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Analyse non linéaire

A new degree for S^1-invariant gradient mappings and applications

by

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ABSTRACT. — We construct a new degree for S^1-invariant gradient maps where the classical degree gives little information. The main technical result needed is a new result on generic homotopies. We apply this degree to obtain a global bifurcation theorem which applies to cases where classical results give limited information.

We apply our results to obtain a bifurcation theorem for periodic solutions of Hamiltonian systems and for a problem in elasticity. We also obtain new results on bifurcation for elliptic equations on domains with an S^1 symmetry.

Keywords: S^1-invariance, gradient maps, degree, global bifurcation, Hamiltonian systems, elliptic equations.

RÉSUMÉ. — Nous construisons un degré adapté à des applications S^1-invariantes de type gradient où la notion classique de degré ne donne que peu d'informations. On utilise pour cela un résultat nouveau sur les homotopies génériques. On applique ce degré pour obtenir un résultat de bifurcation globale.

Nous appliquons nos résultats pour obtenir un théorème de bifurcation de solutions périodiques de systèmes hamiltoniens, et à un problème d'élasticité. Nous obtenons également de nouveaux résultats sur la bifurcation d'équations elliptiques dans des domaines pourvus d'une symétrie S^1.

In this paper, we construct a degree for S^1-invariant gradient mappings and consider its application to some bifurcation problems in Elasticity and Hamiltonian Mechanics.

In more detail, we assume that E is a finite-dimensional normed linear space and $\{ T_g \}_{g \in G}$ is a continuous linear representation of $G = S^1$ on E. We consider maps $F : E \to E$ which are S^1-invariant and also are gradient maps. Let E_{S^1} denote the set of fixed points of the group action. If $\Omega \subseteq E$ is bounded, open and S^1-invariant and if $x \neq F(x)$ on $\partial\Omega \cup (\Omega \cap E_{S^1})$, we construct a special degree $\deg_n^{S^1}(I - F, \Omega)$. The n is used to denote normal. The reason for this will be apparent from the definition of the degree. This degree does not agree with the usual Brouwer degree. In fact, the Brouwer degree is necessarily zero for such maps. Our degree resembles the Fuller index for periodic solutions of autonomous ordinary differential equations in that it is only defined if F has no fixed points in $\overline{\Omega} \cap E_{S^1}$. Note that, as we discussed in [*22*], our degree can not be extended to all S^1-invariant maps. We define our degree by generic arguments. In fact, the main technical result needed to construct our degree is a theorem on « generic » homotopies.

We extend our degree to infinite-dimensional Hilbert spaces and we use it to obtain global bifurcation theorems. For suitable S^1-invariant gradient mappings depending on a parameter, we obtain connected sets of non-trivial solutions which bifurcate at an eigenvalue of the linearized equation and continue globally. This contrasts with the case of general gradient mappings where it is that there is bifurcation (by Rabinowitz [*34*]) but there may not be connected sets of solutions (as in Böhme [*7*]).

We apply our results to three examples. Firstly, we apply them to an example of Wolfe [*40*] involving elastic conducting rods. We obtain global branches of solutions. Our results considerably improve those in [*40*]. Indeed, the original motivation for this work was to understand an earlier example [*39*] of Wolfe.

Secondly, we use our degree to study periodic solutions of *fixed* period of autonomous Hamiltonian systems. In particular, we obtain a bifurcation theorem which extends one of Alexander and Yorke [*3*] and is related to one of Fadell and Rabinowitz [*24*]. (We obtain a result under weaker hypotheses; we obtain connected sets of solutions; our result is a global result; but we do not obtain their multiplicity result, which was the main part of their theorem.) Note that Chow, Maller-Paret and Yorke [*11*] proved a closely related bifurcation theorem in a special case. We obtain better global information than [*11*] and our result seems easier to apply than their method in more general situations. Moreover, our proof seems more natural. In particular, for convex Hamiltonians with isolated critical points and a critical point at 0, we prove that the solution branches bifurcating from zero are always unbounded. We also discuss briefly some other applications to Hamiltonian systems.

Thirdly, we apply our results to nonlinear elliptic equations on domains with a circular symmetry and obtain some new results.

It should be noted that the most usual trick when there are symmetries does not work for S^1 symmetries. This is to choose a subgroup A of G and consider $E_A = \{ x \in E : T_g x = x \text{ for all } g \in A \}$. However, since S^1 is abelian, one easily sees that E_A is invariant under the group action. Hence using E_A does not remove the symmetries (unless $E_A = E_{S^1}$).

In § 1, we define our degree for S^1-invariant gradient maps and study some of its properties. However, we defer the proof of the main technical lemma we need till § 7. In § 2, we discuss bifurcation theorems and, in § 3, we consider generalizations to infinite dimensions. In § 4, we consider applications to Elasticity while, in § 5, we consider applications to periodic solutions of Hamiltonian systems. In § 6, we consider applications to nonlinear elliptic equations on domains with an S^1-symmetry.

I should like to thank P. Wolfe for discussions on his elasticity problems. Most of the results were announced in [*22*].

A convenient reference for group actions is Bredon [*8*].

§ 1. THE BASIC DEGREE

In this section we construct our degree in finite dimensions. However, we defer the proof of the main technical lemma still § 7.

Assume that E is a finite-dimensional normed linear space and $\{ T_g \}_{g \in S^1}$ is a continuous linear representation of S^1 on E. By this, we mean that T_g is linear and bounded for $g \in S^1$, $T_{gh} = T_g T_h$ for $g, h \in S^1$, $T_e = I$ and the map $g \to T_g x$ is continuous for each $x \in E$. Note that the boundedness assumption is redundant in finite dimensions. We will often write G instead of S^1. If $z \in E$, let $G_z = \{ g \in S^1 : T_g z = z \}$. This is known as the *isotropy group* and is a closed subgroup of S^1. It follows that G_z must be $\{ e \}$ or S^1 or a finite cyclic group Z_n generated by a rotation through $2\pi n^{-1}$. Here, we think of S^1 as $\{ e^{it} : 0 \leq t < 2\pi \}$. Elements of E with $G_z = G$ are said to be fixed by G. If $A \subseteq G$, let $E_A = \{ x \in E : T_g x = x \text{ for all } g \text{ in } A \}$. We will use $G(z)$ to denote the orbit $\{ T_g z : g \in G \}$. Note that $G(z)$ is a smooth manifold. A map $F : E \to E$ is said to be G-*invariant* if $F(T_g x) = T_g F(x)$ for $g \in G$, $x \in E$. Similarly, a map $F : E \times R \to E$ is said to be G-*invariant* if $F(T_g x, \lambda) = T_g F(x, \lambda)$ for $g \in G$, $x \in E$, $\lambda \in R$ and a map $f : E \to R$ is said to be G-invariant if $f(T_g x) = f(x)$ for $x \in E$, $g \in G$. Sometimes we use invariant instead of G-invariant. Note that, if F is C^1 and G-invariant, then $T_x(G(x)) \subseteq N(F'(x))$, where N denote the kernel and $T_x(M)$ denotes the tangent space to the manifold M at x. This follows by differentiating in directions tangent to the orbit at x. The above results are proved in Bredon [8].

Fix a scalar product $\langle\, , \rangle$ *on* E.

The following theorem summarizes the basic properties of the degree we will construct. A map $F:\overline{\Omega} \to E$ is said to be *admissible* if Ω is a bounded invariant open set in E and $F:\overline{\Omega} \to E$ is an S^1-invariant continuous gradient mapping such that $x \neq F(x)$ on $\partial\Omega \cup (\Omega \cap E_{S^1})$. A continuous map $F:\overline{\Omega} \times [0,1] \to E$ is an *admissible homotopy* if Ω is as above, if F is S^1-invariant, if $F(\,,t)$ is a gradient mapping for each t and if $x \neq F(x,t)$ for $x \in \partial\Omega \cup (\Omega \cap E_{S^1})$ and $t \in [0,1]$. Note that, when we say F is a gradient, we mean with respect to the given scalar product $\langle\, , \rangle$.

THEOREM 1. — *For each admissible map* $F:\overline{\Omega} \to E$, *there is a corresponding rational number* $\deg_n^{S^1}(I - F, \Omega)$ *such that the following properties hold.*

(*i*) *If* $F:\overline{\Omega} \to E$ *is admissible and* $\deg_n^{S^1}(I - F, \Omega) \neq 0$, then there is an x *in* Ω *such that* $x = F(x)$.

(*ii*) If Ω_i, $i = 1, \ldots, k$ *are disjoint open invariant subsets of* Ω *and* $F:\overline{\Omega} \to E$ *is admissible such that* $x \neq F(x)$ *on* $\overline{\Omega}\backslash\bigcup_{i=1}^{k}\Omega_i$ *then*

$$\deg_n^{S^1}(I - F, \Omega) = \sum_{i=1}^{k} \deg_n^{S^1}(I - F, \Omega_i).$$

(*iii*) *If* $F:\overline{\Omega} \times [0,1] \to E$ *is an admissible homotopy, then* $\deg_n^{S^1}(I - F(\,,t), \Omega)$ *is independent of* t.

(*iv*) *If* $F:\overline{\Omega} \to E$ *is* C^1 *and admissible, if* $S^1(z)$ *are the only fixed points of* F *in* Ω *and if* $N \equiv N(F'(z))$ *is one-dimensional, then*

$$\deg_n^{S^1}(I - F, \Omega) = (\# G_z)^{-1} \operatorname{sgn} \det ((I - F'(z))|_{N^\perp}),$$

where $\#$ K *denotes the number of elements in* K *and det denotes the determinant.*

Assume now that $F:E \to E$ is S^1-invariant and F is a continuous gradient mapping with respect to the scalar product $\langle\, , \rangle$ on E. As in [15], we can assume without loss of generality that $\langle T_g x, T_g y \rangle = \langle x, y \rangle$ for x, $y \in E$, $g \in S^1$ and that F is the gradient of f with respect to $\langle\, , \rangle$ where $f(T_g x) = f(x)$ for $g \in S^1$, $x \in E$. It follows by differentiating the equation $f(T_g x) = f(x)$ with respect to g at $g = e$, that

$$\langle F(x), Ax \rangle = 0 \tag{1}$$

for $x \in E$, where A is the infinitesimal generator of the representation

$g \to T_g$. ($A = \lim_{t \to 0} t^{-1}(T_{\alpha(t)} - I)$, where $\alpha(t) = e^{it}$.) In addition since $\langle T_g x, T_g y \rangle = \langle x, y \rangle$, we see by differentiation that $\langle Ax, y \rangle + \langle x, Ay \rangle = 0$. Thus A is skew-adjoint. Equation (1) is very important below. Note that (1) is essentially a special case of Noether's theorem (cp. [*29*, Theorem 2.3]) and, in physical examples, (1) corresponds to a conservation law.

Now assume that $S^1(z)$ *is an isolated orbit of zeros of* $I - F$, *where* $z \notin E_{S^1}$. Since A is the infinitesimal generator of $\{T_g\}_{g \in S^1}$, it follows that $Az \neq 0$.

LEMMA 1. — *Assume that* U *is an* S^1*-invariant closed neighbourhood of* $G(z)$ *such that* $Av \neq 0$ *if* $v \in U$. *Then there is a neighbourhood* W *of* e *in* S^1 *such that, if* H *is an* S^1*-invariant gradient mapping* (*with respect to the same scalar product as* F) *and if* $T_g x = H(x)$ *where* $g \in W$ *and* $x \in U$, *then* $g = e$.

Proof. — First note that every element of S^1 near e can be written uniquely as $\alpha(t) \equiv \exp(it)$ where t is small. By (1), $\langle H(x), Ax \rangle = 0$. Thus, if $T_g w = H(w)$, $\langle T_g w, Aw \rangle = 0$. Since A is skew-adjoint, it follows that $\langle T_g w - w, Aw \rangle = 0$. Now, if g is near e, $g = \alpha(s)$ where s is small. Hence, if $s \neq 0$,

$$s^{-1} \langle T_{\alpha(s)} w - w, Aw \rangle = 0. \tag{2}$$

As $t \to 0$, $t^{-1}(T_{\alpha(t)} - I)x \to Ax$ uniformly on U and thus

$$t^{-1} \langle T_{\alpha(t)} x - x, Ax \rangle \to \langle Ax, Ax \rangle$$

uniformly on U. Since $Ax \neq 0$ on U and U is compact, it follows that $t^{-1} \langle T_{\alpha(t)} x - x, Ax \rangle \neq 0$ if $x \in U$ and t is small. This contradicts (2) if s is small and hence the result follows.

Note that $Ax \neq 0$ on U if U is a sufficiently small neighbourhood of $S^1(z)$. In general, there may be points close to z where z and $F(z)$ lie on the same orbit.

Now assume that T is a smooth manifold transverse to $S^1(z)$ at z. By the proof of the tubular neighbourhood theorem (cp. Bredon [*8*, Theorem 6.2.2]), we see that every point x in E near z can be uniquely written in the form $T_\alpha w$ where $w \in T$, w is near z and α is near e. We write $x = T_{\alpha(x)} \bar{y}(x)$. If $w \in T$ and w is near z, define $\bar{F}(w) = \bar{y}(F(w))$. This defines a continuous mapping of a neighbourhood of z in T into T. Note that, if w is near z, $F(w)$ is near $F(z) = z$ and thus $\bar{y}(F(w))$ is defined. Moreover, z is an isolated fixed point of $\bar{F}$. To see this, note that, by the definition of $\bar{F}$, $\bar{F}(y) = y$ with y near z implies that $T_\alpha F(y) = y$ where α is near e. Thus $T_{\bar{\alpha}} y = F(y)$, where $\bar{\alpha}$ is the complex conjugate. By Lemma 1, it follows that $\alpha = e$. Hence $F(y) = y$ and thus $y = z$, since $S^1(z)$ is an isolated orbit of fixed points.

We define

$$\mathrm{index}_n^{S^1}(I - F, S^1(z)) = (\# G_z)^{-1} \, \mathrm{index}_T(I - \bar{F}, z).$$

We call this the normal index of the orbit $S^1(z)$ for F. We need to check it is independent of the choice of T and of the choice of z on the orbit $S^1(z)$.

We first show that it is independent of the choice of T. Suppose that U is an open subset of R^n (where $n = \dim E - 1$) with $0 \in U$ and $\phi : U \times [0,1] \to E$ is a C^2 map such that $\phi(0, t) = z$ for all t and $R(\phi_1^1(0, t)) \cap T_z(S^1(z)) = \{0\}$ for all t. Here ϕ_1^1 denotes the partial derivative with respect to the first variable and R denotes the range. If V is a small neighbourhood of 0 in R^n, $\phi(V, t)$ is a possible choice for T. Now, by an examination of the proof of the tubular neighbourhood theorem, there is a neighbourhood Y of z in E and jointly continuous functions $\alpha_t(x)$, $\bar{y}_t(x)$ such that, for every $t \in [0,1]$, any point x in Y can be uniquely expressed in the form $T_{\alpha_t(x)}\bar{y}_t(x)$. Here $\bar{y}_t(x) \in T_t \equiv \phi(V, t)$. The point is that the part of the proof of the tubular neighbourhood theorem we require reduces to the implicit function theorem. Let $\bar{F}_t = \bar{y}_t F$. By the commutativity theorem for the degree (cp. Nussbaum [32]), $\mathrm{Index}_{T_t}(I - \bar{F}_t, z) = \mathrm{index}_{R^n}(I - \phi_t^{-1}\bar{F}_t\phi_t, 0)$, where $\phi_t(x) = \phi(x, t)$. By the homotopy invariance of the degree, it follows that the right hand side is independent of t. (It is easily checked that there is a neighbourhood X of 0 in R^n such that the only solution of $x = \phi_t^{-1}\bar{F}_t\phi_t x$ in X is 0 for all t.) Thus $\mathrm{index}_{T_t}(I - \bar{F}_t, z)$ is independent of t. It follows easily by a suitable choice of ϕ that it suffices to prove our result for ϕ linear, that is, T a hyperplane. We finally obtain that the original definition is independent of the choice of T from the above result if we note that the set of hyperplanes transverse to $T_z(S^1(z))$ form a connected open set.

To see that our definition is independent of the choice of z on $S^1(z)$, we use the commutativity theorem for the degree as before. (If T is a manifold transverse to $S^1(z)$ at z and if $m = T_b z$, then $T_b T$ is transverse to $S^1(z)$ at m and the S^1-invariance ensures that the corresponding $\bar{F}$'s are conjugate.)

Hence our local index is well-defined. We now show that it has a local homotopy invariance property. Suppose that we choose a closed neighbourhood U of $S^1(z)$ as in Lemma 1 such that $x \neq F(x)$ on $U \backslash S^1(z)$. We prove that, if F_1 is an S^1-invariant gradient map sufficiently close to F in the C^0-norm and if $I - F_1$ only vanishes on a finite number of orbits $S^1(z_1), \ldots, S^1(z_k)$ in U, then

$$\mathrm{index}_n^{S^1}(I - F, S^1(z)) = \sum_{i=1}^{k} \mathrm{index}_n^{S^1}(I - F_1, S^1(z_i)). \tag{3}$$

To prove this result, choose a transversal manifold T to $S^1(z)$ at z. Fix T. Note that, if F_1 is close to F, the orbits $S^1(z_i)$ will be near $S^1(z)$. Thus T will also be transversal to the $S^1(z_i)$. Now if $\bar{y}$ is defined as earlier, $\bar{F}_1 \equiv \bar{y}F_1$ will be close to $\bar{F} = \bar{y}F$ in the C^0 norm (since F_1 is close to F). Thus, by the homotopy invariance of the ordinary degree,

$$\mathrm{index}_T(I - \bar{F}, z) = \sum_{j=1}^{m} \mathrm{index}_T(I - \bar{F}_1, w_j). \tag{4}$$

where w_j, $j = 1, \ldots, m$ are the fixed points of $\overline{F}_1$ in $U \cap T$. Here we are assuming that $\overline{F}_1$ has only a finite number of fixed points in $T \cap U$. To see that this is true, note that $\overline{F}_1$ is near $\overline{F}$ and hence its fixed points must be near z. Thus $F_1(w_j)$ is near z and so $F_1(w_j) = T_{\alpha_j}\overline{F}_1(w_j)$, where α_j is near e. Hence, by Lemma 1, we see that, if $\overline{F}_1(w_j) = w_j$, then $\alpha_j = e$ and $F_1(w_j) = w_j$ Thus the w_j are the intersection of the $S^1(z_i)$'s with T. Note that $S^1(z_i)$ may intersect T several times. In fact, by the tubular neighbourhood theorem (cp. Bredon [*8*, Theorem 6.2.2]), $S^1(z_i)$ will intersect T, $\# G_z (\# G_{z_i})^{-1}$ times. Moreover, by our earlier results on the independence of the choice of T or z in the definition of our index, each of these points (for fixed i) will have the same value of $\text{index}_T(I - \overline{F}_1, w_j)$. Thus (4) becomes

$$\text{index}_T(I - \overline{F}, z) = \sum_{i=1}^{k} \# G_z (\# G_{z_i})^{-1} \, \text{index}_T(I - \overline{F}_1, z_i).$$

If we multiply the equation by $(\# G_z)^{-1}$ and use the definition of $\text{index}_n^{S^1}$, (3) follows.

