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Analyse non linéaire

A class of convex non-coercive functionals and masonry-like materials

by

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ABSTRACT. — We consider a class of functionals of the strain (or of the gradient), whose main feature is that they are not coercive when the forces are zero, while they are coercive under suitable assumptions for the load. The main application is to the problem of static equilibrium for a class of elastic materials in which the stress is constrained to be negative semi-definite. Functions of bounded deformation and measure theory are a basic technical tool in the paper.

RÉSUMÉ. — Nous considérons une classe de fonctionnelles de la tension (ou du gradient) dont la caractéristique principale est qu'elles ne sont pas coercives lorsque les forces sont nulles, alors qu'elles le sont sous des hypothèses convenables pour la charge. On applique principalement ces résultats au problème de l'équilibre statique pour une classe de matériaux élastiques lorsque la tension est supposée semi-définie négative. Les fonctions de déformation bornée et la théorie de la mesure sont les outils techniques de base dans cet article.

INDEX

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1. Introduction.

In this paper we consider the problem of minimizing functionals of the type

$$(0.1)\quad E(u)=\int_\Omega \| P_K\varepsilon(u)\|^2 dx - \int_\Omega f(x)u(x)dx - \int_{\partial\Omega} F(x)u(x)d\mathscr{H}^{n-1}$$

where Ω is an open set in $\mathbb{R}^n$, $u:\Omega \to \mathbb{R}^n$; $f:\Omega \to \mathbb{R}^n$ and $F:\partial\Omega \to \mathbb{R}^n$ are given;

$$\varepsilon(u) = \{ \varepsilon_{ij}(u) \}_{i,j=1,\dots,n}, \qquad \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right);$$

K is a closed convex cone in the space V of the $n \times n$ symmetric real matrices; $P_K : V \to K$ is the orthogonal projection on K, with respect to a given scalar product $\langle , \rangle$ in V, whose associated norm is denoted $\| \ \|$.

The main motivation for this work is that it provides the existence of a solution to the following problem 0.1, compare [*14*] [*10*].

PROBLEM 0.1. — *Given a bounded open set* $\Omega \subset \mathbb{R}^3$ *and functions* $f:\Omega \to \mathbb{R}^3$, $F:\partial\Omega \to \mathbb{R}^3$, *find functions* $v:\Omega \to \mathbb{R}^3$

$$\sigma = \{ \sigma_{ij} \}_{i,j=1,2,3}, \quad \sigma_{ij} = \sigma_{ji}, \quad \sigma_{ij}:\Omega \to \mathbb{R}$$

such that (0.2) *and* (0.3) *below hold:*

$$(0.2)\quad \begin{cases} \sigma(x) \text{ is a negative semi-definite matrix, for all } x \in \Omega \\ \varepsilon(v)(x) = A^{-1}\sigma(x) + \lambda(x) \\ \lambda_{ij}(x)(\sigma_{ij}(x) - \tau_{ij}) \geqslant 0 \text{ for all negative semi-definite matrices } \tau, \\ \text{for all } x \in \Omega \end{cases}$$

where A *is a symmetric positive definite linear operator from the space of symmetric matrices to itself, and we have used the convention of summation over repeated indices;*

$$(0.3)\qquad \begin{cases} \dfrac{\partial}{\partial x_j}\sigma_{ij}(x) + f_i(x) = 0 & \text{in } \Omega \\ \sigma_{ij}(x)\nu_j(x) = \mathrm{F}_i & \text{on } \partial\Omega \end{cases}$$

where $\nu(x)$ *is the outward unit normal to* $\partial\Omega$ *at* x.

If we set

$$\mathrm{K}_0 = \{\, \sigma \in \mathrm{V} \mid \sigma \text{ is negative semi-definite} \,\}$$

the last line in (0.2) means that

$$\lambda(x) \begin{cases} = 0 \text{ if } \sigma(x) \text{ is in the interior of } \mathrm{K}_0, \text{ i. e. if } \sigma(x) \text{ is negative definite} \\ \text{is a normal vector to the convex set } \mathrm{K}_0 \text{ at } \sigma(x) \text{ if } \sigma(x) \in \partial\mathrm{K}_0. \end{cases}$$

In particular, if $\sigma(x) = 0$, $\lambda(x)$ can be any positive semidefinite matrix.

The mechanical interpretation of the objects involved in problem 0.1 is as follows: Ω is the portion of space occupied by a body in its underformed state; the vector field $v(x)$ is a displacement field which describes a (infinitesimal) deformation of the body, in the sense that $x + v(x)$ is the position in the deformed state of that particle of the body that was in x before the deformation; the function f is a density of body force and F is a density of force at the boundary; the matrix $\sigma(x)$ is the stress tensor at the point x; the matrix $\varepsilon(v)(x)$ is the strain tensor at x, associated to the displacement v.

The mechanical interpretation of conditions (0.2) and (0.3) is as follows: condition (0.3) states that the stress σ balances the load f, F, i. e. that the body is in static equilibrium; condition (0.2), which relates the stress to the strain, is the constitutive law of the body. Condition (0.2) has a few distinctive features. First, it says that the stress can only be negative semi-definite; this means that for each point $x \in \Omega$ and for each surface element S, having normal a, through x, the tension $\mathrm{F} = \sigma(x).a$ through S is directed against S, because $\mathrm{F}.a = \sigma_{ij}(x)a_j a_i \leqq 0$; in other words, the material is incapable of sustaining tensile stress. Second, the strain is the sum of two parts: one part λ, which is positive semi-definite and orthogonal to the stress, and another « elastic » part which is proportional to the stress through the operator A^{-1}.

Condition (0.2) is a possible mathematical model of the behaviour exibited by rock, concrete, walls, and we shall refer to it as to the *constitutive law for masonry-like materials*, compare [*10*] [*14*] [*18*; chapter 4]. Actually condition (0.2), in two dimensions, is exactly equivalent to (2.13), (2.14) in [*14*], however our formulation has the advantage of making clearer the role played by the convexity of K_0. In fact it is clear that (0.2) is a special case of the constitutive law V, 6.1 in [*11*], which is usually called Hencky's law. We remind to the reader that Hencky's law was introduced as a possible model for perfect plasticity, in which case the convex set K of the admissible stresses is a cylinder that contains 0 in its interior. For some recent work on Hencky plasticity we refer to [*26*] [*6*] [*7*] [*3*] [*28*] [*17*]. In our problem 0 is *not* an interior point of the cone K_0 of the admissible stresses, and this fact is precisely the source of the lack of coerciveness of the deformation energy.

To our knowledge, the first general treatment of the existence problem for the constitutive law (0.2) is given by M. Giaquinta and E. Giusti [*14*], in the isotropic case in two dimensions. In the paper [*14*], the existence of a displacement field v is proved by minimizing a suitable energy functional, and σ is then recovered from v. This method is similar to the one adopted in [*7*] [*2*], for other problems, and it will be followed here, too. On the other hand, in our opinion, the particular case considered in [*14*] hinders the structure of the problem, so that the extension of the results, for instance to the three dimensional case, seems to be at least rather complicated. The larger class of problems considered in the present paper seems to have interesting and simple properties on its own and includes in particular non isotropic masonry-like materials in two or three dimensions.

The energy functional that we shall consider to solve problem 0.1 is functional (0.1), with the following particular choices:

$$\text{(0.4)} \qquad K = \{ A^{-1}\alpha \mid \alpha \in K_0 \}, \langle \alpha, \beta \rangle = (A\alpha, \beta)$$

where $(\alpha, \beta) = \alpha_{ij}\beta_{ij} = \operatorname{tr}(\alpha \circ \beta^t)$ is the standard scalar product in V.

The term $\frac{1}{2}\int_\Omega \| P_K\varepsilon(u) \|^2 dx$ is the stored deformation energy and one of its main features is that it is *not* coercive on any resonable normed space of admissible competing displacement fields. On the other hand, if one considers also the contribution of the virtual work of the forces, under suitable safety conditions for the load, for all the functions $u \in C^1(\overline{\Omega}, \mathbb{R}^3)$, one has

$$E(u) \geqq c_0 \left\{ \int_\Omega \| P_K\varepsilon(u) \|^2 dx + \int_K | \varepsilon(u) | \right\} - c_1$$

where $c_0 > 0$, $c_1 \geqq 0$. This suggests to minimize $E(u)$ on the space

$$\mathscr{U}(\Omega) = \{ u \in L^1(\Omega, \mathbb{R}^3) \mid \varepsilon(u) \text{ is a measure of bounded total variation in } \Omega,\ P_K\varepsilon(u) \in L^2(\Omega, V) \}$$

where $P_K\varepsilon(u)$ is the « projection on K » (see section 2) of the measure $\varepsilon(u)$. Using the compactness and the other known [*23*] [*19*] properties of the space $BD(\Omega) = \{ u \in L^1(\Omega, \mathbb{R}^3) \mid \varepsilon(u)$ is a measure of bounded total variation $\}$ one can obtain the existence of a minimum point for $E(u)$ in $\mathscr{U}(\Omega)$ by the direct method of Calculus of Variation, as soon as one has suitable semicontinuity and coerciveness properties for E(u). Sufficient conditions on the load for the existence of a minimum of functional E(u) are given in section 8.

Once a minimum point v of E(u) is found, one sets

$$\sigma = \mathrm{A}\mathrm{P}_{\mathrm{K}}\varepsilon(v) \tag{0.5}$$

and uses the first variation formula for E(u) at v to get the equilibrium condition (0.3) satisfied, at least in the sense of distributions, while conditions (0.2) are satisfied automatically.

We do not have the uniqueness of v.

The stress field defined in (0.4) is proved to be (section 7) a minimum point for the complementary energy functional

$$\mathrm{I}(\tau) = \frac{1}{2}\int_\Omega (\mathrm{A}^{-1}\tau, \tau)dx$$

among a suitable class of competing admissible stress fields. Since the functional I(τ) is strictly convex, it follows that one gets the same stress σ, no matters which minimum point v of E(u) one starts with.

Problem (0.1) is a problem with Neumann boundary conditions. One might like to consider instead a problem where the displacement at the boundary is prescribed, and it is, for instance, zero. Actually, one finds that Dirichlet boundary conditions are not natural for our energy functional (compare with the situation in minimal surfaces [*16*] [*20*] or in Hencky plasticity [*6*] [*26*]); what one actually obtains, when trying to prescribe a zero displacement on $\partial\Omega$, is a solution pair $\{ v, \sigma \}$ that satisfies a *unilateral* condition at the boundary of the type

$$\begin{aligned} u(x) = -\, t(x)\nu(x), \qquad & t(x) \geqq 0 \\ (\sigma(x)\nu(x)).\nu(x) & \leqq 0 \\ \sigma(x)\nu(x) = 0 \qquad \text{if} \qquad & t(x) > 0 \end{aligned}$$

We remark that the whole theory that we have described applies to the case of general convex cones K and general operators A in n dimensions, though some restrictions on the choice of Ω have to be made in the existence theorems, because of the need of a certain approximation result which is available (section 10) for all Lipschitz domains in $\mathbb{R}^2$, but, so far, only for star shaped Lipschitz domains in $\mathbb{R}^n$. The general theory is developed in parts I and II, while in part III we prove some special results which are particular to the choice of K made in (0.4).

The problem of regularity of v and σ seems to be open. Actually, it seems reasonable to expect the displacement v to be possibly singular, and the measure λ not to be absolutely continuous. In the case of masonry-like materials, a possible typical singularity is given by a normal discontinuity of v across a C^1 surface. I am inclined to hope for some more regularity for σ than for v. A first conjecture might be that σ has in some sense a trace on the singular set of the measure λ and that this trace is zero. This behaviour would resemble the one described in [3] for Hencky plasticity.

The theory developed in part II, under suitable assumptions, should extend to the case of non-homogeneous materials, i. e. to the case of an operator A which depends on the point $x \in \Omega$. This way, keeping the cone K_0 fixed, one would have a variable cone K. The case of a constant operator A and a variable cone K may also be considered. The work done in [2] deals precisely with such a case, where the cone K is the whole space V or a half-space in V, depending on the given instantaneous stress state σ. The particularity of the problem considered in [2] hindered, at that time, the understanding of the underlying structure.

A final special mention can be made to the fact that one may consider functionals of the type

$$\int_\Omega \| P_K Du \|^2 dx - \int_\Omega fu dx \tag{0.6}$$

where $u : \Omega \to \mathbb{R}$ and K is a convex closed cone in $\mathbb{R}^n$. Using the results in part I, a parallel theory to the one developed in part II can be made also for functionals (0.6), but I do not know of any physical phenomenon in which such functionals of the gradient arise naturally.

I would like to thank S. Luckhaus for some useful hints about the coerciveness in the case of prescribed displacement at the boundary. I am also grateful to S. Di Pasquale and P. Villaggio for some useful conversations and I am especially grateful to M. Giaquinta and E. Giusti for drawing my attention to the problem here considered.

PART I

GENERAL RESULTS

2. Vector measures with values in a convex cone

Let Ω be an open set in $\mathbb{R}^n$ and let V be a N-dimensional vector space. We shall denote by $M(\Omega, V)$ the space of the V-valued (Radon) measures in Ω and we shall denote by $M^+(\Omega)$ the space of the positive (Radon) measures in Ω. If $\alpha \in M(\Omega, V)$ and $\mu \in M^+(\Omega)$, we denote by

$$\frac{d\alpha}{d\mu}(x) = \lim_{\rho \to 0^+} \frac{\alpha(B_\rho(x))}{\mu(B_\rho(x))}$$

the density function of the measure α with respect to μ, where

$$B_\rho(x) = \{ y \in \mathbb{R}^n \mid |x - y| < \rho \}.$$

Let $\langle \cdot, \cdot \rangle$ be a given scalar product in V. If $\alpha \in M(\Omega, V)$, we denote by

$$\| \alpha \| (B) = \sup \left\{ \sum_{i=1}^{n_0} \| \alpha(B_i) \| \,\middle|\, \bigcup_{i=1}^{n_0} B_i \subset B, \quad B_i \cap B_j = \emptyset \quad \text{if} \quad i \neq j \right\}$$

the absolute variation measure associated to α. The measure α is absolutely continuous with respect to $\|\alpha\|$ (which fact is denoted $\alpha \ll \|\alpha\|$) and one has $\alpha(B) = \int_B \frac{d\alpha}{d\|\alpha\|}(x) d\|\alpha\|$ for all Borel sets B in Ω.

Throughout this section, K will be a closed convex cone in V.

DEFINITION 2.1. — *We say that a measure $\alpha \in M(\Omega, V)$ is a K-valued measure, and we write $\alpha \in M(\Omega, K)$, if one has*

$$(2.1) \qquad \frac{d\alpha}{d\|\alpha\|}(x) \in K \qquad \text{for} \quad \|\alpha\|\text{-almost all } x \in \Omega$$

We have to remark that, though the measure $\|\alpha\|$ depends on the scalar product given in V, condition (2.1) does not, because K is a cone and the density $\frac{d\alpha}{d\|\alpha\|}(x)$ is determined up to a positive multiplicative factor depending on x. In fact, if μ is any positive measure such that $\|\alpha\| \ll \mu$, by (1.7) and by the standard results on the differentiation of measures, one has

$$\frac{d\alpha}{d\mu}(x) = \frac{d\alpha}{d\|\alpha\|}(x) \quad \frac{d\|\alpha\|}{d\mu}(x) \quad \|\mu\| - \text{a. e.}$$

Clearly, the set $M(\Omega, K)$ of the K-valued measures in Ω is a convex cone in $M(\Omega, V)$. Later we shall see (lemma 2.5) that $M(\Omega, K)$ is closed with respect to the weak convergence of measures.

We shall consider the map $P_K : M(\Omega, V) \to M(\Omega, K)$ defined as

$$P_K\alpha = p_K\left(\frac{d\alpha}{d\|\alpha\|}(x)\right)\|\alpha\|$$

where $p_K : V \to K$ is the orthogonal projection on K with respect to the scalar product $\langle\, , \rangle$.

