ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 2, nº 2 (1985), p. 101-118
<http://www.numdam.org/item?id=AIHPC_1985__2_2_101_0>

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Ann. Inst. Henri Poincaré,
Vol. 2, n° 2, 1985, p. 101-118 *Analyse non linéaire*

Global existence for the Vlasov-Poisson equation in 3 space variables with small initial data

by

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ABSTRACT. — In this paper we consider the Vlasov Poisson equation in three space variables in the whole space. We show the existence of dispersion property. With this dispersion property we are able to prove the existence of a smooth solution for all times under the following assumption: the initial data are localised and small enough.

RÉSUMÉ. — On considère l'équation de Vlasov Poisson en dimension 3, dans l'espace entier. On dégage une propriété de dispersion. L'utilisation de cette propriété permet de prouver l'existence d'une solution régulière pour tout temps, pourvu que les données initiales soient localisées et assez petites.

I. INTRODUCTION

We consider the problem of the existence of a smooth solution of the Vlasov Poisson equation in 3 space variables. The existence of a smooth solution in one space variable has been proved by Iordanskii [*2*], and in two space variables by Ukai and Okabe [*6*]. The results of [*2*] and [*6*] rely on Sobolev type estimates and cannot be extended to higher dimensions. Theses methods do not need any restriction on the size of the initial data, which have only to be smooth enough but on the other hand they

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give no information on the asymptotic behaviour of the solution. Up to now there was no results concerning the existence of a global smooth solution for the Vlasov Poisson equation in three space variables. The method of [*6*] gives the existence of a local, in time, smooth solution, and one can prove the existence of a global weak solution (This is due to Arsenev [*1*]). This weak solution may behave badly for large time and this may be related to the appearance of some kind of turbulence.

On the other hand many authors (Klainerman [*3*], Klainerman and Ponce [*4*] and Shatah [*5*]) have proved the existence of a global smooth solution for the non linear wave equation in high dimensions with small initial data. Their method uses basically the dispersive effect of the linearized wave equation, to balance the effect of the non linear term.

We will follow a similar route for the Vlasov-Poisson equation: this equation describes the evolution of the density of particules $u(x, v, t)$, and reads

$$\frac{\partial u}{\partial t} + v.\nabla_x u + \nabla_x \phi . \nabla_v u = 0 \qquad u|_{t=0} = u_0 \tag{1}$$

where the potential $\phi(x, t)$ is given by the equations

$$-\Delta\phi(x, t) = \rho(x, t); \qquad \rho(x, t) = \int_{\mathbb{R}^3} u(x, v, t)dv \tag{2}$$

and $\rho(x, t)$ denotes the total density of charge.

Now the linearized equation is the transport equation:

$$\frac{\partial u}{\partial t} + v.\nabla_x u = 0 \qquad u|_{t=0} = u_0 \tag{3}$$

whose solution is

$$u(x, v, t) = u_0(x - vt, v). \tag{4}$$

If we assume that $u_0(x, v)$ is bounded by an integrable function $h(x)$:

$$|u_0(x, v)| \leqq h(x); \qquad \int_{\mathbb{R}^3} h(x)dx < +\infty, \tag{5}$$

we deduce the decay estimate

$$|\rho(x, t)| \leqq \frac{1}{t^3} \int_{\mathbb{R}^3} h(x)dx, \tag{6}$$

thanks to the change of variables $v \to \xi = x - vt$, x and t being constant. The decay of order t^{-3} (more generally, of order t^{-d} in any $\mathbb{R}^d$ space) in formula (6), will be the basic ingredient of our proof.

Finally, our result shows that, if the initial data are small enough, the electric field

$$\mathrm{E} = -\nabla_x \Phi \tag{7}$$

decays in $L^\infty(\mathbb{R}^3)$ like t^{-2}. No phenomena of turbulence nor solitary wave may appear in this case. We notice that the time is reversible, and in our proofs, we only consider the case $t > 0$.

This paper is organized as follows: in section II, we describe the equations and the classical estimates. In section III, we prove some results concerning the Hamiltonian systems which govern the trajectories of the particules. In section IV, we build up an iterative scheme which leads to the existence of a smooth solution. Section V is devoted to the proof of the uniqueness.