Suppose now that Ω is a bounded open S^1-invariant subset of E and $F: \overline{\Omega} \to E$ is an S^1-invariant gradient mapping such that $x \neq F(x)$ on $\partial\Omega \cup (\Omega \cap E_{S^1})$ and such that $I - F$ only vanishes on a finite number of orbits $\{ S^1(x_i) \}_{i=1}^{m}$ in Ω. We then define

$$\deg_n^{S^1}(I - F, \Omega) = \sum_{i=1}^{m} \text{index}_n^{S^1}(I - F, S^1(x_i)).$$

We call this an S^1-invariant normal degree. It follows easily from our earlier results that this degree is homotopy invariant if we deform F through S^1-invariant gradient mappings F_t such that F_t has no fixed points on $\partial\Omega \cup (\Omega \cap E_{S^1})$ and such that $I - F_t$ only vanishes on a finite number of orbits for each t. However, to obtain a useful degree, we have to define the degree for maps which vanish on an infinite number of orbits and prove a stronger homotopy invariance property. To do this, first note that, by examining the proof of Lemma 4.8 in Wasserman [*38*], we see that an S^1-invariant gradient map F can be approximated in the C^0 norm by a smooth S^1-invariant gradient map F_1 such that $I - F_1'(x)$ has only a one dimensional kernel whenever $x = F_1(x)$ and such that $I - F_1$ only vanishes on a finite number of orbits in Ω. (In fact, by the results in [*17*], the second assumption is a consequence of the first. Note also that such maps F_1, which we call generic maps, form an open set in the C^1 norm in the invariant gradient maps.) Now define $\deg_n^{S^1}(I - F, \Omega) = \deg_n^{S^1}(I - F_1, \Omega)$. Note that, since F_1 is near F on $\overline{\Omega}$, our assumptions ensure that F_1 has no fixed points in $\partial\Omega \cup (E_{S^1} \cap \Omega)$. We have to show that this definition is independent of the choice of F_1. The weak homotopy invariance result we proved earlier ensures that this definition agrees with the earlier one when $I - F$ only vanishes on a finite number of orbits. The proof that the above definition is independent of the choice of F_1 and of homotopy invariance will follow easily from the following proposition. Here ∇_x denotes the gradient with respect to x.

PROPOSITION 1. — *Assume that* Ω *is a bounded open invariant subset of* E *and* $H : \bar{\Omega} \times [0, 1] \to R$ *is a continuous* S^1*-invariant map such that* $\nabla_x H(x, t)$ *exists and is continuous and* $\nabla_x H(x, t) \neq 0$ *if* $x \in \partial\Omega \cup (E_{S^1} \cap \Omega)$ *and* $t \in [0, 1]$. *Then there is a smooth invariant mapping* $\tilde{H}$ *such that* $\nabla_x \tilde{H}$ *is close to* $\nabla_x H$ *in the* C^0 *norm and* $\nabla_x \tilde{H}(, t)$ *only vanishes on a finite number of orbits for each* t *in* $[0, 1]$. *Moreover if* H *is* C^1, *we can approximate* H *by such an* $\tilde{H}$ *in the* C^1 *norm.*

The proof of this is rather technical and we defer it till § 7. However, we now use it to complete the construction of our degree. Firstly, suppose that F_1 and F_2 are generic maps which approximate F closely. Then $tF_1 + (1 - t)F_2$ is a homotopy with no fixed points on $\partial\Omega \cup (\Omega \cap E_{S^1})$ for $0 \leq t \leq 1$. It follows easily from Proposition 1 that there is an homotopy $Z(,)$ joining F_1 and F_2 such that $I - Z(, t)$ only vanishes on a finite number of orbits in Ω for each t and $x \neq Z(x, t)$ if $t \in [0, 1]$ and $x \in \partial\Omega \cup (\Omega \cap E_{S^1})$. (We construct Z as the composite of the obvious homotopy joining F_1 to $I - \nabla_x \tilde{H}(, 0)$, $I - \nabla_x \tilde{H}$ and the obvious homotopy joining $I - \nabla_x \tilde{H}(, 1)$ to F_2. We also use here that the generic maps form an open set in the C^1 norm.) Thus, by our earlier results, $\deg_n^{S^1}(I - F_1, \Omega) = \deg_n^{S^1}(I - F_2, \Omega)$, as required. Homotopy invariance can be proved by a similar argument.

We now complete the proof of Theorem 1. — Part (*iii*) has been proved above while part (*i*) follows from the definition of the degree. Part (*ii*) follows easily by approximating F by « generic » maps. To prove (*iv*), note that, since $I - F'(z)$ is self adjoint (cp. Vainbert [*37*]), $N^\perp$ is invariant under $I - F'(z)$. We take $T = z + N^\perp$ in the definition of the local index. It is easy to see that $\bar{F}$ is C^1 and $(\bar{F})'(z) = (I - F'(z))|_{N^\perp}$. The result follows from this, our earlier definition of $\text{index}_n^{S^1}$ and the classical formula for the index at a point where the derivative is invertible (cp. Lloyd [*28*, § 1.1]). This completes the proof of Theorem 1.

The proof of (*iv*) shows that the assumption that $N(F'(z))$ is one-dimensional implies that $S^1(z)$ is an isolated orbit of fixed points of F, and that we could assume that F is differentiable at z rather than F is C^1.

To complete this section, we obtain two additional properties of our degree. The results will be used in later sections.

PROPOSITION 2. — (*i*) *Assume* $F : \bar{\Omega} \to E$ *is admissible and* $H : K \to K$

is an invariant self-adjoint linear operator on the linear space K *such that* 1 *is not an eigenvalue of* K. *Then*

$$\deg_n^{S^1}(\mathrm{I}-(\mathrm{F},\mathrm{H}),\Omega\times\mathrm{B}) = \deg_n^{S^1}(\mathrm{I}-\mathrm{F},\Omega)\,\mathrm{sgn}\det(\mathrm{I}-\mathrm{H}),$$

where B *is a ball in* K *with centre* 0.

(*ii*) *Assume that* $\mathrm{F}:\Omega_1\oplus\mathrm{H}_\alpha\subseteq\mathrm{E}\oplus\mathrm{H}\to\mathrm{E}\oplus\mathrm{H}$ *is an admissible map such that* F *is* C^1, *the orbit* $S^1((u_0, V_0))$ *is an isolated orbit of solutions and* $\mathrm{P}(\mathrm{I}-\mathrm{F}_2'(u,v))|_{\mathrm{H}}$ *is invertible for* $(u,v)\in\Omega_1\oplus\mathrm{H}_\alpha$. *Here* P *is the natural projection onto* H *and* H_α *denotes the ball of radius* α *in* H. *Finally, assume that* E *and* H *are* S^1*-invariant and that the equation* $\mathrm{P}(\mathrm{I}-\mathrm{F}(u,v))=0$ *has a unique solution* $v=g(u)$ *near* $S^1(v_0)$ *for* u *near* $S^1(u_0)$. *Then*

$$\mathrm{index}_n^{S^1}(\mathrm{I}-\mathrm{F}, S^1((u_0,v_0))) = \mathrm{sgn}\det(\mathrm{P}(\mathrm{I}-\mathrm{F}_2'(u_0,v_0))|_{\mathrm{H}})\,\mathrm{index}_n^{S^1}(\mathrm{I}-\tilde{\mathrm{F}}, S^1(u_0)),$$

where

$$\tilde{\mathrm{F}}(u) = (\mathrm{I}-\mathrm{P})\mathrm{F}(u, g(u)).$$

Remarks. — The product theorem is not true for this degree. The existence and local uniqueness of g follow from the implicit function theorem. It is possible to prove more general versions of (*ii*) by only assuming the invertibility condition on F_2' for (u, v) near (u_0, v_0).

Proof of Proposition 2. — (*i*) This is proved by reducing to the case where F is a « generic » map. We then note that, if z is a fixed point of F and if T is a manifold in E transverse to $S^1(z)$ at z, then $\mathrm{T}\times\mathrm{K}$ is transverse to the orbit through $(z, 0)$.

(*ii*) We only sketch the proof because we do not use it for our main theorems. It suffices to prove the result for F smooth and under the assumption that $\mathrm{N}((\mathrm{I}-\mathrm{F}')(u_0, v_0))$ is 1-dimensional. A similar argument to that in the proof of Theorem 3 in [*23*] reduces our problem to the case where $v_0=0$ and the assumptions of part (*i*) hold. Part (*ii*) then follows from part (*i*).

Note that an isolated orbit $S^1(z)$ of fixed points of F need not correspond to an isolated fixed point of the mapping induced by F on the orbit space. Thus it does not seem convenient to work with orbit spaces.

Finally if $\mathrm{F}:\bar{\Omega}\to\mathrm{E}$ is an admissible map and K is a subgroup of S^1, then $\mathrm{F}|_{\mathrm{E_K}\cap\Omega}$ is an S^1-invariant gradient map of $\bar{\Omega}\cap\mathrm{E_K}$ into E_k and thus its index is also defined. Here

$$\mathrm{E_K} = \{x\in\mathrm{E}:\mathrm{T}_g x = x \quad \text{for all } g \quad \text{in K}\}.$$

Thus we have a whole series of indices which are sometimes convenient.

There is an alternative way of defining our degree. If $S^1(z)$ is an isolated orbit of zeros of $\tilde{\mathrm{F}}=\mathrm{I}-\mathrm{F}$, we choose a transverse manifold T as before. One can then use [*21*, § 3] to prove that z is an isolated solution of $\nabla_{\mathrm{T}}\tilde{f}=0$ on T, where $\tilde{\mathrm{F}}=\nabla\tilde{f}$ and ∇_{T} denotes the gradient on the manifold T. Then

$\text{index}_n^{S^1}(I - F, S^1(z))$ is defined to be $(\# G_z)^{-1}\,\text{index}_T(\nabla_T \tilde{f}, z)$. This method seems slightly more tedious to justify but is more convenient for more general group actions.

§ 2. BIFURCATION THEOREMS

In this section, we first prove a global bifurcation theorem on a finite dimensional space. This theorem, or variants of it, will be applied in § 4-6.

We first *assume* as before that E *is a finite-dimensional linear space with a continuous linear representation* $\{T_g\}_{g\in G}$ *on* E. *Assume that* $A : E \times R \to E$ *is a continuous* S^1*-invariant map such that* $A(\,,\lambda)$ *is a gradient map on* E *for each* λ *in* R *with respect to the same scalar product* $\langle , \rangle$. As in § 1, we can assume without loss of generality that each T_g is unitary. Moreover, we *assume that* $A(0, \lambda) = 0$ *for* $\lambda \in R$ *and* $A(x, \lambda) = \lambda Bx + K(x, \lambda)$, *where* B *is linear and* $\|x\|^{-1}K(x, \lambda) \to 0$ as $\|x\| \to 0$ *locally uniformly in* λ. We will return to the case where B depends more generally on λ later.

If μ is a characteristic value of B, let $N_\mu = N(I - \mu B)$. If $\mu = \lambda_i$, we write N_i for N_{λ_i}. Now, it is easy to see that B is S^1-invariant and thus N_i is S^1-invariant. Hence $\{T_g|_{N_i}\}_{g\in G}$ is a representation of S^1 on N_i. Hence we can write $N_i = \oplus \sum_{j=1}^{p} V_j$ where V_j is S^1-invariant and V_j is *real* irreducible (that is, irreducible as a real representation). Now, it is well known that each V_j is 1 or 2-dimensional and, if $\dim V_j = 1$, each T_g acts trivially on V_j. For example, these follow easily since the commutativity of S^1 ensures that each *complex* irreducible representation is 1-dimensional and since S^1 is connected. *Now assume that* $N_i \cap E_{S^1} = \{0\}$. Since each T_g is unitary, it follows easily that $E_{S^1} \subseteq N_i^\perp$. Since $N_i \cap E_{S^1} = \{0\}$, each V_j is 2-dimensional. Let G_j denote the isotropy group for non-zero elements of V_j. (It is independent of the choice of element since S^1 acts transitively on the unit sphere in V_j.) Define

$$n(\lambda_i) = \text{sgn}\,[\lambda_i \det((I - \lambda_i B)|_{E_{S^1}}] \sum_{j=1}^{p} (\# G_j)^{-1}.$$

We could replace E_{S^1} by $N_i^\perp$ since

$$\text{sgn}\det((I - \lambda_i B)|_{N_i^\perp}) = \text{sgn}\det((I - \lambda_i B)|_{E_{S^1}}).$$

One way to see this is to note that an invariant self-adjoint operator on a 2-dimensional V_j must have positive determinant and to use the eigenspace decomposition for B. The first formula for $n(\lambda_i)$ is more convenient in applications while the second is more convenient in proofs. The first formula is useful because it shows that $n(\lambda_i)$ tends to have fixed sign. It can only have different sign at two characteristic values λ_i, λ_j if there is another characteristic value of B between λ_i, λ_j which has eigenvectors in E_{S^1}.

Let $\mathcal{D}$ denote the closure in $E \times R$ of $\{ (x, \lambda) \in E \times R : x = A(x, \lambda), x \neq 0 \}$. As is well-known, $(0, v) \in \mathcal{D}$ implies that v is a characteristic value of B.

THEOREM 2. — *Assume that the above assumptions hold and that λ_i is a characteristic value of* B *such that* $N_i \cap E_{S^1} = \{ 0 \}$. (a) *Then* $(0, \lambda_i) \in \mathcal{D}$ *and the component* C_i *of* $\mathcal{D}$ *containing* $(0, \lambda_i)$ *is* (*i*) *unbounded or* (*ii*) *contains* $(0, \lambda_j)$ *where* $\lambda_j \neq \lambda_i$ *or* (*iii*) $(a, \mu) \in \mathcal{D}$ *where* $a \in E_{S^1} \backslash \{ 0 \}$. (b) *Moreover, if* (*i*) *and* (*iii*) *do not hold, let* $C_i \cap (\{ 0 \} \times R) = \{ (0, \alpha_j) \}_{j=1}^{t}$. *If* $N_{\alpha_j} \cap E_{S^1} = \{ 0 \}$ *for* $1 \leq j \leq t$, *then* $\sum_{j=1}^{t} n(\alpha_j) = 0$.

Remarks. — Note that the last part gives considerably more information than the first part if C_i is bounded and does not intersect $(E_{S^1} \backslash \{ 0 \}) \times R$. For example, (*b*) cannot occur if $N_{\alpha_j} \cap E_{S^1} = \{ 0 \}$ for each α_j and if each $n(\alpha_j)$ has the same sign. As a second example, assume that each N_{α_j} is 2-dimensional and does not intersect E_{S^1}. Let G_j denote the isotropy group of non-zero elements of N_{α_j}. Then Theorem 2(*b*) implies that $\sum_{j=1}^{t} \varepsilon_j (\# G_j)^{-1} = 0$ where $\varepsilon_j = \pm 1$. This is quite a strong restriction, especially if $\# G_j$'s are all distinct.

Before proving Theorem 2, we need the following lemma.

LEMMA 2. — *Assume that the conditions of Theorem 2 hold and that* D *is a bounded open invariant neighbourhood of zero such that* $x \neq A(x, \lambda_i)$ *if* $x \in \partial D$ *or if* $x \in (D \cap E_{S^1}) \backslash \{ 0 \}$. *Choose* $\mu > 0$ *such that* $x \neq A(x, \lambda)$ *if* $| \lambda - \lambda_i | \leq \mu$ *and* $x \in \partial D$ *or* $x \in (D \cap E_{S^1}) \backslash \{ 0 \}$ *and then choose* $\delta > 0$ *such that* $x \neq A(x, \lambda)$ if $0 < \| x \| \leq \delta$ *and* $\lambda = \lambda_i \pm \mu$. *Then*

$$\deg_n^{S^1} (I - A(\,, \lambda_i + \mu), D \backslash \overline{E}_\delta) - \deg_n^{S^1} (I - A(\,, \lambda_i - \mu), D \backslash \overline{E}_\delta) = n(\lambda_i).$$

(*Here* E_δ *is the open ball in* E *with centre* O *and radius* δ.)

Remarks 1. — If μ is sufficiently small, $x \neq A(x, \lambda)$ for $| \lambda - \lambda_i | \leq \mu$ and $x \in \partial D$ or $x \in (D \cap E_{S^1}) \backslash \{ 0 \}$. This is obvious except to exclude the possibility that x is small, $x \in E_{S^1}$ and $x = A(x, \lambda)$. To see that this cannot happen, note that, by the Liapounov-Schmidt reduction, such an x must be of the form $u +$ h.o.t., where $u \in N(I - \lambda_i B)$. Thus, since $N(I - \lambda_i B) \cap E_{S^1} = \{ 0 \}$, x cannot be in E_{S^1}. Note also that we must use $D \backslash \overline{E}_\delta$ rather than D because the index is not defined on D.

2. — Thus we can think of $n(\lambda_i)$ as the change in the index as we cross λ_i. This is the best way to think of $n(\lambda_i)$.

Proof. — The proof is in two parts. We first show that the left hand side of our equality is independent of the nonlinear terms K and on D. We can calculate the expression by choosing a suitable D and K and then doing a bifurcation analysis.

We first show that the expression is independent of D and K. Note that it is independent of δ by property (*ii*) of our degree. Moreover, by homotopy invariance, it is independence of μ. (Note that we can choose δ uniformly in μ for μ small but bounded away from zero.) We now show the expression is independent of D. Suppose that $0 \in D_1 \subset D$, where D_1 is open and invariant, such that $x \neq A(x, \lambda)$ if $x \in \partial D_1$ and $\lambda = \lambda_i$. As before this still holds if $| \lambda - \lambda_i | \leq \mu$, where μ is small. By homotopy invariance

$$\deg_n^{S^1}(I - A(\,, \lambda_i + \mu), D \backslash \overline{D}_1) = \deg_n^{S^1}(I - A(\,, \lambda_i - \mu), D \backslash \overline{D}_1). \quad (5)$$

By the additivity of our index, it follows that our expression is the same for D and D_1, as required. Obviously, our expression is unchanged if we change the nonlinear term K in such a way that it is still the same for $\lambda = \lambda_i \pm \mu$. We choose a smooth invariant function r such that $r = 0$ except near $(0, \lambda_i)$ and $r = 1$ in a smaller neighbourhood of $(0, \lambda_i)$. We then take $\tilde{K}(x, \lambda)$ to be $\nabla(r(x, \lambda)k_0(x) + (1 - r(x, \lambda))k_1(x, \lambda))$, where ∇ denotes gradient (with respect to x), $K(x, \lambda) = \nabla k_1(x, \lambda)$ and $k_0(x)$ will be chosen later such that $k_0(0) = k_0'(0) = k_0^2(0) = 0$ and zero is an isolated solution of $x = \lambda_i Bx + \nabla k_0(x)$. Since $\tilde{K}(x, \lambda) = K(x, \lambda)$ if $\lambda = \lambda_i \pm \mu$, our expression is unchanged. On the other hand, near $(0, \lambda_0)$, $\tilde{K}(x, \lambda) = \nabla k_0(x)$ and thus zero is an isolated solution of $x = \lambda_i Bx + \tilde{K}(x, \lambda_i)$. Thus we can calculate our expression by choosing $D = E_\varepsilon$ where ε is small. Since we can also choose δ small, it follows that it suffices to prove our lemma for $K(x, \lambda) = \nabla k_0(x)$.