Obviously, if $\alpha \in M(\Omega, V)$, then α is a K-valued measure if and only if $p_K\alpha = \alpha$, which in turn is true if and only if $P_{K^\perp}\alpha = 0$ where

$$K^\perp = \{ \beta \in V \mid \langle \alpha, \beta \rangle \leqslant 0 \quad \text{for all} \quad \alpha \in K \}$$

is again a closed convex cone, which is called the orthogonal cone to K with respect to the scalar product $\langle , \rangle$.

If μ is any positive measure such that $\| \alpha \| \ll \mu$ one has

$$\mathrm{P}_{\mathrm{K}}\alpha = p_{\mathrm{K}}\left(\frac{d\alpha}{d\mu}(x)\right)\mu$$

Now we collect a few results that will be used later.

LEMMA 2.2. — *Let $\mu \in \mathrm{M}^{+}(\Omega)$ be a bounded positive measure and let $f : \Omega \to \mathrm{K}$ be a μ-measurable function. Then, for all $\varepsilon > 0$, there exists a function $g \in \mathrm{C}_0^0(\Omega, \mathrm{K})$ such that*

$$\mu(\{ x \in \Omega \mid f(x) \neq g(x) \}) < \varepsilon \tag{2.2}$$

If in addition one has that if $\| f(x) \| \leqq \mathrm{M}$ for μ-almost all $x \in \Omega$, then one can find the function g such that $\| g(x) \| \leqq \mathrm{M}$ μ-almost everywhere in Ω.

Proof. — Let $f : \Omega \to \mathrm{K}$ be μ-measurable and take a number $\varepsilon > 0$. By Lusin and Tietze theorems there exists a function $g_1 \in \mathrm{C}_0^0(\Omega, \mathrm{V})$ such that

$$\mu(\{ x \in \Omega \mid g_1(x) \neq f(x) \}) < \varepsilon$$

If we set

$$g(x) = p_{\mathrm{K}}(g_1(x))$$

we have immediately

$$g \in \mathrm{C}_0^0(\Omega, \mathrm{K})$$
$$g_1(x) = f(x) \Rightarrow g(x) = f(x)$$

and (2.2) follows.

If in addition one assumes that $\| f(x) \| \leqq \mathrm{M}$ μ-a. e. in Ω, one can find the function g_1 so that $\| g_1 \|_{\mathrm{C}^0} \leqq \mathrm{M}$ and it follows that

$$\| g(x) \| \leqq \| g_1(x) \| \leqq \mathrm{M} \quad \text{for all } x .$$

Q. E. D.

LEMMA 2.3. — *Let $\mu \in \mathrm{M}^{+}(\Omega)$ be a bounded positive measure and let $f : \Omega \to \mathrm{K}$ be a μ-measurable function such that*

$$\int_{\Omega} \| f \| \, d\mu < + \infty$$

then for any number $\varepsilon > 0$ there exists a function $g \in \mathrm{C}_0^0(\Omega, \mathrm{K})$ such that

$$\int_{\Omega} \| f - g \| \, d\mu < \varepsilon$$

Proof. — Take a number $\varepsilon > 0$ and let $\mathrm{M} > 0$ be such that

$$\int_{\mathrm{E_M}} \| f \| \, d\mu < \frac{\varepsilon}{2}$$

where $E_M = \{ x \in \Omega \mid \| f(x) \| > M \}$. Consider then the function $f_1 : \Omega \to K$ defined as

$$f_1(x) = \begin{cases} f(x), & \text{if } \| f(x) \| \leqq M \\ M \dfrac{f(x)}{\| f(x) \|}, & \text{if } f(x) > M \end{cases}$$

By lemma 2.2 one can find a function $g \in C_0^0(\Omega, K)$ such that

$$\mu(\{ x \in \Omega \mid g(x) \neq f_1(x) \}) < \frac{\varepsilon}{4M}$$

$$\| g(x) \| \leqq M \quad \text{for all } x$$

and one has

$$\int_\Omega \| f-g \| \, d\mu \leqq \int_\Omega \| f_1 - g \| \, d\mu + \int_\Omega \| f - f_1 \| = \int_{\{x\in\Omega \mid f_1(x) \neq g(x)\}} \| f_1 - g \| \, d\mu$$

$$+ \int_{E_M} \| f - f_1 \| \leqq \varepsilon \qquad \text{Q. E. D.}$$

LEMMA 2.4. — *Assume that* $\alpha \in M(\Omega, V)$, *then one has* $\alpha \in M(\Omega, K)$ *if and only if*

$$(2.3) \qquad \int_\Omega g d\alpha \leqq 0$$

for all $g \in C_0^0(\Omega, K^\perp)$.

Proof. — If $\alpha \in M(\Omega, K)$ and $g \in C_0^0(\Omega, K^\perp)$ one has

$$\int_\Omega g d\alpha = \int_\Omega \left\langle g(x), \frac{d\alpha}{d\|\alpha\|}(x) \right\rangle d\|\alpha\| \leqq 0$$

On the other hand, consider the function $\phi(x) = p_{K^\perp}\left(\dfrac{d\alpha}{\|\alpha\|}(x) \right)$ which is $\|\alpha\|$-summable in Ω. By lemma 2.3 there exists a sequence of functions $g_j \in C_0^0(\Omega, K^\perp)$ such that $g_j \to \phi$ in $L^1(\Omega, \|\alpha\|)$, hence one has

$$\int_\Omega \left\langle g_j(x), \frac{d\alpha}{d\|\alpha\|}(x) \right\rangle d\|\alpha\| \to \int_\Omega \left\langle \phi(x), \frac{d\alpha}{d\|\alpha\|}(x) \right\rangle d\|\alpha\|$$

$$= \int_\Omega \left\| p_{K^\perp}\left(\frac{d\alpha}{d\|\alpha\|}(x) \right) \right\|^2 d\|\alpha\|$$

and it is clear that if (2.3) holds for all g_j then one must have

$$p_{K^\perp}\left(\frac{d\alpha}{d\|\alpha\|}(x) \right) = 0 \quad \|\alpha\|\text{-a. e. in } \Omega. \qquad \text{Q. E. D.}$$

LEMMA 2.5. — *If* $\alpha_j \in M(\Omega, K)$, $\alpha \in M(\Omega, V)$ *and one has*

$$(2.4) \qquad \alpha_j \to \alpha \quad \textit{weakly in } M(\Omega, V)$$

then one has also $\alpha \in M(\Omega, K)$.

Proof. — For any $g \in C_0^0(\Omega : K^\perp)$ and for all j, by lemma 2.4, one has

$$\int_\Omega g d\alpha_j \leqq 0$$

then, by (2.4), one gets

$$\int_\Omega g d\alpha \leqq 0$$

and, again by lemma 2.4, one has that $\alpha \in M(\Omega, K)$. Q. E. D.

The next lemma states that $\alpha = P_K\alpha + P_{K^\perp}\alpha$ is the « smallest » decomposition of the measure α in a pair of K and $K^\perp$-valued measures.

LEMMA 2.6. — *Assume that* $\alpha \in M(\Omega, V)$, $\alpha_1 \in M(\Omega, K)$, $\alpha_2 \in M(\Omega, K^\perp)$ *and assume also that* $\alpha_1 + \alpha_2 = \alpha$. *Then one has*

$$\| \alpha_1 \| \geqq \| P_K\alpha \| \tag{2.5}$$

$$\| \alpha_2 \| \geqq \| P_{K^\perp}\alpha \| \tag{2.6}$$

as measures. Moreover, if the equality (as measures) *holds in either one of* (2.5), (2.6) *one has that*

$$\alpha_1 = P_K\alpha, \qquad \alpha_2 = P_{K^\perp}\alpha \tag{2.7}$$

Proof. — Consider the positive measure $\mu = \| \alpha_1 \| + \| \alpha_2 \|$. Clearly one has

$$\alpha = \frac{d\alpha}{d\mu} d\mu = \left(\frac{d\alpha_1}{d\mu} + \frac{d\alpha_2}{d\mu}\right) d\mu$$

where

$$\frac{d\alpha_1}{d\mu}(x) \in K, \qquad \frac{d\alpha_2}{d\mu}(x) \in K^\perp \qquad \mu\text{-a. e. in } \Omega$$

By the properties of minimum distance of the projections p_K and $p_{K^\perp}$ one has that

$$\left\| p_K\left(\frac{d\alpha}{d\mu}(x)\right) \right\| \leqq \left\| \frac{d\alpha_1}{d\mu}(x) \right\|$$

$$\left\| p_{K^\perp}\left(\frac{d\alpha}{d\mu}(x)\right) \right\| \leqq \left\| \frac{d\alpha_2}{d\mu}(x) \right\|$$

for μ-almost all $x \in \Omega$. On the other hand, recalling also that $p_K(t\beta) = tp_K(\beta)$ for all $t \geqslant 0$, $\beta \in V$, one has that

$$\frac{dP_K\alpha}{d\mu}(x) = \frac{dP_K\alpha}{d\|\alpha\|}(x) \quad \frac{d\|\alpha\|}{d\mu}(x) = p_K\left(\frac{d\alpha}{d\|\alpha\|}(x)\right) \quad \frac{d\|\alpha\|}{d\mu}(x)$$

$$= p_K\left(\frac{d\alpha}{d\|\alpha\|}(x) \quad \frac{d\|\alpha\|}{d\mu}(x)\right) = p_K\left(\frac{d\alpha}{d\mu}(x)\right)$$

hence it follows that

$$\left\| \frac{d\mathrm{P_K}\alpha}{d\mu}(x) \right\| \leqq \left\| \frac{d\alpha_1}{d\mu}(x) \right\| \qquad \mu\text{-a. e. in } \Omega$$

and one gets

$$\int_{\mathrm{B}} \| \mathrm{P_K}\alpha \| = \int_{\mathrm{B}} \left\| \frac{d\mathrm{P_K}\alpha}{d\mu}(x) \right\| d\mu \leqq \int_{\mathrm{B}} \left\| \frac{d\alpha_1}{d\mu}(x) \right\| d\mu = \int_{\mathrm{B}} \| \alpha_1 \|$$

for all Borel sets B, which is equivalent to (2.5). Similarly one gets (2.6).

Now, to prove the second part of the lemma, assume for instance that

$$\| \alpha_1 \| = \| \mathrm{P_K}\alpha \| \quad \text{as measures}$$

Then one has

$$\left\| \frac{d\alpha_1}{d\mu}(x) \right\| = \left\| \frac{d\mathrm{P_K}\alpha}{d\mu}(x) \right\| = \left\| p_{\mathrm{K}}\left(\frac{d\alpha}{d\mu}(x) \right) \right\| \qquad \mu\text{-a. e. in } \Omega$$

Recalling the properties of the projections it follows that

$$\frac{d\alpha_1}{d\mu}(x) = p_{\mathrm{K}}\left(\frac{d\alpha}{d\mu}(x) \right) = \frac{d\|\alpha\|}{d\mu}(x) \quad p_{\mathrm{K}}\left(\frac{d\alpha}{d\|\alpha\|}(x) \right)$$

and similarly for α_2, from which (2.7) follows. Q. E. D.

LEMME 2.7. — *Assume that* $\alpha_j, \alpha \in \mathrm{M}(\Omega, \mathrm{V})$ *and that*

(2.8) $$\alpha_j \rightharpoonup \alpha$$

(2.9) $$\int_\Omega \| \alpha_j \| \to \int_\Omega \| \alpha \|$$

then one has

(2.10) $$\mathrm{P}_{\mathrm{K}_j}\alpha \rightharpoonup \mathrm{P_K}\alpha$$

(2.11) $$\int_\Omega \| \mathrm{P_K}\alpha_j \| \to \int_\Omega \| \mathrm{P_K}\alpha \|$$

Proof. — The function $\beta \to \| p_{\mathrm{K}}(\beta) \|$ is continuous and positively homogeneous and by theorem 3 of [*22*] one obtains (2.11). To prove (2.10), we shall see that for any sequence $j_h \to \infty$ there exists a subsequence j_{h_i} such that

$$\mathrm{P_K}\alpha_{j_{h_i}} \rightharpoonup \mathrm{P_K}\alpha$$

In fact if $j_h \to \infty$, as the measure $\| \mathrm{P_K}\alpha_j \|$, $\| \mathrm{P_{K^\perp}}\alpha_j \|$ are equibounded, there exists a subsequence j_{h_i} such that

$$\mathrm{P_K}\alpha_{j_{h_i}} \rightharpoonup w_1$$

$$\mathrm{P_{K^\perp}}\alpha_{j_{h_i}} \rightharpoonup w_2$$

where, using also lemma 2.5, $w_1 \in M(\Omega, K)$ and $w_2 \in M(\Omega, K^\perp)$ and

$$\alpha = w_1 + w_2$$

Now, by (2.11) and the lower-semicontinuity of the total variation with respect to the weak convergence, one has

$$\int_\Omega \| w_1 \| \leqq \lim_{i \to \infty} \int_\Omega \| P_K \alpha_{j_{h_i}} \| = \int_\Omega \| P_K \alpha \| \tag{2.12}$$

On the other hand, by lemma 2.6, the measure

$$\lambda = \| w_1 \| - \| P_K \alpha \|$$

is a positive measure and by (2.12) one must have

$$\lambda(\Omega) = \int_\Omega \| w_1 \| - \int_\Omega \| P_K \alpha \| = 0$$

In conclusion, one has $\| w_1 \| = \| P_K \alpha \|$ as measures and by the second part of lemma 2.6 one obtains $w_1 = P_K \alpha$. Q. E. D.

3. The functionals $\int_\Omega \| P_K(\alpha)(x) \|^2 dx$ and their lower-semicontinuity

Let V be a N-dimensional vector space with a scalar product $\langle .,. \rangle$, let K be a closed convex cone in V and let us denote by $\mathscr{L}^n$ the n-dimensional Lebesgue measure. We shall use the notation introduced in section 2.

For any measure $\alpha \in M(\Omega, V)$ we shall denote by

$$\alpha = \alpha^a + \alpha^s = \alpha^a(x)dx + \alpha^s$$

the Lebesgue decomposition of α in an absolutely continuous and a singular part with respect to $\mathscr{L}^n \equiv dx$.

We shall consider the space

$$L_K(\Omega, V) = \left\{ \alpha \in M(\Omega, V) \mid P_K\alpha \ll \mathscr{L}^n \quad \text{and} \quad \int_\Omega \| P_K\alpha(x) \|^2 dx < +\infty \right\}$$

where, for brevity, we have set $P_K\alpha(x) = \dfrac{dP_K\alpha}{d\mathscr{L}^n}(x)$.

Clearly, one has $\alpha \in L_K(\Omega, V)$ if and only if $\alpha \in M(\Omega, V)$, $(P_K\alpha)^s = 0$ and $(P_K\alpha)^a(x) = P_K\alpha(x) \in L^2(\Omega, V)$.

LEMMA 3.1. — *One has* $P_K\alpha \ll \mathscr{L}^n$ *if and only if*

$$p_K\left(\frac{d\alpha}{d\mu}(x)\right) = 0 \qquad \mu^s\text{-a. e.} \quad in \quad \Omega \tag{3.1}$$

for any positive measure μ *such that* $\|\alpha\| \ll \mu$. *Moreover, if* $\alpha \in L_K(\Omega, V)$, *one has*

$$P_K\alpha(x) = p_K(\alpha^a(x)) \tag{3.2}$$

Proof. — If $\|\alpha\| \ll \mu$ then one has $(P_K\alpha)^s = p_K\left(\frac{d\alpha}{d\mu}(x)\right)\mu^s$ so that $(P_K\alpha)^s = 0$ if and only if (3.1) holds. Moreover, if $\alpha \in L_K(\Omega, V)$, using also (3.1) for $\mu = \|\alpha\|$, one has

$$P_K\alpha = p_K\left(\frac{d\alpha}{d\|\alpha\|}(x)\right)\|\alpha\| = p_K\left(\frac{d\alpha}{d\|\alpha\|}(x)\right)\|\alpha^a\|$$

where

$$\frac{d\alpha}{d\|\alpha\|}(x) = \frac{d\alpha^a}{d\|\alpha\|^a}(x) = \frac{d\alpha^a}{d\mathscr{L}^n}(x)\quad\left(\frac{d\|\alpha\|^a}{d\mathscr{L}^n}(x)\right)^{-1}\qquad \|\alpha\|\text{-a. e.}$$

and (3.2) follows. Q. E. D.