II. DESCRIPTION OF THE VLASOV-POISSON EQUATIONS

We will denote by u_α, $(1 \leqq \alpha \leqq N)$, N positive functions which describe the density at the point x, and at the time t, of particules of the type α, which have the velocity v. We denote by q_α and m_α their masse and charge. The variation of u_α is described by the equations

$$\frac{\partial u_\alpha}{\partial t} + (v.\nabla_x)u_\alpha + \frac{q_\alpha}{m_\alpha}(E.\nabla_v)u_\alpha = 0\,; \qquad 1 \leqq \alpha \leqq N. \tag{8}$$

As usual we have

$$v.\nabla_x = \sum_{i=1}^{3} v_i \frac{\partial}{\partial x_i} \qquad \text{and} \qquad E.\nabla_v = \sum_{i=1}^{3} E_i \frac{\partial}{\partial v_i}.$$

The electric field E is related to the variation of the functions u_α by the Poisson equation:

$$E = -\nabla_x \Phi\,; \qquad -\Delta\phi = 4\pi\rho\,; \qquad \rho(x,t) = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} u_\alpha(x, v, t)dv \tag{9}$$

Or

$$E = 4\pi \sum_\alpha q_\alpha \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3}\left(\int_{\mathbb{R}^3} u_\alpha(y, v, t)dv\right)dy. \tag{10}$$

If we assume that the functions $u_\alpha(x, v, t)$ are smooth, we deduce from (8) that the function

$$t \to u_\alpha(x(t), v(t), t)$$

is constant whenever $(x(t), v(t))$ is a solution of the Hamiltonian system:

$$\dot{x}(t) = v(t), \qquad \dot{v}(t) = \frac{q_\alpha}{m_\alpha} E(x(t), t). \tag{11}$$

Therefore, the positivity and the $L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)$-norm of the functions $u_\alpha(.,.,t)$ is preserved. Furthermore, since the divergence of the vector

field $\left(v, \frac{q_\alpha}{m_\alpha} E(x, t)\right)$ is zero, we deduce from the Liouville theorem that the $L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)$-norm of u_α is also preserved:

$$\frac{d}{dt}\left(\iint_{\mathbb{R}^3_x \times \mathbb{R}^3_v} u_\alpha(x, v, t) dx dv\right) = 0. \tag{12}$$

III. SOME LEMMATA CONCERNING THE ELECTRIC FIELD E AND THE HAMILTONIAN SYSTEM (11)

We begin with some notations. For functions $\rho(x)$ and $u(x, v)$, we shall denote by $\| \rho \|_p$ and $\| u \|_p$ the usual L^p norms:

$$\| \rho \|_p = \left(\int_{\mathbb{R}^3} | \rho(x) |^p dx\right)^{1/p}, \qquad \| u \|_p = \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} | u(x, v) |^p dx dv\right)^{1/p}$$

$$\| \rho \|_\infty = \operatorname{Sup} | \rho(x) | ; \qquad \| u \|_\infty = \operatorname*{Sup}_{(x,v)} | u(x, v) | .$$

For a function $\rho(x, t)$ we will write:

$$||| \rho ||| = \operatorname*{Sup}_{t \in \mathbb{R}_+} \| \rho(., t) \|_\infty .$$

LEMMA 1. — *Let $\rho(x)$ be a smooth function belonging to the space $L^1(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$ then for the function Φ given by the formula:*

$$\Phi(x) = \int_{\mathbb{R}^3} \frac{\rho(y) dy}{| x - y |} \tag{13}$$

one has the following estimates

$$\| \Phi \|_\infty \leqq C \| \rho \|_\infty^{1/3} \| \rho \|_1^{2/3} \tag{14}$$

$$\| \nabla \Phi \|_\infty \leqq C \| \rho \|_\infty^{2/3} \| \rho \|_1^{1/3} \tag{15}$$

$$\| D^2 \Phi \|_\infty \leqq C_\theta \| \rho \|_\infty^{3(1-\theta)/(3+\theta)} \| \nabla_x \rho \|_\infty^{3\theta/(3+\theta)} \| \rho \|_1^{\theta/(3+\theta)}, \tag{16}$$

(In (14) and (15) the constant C is « universal »; in (16) D^2 denotes any second order derivative and the constant C_θ depends on $\theta \in]0, 1[$).