We now find a suitable k_0 and then use a bifurcation argument to complete the proof. For simplicity, assume $\lambda_i > 0$. Suppose $N_i = \oplus \sum_{j=1}^{p} V_j$, where V_j is real irreducible. Define $K_0(x) = \left(\sum_{j=1}^{p} a_i \| P_j x \|^2 \right)^2 \equiv (L(x))^2$, where P_j is the orthogonal projection onto V_j, $a_i > 0$ and the a_i are distinct. If $x = \lambda Bx + \nabla k_0(x)$, if x is small and λ is near λ_i, it follows easily from our choice of k_0 that $x \in N_i$. Moreover, by a simple calculation, at most one $P_j x$ is non-zero, $\lambda < \lambda_i$ and $4 \| P_j x \|^2 a_j^2 = 1 - \lambda \lambda_i^{-1}$. (Thus we have p orbits of solutions.) Another simple calculation using Theorem 1 (*iv*) shows that the orbit of solutions with $P_j x \neq 0$ has index

$$-(\# G_j)^{-1} \operatorname{sgn} \det ((I - \lambda B)|_{N_i^\perp}).$$

Note that the non-trivial eigenvalue of the linearization which corresponds to an eigenvector in V_j is negative. Since $\det((I - \lambda B)|_{N_i^\perp}) = \det((I - \lambda_i B)|_{N_i^\perp})$ if λ is near λ_i, the result now follows easily by the additivity of the index and our comment after the definition of $n(\lambda_i)$.

Proof of Theorem 2. — The proof is a straightforward modification of Rabinowitz's bifurcation theorem [33] except that we use our degree rather than the Leray-Schauder degree and we use Lemma 2 here. We sketch it rather briefly. It is convenient in the proof to replace $\mathscr{D}$ by $\mathscr{D} \cup \{(0, \lambda_i)\}$. (If $(0, \lambda_i) \notin \mathscr{D}$, set $C = \{(0, \lambda_i)\}$. Of course, we eventually prove this cannot happen.) Suppose that $C \subseteq E_n \times [-n, n]$, that $C \cap (E_{S^1} \times R) \subseteq \{0\} \times R$, and that $(0, \alpha) \in C$ (with $\alpha \neq 0$) implies that $N(I - \alpha B) \cap E_{S^1} = \{0\}$. We easily see (cp. [15]) that $\mathscr{D} \cap (\overline{E}_n \times [-n, n]) = C_1 \cup C_2$, where C_1 and C_2 are closed and disjoint, $C \subseteq C_1$ and $(x, \lambda) \in C_2$ if $(x, \lambda) \in \mathscr{D}$ and (*a*) $\| x \| = n$ or $|\lambda| = n$ or (*b*) $x \in E_{S^1} \backslash \{0\}$ or (*c*) $(0, \alpha) \in \mathscr{D} \backslash C$. Since any connected subset of $\mathscr{D} \cap (\overline{E}_n \times [-n, n])$ must be wholly contained in C_1 or C_2 and since orbits are connected, it follows that C_1 and C_2 are invariant. We can now find an invariant open subset θ of $E_n \times (-n, n)$ such that $C_1 \subseteq \theta$, $C_2 \cap \overline{\theta} = \emptyset$ and $(0, \alpha) \in \overline{\theta}$ implies that α is near one of the α_j's, where α_j was defined in the statement of Theorem 2. If λ is near n or $-n$, $\deg_n^{S^1}(I - A(\ ,\lambda), \theta_\lambda) = 0$ since there are no solutions of the equation in θ_λ. Here $\theta_\lambda = \{x \in E : (x, \lambda) \in \theta\}$. Moreover, if μ is small, Lemma 2 implies that

$$\deg_n^{S^1}(I - A(\ ,\alpha_j + \mu), \theta_{\alpha_j} \backslash \overline{E}_\delta) - \deg_n^{S^1}(I - A(\ ,\alpha_j - \mu), \theta_{\alpha_j} \backslash \overline{E}_\delta) = n(\alpha_j).$$

The result now follows easily.

The above ideas can be used in greater generality. In particular, we can replace λB by $B(\lambda)$. Assume B depends real analytically upon λ and there is at least one λ for which $I - B(\lambda)$ is invertible. (More generally, we could assume that $I - B(\lambda)$ is C^1 and is invertible except at isolated points.) As before, let λ_i denote the characteristic values of $B(\lambda)$. When we say λ_i is a characteristic value of $B(\lambda)$, we now mean that $I - B(\lambda_i)$ is not invertible. Assume that $Q_i B'(\lambda_i)|_{N_i}$ is invertible on N_i for each i where $N_i = N(I - B(\lambda_i))$ and Q_i is the orthogonal projection onto N_i. Otherwise our assumptions are the same as for Theorem 2.

In this case, define $n(\lambda_i) = \operatorname{sgn}\det((I - B(\lambda_i))|_{E_{S^1}}) q(\lambda_i)$, where $q(\lambda_i)$ is defined below. We write $N_i = M_1 \oplus M_2$, where $M_1(M_2)$ is the subspace on which $Q_i B'(\lambda_i) Q_i$ is positive (negative). If $\{G_i\}_{i=1}^m$ are the isotropy groups for the real irreducible sub-representations of M_1 and if $\{G_i\}_{i=m+1}^n$ are the isotropy groups for the real irreducible sub-representations of M_2, let

$$q(\lambda_i) = \sum_{j=1}^{m} (\# G_j)^{-1} - \sum_{j=m+1}^{n} (\# G_j)^{-1}.$$

THEOREM 2'. — *Assume that the above assumptions hold where*

$$A(x, \lambda) = B(\lambda)x + F(x, \lambda).$$

In particular, assume that $Q_i B'(\lambda_i) Q_i$ *is invertible on* N_i *for each i. Suppose that* $N_j \cap E_{S^1} = \{0\}$. *Then* $(0, \lambda_j) \in \mathscr{D}$ *and the component* C_j *of* $\mathscr{D}$ *containing* $(0, \lambda_j)$ *is (i) unbounded or (ii) contains* $(0, \lambda_s)$ *where* $\lambda_s \neq \lambda_j$ *or (iii)* $(a, \mu) \in \mathscr{D}$ *where* $a \in E_{S^1} \setminus \{0\}$. *Moreover, if (i) and (iii) do not hold, let*

$$C_j \cap (\{0\} \times R) = \{(0, \alpha_k)\}_{k=1}^{t}.$$

If $N_{\alpha_k} \cap E_{S^1} = \{0\}$ *for* $1 \leq k \leq t$, *then* $\sum_{k=1}^{t} n(\alpha_k) = 0.$

The proof of this is very similar to the proof of Theorem 2. The only changes occur in the proof of Lemma 2. Before proving this variant of Lemma 2, we make one very useful comment. Suppose one can find a family of invariant self-adjoint linear operators $B_t(\lambda)$ such that $B_0(\lambda) = B(\lambda)$, $B_t(\lambda_i) = B_0(\lambda_i)$ for all t and $I - B_t(\lambda)$ is invertible if $\lambda \in [\lambda_i - \mu, \lambda_i + \mu] - \{\lambda_i\}$ for all t. Then the degree difference as in Lemma 2 is independent of t. This follows easily from homotopy invariance. (Much more general versions could be proved.) In our case, it is easy to construct such a deformation of $B(\lambda)$ to $B(\lambda_i) + (\lambda - \lambda_i) Q_i B'(\lambda_i) Q_i$, where Q_i is the orthogonal projection onto N_i. (We first deform $B(\lambda)$ to $B(\lambda_i) + (\lambda - \lambda_i) B'(\lambda_i)$ and then use the obvious deformation to $B(\lambda_i) + (\lambda - \lambda_i) Q_i B'(\lambda_i) Q_i$. Rather similar arguments appear in Magnus [*30*].) Thus it suffices to prove the result for this linear term. If we note that we can choose our decomposition of N_i into real irreducible representations such that $Q_i B'(\lambda_i) Q_i$ is a diagonal operator on each V_j, the proof of the formula is essentially the same as the last part of the proof of Lemma 2. We choose k_0 similarly. One difference is that at λ_i, there is in general bifurcation in both directions.

Remarks 1. — If the invertibility assumption on $Q_i B'(\lambda_i) Q_i$ fails but there are only isolated eigenvalues, Theorem 2' still holds if we define $n(\lambda_i)$ differently. We *define* $n(\lambda_i)$ to be the change in index as we cross λ_i as in the statement of Lemma 2, provided that zero is an isolated solution of $x = A(x, \lambda_i)$ and $N_i \cap E_{S^1} = \{0\}$. We choose D a ball of small radius.
If zero is a non-isolated solution, we perturb the higher order terms (in x) so that the isolatedness assumption holds. The proof of the first part of Lemma 2 shows that $n(\lambda_i)$ is independent of the higher order terms. We choose a family of invariant self-adjoint linear operators $\tilde{B}(\lambda)$ close to $B(\lambda)$ depending smoothly on λ. In general, $I - \tilde{B}(\lambda)$ will fail to be invertible at a finite number of points $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ near λ_i. It is not difficult to show that we can choose $\tilde{B}(\lambda)$ such that $\tilde{Q}_j \tilde{B}'(\tilde{\alpha}_j) \tilde{Q}_j$ is invertible for each j, where Q_j

is the orthogonal projection onto $N(I - \tilde{B}(\tilde{\alpha}_j))$. Now what we are trying to calculate is the change in our index (on $D \backslash E_\delta$) from $\lambda_i - \mu$ to $\lambda_i + \mu$. This will simply be the sum of the changes at each $\tilde{\alpha}_j$. Thus $n(\lambda_i) = \Sigma \tilde{n}(\tilde{\alpha}_j)$, where $\tilde{n}(\alpha_j)$ is the index change at $\tilde{\alpha}_j$ for $\tilde{B}(\lambda)$. Since we can use the formula in the paragraph before Theorem 2′ to evaluate $\tilde{n}(\tilde{\alpha}_j)$, we obtain $n(\lambda_i)$. Of course, the problem is in doing the calculations. However, we will do this in detail in one example in § 5. Lastly, our methods can sometimes be used even if $N_i \cap E_{S^1} \neq \{ 0 \}$, especially if no solutions bifurcate in E_{S^1} at λ_i. We discuss this briefly for Hamilton systems in § 5.

2. — Theorems 2 and 2′ admit many variants. We could permit A to be defined on only part of $E \times R$ with the statement modified accordingly. Moreover, the results can sometimes be improved. Assume that A is C^1, that $\{ (x, \lambda) \in E_{S^1} \times R : x = A(x, \lambda) \}$ consists of smooth curves $(x_i(\lambda), \lambda)$, and that non-symmetric solutions (that is, solutions not in E_{S^1}) can only bifurcate from these curves at isolated points (z, α). Note that non-symmetric solutions can only bifurcate at (z, α) if $N(I - A'_1(z, \mu)) \nsubseteq E_{S^1}$. Under suitable hypotheses, one can prove a variant of Lemma 2 on the change of our index, $n((z, \alpha))$, as we cross (z, α) on the curve $(x_i(\lambda), \lambda)$. We could then prove a strengthened version of Theorem 2′ for $\tilde{D}$, where $\tilde{D}$ is the closure of

$$\{ (x, \lambda) \in E \times R : x \notin E_{S^1}, x = A(x, \lambda) \} .$$

Here solutions $(x, \lambda) \in E_{S^1} \times R$ are serving as the « trivial » solutions. Of course difficulty is in understanding $D \cap (E_{S^1} \times R)$ and in calculating the $n((z, \alpha))$'s. However, we meet an example in § 5.

§ 3. THE DEGREE ON HILBERT SPACES

In this section, we generalize the results of § 1 and § 2 to Hilbert spaces. We prove that the degree of § 2 can be extended to completely continuous mappings on a Hilbert space. Indeed, Theorem 1 (*i*)-(*iii*) remain valid as stated if we assume that our maps are completely continuous. We assume that H is separable though this could be avoided. *Assume that* $\{ T_g \}_{g \in S^1}$ *is a continuous linear representation of* S^1 *on* H. (This is defined as in the finite-dimensional case.) As before, we can use integration with respect to an invariant Haar measure to ensure that each T_g is unitary, that is, $\langle T_g x, T_g y \rangle = \langle x, y \rangle$ for $g \in S^1$, x, $y \in H$. We have to be a little careful because the usual Schauder projections do not preserve gradient structure. Finally, assume that there exist finite-dimensional invariant orthogonal projections P_n such that $P_n \to I$ strongly as $n \to \infty$, that is $P_n x \to x$ as $n \to \infty$ for each x in H. Later in the section we will prove that this assumption is redundant.

Assume now that W *is a bounded open invariant subset of* H *and* $A : \overline{W} \to H$ *is a completely continuous invariant gradient mapping such that* $x \neq A(x)$ *on* $\partial W \cup (W \cap H_{S^1})$. Essentially, we want to define $\deg_n^{S^1}(I - A, W)$ to be $\deg_n^{S^1}(I - P_nAP_n, R(P_n) \cap W)$ for large n but we have to be a little more careful. Note that $Z = \{ x \in W : x = A(x) \}$ is a compact subset of W. Hence we easily see that $P_n x \to x$ as $n \to \infty$ uniformly for $x \in Z$. Thus we can choose an invariant neighbourhood Z_1 of Z and an $\varepsilon > 0$ such that $P_n Z_1 + \overline{B_{2\varepsilon}(0)} \subseteq D$ for all large n. For the moment it is convenient to change our notation slightly and write $\deg_n$ for our normal degree on the subspace $R(P_n)$. If A is the gradient of f, it is easy to see that P_nAP_n is the gradient of fP_n. Note that, if $m > n$ and both are large,

$$\deg_n(I_n - P_nAP_n, P_nZ_1) = \deg_m(I_m - P_nAP_n, P_nZ_1 \times B_{\varepsilon,n,m}). \tag{6}$$

Here I_n denotes the identity map on $R(P_n)$ and $B_{\varepsilon,n,m}$ denotes the open ball of radius ε in the orthogonal complement to $R(P_n)$ in $R(P_m)$. This follows from the product theorem of § 1. We have not shown that $x \neq P_nAP_nx$ if $x \in \partial(P_nZ_1)$ or if $x \in P_nZ_1 \cap H_{S^1}$. We will prove this in a moment in greater generality. We prove that

$$t_m(x_m - P_nAP_nx_m) + (1 - t_m)(x_m - P_mAP_mx_m) \neq 0 \tag{7}$$

if $m > n$, $0 \leq t_m \leq 1$, m and n are both large and $x_m \in \partial(P_mZ_1)$. To prove this, suppose not. We first note that $\{ x_m \}_{n=1}^{\infty}$ lies in a compact subset of $\overline{W}$ since A is compact. Thus, by choosing a subsequence if necessary, we may assume that $x_m \to y \in \overline{W}$ as $m \to \infty$. As m and n tend to ∞, $P_mx_m \to y$, $P_nx_m \to y$ and $AP_nx_m \to Ay$. Thus we eventually find that $y = Ay$. Hence $y \in Z$. Now it is easy to show that $x_m \to y \in Z$ as $m \to \infty$ implies that $x_m \notin \partial(P_mZ_1)$ for large m. Remember that Z is a compact subset of int Z_1. Hence we have a contradiction. Thus we have proved (7). We could use a similar but easier argument to show that (7) holds if $x_m \in P_mZ_1 \cap H_{S^1}$. Thus we find by homotopy invariance that

$$\deg_m(I_m - P_nAP_n, P_mZ_1) = \deg_m(I - P_mAP_m, P_mZ_1) \tag{8}$$

if m and n are both large. Moreover, by a similar argument, any fixed point of P_nAP_n in $P_mZ_1 \cup (P_nZ_1 \times B_{\varepsilon,n,m})$ lies in $P_mZ_1 \cap (P_nZ_1 \times B_{\varepsilon,n,m})$. Thus, we can replace P_mZ_1 by $P_nZ_1 \times B_{\varepsilon,n,m}$ in the left hand side of (8). Hence (6) and (8) imply that

$$\deg_n(I - P_nAP_n, P_nZ_1) = \deg_m(I - P_mAP_m, P_mZ_1)$$

if m and n are both large. Thus we define $\deg_n^{S^1}(I - A, W) = \deg_n(I - P_nAP_n, P_nZ_1)$ for n large. It is easy to check that our definition is independent of the choice of Z_1.

Analogues of parts (i)-(iii) of Theorem 1 and Proposition 2 (i) can be proved by similar arguments. This establishes the basic properties of our degree in infinite dimensions. However, we have only been able to prove a version of Theorem 1 (*iv*) when $\dim N(I - A'(z))$ is 1 under the assumption that the map $g \to T_g z$ is C^2. (This is a restriction on z, which is discussed in [*16*]. Indeed the proof depends upon some of the results in [*16*], as well as finite-dimensional approximations.)

It remains to construct the P_n's. If P_n is orthogonal, it is easy to see that P_n is invariant if and only if $R(P_n)$ is invariant. Now (cp. [*16*]) there is a dense subset $\mathscr{A}$ of H such that $\{ T_g x : g \in S^1 \}$ is contained in a finite-dimensional subspace of H for each $x \in \mathscr{A}$. It is easy to see that $\mathscr{A}$ is a subspace. Choose a countable subset $\{ x_i \}_{i=1}^{\infty}$ of $\mathscr{A}$ which is dense in H. Define an orthogonal projection P_n by $R(P_n) = \operatorname{span} \{ T_g x_i : g \in S^1, 1 \le i \le n \}$. Note that $R(P_n) \subseteq \mathscr{A}$, $R(P_n)$ is finite-dimensional and $R(P_n)$ is invariant. Thus, by our earlier comments, P_n is a finite-dimensional invariant orthogonal projection. Since $\bigcup_{n=1}^{\infty} R(P_n)$ is dense in H, and since $P_n x_i \to x_i$ as $n \to \infty$ for each i it follows easily that $P_n \to I$ strongly.

Moreover, *natural analogues of Lemma 2 and Theorem 2 can be proved for* A *completely continuous. One change is that* $n(\lambda_i)$ *is now defined to be*

$$\operatorname{sgn} [\lambda_i \operatorname{index}_{H_{S^1}} ((I - \lambda_i B) |_{H_{S^1}}, 0)] \sum_{j=1}^{p} (\# G_j)^{-1},$$ *with other formulae changed in a corresponding way.* The proof of Lemma 2 needs to be changed, because of compactness difficulties. We first approximate K by $P_n K(P_n x, \lambda)$, where n is large and P_n is as before except that we construct P_n from the eigenvectors of B. (Thus B and P_n commute.) We can then calculate our indices by using our analogue of Proposition 2 (*i*) in infinite dimensions to reduce to finite dimensions. (Alternatively, we could approximate $P_n K(P_n x, \lambda)$ so that zero is an isolated solution for $\lambda = \lambda_i$.) The remainder of the proof of Theorem 2 is unchanged. The infinite-dimensional version of Theorem 2 can also be proved by finite-dimensional approximations.