For later use, we list here a few simple properties of the projection P_K.

FACT 3.2. — *For any* $\alpha, \beta \in L_K(\Omega, V)$ *and* $t > 0$, *one has*

$$t\alpha \in L_K(\Omega, V) \tag{3.3}$$

$$P_K(t\alpha)(x) = tP_K\alpha(x) \tag{3.4}$$

$$\alpha + \beta \in L_K(\Omega, V) \tag{3.5}$$

$$P_K(\alpha + \beta)(x) = p_K(\alpha^a(x) + \beta^a(x)) \tag{3.6}$$

$$\|P_K(\alpha + \beta)(x)\| \leqq \|P_K\alpha(x)\| + \|P_K\beta(x)\| \tag{3.7}$$

We have the following semicontinuity result.

THEOREM 3.3. — *Assume that* $\alpha_j \in L_K(\Omega, V)$, $\alpha \in M(\Omega, V)$ *and that*

$$\alpha_j \to \alpha \quad \textit{weakly in } M(\Omega, V) \tag{3.8}$$

$$\int_\Omega \|P_K\alpha_j(x)\|^2 dx \leqq c \quad \text{for } \textit{all } j \tag{3.9}$$

where C *is some positive number. Then one has* $\alpha \in L_K(\Omega, V)$ *and*

$$\int_K \|P_K\alpha(x)\|^2 dx \leqq \min_{j\to\infty}\lim \int_\Omega \|P_K\alpha_j(x)\|^2 dx \tag{3.10}$$

Proof. — It is sufficient to show that for any subsequence $\gamma_h = \alpha_{j_h}$ there exists a further subsequence $\beta_i = \gamma_{h_i}$ such that

$$\min_{i\to\infty}\lim \int_\Omega \|P_K\beta_i(x)\|^2 dx \geqq \int_\Omega \|P_K\alpha(x)\|^2 dx \tag{3.11}$$

Let γ_n be a subsequence of α_j. By (3.9) there is a subsequence $\beta_i = \gamma_{h_i}$ such that

$$P_K \beta_i \rightharpoonup g \qquad \text{in} \quad L^2(\Omega, V)$$

for some function $g \in L^2(\Omega, V)$, and one has

$$(3.12) \qquad \int_\Omega \| g(x) \|^2 dx \leqq \min_{i \to \infty} \lim \int_\Omega \| P_K \beta_i(x) \|^2 dx$$

On the other hand, one has the decompositions $\beta_i = P_K \beta_i + P_{K^\perp} \beta_i$ and $\alpha = g(x)dx + (\alpha - g(x)dx)$, where

$$P_{K^\perp} \beta_i \rightharpoonup \alpha - g(x)dx \qquad \text{in} \quad M(\Omega, V)$$

By lemma 2.5, it follows that $g(x)dx \in M(\Omega, K)$ and $(\alpha - g(x)dx) \in M(\Omega, K^\perp)$ and, by lemma 2.6, one has in particular

$$\| g(x) \| dx \geqq \| P_K \alpha \| \quad \text{as measures}$$

It follows that $P_K \alpha = \phi(x)dx$ for some function $\phi \in L^1(\Omega, V)$, which satisfies also $\| \phi(x) \| \leqq \| g(x) \|$ $\mathscr{L}^n$-a. e. In conclusion we have

$$\int_\Omega \| g(x) \|^2 dx \geqq \int_\Omega \| \phi(x) \|^2 dx$$

and, recalling (3.12), one has (3.11) and the proof is completed. Q. E. D.

LEMMA 3.4. — *Assume that* $\alpha \in L_K(\Omega, V)$, $\eta \in C_0^0(\mathbb{R}^n)$, $\eta \geqslant 0$ $\int \eta = 1$, and let A be an open set in $\mathbb{R}^n$ such that $A + \operatorname{spt} \eta \subset \Omega$. *If we consider the measure* $\alpha * \eta \in M(A, V)$ *defined as*

$$(3.13) \qquad \langle \alpha * \eta, g \rangle = \langle \alpha, \eta * g \rangle \quad \text{for all} \quad g \in C_0^0(A, \mathbb{R}^n)$$

then we have $\alpha * \eta \in L_K(A, V)$ *and*

$$\int_A \| P_K(\alpha * \eta)(x) \|^2 dx \leqslant \int_\Omega \| P_K \alpha(x) \|^2 dx$$

Proof. — Clearly, we have $\alpha * \eta = (P_K \alpha) * \eta + (P_{K^\perp} \alpha) * \eta$ and we claim that

$$(3.14) \qquad (P_K \alpha) * \eta \in M(\Omega, K)$$

$$(3.15) \qquad (P_{K^\perp} \alpha) * \eta \in M(\Omega, K^\perp)$$

In fact, for all $g \in C_0^0(A, K^\perp)$ one has

$$(3.16) \qquad \langle (P_K \alpha) * \eta, g \rangle = \langle P_K \alpha, \eta * g \rangle \leqq 0$$

because $(\eta * g)(x) = \int_\Omega \eta(x-y)g(y)dy \in K^\perp$ for all $x \in \Omega$, and (3.14) follows from lemma 2.4. Similarly one gets (3.15).

Now, by lemma 2.6 one has that

$$\| P_K(\alpha * \eta) \| \leqq \| (P_K\alpha) * \eta \| \tag{3.17}$$

as measures, and also $\mathscr{L}^n$-almost everywhere, if we think of their densities. Finally, using Jensen's inequality and the standard properties of convolutions, one has

$$\int_A \| (P_K\alpha) * \eta \|^2 \leqslant \int_\Omega \| P_K\alpha \|^2$$

and the proof is concluded by (3.17). Q. E. D.

4. Directional derivatives of the functional $\int_\Omega \| P_K\alpha \|^2$.

We use the notation introduced in section 3.

THEOREM 4.1. — *Assume that* $\alpha \in L_K(\Omega, V)$, $g \in L^2(\Omega, V)$, *and consider the measure* $\beta = g(x)dx$; *then, for all* $t \in \mathbb{R}$, *one has* $\alpha + t\beta \in L_K(\Omega, V)$ *and*

$$\frac{d}{dt} \int_\Omega \| P_K(\alpha + t\beta) \|^2 |_{t=0} = 2 \int_\Omega \langle P_K\alpha(x), g(x) \rangle \, dx \tag{4.1}$$

Proof. — Both β and $-\beta$ belong to $L_K(\Omega, V)$, hence, by 3.2, we have $\alpha + t\beta \in L_K(\Omega, V)$ for all $t \in \mathbb{R}$.

To prove (4.1) we use (3.6) and the fact that

$$\frac{d}{dt} \| p_K(u + tv) \|^2 = 2 \langle p_K(u), v \rangle \qquad \forall u, v \in V$$

which is in turn clear because $\| p_K(u) \| = \text{dist}\,(u, K^\perp)$. Q. E. D.

PART II

FUNCTIONALS OF THE STRAIN

5. General notation for part II.

In this part of the paper we assume that Ω is a Lipschitz domain in $\mathbb{R}^n$, i. e. Ω is a bounded connected open set with Lipschitz boundary. We denote by $\nu(x)$ the outward unit normal vector to $\partial\Omega$ at x.

We shall consider the vector space

$$V = \{ \alpha = \{ \alpha_{ij} \}_{i,j=1,\dots,n} \mid \alpha_{ij} \in \mathbb{R}, \quad \alpha_{ij} = \alpha_{ji} \}$$

of the $n \times n$ symmetric matrices with real entries, endowed with the standard scalar product of matrices (we use the convention of summation over repeated indices):

$$(\alpha, \beta) = \alpha_{ij}\beta_{ij} = \text{trace } (\alpha \circ \beta^t)$$

and the associated norm

$$| \alpha | = (\alpha, \alpha)^{1/2}$$

Moreover, we consider a symmetric linear operator $A : V \to V$ such that the quadratic form $\alpha \mapsto (A\alpha, \alpha)$ is positive definite, and the scalar product and the norm

$$\langle \alpha, \beta \rangle = (A\alpha, \beta)$$
$$\| \alpha \| = \langle \alpha, \alpha \rangle^{1/2}$$

We consider also a fixed closed convex cone K in V and we denote by $K^\perp$ the orthogonal cone to K with respect to $\langle \cdot, \cdot \rangle$. We denote by $p_K, p_{K^\perp}, p_K, p_{K^\perp}$ the projections (with respect to the scalar product $\langle \cdot, \cdot \rangle$) introduced in sections 1 and 2.

We define the space

$$\mathscr{U}(\Omega) = \{ u \in BD(\Omega) \mid P_K \varepsilon(u) \in L^2(\Omega, V) \}$$

where $BD(\Omega) = \{ u \in L^1(\Omega, \mathbb{R}^n) \mid \varepsilon(u)$ is a measure of finite total variation in $\Omega \}$ is the space of the vector fields of bounded deformation in Ω [*24*] [*23*] [*19*] [*6*].

We consider also the set

$$K_0 = \{ A\alpha \mid \alpha \in K \}$$

which is clearly a closed convex cone in V. We notice that, by definition, one has $\beta \in K^\perp$ if and only if $\langle \alpha, \beta \rangle \leqq 0$ for all $\alpha \in K$; that is, if $(A\alpha, \beta) \leqq 0$ for all $\alpha \in K$; that is, if $(\tau, \beta) \leqq 0$ for all $\tau \in K_0$. In conclusion, if we denote by K_0^* the orthogonal cone to K_0 with respect to the standard scalar product in V, we have

(5.1) $$K_0^* = K^\perp$$

We shall consider functionals of the type

(5.2) $$E(u) = \frac{1}{2}\int_\Omega \| P_K(\varepsilon(u))(x) \|^2 dx - \int_\Omega f(x)u(x)dx - \int_{\partial\Omega} F(x)u(x)d\mathscr{H}^{n-1}$$

where $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and

(5.3) $$f \in L^n(\Omega, \mathbb{R}^n), \qquad F \in L^\infty(\partial\Omega, \mathbb{R}^n)$$

A pair (f, F) as in (5.1) will be called a *load*. We notice that the functional $E(u)$ is defined for all the functions $u \in \mathcal{U}(\Omega)$, because $BD(\Omega) \subset L^{\frac{n}{n-1}}(\Omega, \mathbb{R}^n)$ and $u|_{\partial\Omega}$ is a well defined summable function on $\partial\Omega$ for all $u \in BD(\Omega)$.

6. The Neumann problem. Equilibrium and constitutive equations.

We shall consider the following

PROBLEM 6.1. — *Given a load* (f, F), *find a minimum point* v *for the functional* $E(u)$ *in the space* $\mathcal{U}(\Omega)$.

In this and the next sections we shall derive the properties of the minima of the functional $E(u)$. The existence of solutions to problem 6.1 will be proved in section 8, under suitable conditions for the load.

It is convenient to give the following

DEFINITION 6.2. — *Assume that* $\sigma \in L^1(\Omega, V)$, $f \in L^1(\Omega, \mathbb{R}^n)$, $F \in L^1(\partial\Omega, \mathbb{R}^n)$. *We shall say that*

$$\begin{aligned} \operatorname{div} \sigma + f = 0 \quad & \text{in} \quad \Omega \\ \sigma . \nu = F \quad & \text{on} \quad \partial\Omega \end{aligned}$$

in the sense of distributions in $\overline{\Omega}$, *if one has*

$$\int_{\Omega} \sigma_{ij}(x)\varepsilon_{ij}(\phi)(x)dx - \int_{\Omega} f(x)\phi(x)dx - \int_{\partial\Omega} F(x)\phi(x)d\mathcal{H}^{n-1} = 0$$

for all $\phi \in C^1(\overline{\Omega}, \mathbb{R}^n)$.

Using the notation introduced in section 5 we have:

THEOREM 6.3. — *Let* $v \in \mathcal{U}(\Omega)$ *be a minimum point for the functional* $E(u)$ *in* $\mathcal{U}(\Omega)$ *and consider the function* $\sigma : \Omega \to V$ *defined as*

$$\sigma(x) = AP_K(\varepsilon(v))(x)$$

then one has

(6.1, *i*) $$\sigma \in L^2(\Omega, K_0)$$

(6.1, *ii*) $$\varepsilon(v) = A^{-1}\sigma + \lambda$$

(6.1, *iii*) $$\lambda = P_{K^\perp}\varepsilon(v) \in M(\Omega, K_0^*)$$

and

(6.2) $$\begin{aligned} \operatorname{div} \sigma + f = 0 \quad & \text{in} \quad \Omega \\ \sigma . \nu = F \quad & \text{on} \quad \partial\Omega \end{aligned}$$

in the sense of distribution in $\overline{\Omega}$ (*definition* 6.2).

Proof. — Let $v \in \mathcal{U}(\Omega)$ be a minimum point for $E(u)$ in $\mathcal{U}(\Omega)$. Then, in view of theorem 4.1, for any function $\phi \in C^1(\Omega, \mathbb{R}^n)$ and for any $t \in \mathbb{R}$ one has that $v + t\phi \in \mathscr{U}(\Omega)$, and

$$(6.3)\quad 0 = \frac{d}{dt}E(u+t\phi)\,|_{t=0} = \int_\Omega \langle P_K(\varepsilon(v))(x), \varepsilon(\phi)(x) \rangle\, dx - \int_\Omega f\phi dx - \int_\Omega F\phi d\mathscr{H}^{n-1}$$

that is
$$\int_\Omega (AP_K\varepsilon(v)(x), \varepsilon(\phi)(x))dx - \int_\Omega f\phi dx - \int_{\partial\Omega} F\phi d\mathscr{H}^{n-1} = 0$$

and the equilibrium conditions (6.2) follow. On the other hand one has

$$\varepsilon(v) = P_K\varepsilon(v) + P_{K^\perp}\varepsilon(v) = A^{-1}\sigma + \lambda$$

where $\lambda = P_{K^\perp}\varepsilon(v) \in M(\Omega, K^\perp) = M(\Omega, K_0^*)$. Q. E. D.

The constitutive law (6.1) is the general kind of law that one can have satisfied by minimizing functionals of the type (5.2).

We want to remark that conditions (6.1) are equivalent to (0.2) if λ is an absolutely continuous measure; on the other hand if λ has a non zero singular part, we have the somewhat unpleasant fact that the stress field σ, which is *a priori* defined just $\mathscr{L}^n$-a. e. in Ω, need not be defined in the set Σ where the measure λ^s is concentrated. However it seems not unreasonable to expect the stress to have a trace on Σ, and this trace to be zero (compare [*3*]).

In part III, we shall see that in the case of masonry-like materials, if Σ is a C^1 surface, then the discontinuity of the displacement has to be normal to the surface. The problem of the structure of the set Σ seems to be open. For a first study of the structure of the singular set of general functions in $BD(\Omega)$, we refer to [*19*]. For functions in $BV(\Omega)$, the structure of the set where the $(n-1)$-dimensional part of the measure $(Du)^s$ is concentrated is known [*15*; theorem 4.5.9].

The following proposition will be needed in proving the principle of the minimum complementary energy for the stress.

PROPOSITION 6.4. — If $v \in \mathscr{U}(\Omega)$ is a minimum point for E, then one has

$$(6.4)\qquad \int_\Omega \| P_K\varepsilon(v)(x) \|^2 dx - \int_\Omega fv dx - \int_\Omega Fv d\mathscr{H}^{n-1} = 0$$

Proof. — Clearly, the function $h(t) = E(v + tv)$ has a minimum in $t = 0$ and (6.4) is just $h'(0) = 0$.

7. Admissible stress fields. Complementary energy. Uniqueness of the stress.

The main result in this section is theorem 7.7, from which the uniqueness of the stress follows. We have to start with a bunch of definitions.

DEFINITION 7.1. — *Given a load* (f, F) *as in (5.3) and a stress field* $\tau \in L^2(\Omega, V)$, *we shall say that* τ *is* (f, F)*-admissible for problem 6.1, or, shortly, admissible, if one has*

$$\tau(x) \in K_0 \quad \text{for } \mathscr{L}^n\text{-almost all } x \in \Omega$$

and

$$\begin{cases} \operatorname{div} \tau + f = 0 & \text{in } \Omega \\ \tau . \nu = F & \text{on } \partial\Omega \end{cases}$$

in the sense of distributions in $\overline{\Omega}$.

