17) \qquad B^2 = \sum_{j=0}^{m} b_j^2 = \|v\|^2_{\sigma,2;m} ,$$

$$\tilde{B}^2 = \sum_{j=1}^{m} jb_j^2 = (1/2)\partial_\sigma \|v\|^2_{\sigma,2;m} .$$

Applying Lemma 3.1 to (3.14) and using (3.17), we finally obtain

$$(3.18)\quad -\sum_{j=0}^{m}(j!)^{-2}e^{2j\sigma}Q_j(v)$$
$$\leqslant \gamma\left[\sum_{p=1}^{m}(p+1)(\gamma\gamma')^p e^{(p-1)\sigma}M(|v|_2)^p \| v \|^p_{\sigma,2;m}\right]\partial_\sigma \| v \|^2_{\sigma,2;m}.$$

In view of the remark following (3.5), this shows that (2.3) is true with

$$(3.19)\qquad \overline{\alpha}(\sigma,\mu,\nu) = 2\gamma\sum_{p=1}^{\infty}(p+1)(2\gamma\gamma')^p e^{(p-1)\sigma}M(\mu)^p\nu^p.$$

Since the series (3.19) converges absolutely for every μ, $\nu \geqslant 0$ provided that σ is sufficiently small, we have proved Lemma 2.3.

Proof of Lemma 3.1. — We begin with (3.16). We first sum in j for $k \leqslant j \leqslant m$. Then $\Sigma(j-k+1)b_jb_{j-k+1} \leqslant \tilde{B}^2$ by the Schwarz inequality. The remaining factor does not exceed a sum, taken for $1 \leqslant k_\nu \leqslant m$ independently for $\nu = 1, \ldots, p$, which separates into the product of p single series $\Sigma b_{k-1}/k \leqslant \gamma' B$ (again by Schwarz), where $\gamma' = (\Sigma k^{-2})^{1/2} < 2$. This proves (3.16). To prove (3.15), we multiply the summand by $(j/k_1)^{1/2}$, which is not smaller than one. We sum the resulting majorizing series again in j first, obtaining $\Sigma j^{1/2}b_jb_{j-k} \leqslant B\tilde{B}$ by Schwarz. The remaining factor is majorized by the product of $p-1$ identical series considered above (which is smaller than $\gamma' B$ each), and another series $\Sigma b_{k_1}/k_1{}^{1/2} \leqslant \gamma'\tilde{B}$ (Schwarz again). The result is again smaller than $\gamma'^p\tilde{B}^2$.

4. PROOF OF LEMMA 2.2

We first prove the second part of the lemma. Let $f \in H^\infty$ with $\| f \|_{\sigma,s} < \infty$, and let $r < e^\sigma$. We have to show that f has analytic continuation into $S(r)$ with L^2-norm finite. For this purpose we may assume that $s = 0$. (This is obvious if $s \geqslant 0$; if $s < 0$, note that there is σ' such that $r < e^{\sigma'} < e^\sigma$ and $\| f \|_{\sigma',0} < \infty$.)

In what follows we write $| f |$ for the L^2-norm $| f |_0$, to avoid confusion with the pointwise values $| f(x) |$. Set

$$(4.1)\qquad M^2 = \| f \|^2_{\sigma,0} = \sum_{j=0}^{\infty} j!^{-2}e^{2j\sigma}| \partial^j f |^2.$$

Then we have $|\partial^j f| \leqslant j!e^{-j\sigma}\mathrm{M}$, hence

$$|\partial^j f(x)| \leqslant (|\partial^j f|\,|\partial^{j+1} f|)^{1/2} \leqslant (j+1)^{1/2} j! e^{-(j+1/2)\sigma}\mathrm{M}.$$

Thus the Taylor series for f about x has radius of convergence at least equal to e^σ, and f has analytic continuation into $\mathrm{S}(r)$, which we denote by the same letter f.

The Taylor series $f(x+iy) = \sum_{j=0}^{\infty} \partial^j f(x)(iy)^j/j!$ gives, by the Schwarz inequality,

$$(4.2)\qquad |f(x+iy)|^2 \leqslant \left(\sum_{j=0}^{\infty} j!^{-2} e^{2j\sigma} |\partial^j f(x)|^2\right)\left(\sum_{j=0}^{\infty} e^{-2j\sigma}|y|^{2j}\right)$$

for $|y| < e^\sigma$. It follows on integration that

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 dx \leqslant \mathrm{M}^2(1 - e^{-2\sigma}r^2)^{-1} \qquad \text{for} \quad |y| \leqslant r < e^\sigma.$$

Integration in y shows that $f \in \mathrm{L}^2(\mathrm{S}(r))$, as required.

To prove the first part of the lemma, assume that f is analytic and belongs to $\mathrm{L}^2(\mathrm{S}(r))$, where $r > e^\sigma$. The Cauchy integral theorem gives for $0 < y < r$,

$$(4.3)\qquad \partial^j f(x) = (2\pi i)^{-1} j! \int_{-\infty}^{\infty} [(\xi - x - iy)^{-j-1} f(\xi - iy) - (\xi - x + iy)^{-j-1} f(\xi + iy)] d\xi.$$

We apply the Schwarz inequality to (4.3) by factoring each $|\xi - x \pm iy|^{-j-1}$ into equal parts. Using the estimate

$$(4.4)\qquad \int_{-\infty}^{\infty} |x \pm iy|^{-j-1} dx \leqslant cj^{-1/2} y^{-j} \qquad (j \geqslant 1),$$

we thus obtain (note that $|x - \xi - iy| = |x - \xi + iy|$)

$$(4.5)\qquad |\partial^j f(x)|^2 \leqslant cj!^2 j^{-1/2} y^{-j} \int_{-\infty}^{\infty} |x - \xi + iy|^{-j-1}(|f(\xi+iy)|^2 + |f(\xi - iy)|^2) d\xi.$$

Integrating (4.5) in x and again using (4.4), and then multiplying with jy^{2j}, we obtain for $j \geqslant 1$, $0 \leqslant y < r$,

$$(4.6)\qquad jy^{2j}|\partial^j f|^2 \leqslant cj!^2 \int_{-\infty}^{\infty} (|f(\xi+iy)|^2 + |f(\xi-iy)|^2) d\xi.$$

Integration in $y \in (0, r)$ gives

$$(4.7)\qquad r^{2j+1}|\partial^j f|^2 \leqslant cj!^2 \mathrm{K}^2, \qquad \mathrm{K}^2 = \int_{\mathrm{S}(r)} |f(x+iy)|^2 dxdy.$$