Finally, note that we do not really need a Hilbert space but only a Banach space E with a continuous scalar product $\langle \, , \rangle$ such that A is a gradient with respect to $\langle \, , \rangle$. Here a scalar product is a continuous real valued symmetric bilinear map $\langle \, , \rangle$ such that $\langle x, x \rangle > 0$ if $x \neq 0$. However, it now seems necessary to *assume* the existence of invariant finite-dimensional projections P_n's such that $P_n \to I$ strongly as $n \to \infty$ and $\langle P_n x, y \rangle = \langle x, P_n y \rangle$ for all $x, y \in E$ and $n \ge 1$. This can sometimes be verified by first working on Hilbert spaces and then using interpolation theory. This generalization may be convenient because of the technical problem of keeping the nonlinear terms in the equation well behaved. Note that, in generalizing Theorem 2 to this case, we have to generalize some standard results for self-adjoint linear operators.

§ 4. ON A PROBLEM IN ELASTICITY

Here we study the bifurcation of solutions of an elastic conducting rod. More precisely, following Wolfe [*40*], we study the following problem on R^3

$$[N(v(s))(v(s))^{-1}\underline{r}'(s)]' + \lambda\underline{r}'(s) \times \underline{k} + \omega^2\tilde{P}\underline{r}(s) = 0 \qquad (9)$$
$$\underline{r}(0) = 0, \qquad \underline{r}(1) = b\underline{k}.$$

Here $v(s) = |\underline{r}'(s)|$, $\tilde{P} : R^3 \to R^3$ is the natural projection on the $x-y$ plane and N *is a* C^1 *function on* R *such that* $N'(t) > 0$ *on* R, $N(1) = 0$ *and* $N(t) \to \infty$ *as* $t \to \infty$. Note that we use N where Wolfe uses $\hat{N}$. *Moreover b is a fixed number such that* $b > 1$, $\underline{k}$ is the unit vector in the z direction and $\lambda > 0$. Rather than study this problem directly, we consider a modified problem. Choose a C^1 function $\tilde{N}$ on $[0, \infty)$ such that $\tilde{N}'(t) > 0$ on R, $\tilde{N}(0) = 0$, $\tilde{N}(t) = N(t)$ if $1 + \varepsilon \leq t \leq \varepsilon^{-1}$ and $t^{-1}\tilde{N}(t)$ is constant near 0 and ∞. We denote by Problem P the problem when in the original equations N is replaced by $\tilde{N}$. Note that any solutions $r(s)$ of Problem P such that $1 + \varepsilon \leq v(s) \leq \varepsilon^{-1}$ on $[0, 1]$ is a solution of our original problem. It is easy to see that $\hat{\underline{r}}(s) = bs\underline{k}$ is a solution (of both Problem P and the original problem) for all λ. We call these solutions the trivial solutions. Let D denote the closure in $C^1[0, 1] \times [0, \infty)$ of the non-trivial solutions of (9). (As usual, we are thinking of solutions as pairs $(\underline{r}, \lambda)$.)

The linearized problem has eigenvalues determined by the formula $\lambda^2 b(4N(b))^{-1} + \omega^2 = \pi^2 m^2 N(b) b^{-1}$ for m a positive integer such that $\pi^2 m^2 N(b) \geq \omega^2 b$. Let $\lambda_1 < \lambda_2 < \ldots$ denote the *positive* eigenvalues.

THEOREM 3. — $(0, \lambda_m) \in D$ *and the component* C *of* D *containing* $(\hat{\underline{r}}, \lambda_m)$ *is unbounded in* $C^1[0, 1] \times R$ *or* C *contains an element with* $\lambda = 0$.

Remark. — Note that this considerably improves the result in the announcement [*22*].

Proof. — The idea is to apply the results of § 2. We first prove Theorem 3 with the third possibility that there exist $(\underline{r}_n, \lambda_n) \in C$ such that $\inf_{s \in [0,1]} |\underline{r}_n'(s)| \to 1$ as $n \to \infty$. We remove this extra possibility at the end of the proof. It suffices to prove this weakened version of Theorem 3 for the modified problem, Problem P. For technical reasons, it is convenient to reduce this problem to a finite-dimensional problem.

STEP 1. — *Reduction of Problem* P *to a finite-dimensional problem.* We use the space $Z = \{ \underline{w} \in W_2^1[0, 1] : \underline{w}(0) = 0, \ \underline{w}(1) = 0 \}$. Note that our functions take values in R^3. We use monotone operator theory to do our reduction.

Define $A: R^3 \to R^3$ by $A(\underline{z}) = \tilde{N}(|\underline{z}|)|\underline{z}|^{-1}\underline{z}$. We prove that there exist K, $k > 0$ such that

$$K|\underline{z}_1 - \underline{z}_2|^2 \geq \langle A(\underline{z}_1) - A(\underline{z}_2), \underline{z}_1 - \underline{z}_2 \rangle \geq k|\underline{z}_1 - \underline{z}_2|^2, \tag{10}$$

where $\langle\, , \rangle$ denotes the usual scalar product on R^3. Now A is C^1. (Remember that $\tilde{N}(|\underline{z}|)|\underline{z}|^{-1}$ is constant near 0.) Thus it suffices to prove that $KI \geq A'(\underline{z}) \geq kI$ on R^3. Now $A'(\underline{z})$ is a positive multiple of the identity if $|\underline{z}|$ is small or $|\underline{z}|$ is large since $\tilde{N}(|\underline{z}|)|\underline{z}|^{-1}$ is constant there. Thus it suffices to prove that $A'(\underline{z})$ is positive definite for each $\underline{z}$. Now

$$A'(\underline{z})\underline{h} = \tilde{N}(|\underline{z}|)|\underline{z}|^{-1}\underline{h} + \tilde{N}'(|\underline{z}|)|\underline{z}|^{-2}\langle \underline{z}, \underline{h}\rangle h - \tilde{N}(|\underline{z}|)|\underline{z}|^{-3}\langle \underline{z}, \underline{h}\rangle \underline{h}.$$

It is easy to see that the two subspaces $\{\alpha\underline{z} : \alpha \in R\}$ and $\{\underline{h} : \langle \underline{h}, \underline{z}\rangle = 0\}$ are invariant under $A'(\underline{z})$. Thus it suffices to prove that $A'(\underline{z})$ is positive definite on each. Since, $\tilde{N}(|\underline{z}|)|\underline{z}|^{-1} > 0$, this is trivial for the second space. Note that the second and third terms vanish on this space. Now $A'(\underline{z})\underline{z} = \tilde{N}'(|\underline{z}|)\underline{z}$. Since $\tilde{N}'(|\underline{z}|) > 0$, it follows that $A'(\underline{z})$ is positive definite on the first space and the result follows.

Now consider the nonlinear operator B_λ on Z defined by

$$(B_\lambda(\underline{r}), \underline{w})_1 = (\tilde{N}(|\underline{\hat{r}}' + \underline{r}'|)|\underline{\hat{r}}' + \underline{r}'|^{-1}(\underline{\hat{r}} + \underline{r}'), \underline{w}') - \lambda((\underline{\hat{r}}' + \underline{r}') \times \underline{k}, \underline{w}) - \omega^2(\tilde{P}\underline{r}, \underline{w}).$$

Here $(\, ,)$ is the usual scalar product on $L^2[0,1]$ and $(u, v)_1 = (\underline{u}', \underline{v}')$ for $\underline{u}$, $\underline{v} \in Z$. By (10) and a simple estimation,

$$(B_0(\underline{r}_1) - B_0(\underline{r}_2), \underline{r}_1 - \underline{r}_2)_1 \geq k\|\underline{r}_1 - \underline{r}_2\|_{1,2}^2 - \omega^2\|\underline{r}_1 - \underline{r}_2\|_2^2 \tag{11}$$

if $\underline{r}_1, \underline{r}_2 \in Z$. Here $\|\underline{r}\|_{1,2} = \|\underline{r}'\|_2$. On the other hand, if F is defined by the formula

$$(F(\underline{r}), \underline{w})_1 = ((\underline{\hat{r}}' + \underline{r}') \times \underline{k}, \underline{w}),$$

we easily see that

$$|(F(\underline{r}_1) - F(\underline{r}_2), \underline{r}_1 - \underline{r}_2)_1| \leq C\|\underline{r}_1 - \underline{r}_2\|_{1,2}\|\underline{r}_1 - \underline{r}_2\|_2 \tag{12}$$

if $\underline{r}_1, \underline{r}_2 \in Z$. Since $B_\lambda = B_0 + \lambda F$, it follows from (11) and (12) that

$$(B_\lambda(\underline{r}_1) - B_\lambda(\underline{r}_2), \underline{r}_1 - \underline{r}_2)_1 \geq k\|\underline{r}_1 - \underline{r}_2\|_{1,2}^2 - \omega^2\|\underline{r}_1 - \underline{r}_2\|_2^2 - |\lambda|C\|\underline{r}_1 - \underline{r}_2\|_{1,2}\|\underline{r}_1 - \underline{r}_2\|_2 \dots \tag{13}$$

if $\underline{r}_1, \underline{r}_2 \in Z$. Let Z_n be the subspace of Z such that each component is a linear combination of $\sin k\pi x$ for $1 \leq k \leq n - 1$. As is well-known,

$$\|\underline{r}\|_{1,2}^2 \geq n^2\pi^2\|\underline{r}\|_2^2 \tag{14}$$

if $\underline{r} \in Z_n^\perp$. Here $Z_n^\perp$ means the orthogonal complement in $\dot{W}_2^1[0,1]$. We simply prove the result for each component and then add. Note that we obtain the same orthogonal complement whether we use $(\, ,)$ or $(\, ,)_1$ as the scalar product. By (13), (14) and a simple calculation we find that, if $K > 0$, then there exist $k, n > 0$ such that

$$(B_\lambda(u + v_1) - B_\lambda(u + v_2), v_1 - v_2)_1 \geq k_1 \| v_1 - v_2 \|_{1,2}^2$$

if $v_1, v_2 \in Z_n^\perp$, $u \in Z_n$ and $|\lambda| \leq K$. A similar but easier argument shows the map $w \to B_\lambda(w)$ is Lipschitz continuous. It follows from a well-known result in monotone operator theory (cp. Brezis [9, Theorem 2.3]) that, for each $u \in Z_n$, the equation $PB_\lambda(u + v) = 0$ has a unique solution $\tilde{r}(u, \lambda)$ in $Z_n^\perp$. Here P is the orthogonal projection onto $Z_n^\perp$. Since the map $(w, \lambda) \to B_\lambda(w)$ is Lipschitz continuous on bounded sets, it follows easily that the map $(u, \lambda) \to \tilde{r}(u, \lambda)$ is locally Lipschitz continuous, as a map of $Z_n \times [-K, K]$ to Z.

We now prove that $\tilde{r}(u, \lambda)$ is a C^1 function of s. Now $B_\lambda(\tilde{r}(u, \lambda) + u) \in Z_n$ and hence is smooth. It follows easily from the definition of B_λ that $w = \tilde{r}(u, \lambda) + u$ has the property that $\tilde{N}(|w'(s)|)|w'(s)|^{-1}w'(s)$ is continuous. Hence $\tilde{N}(|w'(s)|)$ is continuous and thus, since $\tilde{N}$ is strictly monotone, $|w'(s)|$ is continuous. It follows that w' is continuous. Hence $\tilde{r}(u, \lambda) + u$ is C^1 as required. It is easy to then deduce from the equation that $\tilde{r}(u, \lambda) + u$ is C^2.

We next prove that $\tilde{r}(u, \lambda) + u$ is small in the C^1 norm if u is small in $W_2^1[0, 1]$ and λ is bounded. As before, let $w = \tilde{r}(u, \lambda) + u$. By what we have already proved, w is small in $W_2^1[0, 1]$. (Note that $r(0, \lambda) = 0$ for all λ.) Since B_λ is continuous as a map of Z into Z, it follows that $B_\lambda(w)$ is small in Z. Now $B_\lambda(w) \in Z_n$, by the definition of $\tilde{r}(u, \lambda)$. Since all norms on a finite-dimensional space are equivalent, it follows that $B_\lambda(w)$ is small in any norm. Since w is small in Z, it follows easily from the equation that $\frac{d}{ds}(\tilde{N}(|w' + b\underline{k}|)|w' + b\underline{k}|^{-1}(w' + b\underline{k}))$ is small in $L^2[0, 1]$. Hence $\tilde{N}(|Y'|)|Y'|^{-1}Y'$ is close to a constant on all of $[0, 1]$, where $Y(s) = w(s) + bs\underline{k}$. Hence $\tilde{N}(|Y'|)$ is close to a constant on $[0, 1]$. It follows from our monotonicity assumptions on $\tilde{N}$ that $|Y'|$ is close to a constant on all of $[0, 1]$. Since Y' is near $b\underline{k}$ in $L^2[0, 1]$, the constant must be near b. Since $\tilde{N}(|Y'|)|Y'|^{-1}Y'$ is close to a constant, it follows that Y' is close to a constant on all of $[0, 1]$. Thus $Y'(s)$ is near $b\underline{k}$ on $[0, 1]$. This proves the result.

Now Problem P reduces to the equation $(I-P)B_\lambda(\tilde{r}(u, \lambda) + u) = 0$ (for $|\lambda| \leq K$). Now $u = 0$ is a solution for all λ. Moreover

$$\| B_\lambda(\tilde{r}(u, \lambda) + u) - L_\lambda(u + \tilde{r}(u, \lambda)) \|_{1,2} = o(\| u \|_{1,2}) \tag{15}$$

as $\| u \| \to 0$ locally uniformly in λ, where L_λ is defined by

$$(L_\lambda \underline{r}, \underline{w})_1 = (b^{-1}\tilde{N}(b)\underline{r}', \underline{w}') - \lambda((\underline{r}' \times \underline{k}), \underline{w}) + \omega^2(\tilde{P}\underline{r}, \underline{w}).$$

(15) follows by a simple estimation once one recalls that $\tilde{r}(u, \lambda) + u$ is

small in $C^1[0,1]$ and $\| \tilde{r}(u,\lambda) + u \|_{1,2} \leq K \| u \|_{1,2}$. The first of these results is proved in the previous paragraph while the second holds because $\tilde{r}(0,\lambda)=0$ and $\tilde{r}$ is locally Lipschitz in u as a map of Z_n into $W_2^1[0,1]$.

We need to prove that $\| r_1(u,\lambda) - \tilde{r}(u,\lambda) \|_{1,2} = o(\| u \|_{1,2})$ as $\| u \|_{1,2} \to 0$, where $r_1(u,\lambda)$ is the unique solution of $PL_\lambda(u+v)=0$ in $Z_n^\perp$. That is this equation has a unique solution follows by the same argument used to prove the existence and uniqueness of $\tilde{r}(u,\lambda)$. By (15),

$$\| PL_\lambda(u + \tilde{r}(u,\lambda)) \|_{1,2} = o(\| u \|_{1,2})$$

and hence

$$\| PL_\lambda(\tilde{r}(u,\lambda) - r_1(u,\lambda)) \|_{1,2} = o(\| u \|_{1,2}) .$$

Since PL_λ is coercive on $Z_n^\perp$, it follows that $\| r(u,\lambda) - r_1(u,\lambda) \|_{1,2} = o(\| u \|_{1,2})$, as required. Since r_1 is linear in u, it follows that $(I-P)L_\lambda(u+r_1(u,\lambda))$ is the linearization of $(I-P)B_\lambda(u+\tilde{r}(u,\lambda))$ at $u=0$.

STEP 2. — *We verify the hypotheses of Theorem 2′ for our finite-dimensional problem.* Now, our original problem has an S^1-symmetry due to rotations in the $x-y$ plane. It is easy to see that equation is invariant under this symmetry group which acts orthogonally. Now Z_n is invariant under this symmetry group. Thus P is S^1-invariant. By uniqueness, it follows that $T_g\tilde{r}(u,\lambda) = \tilde{r}(T_g u,\lambda)$, where $\{ T_g \}_{g\in S^1}$ denotes the action of our symmetry group. (Note that $\hat{\underline{r}}(s)$ is fixed by the action of the symmetry group.) Hence the equation $(I - \underline{P})B_\lambda(u + \tilde{r}(u,\lambda)) = 0$ is S^1-invariant. Next we show that we have a gradient structure. Wolfe notes that formally our equation is the gradient of a functional Φ (defined in [*40*]. This is easy to prove rigorously on Z once we check that the term $\int_0^1 \tilde{\Gamma}(|\underline{r}'(s)|)ds$ is Gâteaux differentiable, where $\tilde{\Gamma}(y) = \int_0^Y \tilde{N}(m)dm$. Note that $\tilde{N}$ is linear near 0 and ∞ and $\tilde{N}$ is C^1. It follows easily that $\tilde{\Gamma}(y) = S(y^2)$, where S is C^1 with bounded derivative on $[0,\infty)$. Then $\int_0^1 \tilde{\Gamma}(|\underline{r}'(s)|)ds = \int_0^1 S(|\underline{r}'(s)|^2)ds$. It is easy to check that this last expression is Gâteaux-differentiable (cp. Vainberg [*37*, Theorem 21.1]). Note that the map $\underline{r} \to |\underline{r}'|^2$ is a polynomial (and hence smooth) map of Z into $L^1[0,1]$. Hence $B_\lambda(\underline{r})$ is the gradient of $\Phi(\hat{\underline{r}}+\underline{r})$ on $\dot{W}_2^1[0,1]$. One can then use a standard argument (cp. Amann and Zehnder [*4*]) to check that, for fixed λ, $(I-P)B_\lambda(u + \tilde{r}(u,\lambda))$ is the gradient of $\Phi(\hat{\underline{r}} + u + \tilde{r}(u,\lambda))$ provided that we know that the map $u \to \tilde{r}(u,\lambda)$ is Fréchet differentiable as a map of Z_n into Z. This can be proved by similar arguments to the earlier proof that $\tilde{r}(\ ,\lambda)$ has a derivative at 0. (We note that the map $u \to \tilde{r}(u,\lambda)$ is Lipschitz as a map of $Z_n \to Z$, prove that the same map is continuous as a map of Z_n to $C^1[0,1]$ and use the obvious linearization of $B_\lambda(w)$ at $u + \tilde{r}(u, \lambda)$.) Thus we have checked all the basic assumptions of § 2.