REMARK 7.2. — By theorem 6.3, if v is a minimum point for $E(u)$ then the function $\sigma(x) = AP_K(\varepsilon(v))(x)$ is a (f, F)-admissible stress field. In other words, *the existence of an admissible stress field is a necessary condition in order for problem 6.1 to have a solution.*

DEFINITION 7.3. — *Assume that* $u, u_j \in \mathscr{U}(\Omega)$. *We shall say that the sequence* u_j *is strongly convergent to* u *in* $\mathscr{U}(\Omega)$ *if*

(7.1) $$u_j \to u \quad \text{in} \quad L^{n/n-1}(\Omega, \mathbb{R}^n)$$

(7.2) $$\int_\Omega |\varepsilon(u_j)| \to \int_\Omega |\varepsilon(u)|$$

(7.3) $$P_K\varepsilon(u_j) \to P_K\varepsilon(u) \quad \text{in} \quad L^2(\Omega, V)$$

REMARK 7.4. — If $u_j \to u$ as in (7.1), (7.2), then one has [5] [27]

$$u_j|_{\partial\Omega} \to u|_{\partial\Omega} \quad \text{in} \quad L^1(\partial\Omega, \mathbb{R}^n)$$

Moreover, in view of lemma 2.7, one has also

$$\int_\Omega \| P_K\varepsilon(u_j) \| \to \int_\Omega \| P_K\varepsilon(u) \|$$

DEFINITION 7.5. — *We shall say that an open set* $\Omega \subset \mathbb{R}^n$ *is K-admissible or, shortly, admissible if, for any function* $u \in \mathscr{U}(\Omega)$, *there exists a sequence* $u_j \in C^1(\overline{\Omega}, \mathbb{R}^n)$ *such that* $u_j \to u$ *strongly in* $\mathscr{U}(\Omega)$.

We shall see in section 10 that all the Lipschitz domains in $\mathbb{R}^2$ are admissible and that all the strictly star shaped Lipschitz domains in $\mathbb{R}^n$ are admissible. Most of the results that we are going to give will make use of admissible Lipschitz domains. It would be nice to know whether or not Lipschitz domains in $\mathbb{R}^n$ are admissible, also for $n \geqslant 3$.

The main property, at least for our purposes, of the admissible Lipschitz domains is the following one:

LEMMA 7.6. — *Let* Ω *be an admissible Lipschitz domain in* $\mathbb{R}^n$ *and let*

(f, F) be a load. For any (f, F)-admissible stress field τ and for any $u \in \mathscr{U}(\Omega)$, one has

$$(7.4) \qquad \int_\Omega (\mathrm{P_K}\varepsilon(u), \tau)dx - \int_\Omega fudx - \int_{\partial\Omega} \mathrm{F}ud\mathscr{H}^{n-1} \geqslant 0$$

Proof. — Let $u_j \in \mathrm{C}^1(\overline{\Omega}, \mathbb{R}^n)$ be a sequence of functions such that $u_j \to u$ strongly in $\mathscr{U}(\Omega)$. For each j, one has

$$(7.5) \quad \int_\Omega (\mathrm{P_K}\varepsilon(u_j), \tau)dx - \int_\Omega fu_jdx - \int_{\partial\Omega} \mathrm{F}u_j d\mathscr{H}^{n-1} = -\int_\Omega (\mathrm{P_K}\ \varepsilon(u_j), \tau) \geqslant 0$$

where the last inequality follows because $\mathrm{K}^\perp = \mathrm{K}_0^*$, so that $(P_{\mathrm{K}^\perp}\varepsilon(u_j)(x), \tau(x)) \leqslant 0$ for $\mathscr{L}^n$-almost all $x \in \Omega$. Taking the limit for $j \to \infty$ in formula (7.5), by definition 7.3 and remark 7.4, one obtains (7.4). Q. E. D.

THEOREM 7.7. — *Let Ω be an admissible Lipschitz domain in $\mathbb{R}^n$ and let $v \in \mathscr{U}(\Omega)$ be a minimum point for the functional $\mathrm{E}(u)$ in $\mathscr{U}(\Omega)$; then the stress field $\sigma = \mathrm{AP_K}\varepsilon(v)$ is a minimum point for the complementary energy functional*

$$\mathrm{I}(\tau) = \frac{1}{2}\int_\Omega (\mathrm{A}^{-1}\tau, \tau)dx$$

among all the (f, F)-admissible stress fields.

Proof. — Let τ be an (f, F)-admissible stress field. Taking into account the convexity and homogeneity of the function $\alpha \to \langle \mathrm{A}^{-1}\alpha, \alpha \rangle$ one has

$$\mathrm{I}(\tau) - \mathrm{I}(\sigma) > \int_\Omega (\mathrm{A}^{-1}\sigma, \tau - \sigma)dx = \int_\Omega (\mathrm{P_K}\varepsilon(v), \tau - \sigma) \geqslant 0$$

where the last inequality follows from lemma 7.4 and proposition 6.4. Q. E. D.

We remark that theorem 7.7 above is a rephrasing in our situation, where $\varepsilon(v)$ may be a measure, of a well known result by J. J. Moreau, compare also section V, 6 in [*11*].

An important consequence of theorem 7.7 is that, if v_1 and v_2 are any two minimum points for $\mathrm{E}(u)$, the corresponding stress fields

$$\sigma_1 = \mathrm{AP_K}\varepsilon(v_1), \qquad \sigma_2 = \mathrm{AP_K}\varepsilon(v_2)$$

must coincide; in fact, both σ_1 and σ_2 minimize the complementary energy, which is a strictly convex functional.

8. The Neumann problem: coerciveness, semicontinuity, existence.

In this section we shall prove the existence of a solution to problem 6.1, under suitable assumptions for the load.

First we introduce suitable notions of weak convergence and coerciveness in the space $\mathcal{U}(\Omega)$, then we use the direct method of the Calculus of Variations, the properties of $BD(\Omega)$ [23] [19] and theorem 3.3 to get the existence theorem 8.5. We remark that theorem 8.5 is easy, the difficulty having been moved to the question of deciding whether the functional is coercive and lower semicontinuous. Further in the section we are able to give better sufficient conditions on the load for the existence of a minimum of E. These conditions basically reduce to the existence of a safe (f, F)-admissible stress. However, really simple sufficient conditions, at least for special configurations, are still lacking. A few better results in two dimensions have been obtained in [14].

DEFINITION 8.1. — *The functional* $E(u)$ *is said to be coercive on* $\mathcal{U}(\Omega)$ *if there exist numbers* $c_1 > 0$ *and* $c_2 \geqslant 0$ *such that*

$$E(u) \geqslant c_1 \left\{ \int_\Omega \| P_K \varepsilon(u) \|^2 dx + \int_\Omega | \varepsilon(u) | \right\} - c_2$$

for all $u \in \mathcal{U}(\Omega)$.

A condition on the load, which is sufficient for the coerciveness of $E(u)$, will be given in theorem 8.8.

Because of the Sobolev-Poincaré type inequality for $BD(\Omega)$ and the compact immersion $BD(\Omega) \hookrightarrow L^1(\Omega)$ [33] [19], when the functional $E(u)$ is coercive on $\mathcal{U}(\Omega)$, from any sequence $u_j \in \mathcal{U}(\Omega)$ such that

$$E(u_j) + \int_\Omega | u_j | \, dx \leqslant \text{constant} < +\infty \quad \forall j$$

one can extract a subsequence, let us call it again u_j, such that, for some $u \in L^{n/n-1}(\Omega, \mathbb{R}^n)$ one has

$$(8.1, i) \qquad u_j \to u \quad \text{in} \quad L^1(\Omega, \mathbb{R}^n)$$

$$(8.1, ii) \qquad u_j \rightharpoonup u \quad \text{in} \quad L^{n/n-1}(\Omega, \mathbb{R}^n)$$

$$(8.1, iii) \qquad \int_\Omega | \varepsilon(u_j) | \leqslant M \quad \text{for all } j$$

$$(8.1, iv) \qquad \int_\Omega \| P_K \varepsilon(u_j) \|^2 dx \leqslant M \quad \text{for all } j$$

where M is some positive number.

DEFINITION 8.2. — *When a sequence* $u_j \in \mathcal{U}(\Omega)$ *converges to a function* $u \in L^{n/n-1}(\Omega, \mathbb{R}^n)$ *as in (8.1) we shall say that* u_j *converges to* u *weakly in* $\mathcal{U}(\Omega)$.

PROPOSITION 8.3. — *The space* $\mathcal{U}(\Omega)$ *is a closed subspace of* $L^{n/n-1}(\Omega, \mathbb{R}^n)$, *with respect to the weak convergence in* $\mathcal{U}(\Omega)$.

Proof. — By $=8.1, i)$ and $(8.1, iii)$ one has $u \in \mathrm{BD}(\Omega)$ and $\varepsilon(u_j) \rightharpoonup \varepsilon(u)$ weakly as measures, then, by theorem 3.3 one has $u \in \mathcal{U}(\Omega)$ and

$$(8.2) \qquad \int_\Omega \| \mathrm{P_K}\varepsilon(u) \|^2 \leqslant \min_{j\to\infty} \lim \int_\Omega \| \mathrm{P_K}\varepsilon(u_j) \|^2 \qquad \text{Q. E. D.}$$

DEFINITION 8.4. — *We say that the functional* E(u) *is (sequentially, weakly) lower-semicontinuous (in* $\mathcal{U}(\Omega)$*) if, for any sequence* $u_j \in \mathcal{U}(\Omega)$ *and for any* $u \in \mathcal{U}(\Omega)$ *such that* $u_j \to u$ *weakly in* $\mathcal{U}(\Omega)$*, one has*

$$\min_{j\to\infty} \lim \mathrm{E}(u_j) \geqslant \mathrm{E}(u)$$

A sufficient condition for the lower-semicontinuity of E(u) will be given in theorem 8.10.

Here is the existence theorem:

THEOREM 8.5. — *Let* Ω *be a Lipschitz domain and let* (f, F) *be a load such that there exists an* (f, F)*-admissible stress field* τ*. If the functional* E(u) *is coercive on* $\mathcal{U}(\Omega)$ *and lower-semicontinuous with respect to the weak convergence in* $\mathcal{U}(\Omega)$*, then there exists a solution to problem 6.1.*

Proof. — Clearly one has

$$-\infty < \inf_{u \in \mathcal{U}(\Omega)} \mathrm{E}(u) < +\infty$$

Let $u_j \in \mathcal{U}(\Omega)$ be a sequence such that

$$(8.3) \qquad \lim_{j\to\infty} \mathrm{E}(u_j) = \inf_{u\in\mathcal{U}(\Omega)} \mathrm{E}(u)$$

Consider the space $\mathcal{I}$ of the infinitesimal rigid motions in $\mathbb{R}^n$

$$\mathcal{I} = \{ w : \mathbb{R}^n \to \mathbb{R}^n \mid \varepsilon(w) \equiv 0 \} \subset \{ \text{affine maps: } \mathbb{R}^n \to \mathbb{R}^n \}$$

and take some fixed continuous linear map

$$\mathrm{T} : \mathrm{BD}(\Omega) \to \mathcal{I}$$

such that $\mathrm{T}w = w$ for all $w \in \mathcal{I}$ compare for instance [*19*]. We remark that for all $w \in \mathcal{I}$ we have

$$\int_\Omega fw + \int_{\partial\Omega} \mathrm{F}w = \int_\Omega \tau . \varepsilon(w) = 0$$

because τ is (f, F)-admissible, and it follows that

$$(8.4) \qquad \mathrm{E}(u) = \mathrm{E}(u - \mathrm{T}u)$$

Now we set

$$v_j = u_j - \mathrm{T}u_j \in \mathcal{U}(\Omega)$$

By (8.3) and (8.4) and by the coerciveness of $E(u)$, for all j, we have

$$\int_{\Omega} \| P_K \varepsilon(v_j) \|^2 + \int_{\Omega} | \varepsilon(v_j) | \leqq \text{constant} < +\infty$$

and by the Sobolev-Poincaré type inequality for $BD(\Omega)$ functions [*23*] [*19*] we have also

$$\int_{\Omega} | v_j |^{n/n-1} dx \leqslant \text{constant} < +\infty \quad \text{for all } j$$

Because of the compact immersion $BD(\Omega) \to L^1(\Omega)$ and proposition 8.3, we may find a subsequence v_{j_h} that converges weakly in $\mathscr{U}(\Omega)$ to some function $v \in \mathscr{U}(\Omega)$ and, by the semicontinuity of $E(u)$, we get

$$E(v) \geqslant \inf_{u \in \mathscr{U}(\Omega)} E(u) = \lim_{j \to \infty} E(v_j) \geqslant E(v)$$

hence v is the required solution to problem 6.1. Q. E. D.

We notice that the minimum v that we have found in the preceding theorem satisfies $Tv = 0$.

We remark again that the proof of theorem 8.5 is just a straight forward consequence of the known properties of the space $BD(\Omega)$.

Now we shall consider the more difficult problem of finding reasonable conditions on the load for the coerciveness and the semicontinuity of the functional $E(u)$.

DEFINITION 8.6. — *We shall say that a stress field* $\tau \in L^2(\Omega, V)$ *is safe (with respect to the cone* K_0*) if there exists a number* $c_0 > 0$ *such that*

$$-(\tau(x), \alpha) \geqslant c_0 | \alpha |$$

for $\mathscr{L}^n$*-almost all* $x \in \Omega$ *and for all* $\alpha \in K_0^* = K^\perp$.

Clearly, a safe stress field belongs to K_0 almost everywhere.

REMARK 8.7. — A sufficient condition in order for τ to be safe is that there exists a number $c_3 > 0$ such that, for $\mathscr{L}^n$-almost all $x \in \Omega$ one has

$$-\frac{\tau(x) . \alpha}{| \tau(x) | \, | \alpha |} \geqslant c_3 \qquad \text{for all } \alpha \in K_0^*$$

$$| \tau(x) | \geqslant c_3$$

In fact, in such a case, for almost all $x \in \Omega$ one has

$$-(\tau(x) . \alpha) \geqslant c_3^2 | \alpha | \qquad \text{for all } a \in K_0^*$$

The above condition may be rephrased as follows: *let* $\hat{K}_0$ *be a closed cone strictly contained in* K_0 *and let* B *a neighborhood of the origin, if* $\tau(x) \in \hat{K}_0\text{-}B$ *for* $\mathscr{L}^n$*-almost all* $x \in \Omega$, *then* τ *is safe.*

THEOREM 8.8. — *Let Ω be an admissible Lipschitz domain in $\mathbb{R}^n$ and let (f, F) be a load in Ω. If there exists a safe (f, F)-admissible stress field τ then the functional* $\mathrm{E}(u)$ *is coercive on* $\mathcal{U}(\Omega)$.

Proof. — Let $u \in \mathcal{U}(\Omega)$ and let $u_j \in \mathrm{C}^1(\overline{\Omega})$ be such that $u_j \to u$ strongly in $\mathcal{U}(\Omega)$. For each j one has

$$\mathrm{E}(u_j) = \int_\Omega \| \mathrm{P_K}\varepsilon(u_j) \|^2 - \int_\Omega f u_j - \int_{\partial\Omega} \mathrm{F} u_j = \int_\Omega \| \mathrm{P_K}\varepsilon(u_j) \|^2 - \int_\Omega (\tau, \varepsilon(u_j))$$

$$= \int_\Omega \| \mathrm{P_K}\varepsilon(u_j) \|^2 - \int_\Omega (\tau, \mathrm{P_K}\varepsilon(u_j)) - \int_\Omega (\tau, \mathrm{P_{K^\perp}}\varepsilon(u_j)) \geqslant$$

$$\geqslant \int_\Omega \| \mathrm{P_K}\varepsilon(u_j) \|^2 - \left\{ \int_\Omega |\tau|^2 \right\}^{1/2} \left\{ \int_\Omega |\mathrm{P_K}\varepsilon(u_j)|^2 \right\}^{1/2} + c_0 \int_\Omega | \mathrm{P_{K^\perp}}\varepsilon(u_j) |$$

and, as $|\alpha| \leqslant |\mathrm{P_K}\alpha| + |\mathrm{P_{K^\perp}}\alpha|$, one has also

$$\mathrm{E}(u_j) \geqslant c_1 \left\{ \int_\Omega \| \mathrm{P_K}\varepsilon(u_j) \|^2 + \int_\Omega |\varepsilon(u_j)| \right\} - c_2 \tag{8.5}$$

where c_1 and c_2 depend on A, Ω, τ, c_0. Taking the limit in (8.5) for $j \to \infty$ the proof is concluded. Q. E. D.