Proof. — The proof of these estimates is standard. We describe it here for sake of completeness. Let $r > 0$, we have

$$\begin{aligned} | \Phi(x) | &\leqq \int_{|x-y|<r} \frac{| \rho(y) |}{| x - y |} dy + \int_{|x-y|>r} \frac{| \rho(y) |}{| x - y |} dy \\ &\leqq 4\pi \| \rho \|_\infty \int_0^r \frac{\rho^2}{\rho} d\rho + \frac{1}{r} \| \rho \|_1 \\ &\leqq 2\pi r^2 \| \rho \|_\infty + \frac{1}{r} \| \rho \|_1 \end{aligned} \tag{17}$$

Now if we take the infimum of the last term of (17) with respect to r we obtain (14).

Next for $\nabla_x \Phi$ we have:

$$(18) \qquad |\nabla_x \Phi(x)| = \left| \int \frac{\rho(y)(x-y)}{|x-y|^3} dy \right|$$
$$\leqq \int_{|x-y|<r} \frac{|\rho(y)|}{|x-y|^2} dy + \int_{|x-y|>r} \frac{|\rho(y)|}{|x-y|^2} dy$$
$$\leqq 4\pi \|\rho\|_\infty r + \frac{1}{r^2} \|\rho\|_1 .$$

Once again taking the infimum of the last term of (18) with respect to r we obtain (15).

Finally, we compute $\dfrac{\partial^2 \Phi}{\partial x_i \partial x_j}$ for $i \neq j$ (the computation is similar for $i = j$). We have

$$(19) \qquad \left| \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \right| = \left| \fint \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} \rho(y) dy \right| \quad (^1)$$
$$= \left| \fint \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} (\rho(y) - \rho(x)) dy \right|$$
$$\leqq C \left(\int_{|x-y|<r} \frac{1}{|x-y|^{3-\theta}} \frac{|\rho(y) - \rho(x)|}{|x-y|^\theta} dy \right.$$
$$\left. + \int_{|x-y|>r} \frac{1}{|x-y|^3} \rho(y) dy \right)$$
$$\leqq C_\theta \left(r^\theta \|\rho\|_{0,\theta} + \frac{1}{r^3} \|\rho\|_1 \right).$$

In the last term of (19) C_θ denotes a constant depending on θ and $\|\rho\|_{0,\theta}$ is the sup norm of the Holder quotient

$$(20) \qquad \|\rho\|_{0,\theta} = \sup_{x,y} \frac{|\rho(x) - \rho(y)|}{|x-y|^\theta}$$

Now from (19) we deduce the relation

$$(21) \qquad |D^2_{ij} \Phi| \leqq C_\theta \|\rho\|_{0,\theta}^{3/(\theta+3)} \|\rho\|_1^{\theta/(3+\theta)}$$

Finally from the inequality

$$(22) \qquad (\rho(x) - \rho(y))^{1-\theta} \leqq 2 \|\rho\|_\infty^{1-\theta}$$

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(1) In (19) $\fint$ denotes the Cauchy principal value.

we deduce the relation:

$$\| \rho \|_{0,\theta} \leqq 2 \| \rho \|_\infty^{1-\theta} \| \nabla_x \rho \|_\infty^\theta , \tag{23}$$

which, with (21) gives (16) and the proof of the lemma is complete.

Now, we consider the Hamiltonian system

$$\dot{X}(s) = V(s) ; \qquad \dot{V}(s) = E(X(s), s) . \tag{24}$$

with the Cauchy data:

$$X(t) = x ; \qquad V(t) = v . \tag{25}$$

We assume that the function $E(x, t) = (E_i(x, t))_{i=1,2,3}$ is continuous in (x, t). and twice continuously differentiable with respect to x. Furthermore, denoting by $\nabla_x E$ the gradient matrix and $\nabla_x^2 E$ the second derivative of E with respect to x, we assume that $\nabla_x E$ and $\nabla_x^2 E$ are bounded in $\mathbb{R}_x^3 \times \mathbb{R}_t^+$. Therefore, by the classical Cauchy-Lipschitz theorem, equations (24), (25) have a unique solution

$$s \to (X(s, t, x, v), \qquad V(s, t, x, v)) .$$

defined for (s, t, x, v) in $\mathbb{R}_s^+ \times \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$.