We now need to evaluate $n(\alpha)$ for each eigenvalue α of L_λ. Here our notation follows § 2 and, as there, α is an eigenvalue of L_λ means that $N(L_\alpha) \neq \{0\}$. Note that, by our reduction, $N(L_\alpha) = \{ u + r_1(u, \alpha) : \tilde{L}_\alpha u = 0 \}$, where $\tilde{L}_\alpha u = (I - P)L_\alpha(u + r_1(u, \alpha))$. Now, since L_λ depends linearly on λ, it is easy to see that r_1 and L also depend smoothly on λ. Define $r_\lambda(u) = r_1(u, \lambda)$. By differentiating $PL_\lambda(I + r_\lambda) = 0$ with respect to λ, we see that

$$PL_\lambda r'_\lambda + P\tilde{B}(I + r_\lambda) = 0\,, \tag{16}$$

where $\tilde{B}$ is defined by $\langle \tilde{B}\underline{r}, \underline{w} \rangle = -(\underline{r}' \times \underline{k}, \underline{w})$, for $\underline{r}$, $\underline{w} \in Z$. Now, Theorem 2′ applies if we prove that $Q\tilde{L}'_\alpha|_{N(\tilde{L}_\alpha)}$ is positive or negative definite. Here Q is the orthogonal projection of Z_n onto $N(\tilde{L}_\alpha)$ and L'_λ means the derivative of $\tilde{L}_\lambda$ with respect to λ. By a simple calculation,

$$Q\tilde{L}'_\alpha = Q(I - P)(\tilde{B}(I + r_\alpha) + L_\alpha r'_\alpha)\,.$$

Hence, if $u \in N(\tilde{L}_\alpha)$,

$$\begin{aligned}
(Q\tilde{L}'_\alpha u, u)_1 &= (\tilde{B}(I + r_\alpha)(u) + L_\alpha r'_\alpha(u), u)_1 \\
&= (\tilde{B}(I + r_\alpha)u,\ (I + r_\alpha)u)_1 + (L_\alpha r'_\alpha(u),\ u)_1 - (B(I + r_\alpha)u, r_\alpha(u))_1 \\
&= (\tilde{B}(I + r_\alpha)u, (I + r_\alpha)u)_1 + (L_\alpha r'_\alpha(u), u + r_\alpha u)_1\,.
\end{aligned}$$

Here we have used (16) and that $r_\alpha(u) \in Z_n^\perp$. Now, since L_α is self-adjoint, the last term is

$$(r'_\alpha(u), L_\alpha(u + r_\alpha(u)))_1 = 0 \quad \text{since} \quad r'_\alpha(u) \in Z_n^\perp$$

and $PL_\alpha(u + r_\alpha(u)) = 0$ (by the definition of r_α). Since

$$N(L_\alpha) = \{ u + r_1(u, \alpha) : u \in N(\tilde{L}_\alpha) \}$$

we see that $Q\tilde{L}'_\alpha|_{N(\tilde{L}_\alpha)}$ is negative definite if and only if the form $\langle \tilde{B}v, v \rangle_1$ on $N(L_\alpha)$ is negative definite. It is proved in [*40*] that the latter form is negative definite if $\alpha > 0$. Thus $Q\tilde{L}'_\alpha|_{N(\tilde{L}_\alpha)}$ is negative definite if $\alpha > 0$. We also need to evaluate sgn det $(L_\alpha|_W)$, where $W = (Z_n)_{S^1}$. Now it is easy to see that $(Z_n)_{S^1} = \{ (0, 0, z) : z \in \dot{W}_2^1[0, 1] \}$. Moreover, it is easy to show that $r_1(u, \lambda) = 0$ if $u \in (Z_n)_{S^1}$. Hence, on $(Z_n)_{S^1}$, $(L_\alpha u, u)_1 = (b^{-1}\tilde{N}(b)(z', z') > 0$ if $z \neq 0$ (where $u = (0, 0, z)$). Thus sgn det $(\tilde{L}_\alpha|_W) = 1$. Hence by the definition of $n(\alpha)$ before Theorem 2′, $n(\alpha) > 0$ if $\alpha > 0$. Note that the calculations in [*40*] ensure that no eigenfunction of L_λ lies in Z_{S^1} and that Theorem 2′ applies to a map with linear term $I - B(\lambda)$.

STEP 3. — *Completion of the proof.* We first prove the weaker version of Theorem 3 where the third alternative is included, that is, there exist $(\underline{r}_n, \lambda_n) \in C$ such that $\inf_{s \in [0,1]} |\underline{r}'_n(s)| \to 1$ as $n \to \infty$. As we noted before, it suffices to prove this result for problem P. If the component C is bounded, then, by our reduction, it suffices to consider our finite-dimensional equation. Since the only solutions of our equation fixed by S^1 are the trivial solutions, the result follows from Theorem 2 and the results on $n(\alpha)$ above. There remains one minor point. Our connected set C is connected in $Z \times [0, \infty)$. However, by a slight generalization of an argument in Step 1 of the proof, the map $(u, \lambda) \to \tilde{r}(u, \lambda)$ is continuous as a map into $C^1[0, 1]$. It follows that C is also connected in $C^1[0, 1] \times R$. Note that a set which is unbounded in $W_2^1[0, 1]$ is also unbounded in $C^1[0, 1]$. This completes the proof of the weaker version of Theorem 3.

To complete the proof of Theorem 3, it suffices to prove that, if $0 \leq \lambda_n \leq k$ for all n, $(\underline{r}_n, \lambda_n) \in D$ and $\inf_{s\in[0,1]} | \underline{r}_n'(s) | \to 1$ as $n \to \infty$, then $\| \underline{r}_n' \|_\infty \to \infty$ as $n \to \infty$. By the third equation of (9)

$$N(| \underline{r}_n'(s) |) \, | \underline{r}_n'(s) |^{-1} z_n'(s) = C_n \tag{17}$$

on $[0, 1]$ (where $\underline{r} = (x, y, z)$). Since there exist s_n such that $| \underline{r}_n'(s_n) | \to 1$ and since $| z_n'(s) | \leq | \underline{r}_n'(s)|$, it follows that $C_n \to 0$ as $n \to \infty$. If we can find a subsequence of $\{ \| \underline{r}_n' \|_\infty \}$ which is bounded, then we can rechoose our original sequence such that $\| \underline{r}_n' \|_\infty \leq k$ for all n. Since $z_n(1) - z_n(0) = b > 1$, we see that there exist $\alpha > 1$ and $t_n \in [0, 1]$ such that $| z_n'(t_n) | \geq \alpha$. Now $\alpha \leq | \underline{r}_n'(t_n) | \leq k$ and N has a positive lower bound on $\left[\frac{1}{2} + \frac{1}{2}\alpha, k\right]$. Since $C_n \to 0$ as $n \to \infty$, this contradicts (17) for $t = t_n$. Thus $\| \underline{r}_n' \|_\infty \to \infty$ as $n \to \infty$. This completes the proof of Theorem 3.

Remarks 1. — The argument in the last paragraph of the proof of Theorem 3 has some other uses. If $(\underline{r}_n, \lambda_n) \in D$ and $\inf_{s\in[0,1]} | \underline{r}_n'(s) | \to 1$ as $n \to \infty$, then a similar argument implies that there are points on the rod where $\frac{dx}{dz}$ or $\frac{dy}{dz}$ are large. Hence the rod is showing physical peculiarities. Secondly, if $y^{-1}N(y)$ has a positive lower bound on $[1 + \varepsilon, \infty)$, a similar argument shows that such solutions cannot exist. In fact, one easily shows for the above solutions $\underline{r}_n$ that, whenever $| z_n'(s) | \geq a > 0$, then $| \underline{r}_n'(s) |$ is near 1 or is large. Moreover, both possibilities must occur and a transition between two possibilities can only occur when $z_n'(s)$ is small. If $y^{-1}N(y) \to \infty$ as $y \to \infty$, it can be shown that solutions with λ bounded are bounded in $C^1[0, 1] \times R$. However, it *probably* can be proved that this fails if $y^{-1}N(y) \to C > 0$ (where $C < \infty$) as $y \to \infty$.

2. — If N is C^2, the proof can be simplified by partially working in $C^1[0, 1]$ and using the implicit function theorem. We could allow λ to be negative and then the theorem could still be proved (with $[0, \infty)$ replaced by R) except that our branch could terminate at a negative eigenvalue of

the linearization. Note that, if $\lambda = 0$ is an eigenvalue, the change in our index across it must be zero because this eigenvalue can be eliminated by a small increase in ω. Our methods can be used in N also depends on s (but $N(s, 1) = 0$ for all s). In particular, *if ω^2 is less than the first eigenvalue of*

$$-\frac{d}{ds}(N(s, |\,\underline{\hat{r}}'(s)\,|)(|\,\underline{\hat{r}}'(s)\,|)^{-1}\underline{z}'(s)) = \lambda \underline{z}(s),$$
$$\underline{z}(0) = \underline{z}(1) = 0,$$

we obtain an infinite number of positive eigenvalues and an analogue of Theorem 3 holds. Here $\underline{\hat{r}}(s)$ denotes the corresponding trivial solution of our new problem. If this condition fails, our method becomes difficult to apply because it is unclear if $Q\tilde{L}'_\alpha|_{N(\tilde{L}_\alpha)}$ is invertible. It can be proved that there are an infinite number of positive eigenvalues, that the above invertibility condition is satisfied for the large eigenvalues and that the analogue of Theorem 3 holds for the large eigenvalues.

§ 5. PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS

In this section, we discuss the application of our ideas to the study of periodic solutions of *fixed* period of Hamiltonian systems. We consider the autonomous system

$$x'(t) = J\nabla H(x(t), \lambda) \tag{18}$$

on R^{2n}, where $H : R^{2n+1} \to R$ is C^1, ∇ denotes the gradient with respect to the first variable and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. We look for periodic solutions of fixed period T. Usually, we assume that H is C^2 in x though this could often be avoided in our final results by approximation arguments.

We first obtain a global bifurcation theorem (Theorem 4). *Assume that* $\nabla H(0, \lambda) = 0$ *for all* λ. By a standard argument, λ can only be a bifurcation point (for *non-constant* T-periodic solutions bifurcating from 0) if

$$z' = JM(\lambda)z \tag{19}$$

has a non-trivial T-periodic solution, where $M(\lambda) = D_1^2 H(0, \lambda)$. In fact, since (cp. [*19*, Remark after lemma]) solutions near 0 have the same symmetry as that of a solution of the linearized equation, (19) must have a a non-constant T-periodic solution if λ is a bifurcation point. For the moment, *assume that* $M(\lambda) = \lambda M$. (We discuss some more general problems later.) Note that this is only a restriction on the *linear* terms and that M is self-adjoint.

Suppose that αi is a purely imaginary eigenvalue of JM with $\alpha > 0$. Thus $-\alpha i$ is also an eigenvalue. Gokhberg et al. [*25*] associate with it a

« signature » $\nu(\alpha, M)$. We summarize the properties of the « signature » which are useful to use. Their proofs can be found in [25]. It is an integer. If $i\alpha$ is an eigenvalue of JM for which the geometric multiplicity equals the algebraic multiplicity, it is simply the signature of the form $-(iJx, x)$ (or equivalently of (Mx, x)) on the subspace spanned by the eigenvectors of JM corresponding to αi. In the general case, the definition is the same except that the eigenspace is replaced by the generalized eigenspace.

$\nu(\alpha, M)$ can be calculated from a special Jordan canonical form (one preserving the symplectic structure) for the eigenspace corresponding to $i\alpha$. It turns out that $|\nu(\alpha, M)| \leq \dim N(i\alpha I\text{-}JM)$ and $\nu(\alpha, M) =$ [algebraic multiplicity of αi] (mod 2). If M is perturbed slightly to M′, where M′ is self-adjoint, and if M′ has *purely imaginary* eigenvalues $i\alpha_1, \ldots, i\alpha_k$ near $i\alpha$, then

$$\nu(\alpha, M) = \sum_{r=1}^{k} \nu(\alpha_r, M'). \tag{20}$$

Note that when M is perturbed to M′, there may also be other, not purely imaginary, eigenvalues, near αi but they can be ignored. Moreover, we can always choose such an M′ such that each $i\alpha_r$ is simple and each $\nu(\alpha_r, M')$ has the same sign.

Suppose $\lambda \neq 0$ *and* M *is invertible*. As in [4], we easily see that 0 is an eigenvalue of

$$\begin{aligned} x' &= \lambda JMx \\ x(0) &= x(T) \end{aligned} \tag{21}$$

if and only if there is a purely imaginary eigenvalue $i\mu$ of JM such that $a(\mu\lambda) \in Z$, where $a(w) = \frac{1}{2} Tw\pi^{-1}$. Here Z denotes the integers.

If λ_j is such an eigenvalue, define

$$\hat{n}(\lambda_j) = \operatorname{sgn}(\lambda_j \det M) \sum (a(\mu\lambda_j))^{-1} \nu(\mu, M),$$

where the summation is over the positive numbers μ for which $i\mu$ is an eigenvalue of JM and $a(\mu\lambda_j) \in Z$. Let D denote the closure in $C[0, T] \times R$ of $\{(x(t), \lambda) : x(0) = x(T), x$ is a non (constant solution of (18)$\}$.

THEOREM 4. — *Assume that our above assumptions hold where* $\nabla H(0, \lambda) = 0$ *for all* λ, $D_1^2 H(0, \lambda) = \lambda M$, *and* M *is invertible and assume that* λ_j *is a positive eigenvalue of (21) with* $\hat{n}(\lambda_j) \neq 0$. *Then the component* C *of* D *containing* $(0, \lambda_j)$ *is (i) unbounded or (ii) contains* (a, μ) *where a is a non-zero constant and* $\nabla H(a, \mu) = 0$ *or (iii)* $\sum \hat{n}(\mu) = 0$, *where the summation is over the* $(0, \mu)$'s *in* $C \cap (\{0\} \times R)$.

Proof. — The proof of Theorem 4 consists of two steps. We first reduce our problem to a finite dimensional problem and then apply a variant of Theorem 2′. The only difficulty in the second step is to calculate $n(\alpha)$, where $n(\alpha)$ was defined in Remark 1 after Theorem 2′.

STEP 1. — *Reduction to a finite-dimensional problem.* We work in the space $\tilde{Z} = \{ x \in W_2^1 [0, T] : x(0) = x(T) \}$. Suppose we look for solutions with $| x(t) | \leq \mathscr{R}$ on $[0, T]$ and $| \lambda | \leq \mathscr{R}$. This ensures that $D_1^2 H(x(t), \lambda)$ is bounded. Now, by a standard reduction (cp. Amann and Zehnder [*4*]), our problem reduces to the solution (w, λ) of a finite-dimensional equation

$$F(w, \lambda) = 0 \tag{22}$$

on a subspace $\mathscr{C}$. Here F is C^1 and is a gradient mapping. Our assumption that $| x(t) | \leq \mathscr{R}$ on $[0, T]$ and $| \lambda | \leq \mathscr{R}$ becomes that (w, λ) lies in a compact subset W of $\mathscr{C} \times R$. $\mathscr{C}$ is the subspace spanned by all functions of the form $\cos (2\pi k t T^{-1} + \phi)\underline{C}$ where $\phi \in R$, $\underline{C} \in R^{2n}$, $k \in Z$ and $| k | \leq p$ (for suitable large p). For future reference, note that F is obtained by applying a projection method and contraction mapping to the equation $Jx' + \nabla H(x, \lambda) = 0$. Since our equation is autonomous, our problem on $\tilde{Z}$ has an S^1-symmetry (where the group acts by translations). This is preserved by the reduction and hence F is S^1-invariant. To be more precise, there is an S^1-invariant map: $Y : \mathscr{C} \times R \to \mathscr{C}^{\perp}$ such that

$$\{ (w + Y(w, \lambda), \lambda) : F(w, \lambda) = 0, (w, \lambda) \in W \}$$

gives the T-periodic solutions of (18) with $| x(t) | \leq \mathscr{R}$ on $[0, T]$ and $| \lambda | \leq \mathscr{R}$. It is easy to see that, with this correspondence, solutions of (22) with $w \in \mathscr{C}_{S^1}$ correspond to constant solutions of (18) and *vice versa*. Note that $\tilde{Z}_{S^1}$ is the set of constant functions. Since $\nabla H(0, \lambda) = 0$, $Y(0, \lambda) = 0$ and $F(0, \lambda) = 0$ for $| \lambda | \leq \mathscr{R}$. For future reference, note that it is only the last sentence where the assumption that $\nabla H(0, \lambda) = 0$ is used.

STEP 2. — *Completion of the proof.* To prove Theorem 4, it suffices to assume that C is bounded. By Step 1, it suffices to consider the finite dimensional equation $F(w, \lambda) = 0$ on $\mathscr{C} \times R$. Hence the result will follow from Theorem 2′ if we prove that $\hat{n}(\lambda_i) = n(\lambda_i)$, where $n(\lambda_i)$ was defined in Remark 1 after Theorem 2′ (for the map F on $\mathscr{C} \times R$). Thus the following lemma completes the proof of Theorem 4.

LEMMA 3. — *Under the assumptions of Theorem 4, $n(\lambda_i) = \hat{n}(\lambda_i)$, where $n(\lambda_i)$ is defined for the map* F *on* $\mathscr{C}$.

Proof. — We first prove the result under the assumption that there is a unique eigenvalue $i\mu$ of JM (where $\mu > 0$) such that $a(\mu\lambda_j) \in Z$ and that μ is simple. Let $k = a(\mu\lambda_j)$. As in [*4*], we easily see that the kernel of

$x'(t) - \lambda_j \mathrm{JM}x(t)$ on $\bar{\mathrm{Z}}$ is spanned by the real and imaginary parts of $\underline{\mathrm{C}} \exp(2\pi ikt\mathrm{T}^{-1})$, where $\underline{\mathrm{C}}$ is the eigenvector corresponding to the eigenvalue $i\mu$ of JM. Since these are contained in $\mathscr{C}$, it follows easily that these also span the kernel of $\mathrm{F}_1^1(0, \lambda_j)$ (if p is large). Now, to apply the formula for $n(\lambda_i)$ before Theorem 2′, we need to prove that $\mathrm{C} = \mathrm{Q}\mathrm{F}_{1,2}^2(0, \lambda_j)\,|_{\mathrm{N}(\mathrm{F}_1^1(0,\lambda_j))}$ is invertible. Here Q is the orthogonal projection of $\mathscr{C}$ onto $\mathrm{N}(\mathrm{F}_1^1(0, \lambda_j)) \equiv \mathrm{N}_j$. We can argue as in the proof of Step 2 of Theorem 3 and find that C is positive (negative) definite if and only if $(\lambda_j \mathrm{M}x, x)$ is positive (negative) definite on the kernel of $\mathrm{J}x' + \lambda_j \mathrm{M}x$ on $\tilde{\mathrm{Z}}$. Hence, by a simple calculation, C is positive (negative) definite if and only if $(\lambda_j \mathrm{M}\underline{\mathrm{C}}, \underline{\mathrm{C}}) > 0 (< 0)$. By our earlier remarks on the calculation of ν, it follows that C is positive definite if and only if $(\operatorname{sgn} \lambda_j)\nu(\mu, \mathrm{M}) > 0$. We also need to calculate $\operatorname{sgn} \det (\mathrm{F}_1^1(0, \lambda_j)\,|_{\mathscr{C}_{\mathrm{S}^1}})$. Now $\mathscr{C}_{\mathrm{S}^1} = \tilde{\mathrm{Z}}_{\mathrm{S}^1}$ = set of constant functions and, on this space,

$$\mathrm{F}_1^1(0, \lambda_j)x = \mathrm{J}x' + \lambda_j \mathrm{M}x = \lambda_j \mathrm{M}x .$$

Hence, since $\mathscr{C}_{\mathrm{S}^1}$ is even-dimensional, $\operatorname{sgn} \det (\mathrm{F}_1^1(0, \lambda_j)\,|_{\mathscr{C}_{\mathrm{S}^1}}) = \operatorname{sgn} \det \mathrm{M}$ Finally, it is easy to see that each non-zero element in our kernel has minimal period $k^{-1}\mathrm{T}$ and this has isotropy group Z_k. Hence the definition of $n(\lambda)$ preceeding Theorem 2′ implies the lemma in this case.