The sufficient condition given in theorem 8.7 reminds one of analogous conditions given in many different problems [*13*] [*15*] [*1*] [*25*] [*2*] [*8*] where L^1 estimates for the gradient or for the strain are needed. This condition seems to be interesting from a theoretical point of view, but it may be difficult to check for a particular given load. For practical purposes it would be important to have also simple and easy-to check sufficient conditions for the coerciveness of $\mathrm{E}(u)$, at least for special configurations of interest.

Now we turn to the problem of semicontinuity. First we give an obvious remark.

REMARK 8.9. — *If* $\mathrm{F} = 0$, *then for any* $f \in \mathrm{L}^n(\Omega, \mathbb{R}^n)$ *the functional* $\mathrm{E}(u)$ *is lower-semicontinuous with respect to the weak convergence in* $\mathcal{U}(\Omega)$.

THEOREM 8.10. — *Assume that Ω is an admissible Lipschitz domain, let (f, F) be a given load, and assume that there exist a function $g \in \mathrm{L}^n(\Omega, \mathbb{R}^n)$ and a stress field $\tau \in \mathrm{L}^2(\Omega, \mathrm{K}_0)$ such that*

$$\tau \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n) \tag{8.6}$$

and

$$\begin{cases} \operatorname{div} \tau + g = 0 & \text{in } \Omega \\ \tau . \nu = \mathrm{F} & \text{on } \partial\Omega \end{cases} \tag{8.7}$$

in the sense of definition 6.2. Then the functional $\mathrm{E}(u)$ *is lower-semicontinuous with respect to the weak convergence in* $\mathcal{U}(\Omega)$.

Proof. — Let $u_j, u \in \mathcal{U}(\Omega)$ be such that $u_j \rightharpoonup u$ weakly in $\mathcal{U}(\Omega)$ and let us prove that

$$\min_{j\to\infty} \lim E(u_j) \geqq E(u) \tag{8.8}$$

For any number $\delta > 0$ set

$$\Omega_\delta = \{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta \}$$

we have

$$\begin{aligned} E(u) - E(u_j) = &\int_\Omega (g - f)(u - u_j)dx + \\ &+ \left\{ \int_{\Omega_\delta} \| P_K\varepsilon(u) \|^2 dx - \int_{\Omega_\delta} \| P_K\varepsilon(u_j) \|^2 dx - \int_{\Omega_\delta} g(u - u_j)dx \right\} + \\ &+ \left\{ \int_{\Omega\setminus\Omega_\delta} \| P_K\varepsilon(u) \|^2 dx - \int_{\Omega\setminus\Omega_\delta} \| P_K\varepsilon(u_j) \|^2 dx - \int_{\Omega\setminus\Omega_\delta} g(u-u_j)dx - \right. \\ &\left. - \int_{\partial\Omega} F(u-u_j)d\mathcal{H}^{n-1} \right\} \end{aligned} \tag{8.9}$$

where, as in remark 8.9, one has

$$\max_{j\to\infty} \lim \left\{ \int_{\Omega_\delta} \| P_K\varepsilon(u) \|^2 dx - \int_{\Omega_\delta} \| P_K\varepsilon(u_j) \|^2 dx - \int_{\Omega_\delta} g(u-u_j)dx \right\} \leqq 0$$

If we call G the second term in parentheses on the right of (8.9) we have

$$\begin{aligned} G = \int_{\Omega\setminus\Omega_\delta} \| P_K\varepsilon(u) \|^2 - \int_{\Omega\setminus\Omega_\delta} \| P_K\varepsilon(u_j) \|^2 dx - \int_{\Omega\setminus\Omega_\delta} (\varepsilon(u-u_j), \tau) \\ - \int_{\partial\Omega_\delta} [\tau . \nu_\delta] . (u-u_j)d\mathcal{H}^{n-1} \end{aligned} \tag{8.10}$$

where ν_δ is the outward normal to $\partial\Omega_\delta$, $[\tau . \nu_\delta]$ is defined as the unique function in $L^\infty(\partial\Omega_\delta, \mathbb{R}^n)$ such that

$$- \int_{\Omega_\delta} (\tau(x), \varepsilon(\phi))dx + \int_{\Omega_\delta} \phi \operatorname{div} \tau dx + \int_{\partial\Omega_\delta} [\tau . \nu_\delta] . \phi d\mathcal{H}^{n-1} = 0$$

for all $\phi \in C^1(\overline{\Omega}_\delta, \mathbb{R}^n)$, see for instance [7, theorem 5.3], and the pairing $(\varepsilon(u - u_j), \tau)$ has to be considered as a measure (compare [*28*] and [*4*]).

Now, arguing as in lemma 7.6, one gets

$$\int_{\Omega\setminus\Omega_\delta} (\varepsilon(u_j), \tau) - \int_{\Omega\setminus\Omega_\delta} (P_K\varepsilon(u_j), \tau) \leqq 0 \tag{8.11}$$

and, adding and subtracting the term $-\int_{\Omega\setminus\Omega_\delta} (P_K\varepsilon(u_j), \tau)$ to 8.10, one gets

$$(8.12)\quad G \leqq \int_{\Omega\setminus\Omega_\delta} \| P_K\varepsilon(u) \|^2 dx + \| \tau \|_\infty \int_{\Omega\setminus\Omega_\delta} | \varepsilon(u) | + c_6 \int_{\Omega\setminus\Omega_\delta} | \tau |^2 dx - \int_{\partial\Omega_\delta} [\tau . \nu_\delta].(u - u_j) d\mathscr{H}^{n-1}$$

At this point, we take a number $\eta > 0$ and we choose Ω_δ such that one has

$$\int_{\Omega\setminus\Omega_\delta} \| P_K\varepsilon(u) \|^2 dx + \| \tau \|_\infty \int_{\Omega\setminus\Omega_\delta} | \varepsilon(u) | + c_6 \int_{\Omega\setminus\Omega_\delta} | \tau |^2 dx < \eta$$

and

$$u_j |_{\partial\Omega_\delta} \to u |_{\partial\Omega_\delta} \quad \text{in} \quad L^1(\partial\Omega_\delta)$$

Taking the limit for $j \to \infty$ in formula (8.12), we obtain $G \leqq \eta$, where η is arbitrary, and it follows that $G \leqq 0$. Recalling (8.9) we have then $\max_{j\to\infty} \lim \{ E(u) - E(u_j) \} \leqq 0$ and (8.8) follows.

Q. E. D.

While assumption (8.7) in theorem 8.10 is natural, assumption (8.6) is not. In two dimensions we are able to relax slightly assumption 8.6, as it is shown in proposition 8.11 and remark 8.12 below.

PROPOSITION 8.11. — *Let Ω be any Lipschitz domain in $\mathbb{R}^n$ and let (f, F) be a given load in Ω. If there exists a (f, F)-admissible stress field τ_0 such that $\tau_0 \in L^\infty(\Omega\setminus\Omega_{\bar\delta}, K_0)$ for some neighborhood $\Omega\setminus\Omega_{\bar\delta}$ of $\partial\Omega$, then there exists another stress field $\tau_1 \in L^\infty(\Omega, K_0)$ and a function $g \in L^n(\Omega, \mathbb{R}^n)$ such that*

$$(8.13)\quad \begin{cases} \operatorname{div} \tau_1 + g = 0 & \text{in } \Omega \\ \tau_1 . \nu = F & \text{on } \partial\Omega \end{cases}$$

Proof. — It is sufficient to consider a function $\phi_1 \in C_0^1(\Omega)$ such that $0 \leqq \phi_1(x) \leqq 1$, $\phi_1(x) = 1$ if dist $(x, \partial\Omega) > \frac{\bar\delta}{2}$, $\phi_1(x) = 0$ if dist $(x, \partial\Omega) < \frac{\bar\delta}{4}$, and to take

$$(8.14)\quad \tau_1 = \tau_0(1 - \phi_1)$$

In fact it is clear that $\tau_1 \in L^\infty(\Omega, K_0)$ and that the divergence of τ_1 in the sense of distributions is a function in $L^n(\Omega, \mathbb{R}^n)$; moreover, one has $\tau_1(x) = \tau_0(x)$ if dist $(x, \partial\Omega) < \frac{\bar\delta}{4}$ and this easily implies (8.13). In fact set $g = \operatorname{div} \tau_1$ and let us prove that

$$(8.15)\quad \int_\Omega \tau_1 . \varepsilon(\phi) dx - \int_\Omega g\phi dx - \int_{\partial\Omega} F\phi dH^{n-1} = 0$$

for all $\phi \in C^1(\overline{\Omega}, \mathbb{R}^n)$. To do that, consider a new function $\psi_1 \in C_0^1(\Omega)$ such that $0 \leqq \psi_1(x) \leqq 1$, $\psi_1(x) = 1$ if $\operatorname{dist}(x, \partial\Omega) > \frac{\overline{\delta}}{8}$ and notice that for all $\phi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ one has

$$\int_\Omega \tau_1 \varepsilon(\phi)dx - \int_\Omega g\phi dx - \int_{\partial\Omega} F\phi dH^{n-1} =$$
$$= \int_\Omega \tau_1 \varepsilon(\phi(1-\psi_1))dx - \int_\Omega g\phi(1-\psi_1)dx - \int_{\partial\Omega} F\phi(1-\psi_1)dH^{n-1} =$$
$$= \int_\Omega (\tau_1 - \tau_0)\varepsilon(\phi(1-\psi_1)) - \int_\Omega (g-f)\phi(1-\psi_1) = 0 \qquad \text{Q. E. D.}$$

Summing up theorems 8.5, 8.8, 8.10, proposition 8.11 and theorem 10.5, we have the following result.

THEOREM 8.12. — *Assume that* Ω is a Lipschitz domain in $\mathbb{R}^n$ and that (f, F) is a load such that there exists a safe (f, F)-admissible stress field τ which is bounded in a neighbourhood of $\partial\Omega$, if $n \geqslant 3$ assume also that Ω is K-admissible; then there exists a solution to problem 6.1.

9. Case of prescribed displacement at the boundary.

In this section we shall consider a particular minimum problem, problem (9.1) below, which is natural for our functionals and which is the closest we can go towards prescribing the displacement at the boundary. The boundary conditions satisfied by the solutions to problem (9.1) are of a unilateral type, and their meaning is especially clear in the case of the masonry-like materials, that we shall study in part III.

Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and take an open ball Ω_1 such that $\overline{\Omega} \subset \Omega_1$.

Consider a fixed function $g \in H^{1,2}(\Omega_1)$, which is going to be our prescribed displacement at the boundary of Ω, and the space

$$\mathscr{U}_g(\Omega_1) = \{\, u \in BD(\Omega_1) \mid P_K\varepsilon(u) \in L^2(\Omega_1, V), \quad u = g \text{ on } \Omega_1 \backslash \Omega \,\}$$

For any given load $f \in L^n(\Omega, \mathbb{R}^n)$ we consider the functional

$$E(u) = \frac{1}{2}\int_\Omega \| P_K\varepsilon(u) \|^2 - \int_\Omega fu dx$$

and the problem

$$(9.1) \qquad \begin{cases} E(u) \to \min \\ u \in \mathscr{U}_g(\Omega_1) \end{cases}$$

First we shall study the equilibrium and constitutive relations, with particular regard to the boundary conditions, which are satisfied by a solution to problem (9.1); then we shall give sufficient conditions for the existence of a solution to the problem.

If v is a solution to problem (9.1) and we set

$$\sigma = AP_K\varepsilon(u)$$

arguing as in theorem 6.3, we get

$$(9.2)\qquad \begin{aligned} &\sigma \in L^2(\Omega, K_0)\\ &\varepsilon(v) = A^{-1}\sigma + \lambda\\ &\lambda = P_{K^\perp}\varepsilon(v) \in M(\Omega, K_0^*) \end{aligned}$$

and, using the Euler equation for test functions $\phi \in C_0^1(\Omega, \mathbb{R}^n)$ we get also

$$(9.3)\qquad \operatorname{div}\sigma + f = 0 \quad \text{in} \quad \Omega$$

in the sense of distributions.

Now we study the conditions satisfied by v and σ at the boundary. For a general function $v \in BD(\Omega_1)$, if we get

$$\begin{aligned} v^- &= \text{trace of } v_{|\Omega} \quad \text{on} \quad \partial\Omega\\ v^+ &= \text{trace of } v_{|\Omega_1\backslash\Omega} \quad \text{on} \quad \partial\Omega \end{aligned}$$

we have [*6*; theorem 1.5, formula (*i*) (where the minus sign is wrong)]

$$(9.4)\qquad \varepsilon(v)_{|\partial\Omega} = ((v^+(x) - v^-(x)) \odot \nu(x))\mathscr{H}^{n-1}{}_{|\partial\Omega}$$

where we denote $a \odot b = \left\{ \frac{1}{2}(a_ib_j + a_jb_i) \right\}_{i,j=1,\dots,n}$ the symmetric tensor product of a, $b \in \mathbb{R}^n$. If $v \in \mathscr{U}_g(\Omega_1)$, then one has $v^+ = g$; moreover, as $P_K\varepsilon(v) \in L^2(\Omega_1, V)$, it follows that the measure $P_K\varepsilon(v)$ is identically zero on $\partial\Omega$, i. e. for all Borel sets $BD(\Omega_1)$, if we get

$$\int_B P_K\varepsilon(v) = \int_B p_K(g(x) - v^-(x)) \odot \nu(x))d\mathscr{H}^{n-1} = 0$$

so that

$$p_K((g(x) - v^-(x)) \odot \nu(x)) = 0 \qquad \mathscr{H}^{n-1}\text{-a. e.} \quad \text{on} \quad \partial\Omega$$

and this is equivalent to saying that

$$(9.5)\qquad (g(x) - v(x)) \odot \nu(x) \in K^\perp = K_0^* \qquad \mathscr{H}^{n-1}\text{-a. e.} \quad \text{on} \quad \partial\Omega$$

where we have written just $v(x)$ for the inner trace of v on $\partial\Omega$, as we shall do from now on.

Formula (9.5) is the boundary condition satisfied by the displacement v: one does not have that $g - v$ is zero on $\partial\Omega$, but just that the difference $g - v$ is such that $(g(x) - v(x)) \odot \nu(x)$ belongs to a certain cone. The meaning of this condition is particularly clear and simple in the case of the masonry-like materials, which we discuss in part III. At this point, the reader may find it useful to start reading part III and to keep on reading it and this section in parallel.

For each unit vector $a \in \mathbb{R}^n$ and for each closed convex cone K, we consider the set

$$(9.6) \qquad (\mathrm{K})_a = \{\, b \in \mathbb{R}^n \mid b \odot a \in \mathrm{K} \,\}$$

As the map $b \mapsto b \odot a$ is linear, the set $(\mathrm{K})_a$ is a closed convex cone in $\mathbb{R}^n$. Formula (9.5) can be written also as $g(x) - v(x) \in (\mathrm{K}_0^*)_{\nu(x)}\ \mathscr{H}^{n-1}$-a. e. on $\partial\Omega$.