We prove the general case by an approximation argument. By Remark 1 after Theorem 2′, if we perturb the operator $\mathrm{F}_1^1(0, \lambda)$ slightly, such that λ_j splits to several eigenvalues $\tilde{\lambda}^r$, $r = 1, \ldots, s$, then

$$n(\lambda_j) = \sum_{r=1}^{s} \tilde{n}(\tilde{\lambda}^r), \tag{23}$$

where $\tilde{n}$ means that it is n for the perturbed operator. In our case, we use our earlier remarks on ν to perturb M to a self-adjoint M′ so that each purely imaginary eigenvalue of M′ is simple. Moreover, it is easy to see that we can perturb M′ a little more to ensure that the quotient of no two purely imaginary eigenvalues with positive imaginary part is an integer. (In fact, we need only avoid a *finite* number of rationals.) We now need to find the eigenvalues of $\mathrm{J}x' + \lambda \mathrm{M}'x$ on $\tilde{\mathrm{Z}}$ near λ_j. Let $\{ i\mu_r \}_{r=1}^{b}$ denote the eigenvalues of M occurring in the statement of the lemma. It is easy to see that λ is an eigenvalue of $\mathrm{J}x' + \lambda \mathrm{M}'x$ on $\tilde{\mathrm{Z}}$ near λ_j if and only if there is an eigenvalue $i\tau$ of M′ near one of the $i\mu_r$'s. The corresponding eigenvalue λ is then $\mu_r \lambda_j \tau^{-1}$, We thus see that each of the eigenvalues satisfies the assumptions of the first part of the proof. Note that $k_r \equiv a(\mu_r \lambda_j) = a(\tau_k \lambda)$ is an integer and that k_r determines the isotropy group of the corresponding kernel. Hence, by (23) and by our proof of the special case, we see that

$$n(\lambda_j) = \sum \tilde{n}(\tau_k) = \sum k_r^{-1} \operatorname{sgn} \det \mathrm{M}' \operatorname{sgn} \lambda_j \nu(\tau_k, \mathrm{M}') ,$$

where the summation is over the purely imaginary eigenvalues $i\tau_k$ of M′ near the $i\mu_r$'s and the k_r is determined by the μ_r which τ_k is close to (as above). Now if we consider the part of the sum over the eigenvalues near $i\mu_r$ for fixed r, we have

$$\begin{aligned} k_r^{-1} \operatorname{sgn} \det \mathrm{M}' \sum \nu(\tau_k, \mathrm{M}') &= k_r^{-1} \operatorname{sgn} \det \mathrm{M}'\nu(\mu_r, \mathrm{M}) \qquad \text{(by (20))} \\ &= k_r^{-1} \operatorname{sgn} \det \mathrm{M}\nu(\mu_r, \mathrm{M}) \end{aligned}$$

since M′ is near M. If we then sum over the μ_r's, the result follows.

Remarks on Theorem 4. 1. — If $\mathrm{H}(x, \lambda) = \lambda \mathrm{T}(x, \lambda)$, it is easy to see that any non-constant solutions $(x_n(t), \lambda_n)$ such that $\lambda_n \neq 0$ and $\lambda_n \to 0$ as $n \to \infty$ have the property that $\| x_n \|_\infty \to \infty$ as $n \to \infty$. We do not claim that the solutions we construct have minimal period T. Indeed there is an example in [*26*, § 3.8] which shows that it can happen that $\hat{n}(\lambda_j) \neq 0$ but no solutions of minimal period T bifurcate at $(0, \lambda_j)$. Note that, if M is positive definite or more generally if each $\hat{n}(\lambda_j)$ (for $\lambda_j > 0$) has the same sign, then possibly (*iii*) can not hold for a component C contained in $\tilde{\mathrm{Z}} \times [0, \infty)$. Sometimes we can apply our theorem in $\tilde{\mathrm{Z}}_{\mathrm{K}}$, where K is a finite subgroup of S^1, and obtain results when $\hat{n}(\lambda_j) = 0$. It follows easily that there is global bifurcation at λ_j if $\nu(\mu, \mathrm{M})$ is non-zero for one of the μ's in the statement of Lemma 3.

2. — For *simplicity*, assume that $\mathrm{H}(x, \lambda) = \lambda \mathrm{H}(x)$ in this remark. In addition, assume that $\mathrm{D}^2\mathrm{H}(c)$ is invertible whenever $\nabla \mathrm{H}(c) = 0$. Let

$$\mathscr{A} = \{ (c, \mu) \in \mathrm{R}^{2n} \times \mathrm{R} : \nabla \mathrm{H}(c) = 0,\ \ z' - \mu \mathrm{J} \nabla \mathrm{H}(c) z = 0 \text{ has a non-constant solution in } \tilde{\mathrm{Z}} \}.$$

If C is a bounded component of D containing $(0, \lambda_j)$, then $\sum \hat{n}(\mu, c) = 0$, where the summation is over the (μ, c)'s in $\mathrm{C} \cap \mathscr{A}$. Here $\hat{n}(\mu, c)$ is defined as before for bifurcation off the constant solution c at μ. (An analogue of Lemma 3 holds.) In particular, if $\mathrm{D}^2\mathrm{H}(c)$ is positive definite whenever $\nabla \mathrm{H}(c) = 0$, then C is unbounded, since all the $\hat{n}(\mu, c)$'s have the same sign. The above result improves Theorem 4 and is proved in the same way. In particular, if H is convex, the above result holds if we delete the positive definiteness assumption but assume that critical points of H are isolated. To prove this, we apply the above result to $\frac{1}{2} \varepsilon \| x \|^2 + \mathrm{H}(x)$ and then pass to the limit.

3. — Our methods can be used if $\mathrm{D}_1^2\mathrm{H}(0, \lambda)$ depends more generally upon λ. We can calculate $n(\lambda_j)$ at an eigenvalue λ_j if QAQ is invertible on the kernel $\tilde{\mathrm{N}}$ of $z' - \mathrm{J}\mathrm{D}_1^2\mathrm{H}(0, \lambda_j) z$ (on $\tilde{\mathrm{Z}}$), where Q is the orthogonal projection of $\tilde{\mathrm{Z}}$ onto $\tilde{\mathrm{N}}$ and $\mathrm{A} = \frac{\partial}{\partial \lambda} \mathrm{D}_1^2\mathrm{H}(0, \lambda) |_{\lambda = \lambda_j}$. This formula for QAQ

can be simplified considerably. In particular, this condition always holds if A is positive definite. A second case where our methods are useful is in the case where $H(x, \lambda) = \lambda H(x)$ but $M = D^2H(0)$ is singular. Assume that $\nabla H(x) = Mx + C(x) + o(|x|^2)$, where C is a quadratic polynomial such that $\tilde{Q}Cx \neq 0$ if $x \in N(M) \setminus \{0\}$. Here $\tilde{Q}$ is the orthogonal projection of R^{2n} onto N(M). Standard bifurcation theory then ensures that 0 is an isolated solution of $\nabla H(x) = 0$ in R^{2n}. By our earlier comments, non-constant solutions can only bifurcate at $(0, \lambda_j)$ if $z' = \lambda_j JD^2H(0)z$ has a non-constant solution in $\tilde{Z}$. In this case, it is possible with care to prove an analogue of Lemma 3 at least if D is a small neighbourhood of zero. It turns out that $\hat{n}(\lambda_j)$, the index change across λ_j, is of the form $\text{index}_{N(M)}\ (\lambda_j \tilde{Q}C|_{N(M)}, 0)\overline{n}(\lambda_j)$, where $\overline{n}(\lambda_j)$ has a similar expression to that in Lemma 3. (Det M is replaced by $\det(M|_{N(M)^\perp})$.) Thus we again obtain bifurcation theorems. Moreover, similar results hold if $\nabla H(x) = Mx + C(x) + o|x|^3$, where C is cubic and satisfies the same non-degeneracy assumptions as before. The idea in the proof of the above results is to vary the nonlinear terms by S^1-invariant gradient maps which do not change the map on $\mathscr{C}_{S^1}$.

4. — We compare our results with those of Fadell and Rabinowitz [*24*], Alexander and Yorke [*3*] and Chow, Mallet-Paret and Yorke [*11*]. They only consider the case where $H(x, \lambda) = \lambda H(x)$. If λ_j is an eigenvalue of the linearized problem, so is $\lambda_j k_r^{-1}$, where k_r is defined in the proof of Lemma 3. Thus if $\hat{n}(k_r^{-1}\lambda_j) \neq 0$ for some k_r, we easily see that there is a connected set of solutions bifurcating from zero for λ near λ_j. With this remark or by the last sentence in Remark 1 above, we see that the assumptions of Theorem 4 are weaker than those in [*24*]. Moreover, unlike [*24*], we obtain connected sets of bifurcating solutions. (An example in Böhme [*7*] implies that this is not always true for gradient mappings.) Moreover, our branches continue globally. It should be stressed that we do not obtain the multiplicity results in [*24*] though their result is natural from our index calculations. Note that this was the main point of [*24*]. Secondly, our results are more general than the one in [*3*] and [*11*] and our proof appears much more natural. (Their proof is by a trick which seems to only work easily for particular problems.) Our method seems to work in more general situations and gives more global information. Moreover, our method seems easier to apply when the eigenvalues of M are rationally related. Note that, as in [*3*], when $H(x, \lambda) = \lambda H(x)$, our result is not as good as it looks. Our unbounded set C may have solutions (x_n, λ_n) such that $\lambda_n \to \infty$ as $n \to \infty$ but the minimal period of the corresponding (scaled) solutions of $x' = J\nabla H(x)$ are bounded.

5. — Lastly, the results on the « signature » in [*25*] could be used to remove the diagonalizability assumptions in [*4*].

To complete this section, we discuss briefly some other uses of our index for Hamiltonian systems. We no longer assume that $\nabla H(0, \lambda) = 0$. However, our reduction of (18) to a finite dimensional problem $F(w, \lambda)=0$ on $\mathscr{C} \times R$ is still valid for $|x(t)| \leq \mathscr{R}$ on $[0, T]$, $|\lambda| \leq \mathscr{R}$. Of course, we no longer have that $F(0, \lambda) = 0$.

In particular, if we have a solution (w_0, λ_0) of (22) in $(\text{int } W)\backslash\mathscr{C}_{S^1}$ such that, for fixed λ_0, the corresponding orbit of solutions is an isolated orbit of solutions, then its index is defined. If this index is non-zero, we have two useful consequences. Let $x_0 = w_0 + Y(w_0, \lambda_0)$, where Y was defined in the reduction. If $\tilde{H}$ is C^1 close to $H(\,,\lambda_0)$ then the equation $x'(t) = J\nabla\tilde{H}(x)$ has a T-periodic solution near x_0. (Technically, we need to assume that $\tilde{H}$ is C^2 but this can be removed by an approximation argument.) To see this, we choose $\varepsilon > 0$ such that (18) for $\lambda = \lambda_0$ has no other T-periodic solution $z(t)$ in the ε neighbourhood U (in $\tilde{Z}$) of the orbit $S^1(x_0)$. It is easy to see that if $\tilde{H}$ is C^1 close to $H(\,,\lambda_0)$ the equation

$$x'(t) = J(\mu\nabla H(x(t), \lambda_0) + (1 - \mu)\nabla\tilde{H}(x(t)))$$

has no T-periodic solution on ∂U_ε for $0 \leq \mu \leq 1$. Now choose $\mathscr{R}$, $K > 0$ such that $U \subseteq \{x : |x(t)| \leq \mathscr{R} \text{ on } [0, T]\}$ and $|\mu D_1^2 H(y, \lambda_0) + (1-\mu)D^2\tilde{H}(y)| \leq K$ if $0 \leq \mu \leq 1$ and $|y| \leq \mathscr{R}$. As earlier, our problem reduces to an equation on a finite-dimensional space $\mathscr{C}$. Let $\tilde{U}$ be the set in $\mathscr{C} \times R$ corresponding to $U \times [0, 1]$. Here μ is the parameter. In $\tilde{U}_0$, there is a unique solution w_0 of non-zero index. Thus, by homotopy invariance, there must be a solution of the reduced equation in $\tilde{U}_\mu$ for $\mu \in [0, 1]$. Here $\tilde{U}_\mu = \{y : (y, \mu) \in \tilde{U}\}$. Hence the result follows. Implicit in the above proof is that whether the index is non-zero is independent of the choice of $\mathscr{C}$ in the reduction up to sign. This follows from Proposition 2 (*ii*) in § 2. In particular if the equation

$$\begin{aligned} y'(t) &= JD_1^2 H(x_0(t), \lambda_0)y(t) \\ y(0) &= y(T) \end{aligned} \tag{24}$$

has only a one-dimensional kernel, it follows easily from the reduction that $F_1'(w_0, \lambda_0)$ has only a one-dimensional kernel. Hence, by the definition of our degree, the index of w_0 is defined and non-zero. The result above generalizes one in Siegel and Moser [*36*, p. 145].

Secondly, if (x_0, λ_0) is as in the previous paragraph, we obtain a branch of T-periodic solutions as we vary λ. More precisely, if

$$\tilde{D} = \{(x(t), \lambda) : x \in \tilde{Z},\ x \text{ is a non-constant solution of (18)}\},$$

then the component of $\tilde{D}$ containing (x_0, λ_0) is (*i*) unbounded or (*ii*) contains (y, μ) in its closure where y is a constant such that $\nabla H(y, \mu) = 0$ or (*iii*) there is a « loop » X of solutions in $\tilde{D}$ around $S^1(x_0) \times \{\lambda_0\}$. More precisely, this means that X is compact, invariant and connected and $X\backslash(S^1(x_0) \times \{\lambda_0\})$ is connected. (Note that, if A is a small neighbourhood of $S^1(x_0) \times \{\lambda_0\}$, then $(\tilde{D} \cap A)\backslash(S^1(x_0) \times \{\lambda_0\})$ is not connected.) This is proved by an easy modification of standard degree arguments (cp. [*20*, § 1]).

For a T-periodic solution x_0, it is easy to see that $\# G_{x_0}$ is the largest positive integer m such that $x_0(t)$ is $m^{-1}T$ periodic. Thus we see that our index contains some limited information on the minimal period of solutions and restricts the bifurcations to solutions such that T is not the minimal period. In general, our index can be used to keep *some* control of bifurcations of non-constant T-periodic solutions as λ varies.

§ 6. ELLIPTIC EQUATIONS ON CERTAIN SYMMETRIC DOMAINS

In this short section, we apply our degree to an infinite-dimensional problem. *Assume that* Ω is a bounded domain in $R^2 \times R^{n-2}$ *and that* Ω *is invariant under rotations in the first two coordinates.* It follows easily that Ω is also invariant under reflections in the first two coordinates. We consider the problem

$$\begin{aligned} -\Delta u &= \lambda f(u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{25}$$

where $f: R \to R$ *is continuous on* R *and differentiable at* 0, $f(0) = 0$ *and* $f'(0) = 1$. Let D denote the closure in $L^\infty(\Omega) \times R$ of the set of non-trivial solutions, that is, solutions (u, λ) where u does not vanish identically. A solution u of (25) is said to be *symmetric* if it is invariant under rotations in the first two coordinates. Otherwise it is said to be non-symmetric. Similarly, an eigenvalue λ_j of the linearized problem

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{26}$$

is symmetric if (26) has a non-trivial symmetric solution for $\lambda = \lambda_j$.

THEOREM 5. — *Assume that λ_j is a non-symmetric eigenvalue of* (26). *Then the component* C *of* D *containing* $(0, \lambda_j)$ *is* (*i*) *unbounded or* (*ii*) *contains a symmetric solution or* (*iii*) *contains* $(0, \mu)$ *where μ is a symmetric eigenvalue of* (26) *or* (*iv*) *contains* $(0, \mu_2)$ *where μ_2 is a non-symmetric eigenvalue of* (26) *and the sum of the multiplicities* (*evaluated in the subspace of symmetric functions*) *of the symmetric eigenvalues between λ_j and μ_2 is odd.*

Proof. — It is easy to see that it suffices to prove the theorem when f is replaced by $f_k(y)$ where

$$f_k(y) = \begin{cases} f(y) & \text{if} \quad |f(y)| \le k \\ k \operatorname{sgn} f(y) & \text{if} \quad |f(y)| > k. \end{cases}$$

Our problem can then be formulated as an equation $\lambda u = \mathrm{F}(u)$ on $\mathrm{E} = \dot{\mathrm{W}}_2^1(\Omega)$ by standard techniques (cp. Berger [5, § 5]). F turns out to be completely continuous, a gradient, 0(2)-invariant (for the obvious action of 0(2)) and Fréchet differentiable at 0. Thus we can apply the infinite-dimensional version of Theorem 2 mentioned near the end of § 3. We only have to evaluate $n(\lambda_j)$ for any non-symmetric eigenvalue λ_j of (26). By the remarks near the end of § 3, it is a_j sgn λ_j sgn $[\mathrm{index}_{\mathscr{S}}((\mathrm{I} - \lambda_j \mathrm{F}'(0))|_{\mathscr{S}}, 0)]$, where $\mathscr{S}$ is the subspace of E of symmetric functions and $a_j > 0$. The result now follows easily when we recall that $\mathrm{index}_{\mathscr{S}}((\mathrm{I} - \lambda_j \mathrm{F}'(0))|_{\mathscr{S}}, 0) = (-1)^w$, where w is the sum of the multiplicities (in the subspace $\mathscr{S}$) of the eigenvalues of $\mathrm{F}'(0)$ in (λ_j^{-1}, ∞).

Remarks. — It seems probable that most eigenvalues are non-symmetric eigenvalues. To see why we expect this to be true, note that (cp. Agmon [*2*]) $\mathrm{N}(\lambda) \sim \lambda^{\frac{1}{2}n}$ for λ large where $\mathrm{N}(\lambda)$ is the number of eigenvalues less than or equal to λ (counting multiplicity). On the other hand, we *expect* that $\mathrm{N}_{\mathscr{S}}(\lambda) \sim \lambda^{\frac{1}{2}(n-1)}$, where $\mathrm{N}_{\mathscr{S}}(\lambda)$ denotes the number of symmetric eigenvalues (with multiplicities counted in $\mathscr{S}$). Thus, provided that symmetric eigenvalues do not have very much larger multiplicities on the whole space than in $\mathscr{S}$, the result would follow. *Assuming this result*, Theorem 5 implies that the branch bifurcating at $(0, \lambda_j)$ meets the symmetric solutions, is unbounded or crosses a number of eigenvalues. If we had more information about the solutions in $\mathscr{S}$, the result could probably be improved.