We have also a boundary condition satisfied by the stress. In fact, let $\psi \in \mathrm{C}^1(\overline{\Omega}, \mathbb{R}^n)$ be such that

$$p_{\mathrm{K}}(-\psi(x) \odot \nu(x)) = 0 \qquad \text{for all} \quad x \in \partial\Omega$$

i. e.

$$(9.7) \qquad -\psi(x) \odot \nu(x) \in \mathrm{K}^{\perp} = \mathrm{K}_0^* \qquad \text{for all} \quad x \in \partial\Omega$$

i. e.

$$-\psi(x) \in (\mathrm{K}_0^*)_{\nu(x)} \qquad \text{for all} \quad x \in \partial\Omega$$

and consider the function $\phi : \Omega_1 \to \mathbb{R}^n$ defined as

$$\phi(x) = \begin{cases} \psi(x) & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \in \Omega_1 \backslash \Omega \end{cases}$$

Clearly, we have $\phi \in \mathrm{BD}(\Omega_1)$ and for all $t > 0$ we have $v + t\phi \in \mathrm{BD}(\Omega_1)$, $v + t\phi = g$ in $\Omega_1 \backslash \Omega$. Moreover, for all $t > 0$ by lemma 3.2 and (9.4)

$$\mathrm{P_K}(\varepsilon(v) + t\varepsilon(\phi)) = \begin{cases} p_{\mathrm{K}}(\varepsilon^a(v) + t\varepsilon(\phi))dx & \text{in} \quad \Omega \\ p_{\mathrm{K}}((g(x) - v(x) - t\psi(x)) \odot \nu(x))d\mathscr{H}^{n-1} & \text{on} \quad \partial\Omega \\ p_{\mathrm{K}}\varepsilon(g) & \text{in} \quad \Omega_1 \backslash \Omega \end{cases}$$

where $p_{\mathrm{K}}((g(x) - v(x)) \odot \nu(x) + t(-\psi(x) \odot \nu(x)) = 0\ \mathscr{H}^{n-1}$-a. e. on $\partial\Omega$ for all $t > 0$, because of (9.5) and (9.7). In conclusion, for all $t \geqslant 0$ we have $v + t\phi \in \mathscr{U}_g(\Omega_1)$ and it follows that (compare theorem 4.1)

$$(9.8) \quad 0 \leqslant \frac{d^+}{dt} \mathrm{E}(v + t\phi)_{|t=0} = \int_\Omega \sigma . \varepsilon(\psi) dx - \int_\Omega f\psi$$

for all $\psi \in \mathrm{C}^1(\overline{\Omega}, \mathbb{R}^n)$ such that (9.7) holds.

Formula (9.8) is the boundary condition satisfied by the stress, and it is to be thought of as being part of the constitutive laws. If the stress is a regular function up to the boundary, one can integrate by parts in (9.8), and one gets

$$\int_{\partial\Omega} (\sigma.\nu).(-\psi)d\mathcal{H}^{n-1} \leqslant 0 \tag{9.9}$$

for all $\psi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ such that $-\psi(x) \in (K_0^*)_{\nu(x)}$ on $\partial\Omega$

This means that the vincolar reaction $\sigma.\nu$ of the boundary satisfies

$$(\sigma(x).\nu(x)).b \leqslant 0$$

for $\mathcal{H}^{n-1}$-almost all $x \in \partial\Omega$ and for all $b \in (K_0^*)_{\nu(x)}$. In other words, if one denotes C^* the orthogonal cone, with respect to the standard scalar product in $\mathbb{R}^n$, to the cone $C \subset \mathbb{R}^n$, one has

$$\sigma(x).\nu(x) \in ((K_0^*)_{\nu(x)})^* \qquad \mathcal{H}^{n-1}\text{-a. e.} \quad \text{on} \quad \partial\Omega \tag{9.10}$$

In fact, if we denote $z(x)$ the projection of $\sigma(x).\nu(x)$ on the cone $(K_0^*)_{\nu(x)}$, we can prove that condition (9.10) is equivalent to

$$\int_{\partial\Omega} |z(x)|\, d\mathcal{H}^{n-1} = 0 \tag{9.11}$$

Now, essentially by lemma 2.3, we can find a sequence of functions $\theta_j \in C^1(\partial\Omega, (K_0^*)_{\nu(x)})$ such that $\theta_j \to \frac{z}{|z|}\mathcal{H}^{n-1}$-a. e. on $\partial\Omega$, $|\theta_j| \leqslant 1$ and we take a sequence $\tilde{\theta}_j \in C^1(\overline{\Omega}, \mathbb{R}^n)$ such that $\tilde{\theta}_j = \theta_j$ on $\partial\Omega$. By (9.9) we have

$$0 \geqslant \int_{\partial\Omega} z.\theta_j + \int_{\partial\Omega} [(\sigma.\nu) - z].\theta_j \qquad \text{for all } j$$

taking the limit for $j \to \infty$, by the dominated convergence theorem, we get

$$0 \geqslant \int_{\partial\Omega} |z|\, d\mathcal{H}^{n-1} + \int_{\partial\Omega} ((\sigma.\nu)-z).\frac{z}{|z|}.d\mathcal{H}^{n-1} = \int_{\partial\Omega} |z|\, d\mathcal{H}^{n-1}$$

and (9.11) follows.

The boundary condition (9.8) for the stress is a weak formulation of (9.10). Again, this condition is especially simple in the case of masonry-like materials (section 12).

Summing up, we have proved the following:

THEOREM 9.1. — *If v is a solution to problem (9.1) and we set $\sigma = AP_K\varepsilon(v)$, then the constitutive equations (9.2) and (9.8) are satisfied, moreover one has the equilibrium condition (9.3) and the boundary condition (9.5) for the displacement.*

Remark. — By mechanical considerations one expects the vincolar reaction to be zero where $g \neq v$. In section 12 we shall prove this result for the case of masonry, under some regularity assumptions.

Now we state our existence theorem for problem (9.1).

THEOREM 9.3. — *Let Ω be a K-admissible Lipschitz domain in $\mathbb{R}^n$, and assume that the interior of K is not empty. Then for all $f \in L^n(\Omega, \mathbb{R}^n)$ there exists a solution to problem (9.1).*

To prove theorem 9.2 we need the following

LEMMA 9.3. — *If K is a convex cone and α is an interior point of K, $\|\alpha\| = 1$, then there is a number $c_1 > 0$ such that, for all $\beta \in K^\perp$ one has*

$$c_1 \|\beta\| \leqslant \langle \beta, -\alpha \rangle \tag{9.12}$$

Proof. — We set

$$c_1 = \min \{ \langle \beta, -\alpha \rangle \mid \beta \in K^\perp, \|\beta\| = 1 \}$$

As α is an interior point of K we have $c_1 > 0$ and (9.12) follows immediately.

Proof of theorem 9.2. — First we shall prove the coerciveness of E(u), more precisely, we shall prove that for any function $u \in \mathcal{U}_g(\Omega_1)$ one has

$$E(u) \geq c_2 \left\{ \int_\Omega \| P_K \varepsilon(u) \|^2 dx + \int_\Omega | \varepsilon(u) | + \int_{\partial\Omega} | u - g | \, d\mathcal{H}^{-1} \right\} - c_3 \tag{9.13}$$

where $c_2 > 0$ and $c_3 \geq 0$ depend on c_1, n, A, Ω, g, f but not on u. Let $\alpha \in \overset{\circ}{K}$ be such that $\|\alpha\| = 1$, let $u \in \mathcal{U}_g(\Omega_1)$ and let $w_j \in C^1(\overline{\Omega}, \mathbb{R}^n)$ be a sequence that converges strongly to $u - g$ in $\mathcal{U}(\Omega)$. For each $j \in \mathbb{N}$, by lemma 9.3 we have

$$\int_\Omega \| P_{K^\perp} \varepsilon(w_j) \| \, dx \leq \frac{1}{c_1} \int_\Omega \langle P_{K^\perp} \varepsilon(w_j), -\alpha \rangle \, dx$$

where

$$\int_\Omega \langle P_{K^\perp} \varepsilon(w_j), -\alpha \rangle \, dx = \int_\Omega \langle \varepsilon(w_j), -\alpha \rangle \, dx - \left\langle \int_\Omega P_K \varepsilon(w_j) dx, -\alpha \right\rangle \leq$$

$$\leq \int_{\partial\Omega} \langle w_j \odot v, -\alpha \rangle \, d\mathcal{H}^{n-1} + \int_\Omega \| P_K \varepsilon(w_j) \| \, dx$$

Hence we get

$$\int_\Omega \| P_{K^\perp} \varepsilon(w_j) \| \, dx + \frac{1}{c_1} \int_{\partial\Omega} \langle w_j \odot v, \alpha \rangle \, d\mathcal{H}^{n-1} \leq \frac{1}{c_1} \int_\Omega \| P_K \varepsilon(w_j) \| \, dx \tag{9.14}$$

and taking the limit in (9.14) for $j \to \infty$, by remark 7.4, we obtain

$$(9.15)\qquad \int_\Omega \| P_{K^\perp}\varepsilon(u-g)\| + \frac{1}{c_1}\int_{\partial\Omega} \langle (u-g)\odot v, \alpha\rangle \, d\mathcal{H}^{n-1} \leq \frac{1}{c_1}\int_\Omega \| P_K\varepsilon(u-g)\| \, dx$$

Now, by (9.5) and by lemma (9.3) we have

$$\frac{1}{c_1}\langle (g-u)(x)\odot v(x), -\alpha\rangle \geq \| (g-u)(x)\odot v(x)\| \geq \gamma\,|(g-u)(x)|$$

$\mathcal{H}^{n-1}$-a. e. on $\partial\Omega$, for some number $\gamma = \gamma(n, A)$, and we conclude that

$$(9.16)\qquad \int_\Omega |\varepsilon(u-g)| + \int_{\partial\Omega} |g-u| \, d\mathcal{H}^{n-1} \leq c_4 \int_\Omega \| P_K\varepsilon(u-g)\|$$

where c_4 depends on c_1, γ and A. On the other hand, by the Sobolev-Poincaré type inequality in $BD(\Omega)$ [*19*] [*23*], we have

$$\left| \int_\Omega fudx \right| \leq \| f\|_{L^n} \| u\|_{L^{n/n-1}} \leq \| f\|_{L^n} c_5 \left\{ \int_\Omega |\varepsilon(u)| + \int_{\partial\Omega} |u-g| \, d\mathcal{H}^{n-1} \right\}$$

where c_5 depends on Ω and g, and by (9.16) we have

$$(9.17)\qquad \left| \int_\Omega fudx \right| \leq \| f\|_{L^n} c_5 \left\{ c_4 \int_\Omega \| P_K\varepsilon(u-g)\| + \int_\Omega |\varepsilon(g)| \right\} \leq \leq \frac{1}{4}\int_\Omega \| P_K\varepsilon(u)\|^2 + c_6$$

where c_6 depends on c_1, γ, A, Ω, g, f but not on u.
By (9.17) we have immediately

$$(9.18)\qquad E(u) \geq \frac{1}{4}\int_\Omega \| P_K\varepsilon(u)\|^2 - c_6 \qquad \text{for all} \quad u \in \mathcal{U}_g(\Omega_1)$$

and by (9.16) it follows also

$$(9.19)\qquad E(u) \geq c_7 \left\{ \int_\Omega |\varepsilon(u)| + \int_{\partial\Omega} |g-u| \, d\mathcal{H}^{n-1} \right\} - c_8$$

for suitable $c_7 > 0$ and $c_8 > 0$. Finally (9.18) and (9.19) imply (9.13).

Now we set

$$l = \inf \{ E(u \mid u \in \mathcal{U}_g(\Omega_1) \}$$

By the coerciveness (9.13) we have $-\infty < l < +\infty$. Let $u_j \in \mathcal{U}_g(\Omega_1)$ be a sequence of functions such that

$$\lim_{j\to\infty} E(u_j) = l$$

By the properties of $BD(\Omega)$ functions and again by the coerciveness of $E(u)$, for all j we have, for some number $M < +\infty$,

$$\int_{\Omega_1} \| P_K\varepsilon(u_j) \|^2 = \int_{\Omega} \| P_K\varepsilon(u_j) \|^2 + \int_{\Omega_1/\Omega} \| P_K\varepsilon(g) \|^2 \leq M$$

$$\int_{\Omega_1} | \varepsilon(u_j) | = \int_{\Omega} | \varepsilon(u_j) | + \int_{\partial\Omega} | (u - g) \odot v \, | d\mathscr{H}^{-1} + \int_{\Omega_1/\Omega} | \varepsilon(g) | \leq M$$

and the Sobolev-Poincaré inequality, possibly changing the constant M, we have also

$$\int_{\Omega_1} | u_j |^{n/n-1} dx \leq M < +\infty \qquad \text{for all } j$$

Hence, possibly taking a subsequence, one has $u_j \to v$ weakly in $\mathscr{U}(\Omega_1)$ for some $v \in \mathscr{U}(\Omega_1)$. Now one has necessarily that $v \equiv g$ in Ω_1/Ω, that is $v \in \mathscr{U}_g(\Omega_1)$, and by the semicontinuity theorem 3.3 one has $E(v) = l$.

Q. E. D.

10. **Approximation theorems.**

In this section we are going to prove various approximation results that we have used in the paper.

If $E \subset \mathbb{R}^n$ and $\rho > 0$ we set

$$\rho E = \{ \rho x \mid x \in E \}$$

DEFINITION 10.1. — *We say that an open set* $A \subset \mathbb{R}^n$ *is strictly star shaped if, for all numbers* $\rho > 1$ *one has* $\overline{A} \subset \rho A$.

If A *is strictly star shaped, also* ρA *is strictly star shaped, for any* $\rho > 0$.

THEOREM 10.2. — *Let* Ω *be a strictly star shaped Lipschitz domain in* $\mathbb{R}^n$. *Let* K *be a convex closed cone in the space* V *of the* $(n \times n)$*-symmetric matrices with real entries, and let* $u \in BD(\Omega)$ *be such that* $P_K(\varepsilon(u)) \in L^2(\Omega, V)$. *Then there exists a sequence of functions* $u_j \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$ *such that* u_j *converge to* u *strongly in* $\mathscr{U}(\Omega)$ *(definition 7.3).*

Before proving theorem 10.2, we have to give two lemmas.

For any function f defined in Ω and for any positive number r we consider the function f_r defined in $(1 + r)\Omega$ as

$$f_r(x) = f\left(\frac{x}{1 + r}\right)$$

LEMMA 10.3. — *Assume that* $u \in BD(\Omega)$. *Then one has* $u_r \in BD((1+r)\Omega)$ *and, for all Borel* $E \subset \Omega$, *one has*

$$\int_{(1+r)E} \varepsilon(u_r) = (1+r)^{n-1} \int_E \varepsilon(u) \tag{10.1}$$

$$\int_{(1+r)\Omega} |\varepsilon(u_r)| = (1+r)^{n-1} \int_\Omega |\varepsilon(u)| \tag{10.2}$$

Moreover, for all $|\varepsilon(u)|$*-measurable function* $f: \Omega \to \mathbb{R}$ *one has*

$$\int_{(1+r)\Omega} f_r(x)\,|\varepsilon(u_r)| = (1+r)^{n-1} \int_\Omega f(x)\,|\varepsilon(u)| \tag{10.3}$$

and for $|\varepsilon(u)|$*-almost all* $x \in \Omega$ *one has*

$$\frac{d\varepsilon(u_r)}{d|\varepsilon(u_r)|}((1+r)x) = \frac{d\varepsilon(u)}{d|\varepsilon(u)|}(x) \tag{10.4}$$

Finally, if $P_K\varepsilon(u) \in L^2(\Omega)$ *then one has also* $P_K\varepsilon(u_r) \in L^2((1+r)\Omega)$ *and*

$$P_K\varepsilon(u_r)((1+r)x) = \frac{1}{1+r} P_K\varepsilon(u)(x) \quad \textit{for } \mathcal{L}^n\textit{-almost all } x \in \Omega \tag{10.5}$$

Proof. — Obviously, one has $u_r \in L^1((1+r)\Omega)$. Formula (10.1) and the fact that $\varepsilon(u_r)$ is a bounded measure both follow because for all functions $g \in C_0^1(\Omega)$ and for all $i, j = 1, \ldots, n$ one has

$$\langle \varepsilon_{ij}(u_r), g_r \rangle = (1+r)^{n-1} \langle \varepsilon_{ij}(u), g \rangle \tag{10.6}$$

which in turn comes from

$$\int_{(1+r)\Omega} D_i g_r(x) u_r^j(x) dx = (1+r)^{n-1} \int_\Omega D_i g(x) u^j(x) dx$$

Formula (10.2) follows immediately from (10.1); (10.3) is equivalent to (10.2); and formula (10.4) follows from (10.1) (10.2) because

$$\frac{d\varepsilon(u)}{d|\varepsilon(u)|}(x) = \lim_{s\to 0^+} \frac{\int_{B_s(x)} \varepsilon(u)}{\int_{B_s(x)} |\varepsilon(u)|} =$$

$$= \lim_{s\to 0^+} \frac{\int_{B_{(1+r)s}((1+r)x)} \varepsilon(u_r)}{\int_{B_{(1+r)s}((1+r)x)} |\varepsilon(u_r)|} = \frac{d\varepsilon(u_r)}{d|\varepsilon(u_r)|}((1+r)x)$$

Finally, as far as (10.5) is concerned, recalling the definition of $P_K\varepsilon(u)$, one has

$$\int_{(1+r)E} P_K\varepsilon(u_r) = \int_{(1+r)E} p_K\left(\frac{d\varepsilon(u_r)}{d\,|\,\varepsilon(u_r)\,|}(y)\right)|\,d\varepsilon(u_r)\,| =$$

$$= (1+r)^{n-1}\int_E p_K\left(\frac{d\varepsilon(u)}{d\,|\,\varepsilon(u)\,|}(\xi)\right)d\,|\,\varepsilon(u)\,| = (1+r)^{n-1}\int_E P_K\varepsilon(u)$$

In particular, it follows that if $P_K\varepsilon(u) \in L^2(\Omega)$ then also $P_K\varepsilon(u_r) \in L^2((1+r)\Omega)$ and formula (10.5) holds. Q. E. D.

LEMMA 10.4. — *Let Ω be a strictly star shaped open set in $\mathbb{R}^n$ and assume that $u \in L^p(\Omega)$, $1 \leqslant p < +\infty$. Then one has*

$$\int_\Omega |\,u_r\,|^p < (1+r)^n \int_\Omega |\,u\,|^p \tag{10.7}$$

$$\lim_{r\to 0^+} \int_\Omega |\,u - u_r\,|^p dx = 0 \tag{10.8}$$

Proof. — Formula (10.7) is an obvious consequence of the change of variables formula, while the proof of (10.8) is completely similar to the proof of the continuity of the translations. Q. E. D.

Proof of theorem 10.2. — For all $r > 0$ we consider the function u_r defined in the set $(1+r)\Omega$. As Ω is strictly star shaped, we have $l(r) = \text{dist}\,(\partial((1+r)\Omega), \Omega) > 0$. Consider a function $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(x) = 0$ if $|\,x\,| \geqslant 1$, $0 \leqslant \eta(x) \leqslant 1$, $\int_{\mathbb{R}^n} \eta = 1$; then, for any $\delta > 0$, consider the function

$$\eta_\delta(x) = \delta^{-n}\eta\left(\frac{x}{\delta}\right)$$

We claim that for any sequence $r_j \downarrow 0$ one can choose a sequence δ_j, with $\delta_j < l(r_j)$, such that the functions

$$u_j = u_{r_j} * \eta_{\delta_j}$$

satisfy (7.1), (7.2), (7.3).

In fact, consider a sequence $r_j \downarrow 0$. By lemma 10.4 we have

$$\lim_{j\to\infty} \|\,u_{r_j} - u\,\|_{n/n-1} = 0$$

For each j we may choose a number $\delta_j < l(r_j)$ such that

$$\|\,u_{r_j} * \eta_{\delta_j} - u_{r_j}\,\|_{n/n-1} < 1/j$$

and (7.1) follows.

By the properties of the convolution and by lemma (10.3) one has

$$\int_\Omega |\,\varepsilon(u_{r_j} * \eta_{\delta_j})\,| \leqslant \int_{(1+r)\Omega} |\,\varepsilon(u_{r_j})\,| = (1+r_j)^{n-1}\int_\Omega |\,\varepsilon(u)\,|$$

and it follows that

(10.9)
$$\max_{j\to\infty} \lim \int_\Omega |\varepsilon(u_j)| \leqslant \int_\Omega |\varepsilon(u)|$$

Now, (7.2) follows from (10.9) and (7.1), taking into account the lower-semicontinuity of $\int_\Omega |\varepsilon(u)|$ with respect to the weak convergence of $\varepsilon(u)$.

Finally, for all j one has $\varepsilon(u_{r_j} * \eta_{\delta_j}) = \varepsilon(u_{r_j}) * \eta_{\delta_j}$, and, recalling lemma 3.4 and (10.5), one obtains

$$\int_\Omega \| P_K\varepsilon(u_j) \|^2(x)dx = \int_\Omega \| P_K(\varepsilon(u_{r_j}) * \eta_{\delta_j}) \|^2(x)dx$$
$$< \int_{(1+r_j)\Omega} \| P_K\varepsilon(u_{r_j}) \|^2(x)dx \leqslant (1+r_j)^{n-1} \int_\Omega \| P_K\varepsilon(u) \|^2(x)dx$$

so that

$$\max_{j\to\infty} \lim \int_\Omega \| P_K\varepsilon(u_j) \|^2 dx \leqslant \int_\Omega \| P_K\varepsilon(u) \|^2 dx$$

but, by (7.1), (7.2) and lemma 2.7 one has

$$P_K\varepsilon(u_j) \rightharpoonup P_K\varepsilon(u) \quad \text{weakly in} \quad L^2(\Omega, V)$$

and (7.3) follows. Q. E. D.

THEOREM 10.5. — *Let Ω be a Lipschitz domain in $\mathbb{R}^2$ and let K be a closed convex cone in the space V of the 2×2 symmetric matrices with real entries; then, for all function $v \in \mathcal{U}(\Omega)$, there exists a sequence of functions $v_j \in C^\infty(\overline{\Omega}, \mathbb{R}^2)$ that converges to u strongly in $\mathcal{U}(\Omega)$.*

Proof. — We argue as in the proof of theorem A.2 in [*17*] (compare also theorem 1 in [*5*]) except for step 4, which is not needed here, because $n = 2$.

First, we claim that for each function $v \in \mathcal{U}(\Omega)$ and for each number $\delta > 0$ there exists a function $w_\delta \in \mathcal{U}(\tilde{\Omega})$, where $\tilde{\Omega}$ is an open neighborhood of $\tilde{\Omega}$, such that

(10.10)
$$\| v - w_\delta \|_{L^2(\Omega, \mathbb{R}^3)} \leqslant \delta$$

(10.11)
$$\int_\Omega | \varepsilon(w_\delta) | < \int_\Omega | \varepsilon(v) | + \delta$$

(10.12)
$$\int_\Omega \| P_K\varepsilon(w_\delta) \|^2 dx < \int_\Omega \| P_K\varepsilon(v) \|^2 dx + \delta$$

Such a function w_δ can be constructed as in formula (A.16) of [*17*], and arguing as in [*17*] one obtains (10.10) and (10.11). Now we shall prove (10.12). For brevity we denote w_δ just by w. Using the same notation as in [*17*], we have first to choose the cylinders Ω_j such that the condition

$$(10.13)\qquad \sum_{j=1}^{N}\int_{\Omega\cap\Omega_j} \| P_K\varepsilon(v)\|^2 < \frac{\delta}{2N}$$

also holds. Then recalling that

$$\varepsilon(w) = \eta_0\varepsilon(v) + v \odot D\eta_0 + \sum_{j=1}^{N} \eta_j\varepsilon(v_{\lambda,j}) + \sum_{j=1}^{N} v_{\lambda,j} \odot D\eta_j$$

by fact 3.2, we have

$$\| P_K\varepsilon(w)\|^2 \leqslant C\left\{ \eta_0^2(x)\| P_K\varepsilon(v)\|^2 + \sum_{j=1}^{N} \eta_j^2(x)\| P_K\varepsilon(v_{\lambda,j})\|^2 + \right.$$
$$\left. + \left\| P_K\left(v \odot D\eta_0 + \sum_{j=1}^{N} v_{\lambda,j} \odot D\eta_j\right)\right\|^2 \right\}$$

for some constant $C > 0$. As $n = 2$ and $v \in L^{n/n-1}(\Omega, \mathbb{R}^2) = L^2(\Omega, \mathbb{R}^2)$, we have immediately that $P_K\varepsilon(v) \in L^2(\tilde{\Omega}, V)$; moreover, as $\sum_{j=0}^{N} D\eta_j = 0$, recalling (10.3), we get

$$\int_\Omega \| P_K\varepsilon(w)\|^2 dx < \int_\Omega \| P_K\varepsilon(v)\|^2 + N\delta + \int_\Omega \left\| P_K\left(\sum_{j=1}^{N} (v - v_{\lambda,j}) \odot D\eta_j\right)\right\|^2 dx$$

where

$$\left\| P_K\left(\sum_{j=1}^{N} (v - v_{\lambda,j}) \odot D\eta_j\right)\right\| \leqslant \sum_{i=1}^{N} \| (v - v_{\lambda,j}) \odot D\eta_j \|$$

Now, for $\lambda \to 0$, we have $(v - v_{\lambda,j}) \to 0$ in $L^2(\Omega, \mathbb{R}^2)$ and λ may be chosen small enough that (10.12) holds.

To conclude the proof, we notice that, by taking the convolution $u_\delta = w_\delta * \psi$ of a function w_δ as in (10.10), (10.11), (10.12) by a suitable mollifier ψ, we get the same formulae (10.10), (10.11), (10.12) satisfied by u_δ in place of v and 2δ in place of δ.

Finally, if we consider the sequence of functions $v_j = u_{1/j}$ these functions satisfy (7.1), moreover, by the lower semicontinuity of the total variation they satisfy (7.2) and, by lemma 2.7 and theorem 3.3 they satisfy also (7.3). Q. E. D.

PART III

MASONRY-LIKE MATERIALS

11. Introduction and notation for part III.

The general theory we have developed in parts I and II applies in particular to the case when

(11.1) $$K_0 = \{ n \times n \text{ negative semi-definite matrices} \}$$

i. e. in the case of masonry-like materials, that we have considered in the introduction. In particular, we recall that we have an existence theorem 8.5 for the Neumann boundary conditions, and an existence theorem 9.2 for the case of a prescribed displacement at the boundary. All of part II can be read thinking of the particular choice (11.1), and there is no need to restate our results in this case. On the other hand, there are a few points of the theory where the choice (11.1) makes possible to get more precise results. For instance, the boundary behaviour of the solution, in the situation of section 9, is especially meaningful for masonry-like materials, and it will be discussed in section 12. Conditions on a stress field in order for it to be safe are given in section 13. Finally, in section 14, we shall compute an explicit formula for the number $\| P_K\alpha \|^2$ where α is a 2×2 symmetric matrix, $K = \{ A^{-1}\alpha \mid \alpha \in K_0 \}$ and A corresponds to the case of isotropic materials. The formula that we obtain shows that, in this particular case, our energy functional coincides to the functional considered in [*14*].

Now we collect a few simple properties of the cone of the negative semi-definite matrices.

FACT 11.1. — *i*) The set K_0 defined in (11.1) is a closed convex cone in V.

ii) If $\overset{\circ}{K}_0$ denotes the interior of K_0, one has

$$\overset{\circ}{K}_0 = \{ \alpha \in V \mid \alpha \text{ is negative-definite} \}$$

iii) $-K_0 = \{ \alpha \in V \mid \alpha \text{ is positive semi-definite} \}$

iv) If $\tau \in V$ is such that $\tau^t = \tau^{-1}$ (i. e. τ is unitary) then the map $m : V \to V$, defined as $m(\alpha) = \tau^{-1}\alpha\tau$, is a linear isometry of V, which leaves invariant the cone K_0 (and obviously also $\overset{\circ}{K}_0$ and $-K_0$).

v) $K_0^* = -K_0$.

12. Boundary behaviour of the solutions in the case of prescribed displacement.

In this section we shall use the general notation introduced in section 9 and we shall make the special choice (11.1). The results are collected in theorem 12.2. First we have to prove a simple lemma.

LEMMA 12.1. — *For each vector* $w \in \mathbb{R}^n - \{0\}$ *one has*

$$(\mathrm{K}_0)_w = \{ -rw \mid r \geqslant 0 \} \tag{12.1}$$

$$(\mathrm{K}_0^*)_w = \{ rw \mid r \geqslant 0 \} \tag{12.2}$$

where the sets $(\mathrm{K}_0)_w$, $(\mathrm{K}_0^*)_w$ *are defined as in (9.6).*

Proof. — By fact 11.1, (*iv*) we may assume that $w = e_1 = (1, 0, 0, \ldots, 0)$. In this case, for any $a \in \mathbb{R}^n$ we have

$$a \odot w = \begin{pmatrix} a_1 & \frac{a_2}{2} & \cdots & \frac{a_n}{2} \\ \frac{a_2}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \frac{a_n}{2} & 0 & \cdots & 0 \end{pmatrix}$$

and if $a \odot w \in \mathrm{K}_0$ one must have in particular that

$$\det(a \odot w) = a_1 \leqslant 0$$

and that

$$\mathrm{I}_2(a \odot w) = \sum_{i=2}^{n} \frac{a_i^2}{4} \leqslant 0$$

where $\mathrm{I}_2(a \odot w)$ is the second invariant of the matrix, i. e. the sum of 2×2 minors constructed on the principal diagonal of the matrix. This way, (12.1) is proved and (12.2) follows immediately. Q. E. D.

THEOREM 12.2. — *Let v be a solution to problem (9.1) and set* $\sigma = \mathrm{AP_K}\varepsilon(v)$. *Then one has*

$$\left.\begin{array}{l} g(x) - v(x) = r(x)v(x) \\ r(x) \geqslant 0 \end{array}\right\} \mathscr{H}^{n-1}\text{-}a.\,e. \quad on\ \partial\Omega \tag{12.3}$$

Assuming that $\partial\Omega$ *is of class* C^2, *for any function* $h(x) \in C^1(\partial\Omega)$, $h \geqslant 0$, *one has*

$$\langle \sigma.\nu, \nu h \rangle_1 \leqslant 0 \tag{12.4}$$

where $\langle , \rangle_1$ *is the pairing between* $H^{-1/2}(\partial\Omega, \mathbb{R}^n)$, $H^{1/2}(\partial\Omega, \mathbb{R}^n)$. *Finally, if one assumes that* $r(x)$ *is continuous and that* $\sigma \in C^1(\overline{\Omega})$, *one has also that*

(12.5) $\sigma(x).\nu(x) = 0$ $\mathcal{H}^{n-1}$*-a. e. in the set* $T = \{ x \in \partial\Omega \mid r(x) > 0 \}$

Proof. — Formula (12.3) follows immediately from (9.5) and (12.1). Now let us prove (12.4). We have $\sigma \in L^2(\Omega, V)$ and, by (9.3), we have $\operatorname{div} \sigma \in L^n(\Omega, \mathbb{R}^n) \subset L^2(\Omega, \mathbb{R}^n)$, hence we get

$$\sigma.\nu \in H^{-1/2}(\partial\Omega, \mathbb{R}^3)$$

We recall that, for any function $\overline{\psi} \in H^{1/2}(\partial\Omega, \mathbb{R}^n)$, one has

$$\langle \sigma.\nu, \psi \rangle_1 = \int_\Omega \sigma.\varepsilon(\psi)dx + \int_\Omega \psi \operatorname{div} \sigma dx \tag{12.5}$$

where ψ is any function in $H^{1,2}(\Omega, \mathbb{R}^n)$ such that $\psi_{|\partial\Omega} = \overline{\psi}$.

If $\partial\Omega$ is of class C^2 then the function $-\nu(x)$ is of class C^1 and, for any function $h(x) \in C^1(\partial\Omega)$, $h(x) \geqslant 0$, there exists a function $\psi_0 \in C^1(\overline{\Omega}, \mathbb{R}^n)$ such that $\psi_0(x) = -\nu(x)h(x)$ on $\partial\Omega$. By (12.2) one has that (9.7) holds for ψ_0; and (9.8) too. Recalling (12.5) one gets then (12.4).