There is an alternative way of studying this problem, where we use the 0(2) symmetry rather than just the S^1 symmetry. It is easy to prove that any eigenfunction of (26) is a linear combination of eigenfunctions of the form $f(r, z) \cos(m\theta + c)$ where m is non-negative integer and we use coordinates (r, θ, z) for points in R^n, where $z \in \mathrm{R}^{n-2}$. If λ_j is an eigenvalue with an eigenfunction of the above form, we look for solutions in E_A where A is the subgroup generated by rotation through an angle $(2\pi)m^{-1}$ and one reflection. If one can choose A such that λ_j has odd multiplicity in E_A (for example, any eigenvalue, if Ω is a 2-dimensional disc) or such that there is an eigenfunction with $f(r, z) \geq 0$, we can apply Rabinowitz [*33*] or Dancer [*15*] to obtain a branch of solutions (unbounded in the latter case). It seems that the results obtained by the two methods are distinct, even in simple cases, as in the case of a two-dimensional disc.

Our normal index can be extended to more general groups, where we only consider solutions whose isotropy groups G_x are a finite extension of those G_y of the principal orbits. In fact, with care, it can be generalized a little further where we only consider solutions for which $\chi(\mathrm{G}_x/\mathrm{G}_y) \neq 0$, where χ denotes the Euler characteristic. It would be interesting to try to use the above ideas to unify some of the results earlier in this section for (25) and to study our problem for other symmetric domains. It would also be interesting to relate our degree to the equivariant homotopy index where the equivariant homotopy index is defined in the same way as the homotopy index (cp. Conley [*12*]) except that everything is done to preserve the symmetries. This is discussed in a little more detail in § 3 of [*21*].

§ 7. GENERIC S^1-INVARIANT GRADIENT HOMOTOPIES

In this section, we prove Proposition 1 of § 1. This was the key technical result we needed to construct our index. Firstly, we need a number of lemmas. We follow the notation of § 1.

LEMMA 4. — *Assume that* $f : \Omega \to \mathrm{R}$ *is* C^i *and* S^1*-invariant where* $i \geq 1$ *and* Ω *is a bounded open invariant subset of* E. *In addition, assume that* K *is a compact invariant subset of* Ω. *Then we can approximate* f *in the* C^i *norm (on* K*) by a real analytic invariant map* $\tilde{f}$.

Proof. — Choose a compact invariant neighbourhood K_1 of K in Ω. By Narasimhan [*31*, Theorem 1.5.2], we can approximate f in the C^i norm (on K_1) by a real analytic map f_1. We replace f_1 by $\tilde{f}$, where

$$\tilde{f}(x) = \int_{\mathrm{G}} f_1(\mathrm{T}_g x) d\mu \,.$$

Here μ is the usual (normalized) measure on S^1. It is easy to see that $\tilde{f}$ is invariant. Thus we have to prove that (*i*) $\tilde{f}$ is analytic and (*ii*) $\tilde{f}$ is near f in the C^i norm. Note that $\tilde{f}$ is clearly C^∞. Let D denote differentiation. Since f is invariant,

$$\begin{aligned} \mathrm{D}(\tilde{f} - f) &= \mathrm{D}\left(\int_{\mathrm{G}} (f_1(\mathrm{T}_g x) - f(\mathrm{T}_g x)) d\mu\right) \\ &= \int_{\mathrm{G}} (\mathrm{D} f_1(\mathrm{T}_g x)\mathrm{T}_g - \mathrm{D} f(\mathrm{T}_g x)\mathrm{T}_g) d\mu. \end{aligned}$$

Since f_1 is near f in the C^1 norm on K, it follows easily that $\mathrm{D}(\tilde{f} - f)$ is small on K. Since we could use a similar argument for higher derivatives, (*ii*) follows. Since $\tilde{f}$ is C^∞, to prove (*i*), it suffices to obtain a bound

$$\| \mathrm{D}^n \tilde{f}(x) \| \leq \mathrm{K} n \,! \alpha^n \tag{27}$$

for all n and for all x in K_1. Since f_1 is real analytic, it can be extended to a complex analytic function in some open neighbourhood of K_1 in the complexification of E (cp. Bochnak and Siciak [*6*, Theorem 7.1]). It then follows easily from Cauchy's theorem that we obtain an estimate of the form (27) for f_1 uniformly on compact subsets of Ω. Since D^n commutes with the averaging and since each T_g is an isometry, it follows easily from the formula for $\mathrm{D}^n \tilde{f}$ that a similar inequality holds for $\tilde{f}$. This completes the proof.

LEMMA 5. — ([*17*, Remark 4 after proof of Theorem]). *Assume that* $f\colon \Omega \to \mathrm{R}$ *is* C^2 *and* S^1*-invariant and that* $\nabla f(x_0) = 0$ *where* $x_0 \in \Omega \backslash \mathrm{E}_{\mathrm{S}^1}$. *If* $\mathrm{N}(\mathrm{D}^2 f(x_0))$ *is one-dimensional, then every* S^1*-invariant map* f C^2 *close to* f *has a unique orbit of critical points close to* $\mathrm{S}^1(x_0)$.

Note that the symmetries imply that $\dim \mathrm{N}(\mathrm{D}^2 f(x_0)) \geq 1$.

LEMMA 6. — *Assume that* Y *is a finite union of smooth disjoint compact* S^1*-invariant submanifolds of* E *and* $f\colon \mathrm{Y} \to \mathrm{R}$ *is* S^1*-invariant and smooth. Then* f *can be extended to a smooth* S^1*-invariant map of* E *into* R.

This follows easily from the Whitney extension theorem (Abraham and Robbins [*1*, Theorem 13.2]). Once we have an extension, we integrate as before to get an invariant extension.

LEMMA 7. — *Assume that* $\mathrm{G} : \mathrm{E} \to \mathrm{E}$ *is an s-homogeneous polynomial where* $s \geq 2$, $\mathrm{G}(x) \neq 0$ *for* $x \neq 0$ *and* $\mathrm{G}(y) = \pm y$ *implies that* $\mathrm{I} \pm \mathrm{G}'(y)$ *is invertible. Finally assume that* $\mathrm{W} : \mathrm{E} \times \mathrm{R} \to \mathrm{E}$ *is* C^1, $\mathrm{W}(0, \lambda) = 0$ *and* $\mathrm{W}'_1(x, \lambda) = o(\| x \|^{s-1} + | \lambda | \| x \|^{s-2})$ *for* (x, λ) *small and that* $\mathrm{L}(\lambda)$ *is a family of linear operators* C^1 *in* λ *such that* $\mathrm{L}(0) = 0$ *and* $\mathrm{L}'(0) = 0$. *Then, for each small* λ, *the equation* $\lambda x + \mathrm{L}(\lambda)x = \mathrm{G}(x) + \mathrm{W}(x, \lambda)$ *has only a finite number of small solutions. Moreover, at each of these solutions except* $(0, 0)$, $\lambda \mathrm{I} + \mathrm{L}(\lambda) - \mathrm{G}'(x) - \mathrm{W}'_1(x, \lambda)$ *is invertible.*

This follows easily by a slight modification of the proofs of the results in § 4-5 of [*14*].

LEMMA 8. — *Assume that* V_x *is the normal plane to* $\mathrm{T}_x(\mathrm{S}^1(x))$ *at* x *and that* f *is a real-valued* C^i G_x*-invariant map defined on a neighbourhood of* x *in* V_x. *Then* f *can be extended uniquely to an* S^1*-invariant* C^i *map of a neighbourhood of* $\mathrm{S}^1(x)$ *in* E. *Moreover, a similar result holds for* C^∞ *maps.*

Proof. — We simply define $\tilde{f}$ on a neighbourhood of $\mathrm{S}^1(x)$ by $\tilde{f}(z) = f(y)$, where $z = \mathrm{T}_g y$ with $y \in \tilde{\mathrm{V}}_x$. Here $\tilde{\mathrm{V}}_x$ denotes a small neighbourhood of x in V_x. The tubular neighbourhood theorem ensures that this defines $\tilde{f}$ in a neighbourhood of $\mathrm{S}^1(x)$ while the G_x invariance of f on V_x ensures that $\tilde{f}$ is well-defined. Now the map $(x, g) \to \mathrm{T}_g x$ is C^∞ (in fact, real analytic) by [*8*, p. 298]. Thus, by invariance, it suffices to prove $\tilde{f}$ is C^i near $(x, 0)$. Now the function $\overline{y}(z)$ used in the construction of the degree in § 1 is C^∞ because it is obtained by applying the implicit function theorem to a smooth map. Now, by construction, $\overline{y}(z)$ and z are on the same orbit. Thus, by invariance, $\tilde{f}(z) = f(\overline{y}(z))$. Thus $\tilde{f}$ is the composite of C^i functions and hence is C^i.

Remark. — The result also holds for real analytic maps.

The following lemma is probably known but we could find no reference.

LEMMA 9. — *Assume that Ω is a bounded open set in R^n. The set of smooth maps $f: \Omega \times [0,1] \to \mathrm{R}$ with the property that, whenever $0 = \nabla_x f(x,t)$, then $\mathrm{D}_x^2 f(x,t)\mathrm{R}^n$ and $\mathrm{D}_{tx}^2 f(x,t)$ span R^n is open and dense in the set of C^∞ maps from $\mathrm{R}^n \times \mathrm{R} \to \mathrm{R}$.*

Proof. — Openness is trivial. It suffices to prove denseness in the C^r topology (for some $r \geq 2$) because we can then use Lemma 4. The proof of denseness is an easy modification of the proof of the theorem in the appendix to §2.11 in Chow and Hale [*10*]. (We consider the map J defined by $(\mathrm{J}f)(x,t) = (x, t, \nabla_x f(x,t), \mathrm{D}_x^2 f(x,t), \mathrm{D}_{tx}^2 f(x,t))$.

Remark. — Note that we have been (and will be later in the section) careless about boundaries. We have not worried about this because, in our applications, we could always avoid them by shrinking the set slightly.

Note that any non-trivial isotropy group for the action of S^1 is a finite cyclic group Z_n generated by a rotation through $2\pi n^{-1}$.

Proof of Proposition 1. — The first statement of Proposition 1 is a consequence of the second since we can use a standard mollifier argument to approximate H by a C^1 function $\tilde{\mathrm{H}}$ such that $\nabla_x \tilde{\mathrm{H}}$ is near $\nabla_x \mathrm{H}$ and then the Haar integral as in the proof of Lemma 4 to make $\tilde{\mathrm{H}}$ invariant. The idea of the proof of the second statement (as in [*38*]) is to successively build up $\tilde{\mathrm{H}}$ on $\mathrm{E}_{\mathrm{G}_i} \times [0,1]$ (where G_i is an isotropy group) in such a way that, when $\nabla_x \tilde{\mathrm{H}}(x,t) = 0$, G_x acts real irreducibly on $\mathrm{N}(\mathrm{D}_x^2 \tilde{\mathrm{H}}(x,t)) \cap (\mathrm{T}_x(\mathrm{S}^1(x)))^\perp$ and such that « nearly always » $\mathrm{X} = \{(x,t) : \nabla_x \tilde{\mathrm{H}}(x,t) = 0\}$ is a finite union of 2-manifolds and such that each orbit in X is isolated in $\mathrm{X} \cap (\mathrm{E} \times \{t\})$. We successively define $\tilde{\mathrm{H}}$ on $\mathrm{E}_{\mathrm{G}_i}$ as $\# \mathrm{G}_i$ decreases. Let G_i, $i = 1, \ldots, k$, denote the isotropy groups for the action of $\mathrm{G} = \mathrm{S}^1$ on E where $\# \mathrm{G}_j \geq \# \mathrm{G}_i$ if $j \leq i$.

STEP 1. — *We first obtain* H *on* $\mathrm{E}_{\mathrm{G}_i}$ *for* i *small.* If $\nabla_x \mathrm{H}(x,t) \neq 0$ for $x \in \mathrm{E}_{\mathrm{G}_j} \cap \Omega$, $t \in [0,1]$, $j \leq i$ (for example, if $\mathrm{G}_i = \mathrm{S}^1$), we simply take $\tilde{\mathrm{H}}$ to be a real analytic approximation to H in the C^2 norm on $\Omega \times [0,1]$. The closeness of the approximation ensures that $\nabla_x \tilde{\mathrm{H}}(x,t) \neq 0$ if $x \in \mathrm{E}_{\mathrm{G}_j} \cap \Omega$, $t \in [0,1]$, $j \leq i$. Thus we eventually arrive at an isotropy group G_i such that $\nabla_x \tilde{\mathrm{H}}(x,t) = 0$ for some $(x,t) \in \mathrm{E}_{\mathrm{G}_i} \times [0,1]$.

STEP 2. — *We modify* $\tilde{\mathrm{H}}$ *on* $\mathrm{E}_{\mathrm{G}_i}$. *We first modify* $\tilde{\mathrm{H}}$ *such that, if*

$$(x,t) \in (\Omega \cap \mathrm{E}_{\mathrm{G}_i}) \times [0,1] \qquad \textit{and} \qquad \nabla_x \tilde{\mathrm{H}}(x,t) = 0,$$

then $\mathrm{D}_x^2 \tilde{\mathrm{H}}(x,t)$ *has at most a 2-dimensional kernel on* $\mathrm{E}_{\mathrm{G}_i}$ *and* $\mathrm{D}_x^2 \tilde{\mathrm{H}}(x,t)\mathrm{E}_{\mathrm{G}_i}$ *and* $\mathrm{D}_t \mathrm{D}_x \tilde{\mathrm{H}}(x,t)$ *span an* $(m_i - 1)$*-dimensional space, where* $m_i = \dim \mathrm{E}_{\mathrm{G}_i}$. Consider the real analytic map $\tilde{\mathrm{H}}$ we have constructed. For each $(\bar{x}, \bar{t})$ in $\mathrm{E}_{\mathrm{G}_i} \times [0,1]$ such that $\nabla_x \tilde{\mathrm{H}}(\bar{x}, \bar{t}) = 0$, we choose a neighbourhood $\mathrm{N}_{\bar{x}}$ in the normal plane $\mathrm{V}_{\bar{x}}$ (in $\mathrm{E}_{\mathrm{G}_i}$) to $\mathrm{T}_{\bar{x}}(\mathrm{S}^1(\bar{x}))$ at $\bar{x}$ and an $\varepsilon_{\bar{x}} > 0$. Now a finite number of the neighbourhoods $\tilde{N}_{\bar{x}} \equiv \{ T_g N_{\bar{x}} : g \in S^1 \} \times [\bar{t} - \varepsilon_{\bar{x}}, \bar{t} + \varepsilon_{\bar{x}}]$ cover the solutions of $\nabla_x \tilde{H}(x, t) = 0$ in $(E_{G_i} \cap \Omega) \times [0, 1]$. Now, by Lemma 8, there is a natural correspondence between smooth S^1-invariant maps on $\tilde{N}_{\bar{x}}$ and smooth maps on $N_{\bar{x}} \times [\bar{t} - \varepsilon_{\bar{x}}, \bar{t} + \varepsilon_{\bar{x}}]$. Note that G_x acts trivially on $V_{\bar{x}} \subseteq E_{G_i}$ since, by our construction, every critical point in E_{G_i} has isotropy group G_i. By Lemma 9, there is a dense open subset $A_{\bar{x}}$ of the smooth maps on $N_{\bar{x}} \times [\bar{t} - \varepsilon_{\bar{x}}, \bar{t} + \varepsilon_{\bar{x}}]$ with the property that, if $g \in A_{\bar{x}}$, then $\tilde{D}_x^2 g(x, t) V_{\bar{x}}$ and $D_t \tilde{D}_x g(x, t)$ span $V_{\bar{x}}$ whenever $\tilde{D}_x g(x, t) = 0$. Here $\tilde{D}_x$ denotes the gradient *on the space* $V_{\bar{x}}$. By our earlier comments, it follows that there is a dense open subset $\tilde{A}$ of the smooth invariant functions on $\tilde{N}_{\bar{x}}$ such that, if $g \in \tilde{A}$ and $\nabla_x g(x, t) = 0$ then $D_x^2 g(x, t)$ has at most a two-dimensional kernel on E_{G_i} and $D_x^2 g(x, t) E_{G_i}$ and $D_t D_x g(x, t)$ span an $(m_i - 1)$-dimensional space, where $m_i = \dim E_{G_i}$. (The symmetries ensure that they cannot span E_{G_i}.) *We call this property* $(\mathrm{Gen})^i_{\bar{x}}$. By Lemma 6, we can extend a smooth invariant map of $\tilde{N}_{\bar{x}} \cup ((E_{G_{i-1}} \cap \Omega) \times [0, 1])$ to a smooth invariant map on $(E_{G_i} \cap \Omega) \times [0, 1]$. Thus we eventually find a dense open subset $\tilde{A}$ of the smooth invariant functions on $(E_{G_i} \cap \Omega) \times [0, 1]$ which equal $\tilde{H}$ on $(E_{G_{i-1}} \cap \Omega) \times [0, 1]$ such that $(\mathrm{Gen})^i_{\bar{x}}$ holds on $\tilde{N}_{\bar{x}}$. Now the *invariant* smooth functions in $(E_{G_i} \cap \Omega) \times [0, 1]$ which equal $\tilde{H}$ on $(E_{G_{i-1}} \cap \Omega) \times [0, 1]$ form a Fréchet space since it is a closed subspace of the smooth functions. The topology is induced by the seminorms $p_{iK} = \sup_{x \in K} \| f^i(x) \|$, where K is a compact subset of Ω. Since a finite number of the $\tilde{N}_{\bar{x}}$'s cover the solutions of $\nabla_x \tilde{H}(x, t) = 0$ in $(E_{G_i} \cap \Omega) \times [0, 1]$, it follows that we can approximate $\tilde{H}$ by smooth invariant functions which have $(\mathrm{Gen})^i_x$ on $(E_{G_i} \cap \Omega) \times [0, 1]$. (Remember that the finite intersection of open dense sets is dense in a metric space.) By Lemma 4, we can choose $\tilde{H}$ to be real analytic and still retain our condition on $(E_{G_i} \cap \Omega) \times [0, 1]$ for $j \leq i$. Remember that our various conditions hold on open sets of functions. Since $\tilde{H}$ has $(\mathrm{Gen})^i_x$, well-known results (cp. Crandall and Rabinowitz [*13*, Theorem 1.7]) imply that the solutions of $\tilde{D}_x \tilde{H}(x, t) = 0$ in a set $N_{\bar{x}} \times [\bar{t} - \varepsilon_{\bar{x}}, \bar{t} + \varepsilon_{\bar{x}}]$ form a real analytic 1-manifold S. Moreover, by the real analyticity, one easily sees that any component $\tilde{S}$ of this *one-manifold* S is contained in $N_{\bar{x}} \times \{ t \}$ for fixed t or $\tilde{S} \cap (N_{\bar{x}} \times \{ t \})$ consists of isolated points for each fixed t. Since the solutions of $\nabla_x \tilde{H}(x, t) = 0$ in $\tilde{N}_{\bar{x}}$ are simply obtained by the group action from the solutions of $\tilde{D}_x \tilde{H}(x, t) = 0$ in $N_{\bar{x}} \times [\bar{t} - \varepsilon_{\bar{x}}, \bar{t} + \varepsilon_{\bar{x}}]$ (cp. [*20*, § 3]), it follows easily that the solutions of $\nabla_x \tilde{H}(x, t) = 0$ in $(E_{G_i} \cap \Omega) \times [0, 1]$ are a finite union of disjoint connected real analytic 2-manifolds l_i such that (*a*) $l_i \subseteq E_{G_i} \times \{ s \}$ for fixed s or (*b*) $l_i \cap (E_{G_i} \times \{ t \})$ consists of isolated orbits for each t in $[0, 1]$.

We perturb $\tilde{H}$ further to ensure that none of the l_i satisfy (a). By our earlier comments, it suffices to consider the equation $\tilde{D}_x\tilde{H}(x, t) = 0$ on $N_{\bar{x}} \times [\bar{t} - \varepsilon_{\bar{x}}, \bar{t} + \varepsilon_{\bar{x}}]$. Suppose $(\bar{x}, \bar{t})$ is a solution such that $\tilde{D}_x^2\tilde{H}(\bar{x}, \bar{t})$ has a one-dimensional kernel on E_{G_i}. (The result is trivial if this map has zero kernel.) Then the bifurcation equation for solutions of $\tilde{D}_x H(x, t) = 0$ near $(\bar{x}, \bar{t})$ (with $x \in V_{\bar{x}}$) is of the form $s(\alpha, t) = 0$ where $s : R^2 \to R$ is real analytic. If (a) holds, zero must be a non-isolated solution of $s(\alpha, \bar{t}) = 0$ on R and hence $s(\alpha, \bar{t})$ must vanish identically. Now it is easy to construct a C^∞ perturbation of our map such that $(\bar{x}, \bar{t})$ is still a critical point but $\bar{s}(, \bar{t})$ does not vanish identically. Here $\bar{s}$ is the bifurcation equation for the perturbed problem. It follows that, $\tilde{l}_i \not\subseteq E_{G_i} \times \{ \bar{t} \}$, where $\tilde{l}_i$ is the component corresponding to l_i for the perturbed problem. (One can argue as in {*17*, § 2] to show that the l_i behave nicely under perturbations.) Finally, we can use Lemma 4 to ensure that $\tilde{H}$ is real analytic.

At this stage, we have constructed a real analytic approximation $\tilde{H}$ to H such that $\nabla_x\tilde{H}(x, t) = 0$ has only a finite number of orbits of solutions in $E_{G_i} \cap \Omega$ for each t and such that $N(D_x^2\tilde{H}(x, t)\, E_{G_i})$ is at most 2-dimensional whenever $\nabla_x\tilde{H}(x, t) = 0$ (and $x \in E_{G_i} \cap \Omega$). *This gives the required* $\tilde{H}$ *on* E_{G_i}.

STEP 3. — We *start* constructing $\tilde{H}$ in a neighbourhood of E_{G_i} in E. We perturb our map $\tilde{H}$ so that $G_{\bar{x}}$ *acts irreducibly as a real representation on the orthogonal complement to* $T_{\bar{x}}(S^1(\bar{x}))$ *in* $N(D_x^2\tilde{H}(\bar{x}, \bar{t}))$ *for each* $(\bar{x}, \bar{t}) \in L_i$. Here $L_i \equiv \{ (x, t) \in (E_{G_i} \cap \Omega) \times [0, 1] : \nabla_x\tilde{H}(x, t) = 0 \}$. Since $G_{\bar{x}}$ is abelian, any such representation is at most 2-dimensional. Thus $N(D_x^2\tilde{H}(\bar{x}, \bar{t}))$ is at most 3-dimensional for $(\bar{x}, \bar{t}) \in L_i$.

Now $D_x^2\tilde{H}(x, t)$ has a one-dimensional kernel on $\mathscr{L}_i$ except at isolated orbits. To see this, it suffices as before to work in $N_{\bar{x}} \times [\bar{t} - \varepsilon_{\bar{x}}, \bar{t} + \varepsilon_{\bar{x}}]$ where $\nabla_x\tilde{H}(\bar{x}, \bar{t}) = 0$. We have to show that $\tilde{D}_x^2\tilde{H}(x, t)$ is invertible when $(x, t) \in L_i$, $x \in V_{\bar{x}}$ and (x, t) is near (but not equal to) $(\bar{x}, \bar{t})$. By Step 2, $L_i \cap (N_{\bar{x}} \times [\bar{t} - \varepsilon_{\bar{x}}, \bar{t} + \varepsilon_{\bar{x}}])$ is locally a real analytic manifold $(x(\mu), t(\mu))$ where t is not constant. By Crandall and Rabinowitz [*13*, Theorem 1.17], $\tilde{D}_x^2\tilde{H}(x(\mu), t(\mu))$ is invertible when $t'(\mu) \neq 0$. Since the zeros of t' are isolated, the result follows.

Consider a component l_i of L_i. Choose $(\bar{x}, \bar{t}) \in l_i$ such that $D_x^2\tilde{H}(\bar{x}, \bar{t})\,|_{E_{G_i}}$ has a one-dimensional kernel on E_{G_i}. It is easy to perturb $\tilde{H}$ so that $D_x^2\tilde{H}(\bar{x}, \bar{t})$ has a one-dimensional kernel on E. (To do this, we first obtain a C^∞ perturbation. We work in $\overline{N}_{\bar{x}} \equiv T_{\bar{x}}(S^1(\bar{x}))^\perp$ (in E). We add a perturbation so that $D_x^2\tilde{H}(\bar{x}, \bar{t})$ is perturbed by $\varepsilon(I - \overline{P})$, where $\overline{P}$ is the orthogonal projection of $\overline{N}_{\bar{x}}$ into $N_{\bar{x}}$. Note that it is easy to do this on $\overline{N}_{\bar{x}}$ and keep $\tilde{H}\,|_{\overline{N}_{\bar{x}}}$ $G_{\bar{x}}$-invariant.) Now, if the map $\lambda \to N(\lambda)$ is real analytic, where each $N(\lambda)$ is a self-adjoint matrix, then the eigenvalues of $N(\lambda)$ depend real analytically upon λ (cp. Kato [27, Theorem II.6.1]). Hence, if $N(\lambda)$ has a non-trivial kernel for all λ and $\dim N(\lambda) = 1$ for some λ, then $\dim N(\lambda) = 1$ for all λ except at isolated points. Thus we see that $D_x^2\tilde{H}(x, t)$ *has a one-dimensional kernel on* E *for* $(x, t) \in L_i$ *except at isolated orbits. We call these the bad orbits.*

Consider one of these bad orbits generated by (y, τ). We can use a similar perturbation argument to that in the previous paragraph to ensure that $D_x^2\tilde{H}(x, t) \mid (E_{G_i})^{\perp}$ is invertible whenever $(x, t) \in L_i$ and $D_x^2\tilde{H}(x, t) \mid E_{G_i}$ has a two-dimensional kernel. Now, if K is a G_y-invariant subspace of $\overline{N}_y$, the orthogonal projection $\tilde{P}$ of $\overline{N}_y$ onto K is G_y-invariant. Hence we can use the G_y-invariant perturbation $\varepsilon \| P(x - y) \|^2$ on $\overline{N}_y$ to lower the dimension of the kernel of $D_x^2\tilde{H}(y, \tau)$. Of course we use Lemma 4 again to make $\tilde{H}$ real analytic. After a finite number of steps, we *find that* G_y *acts real irreducibly on* $N(D_x^2\tilde{H}(y, \tau) |_{\overline{N}_y})$ *whenever* $(y, \tau) \in L_i$.

Suppose $(y, \tau) \in L_i$ and $T \equiv N(D_x^2\tilde{H}(y, \tau)) |_{\overline{N}_y} \not\subseteq E_{G_i}$. (Thus $T \subseteq E_{G_i}{}^{\perp}$.) Then, as before, T is one or two-dimensional. We consider the case where T is 2-dimensional. The other case is similar but easier. Near (y, τ), $L_i \cap (N_y \times (\tau - \varepsilon, \tau + \varepsilon))$ is an analytic 1-manifold $(x(\alpha), \tau(\alpha))$ (where $x(0) = y$, $\tau(0) = \tau$). Since $\tilde{D}_x^2\tilde{H}(y, \tau) |_{N_y}$ is invertible, we can parametrize by t. Without loss of generality $\tau = 0$. Consider $D_x^2\tilde{H}(x(t), t)$. This has zero as an eigenvalue for all t due to the symmetry and an eigenvalue $\mu(t)$ with $\mu(0) = 0$, corresponding to eigenvectors in T for $t = 0$. Since G_y acts real irreducibly on T, the symmetries ensure that $\mu(t)$ has multiplicity 2 for all t. Consider the bifurcation equation on T determining the small eigenvalues of $D_x^2\tilde{H}(x(t), t)$. We eliminate the extra zero eigenvalue by working on $\overline{N}_y$. The bifurcation equation will be of the form $h(t, \mu)w = 0$, where h is linear in w and G_y-invariant and μ is a small eigenvalue of $\tilde{D}_x^2\tilde{H}(x(t), t)$ on $\overline{N}_y$ if and only if $\det h(t, \mu) = 0$. Now $h(t, \mu)$ must be self-adjoint (cp. Rabinowitz [34, p. 412]). Since G_y acts real irreducibly on T, it follows easily that $h(t, \mu)$ is a scalar multiple of the identity, that is, $h(t, \mu) = r(t, \mu)I$. Hence μ is a small eigenvalue if and only if $r(t, \mu) = 0$. It is now easy to see that we can make a small perturbation (basically on r and in essentially the same way as before) to ensure that $r'_1(t, \mu) \neq 0$ when $r(t, \mu) = 0$ and (μ, t) is small. This implies that $\mu'(t) \neq 0$ when $\mu(t) = 0$. In other words, when an eigenvalue crosses zero, it does so transversally. In each of our above constructions, the condition we have ensured remains true under small perturbations. Thus there is no difficulty in doing the steps successively and coping with the finite number of bad orbits on L_i. Thus we have proved what we claimed at the start of Step 3 and also ensured that *when a non-trivial eigenvalue of* $D_x^2\tilde{H}(y, \mu)$ *is zero* (with $(y, \mu) \in L_i$), *then it crosses zero transversally as we move along* L_i, orthogonally to orbits.

STEP 4. — *We change the nonlinear terms near the bad orbits to ensure* *that* $\nabla_x \tilde{H}(x, t) = 0$ *only vanishes on a finite number of orbits near* E_{G_i} *for each* $t \in [0, 1]$. *This will complete the construction of* $\tilde{H}$ *near* E_{G_i}. We have already proved this for orbits in E_{G_i} and thus, we need only consider solutions which bifurcate out of E_{G_i}. By our earlier comments in § 5 (cp. [*19*, Remark after lemma]), this can only happen at a point (y, τ) where $N(D_x^2\tilde{H}(y, \tau)|_{\overline{N}_y}) \not\subseteq E_{G_i}$. We follow the notation of Step 3. Now T is 1- or 2-dimensional. We only consider the case where T is 2-dimensional. The other case is much easier. (If T is one-dimensional, $T_g|_T = \pm I$ for $g \in G_y$ and both cases occur for some g since $T \not\subseteq E_{G_i}$.)

We consider the bifurcation equation on T for solutions of $\nabla_x \tilde{H}(x, t) = 0$ near (y, τ) and where the solutions $(x(t), t)$ in E_{G_i} are the « trivial » solutions. Note that, as before, it suffices to look for solutions of $\tilde{D}_x\tilde{H}(x, t) = 0$ in $\overline{N}_y$ and we may assume, without loss of generality, that $\tau = 0$. Now our bifurcation equation becomes

$$r(t, 0)w = M(w, t),$$

where $w \in T$, r is defined in Step 3, $M(0, t) = 0$ and $M_1'(0, t) = 0$ for all t, M is real analytic, M is G_y-invariant and M is a gradient for each t. (The linear terms must be the same as that for $\mu = 0$ in the bifurcation equation in Step 3.)

Now G_y is a finite cyclic group which acts real irreducibly on T. The most general G_y-invariant real polynomial on T is calculated in Stattinger [*35*, p. 118]. It is a polynomial in $\| w \|^2$ and the real and the imaginary parts of w^p, where $\# G_y = p$ and we are identifying T with $\mathbb{C}$. Note that $p > 2$ since T is 2-dimensional and real irreducible. Now any invariant gradient polynomial P on T with $P'(0) = 0$ and involving $\| w \|^2$ must have $P^2(0) = 0$ and $P^3(0) = 0$, by the result of Stattinger above. Thus we easily see that we can make a C^3-small C^∞ perturbation to eliminate such terms. Hence we see that after such a perturbation and also (if necessary) a perturbation introducing a term $\mathrm{Re}(cw^p)$, we find that

$$M(w, t) = \nabla \mathrm{Re}(cw^p) + o(\| w \|^{p+1} + | t | \| w \|^p),$$

where $c \neq 0$. If $\tilde{p}$ is the perturbation, we replace $\tilde{p}$ by $\tilde{p}\bar{r}$ where $\bar{r}$ is a fixed C^∞ G_y-invariant function such that $\bar{r} = 1$ near (y, τ) but $\bar{r} = 0$ outside some neighbourhood of (y, τ) in $\overline{N}_y \times [0, 1]$. Thus $\tilde{p}\bar{r}$ will still be small in the C^3 norm if $\tilde{p}$ is small. Note that this perturbation only alters our map near $S^1(y) \times \{ \tau \}$. Now $r(t, 0) = kt + o(t)$ where $k \neq 0$ by the construction in Step 3.

Thus our bifurcation equation now satisfies most of the assumptions of Lemma 7. It is an easy but tedious calculation to check the other conditions. Hence we see that, for the perturbed equation, there are a finite number of solutions in $\overline{N}_y$ of $\tilde{D}_x\tilde{H}(x, t) = 0$ for each t near zero. Thus there are only a finite number of orbits of solutions of $\nabla_x\tilde{H}(x, t) = 0$ near $S^1(y)$

for each t near τ. Moreover, for these solutions (except (y, τ)), $N(D_x^2\tilde{H}(x, t))$ is one-dimensional by the second part of Lemma 7 and since the kernel is determined by the bifurcation equation. Now we can do this at each point (y, τ) where $N(\tilde{D}_x^2\tilde{H}(y, \tau)|_{\tilde{N}_y}) \not\subseteq E_{G_i}$. Thus we eventually have a C^∞ map such that $\nabla\tilde{H}(x, t)=0$ only holds on a finite number of orbits near $E_{G_i} \cap \Omega$ for each $t \in [0, 1]$.

Note that, we have to use perturbations with small support (and thus non-analytic) in the last step because our assumptions on the bifurcation equation are non-generic. Thus we have to be careful that the various perturbations do not disturb each other. (In fact, we could use C^∞ small analytic perturbations at the expense of proving a much more complicated version of Lemma 7.) We have now made a C^3 small perturbation such that $\tilde{H}$ is C^∞ and real analytic except close to a finite number of orbits and such that $\nabla_x\tilde{H}(x, t) = 0$ only holds on a finite number of orbits near E_{G_i} for each t. This completes Step 4 of the proof.

STEP 5. — *Completion of the proof.* We explain how to obtain $\tilde{H}$ on a neighbourhood of $E_{G_{i+1}}$ (for i as in Step 2). The other successive extensions to E_{G_j} thereafter are similar. Choose $\varepsilon > 0$ such that all the solutions of $\nabla_x\tilde{H}(x, t) = 0$ in $V_\varepsilon \equiv \{x \in E : d(x, E_{G_i}) < \varepsilon\}\backslash E_{G_i}$ are those given by the bifurcations in Step 4 and such that, for each of these solutions, $N(D_x^2\tilde{H}(x, t))$ is one-dimensional (also by Step 4). We now use a perturbation argument on $E_{G_{i+1}}$ which is a slight modification of the one in Steps 2-4.

For each $(\bar{x}, \bar{t}) \in L_{i+1}\backslash((V_\varepsilon \times [0, 1]) \cup L_i)$, we can choose a neighbourhood $\tilde{N}_{\bar{x}}$ of $S^1(\bar{x}) \times \{\bar{t}\}$ in $E_{G_{i+1}} \times [0, 1]$ such that the invariant C^∞ functions on $E_{G_{i+1}} \times [0, 1]$ which satisfy $\mathrm{Gen}_{\bar{x}}^{i+1}$ on $\tilde{N}_{\bar{x}}$ are open and dense. This follows by the same argument as in Step 2. Note that Gen_x^i was defined in Step 2. As in Step 2, we deduce that the invariant C^∞ functions which satisfy Gen_x^{i+1} on $L_{i+1}\backslash((V_\varepsilon \cup E_{G_i}) \times [0, 1])$ are dense. Thus, as in Step 2, we find a real analytic invariant approximation H_1 to $\tilde{H}$ which is close in C^n (where $n \geq 3$). Now all our conditions in Steps 1-4 on $\tilde{H}$ are stable to C^2 small perturbations except for the conditions on the non-linear terms at the bad orbits of E_{G_i}. We choose a *fixed* invariant C^∞ function $\bar{r}$ which has support near the bad orbits and is 1 very close to the bad orbits. Then $H_r \equiv \bar{r}\tilde{H} + (1 - \bar{r})H_1$ is easily seen to be a C^∞ invariant function which is close to $\tilde{H}$ in the C^n norm if H_1 is close to $\tilde{H}$ in the C^n norm. Moreover, H_r is unchanged near the bad orbits and is real analytic except near the bad orbits. It follows that H_r still has all the properties of Steps 2-4 on E_{G_i} and has Gen_x^{i+1} on $E_{G_{i+1}}\backslash E_{G_i}$.

We now continue much as in Steps 2-4 to obtain the structure of solutions being good on $E_{G_{i+1}} \cap \Omega$. There are a couple of points to be noted. Firstly, on $L_{i+1}\backslash(E_{G_i} \times [0, 1])$, either $N(D_x^2\tilde{H}(x, t))$ is one-dimensional (which are points which behave well under perturbations) or $\tilde{H}$ is real analytic near b Thus we can repeat all the arguments of Steps 2-4 to obtain a perturbec map $\tilde{H}$ which has the property that $\nabla_x \tilde{H}(x, t) = 0$ has only a finite number of orbits of solutions near $E_{G_{i+1}} \cap \Omega$ for each $t \in [0, 1]$. Secondly, whenever we perturb $\tilde{H}$ to H_1 (as in Steps 2-4), we then replace H_1 by $\bar{r}\tilde{H} + (1 - \bar{r})H_1$. This is still a perturbation close in C^n, « mostly » real analytic and the function is unchanged near the bad orbits in E_{G_i}. Moreover, as we continue our process, we will gain extra bad orbits. However, these will be finite. We always choose our perturbations to leave $\tilde{H}$ unchanged near all the earlier bad orbits. After a finite number of steps, we will have completed the proof.

Remark. — It seems probable that the result holds for general Lie groups. The only difficult is the bifurcation part of argument. This certainly can be proved for C^2 maps by using non-polynomial homogeneous mappings. Thus we can obtain an analogue of Proposition 1 for C^2 maps and general Lie groups. Note that, in § 1, our arguments are still valid if we have an infinite number of orbits of critical points provided that they lie in « small » sets. This requires some care.

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