Now we shall prove (12.5). We shall argue as in the proof of (9.8), with some improvements. Let $\psi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ be such that

$$\psi(x) = 0 \qquad \text{for all} \quad x \in (\partial\Omega)\backslash T \tag{12.6}$$

$$|\psi(x)| \leqslant r(x) \qquad \text{for all} \quad x \in T \tag{12.7}$$

Consider the function $\phi \in BD(\Omega)$ defined as

$$\phi(x) = \begin{cases} \psi(x) & \text{in} \quad \Omega \\ 0 & \text{in} \quad \Omega_1\backslash\Omega \end{cases}$$

then there is a number $t_0 > 0$ such that, if $|t| \leqslant t_0$, one has

$$P_K\varepsilon(v + t\phi)_{|\partial\Omega} = p_K((g(x) - v(x) - t\psi(x)) \odot \nu(x))\mathcal{H}^{n-1}{}_{|\partial\Omega} = 0 \tag{12.8}$$

In fact, one has

$$(g(x) - v(x) - t\psi(x)) \odot \nu(x) = r(x)\nu(x) \odot \nu(x) - t\psi(x) \odot \nu(x)$$

where

$$\operatorname{dist}(r(x)\nu(x) \odot \nu(x), \partial K_0^*) = c_1 r(x) \qquad \text{for all} \quad x$$

$$|t\psi(x) \odot \nu(x)| \leqslant |t| r(x) c_2 \qquad \text{for all} \quad x$$

for suitable numbers $c_1, c_2 > 0$. By (12.8) (recall the proof of (9.8)) we get that $v + t\phi \in \mathcal{U}_g(\Omega_1)$ for $|t| < t_0$ and we have

$$0 = \frac{d}{dt}E(u + t\phi)_{|t=0} = \int_\Omega \sigma.\varepsilon(\psi)dx - \int_\Omega f\psi = 0 \tag{12.9}$$

As σ is assumed to be smooth, we can integrate by parts and we get

$$(12.10) \qquad \int_{\partial\Omega} \sigma(x).\nu(x)\psi(x)d\mathcal{H}^{n-1} = 0$$

for all $\psi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ that satisfy (12.6), (12.7). As we assume that $r(x)$ is continuous, if $x_0 \in T$ there is a neighborhood $\mathcal{U}(x_0)$ of x_0 on $\partial\Omega$ where $r(x) > \frac{1}{2} r(x_0) > 0$ and (12.9) (12.10) hold for any ψ with support in $U(x_0)$ (times possibly a constant, that does not matter, however). It follows then that $\sigma(x_0).\nu(x_0) = 0$. Q. E. D.

13. **Safe stress fields.**

Here we use the notation of section 8 and we assume that K_0 is as in (11.1).

PROPOSITION 13.1. — *A stress field τ of the type*

$$\tau = H - \delta I$$

where $H \in L^2(\Omega, K_0)$, and δ is a positive number, is safe.

Proof. — For all x and for all $\alpha \in K_0^* = - K_0$ we have

$$-(\tau(x), \alpha) = -(H(x), \alpha) + \delta \operatorname{tr}(\alpha)$$

where $(H(x), \alpha) \leqslant 0$. If we call $\mu_1, \ldots, \mu_n$ the eigenvalues of α, which are all non negative numbers, we have

$$|\alpha| = \left(\sum_{j=1}^{n} \mu_j\right)^{1/2} < \sqrt{n}\operatorname{tr}(\alpha)$$

and it follows that

$$-(\tau(x), \alpha) \geqslant \frac{\delta}{\sqrt{n}} |\alpha|$$

Q. E. D.

14. **An explicit formula for $\| P_K \varepsilon(u) \|^2$ for isotropic materials in two dimensions.**

In this section we take $n = 2$ and K_0 as in (11.1). For any matrix

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix} \in V$$

we consider the matrices

$$\alpha^s = \begin{pmatrix} \dfrac{\alpha_{11} + \alpha_{22}}{2} & 0 \\ 0 & \dfrac{\alpha_{11} + \alpha_{22}}{2} \end{pmatrix} = \frac{1}{2}(\mathrm{tr}\,\alpha)\mathrm{I}$$

$$\alpha^{\mathrm{D}} = \begin{pmatrix} \dfrac{\alpha_{11} - \alpha_{22}}{2} & \alpha_{12} \\ \alpha_{12} & \dfrac{\alpha_{22} - \alpha_{11}}{2} \end{pmatrix} = \alpha - \alpha^s$$

which are called the spherical and deviatoric part of α. If we set $\mathrm{V}_s = \{ t\mathrm{I} \mid t \in \mathbb{R} \}$, $\mathrm{V}_{\mathrm{D}} = \{ \alpha \in \mathrm{V} \mid \mathrm{tr}\,\alpha = 0 \}$ we have that V_s and V_{D} are orthogonal spaces in V with respect to the standard scalar product, and α^s, α^{D} are just the orthogonal projection of α on V_s and V_{D} with respect to the standard scalar product in V.

In what follows we shall compute an explicit formula for $\| p_{\mathrm{K}}\alpha \|^2$ in the case that the operator A is of the type

$$\mathrm{A}\alpha = \mu_1\alpha^s + \mu_2\alpha^{\mathrm{D}} \tag{14.1}$$

that is $(\mathrm{A}\alpha)^s = \mu_1\alpha^s$, $(\mathrm{A}\alpha)^{\mathrm{D}} = \mu_2\alpha^{\mathrm{D}}$, where μ_1, μ_2 are positive constants. In classical elasticity, the choice (14.1) of the operator A corresponds to an isotropic (and homogeneous, of course) material, and μ_1, μ_2 are proportional to the bulk modulus and to the shear modulus, respectively.

Our starting point is the fact that, by the properties of the projections and by (5.1), one has

$$\| p_{\mathrm{K}}\alpha \|^2 = \| \alpha - p_{\mathrm{K}}\alpha \|^2 = \min_{\beta \in \mathrm{K}^\perp} \| \alpha - \beta \|^2 = \min_{\beta \in \mathrm{K}_0^*} \| \alpha - \beta \|^2 \tag{14.2}$$

where K_0^*, in this case, is the cone of the positive semi-definite matrices in V. We shall compute the minimum in (14.2) in theorem 14.2, but we have first to set some notation and to make a few preliminary remarks.

For any matrix $\alpha \in \mathrm{V}$ we set

$$\mathrm{I}_1 = | \alpha^s |, \qquad \mathrm{I}_2 = | \alpha^{\mathrm{D}} | \tag{14.3}$$

We have

$$\begin{aligned} 2\mathrm{I}_1^2 &= \mathrm{T}^2 \\ 2\mathrm{I}_2^2 &= \mathrm{T}^2 - 4\mathrm{D} \end{aligned} \tag{14.4}$$

where $\quad \mathrm{T} = \mathrm{tr}\,\alpha = \alpha_{11} + \alpha_{22}, \qquad \mathrm{D} = \det\alpha = \alpha_{11}\alpha_{22} - \alpha_{12}^2 .$

With the choice (14.1), one has

$$\begin{aligned} \langle \alpha, \beta \rangle &= (\mathrm{A}\alpha, \beta) = \mu_1(\alpha^s, \beta^s) + \mu_2(\alpha^{\mathrm{D}}, \beta^{\mathrm{D}}) \\ \| \alpha \|^2 &= \mu_1\mathrm{I}_1^2 + \mu_2\mathrm{I}_2^2 \end{aligned} \tag{14.5}$$

LEMMA 14.1. — *If* A *is as in* (*14.1*) *then one has*

$$K_0 = \{ \alpha \in V \mid T \leqslant 0, I_1 - I_2 \geqslant 0 \} \tag{14.6}$$

$$K = \{ \alpha \in V \mid T \leqslant 0, \mu_1 I_1 - \mu_2 I_2 \geqslant 0 \} \tag{14.7}$$

Proof. — Formula (14.6) follows from the simple remark that

$$2 \det \alpha = I_1^2 - I_2^2 = (I_1 - I_2)(I_2 + I_1) .$$

Then we have that $A\alpha \in K_0$ if and only if

$$T(A\alpha) = \mu_1 \operatorname{tr} \alpha \leqslant 0$$
$$I_1(A\alpha) - I_2(A\alpha) = \mu_1 I_1(\alpha) - \mu_2 I_2(\alpha) \geqslant 0$$

and (14.7) follows. Q. E. D.

Now we can state and prove our result.

THEOREM 14.2. — *If* A *is as in* (*14.2*) *one has*

$$\| P_K \alpha \|^2 = \begin{cases} 0 & \textit{if} \quad \alpha \in K_0^* \\ \left(\dfrac{1}{\mu_1} + \dfrac{1}{\mu_2} \right) (T^2 - 2D - T\sqrt{T^2 - 4D}\,) & \textit{if} \quad \alpha \in V - \{ K \cup K_0^* \} \\ \dfrac{\mu_1 + \mu_2}{2} T^2 - 2\mu_2 D & \textit{if} \quad \alpha \in K \end{cases}$$

Proof. — Clearly, if $\alpha \in K_0^* = K^\perp$ one has $P_K\alpha = 0$, while, if $\alpha \in K$ one has $P_K\alpha = \alpha$, and, by (14.5) (14.4) we get our result in these cases.

We are left to prove the middle case. Given a matrix α, we have to find the minimum value of the function

$$F(\beta) = \frac{1}{2} \| \alpha - \beta \|^2$$

in the set

$$K_0^* = \{ \beta \in V \mid T(\beta) \geqslant 0, \det \beta \geqslant 0 \}$$

Certainly, a minimum point $\bar\beta$ exists, because K_0^* is convex and closed; moreover, if $\alpha \notin K$ then $\bar\beta \neq 0$ and $\bar\beta$ is a stationary point of $F(\beta)$ on the surface $\det \beta = 0$. Finally, $\bar\beta$ is the unique stationary point of $F(\beta)$ on the surface

$$\partial K_0^* - \{ 0 \} = \{ \beta \mid \det \beta = 0, T\beta > 0 \}$$

In conclusion: *if we find a stationary point* $\bar\beta$ *for* F *on* $\{ \det \beta = 0 \}$ *and we have that* $T(\bar\beta) > 0$, *then we have*

$$\| P_K \alpha \|^2 = \| \alpha - \bar\beta \|^2$$

Actually, it is convenient to consider the enlarged problem:

(14.8) $$\text{find a critical point for } F(\beta) \text{ on the surface} \quad \{ \det \beta = 0, \beta \neq 0, \beta \text{ non necessarily symmetric} \}$$

For any given $\alpha \in V$, we shall find a solution $\bar{\beta}$ to problem (14.8) such that $\bar{\beta}$ is symmetric and tr $\bar{\beta} > 0$, and it will be also a solution of our starting problem.

It is clear that the solutions to problem (14.8) satisfy the system

(14.9) $$\begin{cases} \nabla F(\bar{\beta}) + t\nabla(\det \bar{\beta}) = 0 \\ \det \bar{\beta} = 0 \end{cases}$$

for some Lagrange multiplier $t \in \mathbb{R}$, and we may rewrite (14.9) as

(14.10) $$\begin{cases} \mu_1(\alpha - \bar{\beta})^s + \mu_2(\alpha - \bar{\beta})^D + t(\bar{\beta}^s - (\bar{\beta}^D)^t) = 0 \\ \det \bar{\beta} = 0 \end{cases}$$

where $(\bar{\beta}^D)^t$ is the transposed matrix of $\bar{\beta}^D$. Now, let us just look for a solution $\bar{\beta}$ which is symmetric, so that $(\bar{\beta}^D)^t = \bar{\beta}^D$. By the mutual orthogonality of spherical and deviatoric parts, (14.10) is equivalent to

(14.11) $$\begin{cases} \mu_1\alpha^s + \bar{\beta}^s(t - \mu_1) = 0 \\ \mu_2\alpha^D - \bar{\beta}^D(t + \mu_2) = 0 \\ \det \bar{\beta} = 0 \end{cases}$$

Now we consider two different cases, namely

(14.11, *a*) $$\alpha^s = 0$$

(14.12, *b*) $$\alpha^s \neq 0$$

First we consider case (14.12, *b*). In this case we have $I_1 > 0$ and we must have also $I_2 > 0$, otherwise, by lemma 14.1 we would have $\alpha \in - K_0 \in K_0^*$ or $\alpha \in K$, which is assumed not to hold. It follows by (14.11) that one has

$$t - \mu_1 \neq 0 \qquad t - \mu_2 \neq 0$$

By (14.11) we get then

(14.12) $$\begin{cases} \beta_{11} + \beta_{22} = (\alpha_{11} + \alpha_{22})\dfrac{\mu_1}{\mu_2 - t} \\ \beta_{11} - \beta_2 = (\beta_{11} - \beta_{22})\dfrac{2}{\mu_2 + t} \\ \beta_{12} = \alpha_{12}\dfrac{\mu_2}{\mu_2 + t} \\ \det \beta = 0 \end{cases}$$

Computing the β_{ij} and substituting in $\det \beta = 0$ we obtain

$$(14.13) \qquad \left(\frac{\mu_2 + t}{\mu_1 - t}\right)^2 = \frac{I_2^2 \mu_2^2}{I_1^2 \mu_1^2}$$

Again we consider two cases:

$$(14.11, a) \qquad T(\alpha) > 0$$

$$(14.14, b) \qquad T(\alpha) < 0$$

In case (14.14, *a*) we take t such that

$$\frac{\mu_2 + t}{\mu_1 + t} = \frac{I_2 \mu_2}{I_1 \mu_1}$$

that is

$$t = \frac{\mu_1 \mu_2 (I_2 - I_1)}{I_1 \mu_1 + I_2 \mu_2}$$

By (14.11) we obtain

$$\bar{\beta}^s = \alpha^s \frac{1}{I_1} \frac{\mu_1 I_1 + \mu_2 I_2}{\mu_1 + \mu_2}$$

$$\bar{\beta}^D = \alpha^D \frac{1}{I_2} \frac{\mu_1 I_1 + \mu_2 I_2}{\mu_1 + \mu_2}$$

and we notice that $\operatorname{tr} \beta > 0$.

Finally, we may compute $\alpha^s - \bar{\beta}^s$ and $\alpha^D - \bar{\beta}^D$ and by (14.5) we get

$$\| \alpha - \bar{\beta} \|^2 = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} (I_1 - I_2)^2 = $$

$$= \left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\left(T^2 - 2D - T\sqrt{T^2 - 4D}\right)$$

Now we go back to case (14.14, *b*). In that case we choose

$$\frac{\mu_2 + t}{\mu_1 + t} = -\frac{I_2 \mu_2}{I_1 \mu_1}$$

that is

$$t = \frac{\mu_1 \mu_2 (I_1 + I_2)}{\mu_2 I_2 - \mu_1 I_1}$$

where $\mu_2 I_2 - \mu_1 I_1 > 0$, otherwise, by 14.7, we would have $\alpha \in K$. It follows that

$$\bar{\beta}^s = -\alpha^s \frac{\mu_2 I_2 - \mu_1 I_1}{I_1(\mu_1 + \mu_2)}$$

$$\bar{\beta}^D = \alpha^D \frac{\mu_2 I_2 - \mu_1 I_1}{I_2(\mu_1 + \mu_2)}$$

and we have again that tr $\bar{\beta} > 0$ and

$$\| \alpha - \bar{\beta} \|^2 = \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2} (I_1 + I_2)^2 = \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) (T^2 - 2D - T\sqrt{T^2 - 4D}).$$

This concludes the proof for case (14.14, *b*) and for the whole case (14.11, *b*). We still have to consider case (14.11, *a*): if one has $\alpha^s = 0$, then, by (14.10), as $\bar{\beta}$ cannot be zero, we have $t = \mu_1$ and

$$\bar{\beta}^D = \frac{\mu_2}{\mu_1 + \mu_2} \alpha^D$$

and, recalling (14.4):

$$| \operatorname{tr} \beta |^2 = 2 | \bar{\beta}^D |^2 + 4 \det \bar{\beta}$$

As det $\beta = 0$, it follows that

$$| \bar{\beta}^s - \alpha^s |^2 = | \bar{\beta} |^2 = | \beta^D |^2 = \left(\frac{\mu_2}{\mu_1 + \mu_2} \right)^2 | \alpha^D |^2$$

and, computing as before, one obtains

$$\| \alpha - \bar{\beta} \|^2 = \mu_1 | \alpha^s - \beta^s |^2 + \mu_2 | \alpha^D - \beta^D |^2 = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} I_2^2 = -2D \frac{\mu_1 \mu_2}{\mu_1 + \mu_2}$$

Q. E. D.

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