Furthermore, for (s, t) fixed, the mapping

$$(x, v) \to (X(s, t, x, v), \qquad V(s, t, x, v))$$

is twice continuously differentiable.

We are concerned with the behaviour for $s \in [0, t]$ and for t large of the quantities:

$$\det \left(\frac{\partial X}{\partial v} \right) ; \quad \frac{\partial V}{\partial v} ; \quad \frac{\partial X}{\partial x} ; \quad \frac{\partial X}{\partial v} ; \quad \frac{\partial^2 X}{\partial x^2} ; \quad \frac{\partial^2 V}{\partial x^2} .$$

We will show that this behaviour can be controlled with some informations on the asymptotic behaviour, for t going to infinity, of the vector field $E(x, t)$.

PROPOSITION 1. — *We assume that* $E(x, t)$ *satisfies the estimates*

$$\| \nabla_x E(., t) \|_\infty \leqq \eta/(1 + t)^{5/2} \tag{26}$$

$$\| \nabla_x^2 E(. \ t) \|_\infty \leqq \eta/(1 + t)^{5/2} \tag{27}$$

with $\eta < 1$. *Then, the following estimates are valid for any* (t, x, v) *in* $\mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$, *and any* s *such that* $0 \leqq s \leqq t$.

$$\left| \frac{\partial X}{\partial v} (s, t, x, v) - (s - t) \, \mathrm{Id} \right| + \left| \frac{\partial V}{\partial v} (s, t, x, v) \right| \leqq C\eta(t - s) \tag{28}$$

$$\left| \frac{\partial X}{\partial x} (s, t, x, v) - \mathrm{Id} \right| + \left| \frac{\partial V}{\partial x} (s, t, x, v) \right| \leqq C\eta \tag{29}$$

$$\left|\frac{\partial^2 X}{\partial x^2}(s, t, x, v)\right| + \left|\frac{\partial^2 V}{\partial x^2}(s, t, x, v)\right| \leqq C\eta \tag{30}$$

where Id *denotes the identity matrix of* $\mathbb{R}^3$, *and* C *any arbitrary constant.*

Proof. — For any fixed (t, x, v) in $\mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$, we write

$$\frac{\partial X}{\partial v}(s) = \frac{\partial X}{\partial v}(s, t, x, v) \qquad \text{and} \qquad R(s) = \frac{\partial X}{\partial v}(s) - (s - t)\mathrm{Id}\,.$$

This matrix satisfies the differential equation, obtained from (24) and (25).

$$\widehat{\frac{\partial \dot{X}}{\partial v}}(s) = \nabla_x E(X(s), s)\,.\,\frac{\partial X}{\partial v}(s)\,; \qquad \frac{\partial X}{\partial v}(t) = 0\,; \qquad \widehat{\frac{\partial \dot{X}}{\partial v}}(t) = \mathrm{Id}\,. \tag{31}$$

Then, thanks to the Taylor formula, we have

$$R(s) = \int_s^t (u - s)\nabla_x E(X(u), u)\,.\,(R(u) + (u - t)\mathrm{Id})du\,. \tag{32}$$

Thus, with (26)

$$|R(s)| \leqq \eta \int_s^t \frac{u - s}{(1 + u)^{5/2}}\,|R(u)|\,du + \eta \int_s^t \frac{(u - s)(t - u)}{(1 + u)^{5/2}}\,du \tag{33}$$

Thanks to integrations by parts, it is easy to see that

$$\int_s^t \frac{(u - s)(t - u)}{(1 + u)^{5/2}}\,du \leqq C(t - s) \tag{34}$$

Now we can apply Gronwall's lemma to inequality (33) and we obtain

$$|R(s)| \leqq C\eta(t-s) + C\eta^2 \int_s^t \frac{(t-u)(u-s)}{(1 + u)^{5/2}} \exp\left(\eta \int_s^u \frac{w - s}{(1+w)^{5/2}}\,dw\right)ds \tag{35}$$

Now, an integration by parts leads to

$$\int_s^u \frac{w - s}{(1 + w)^{5/2}}\,dw \leqq C \tag{36}$$

Then, as $\eta < 1$, (34) and (35) give

$$|R(s)| \leqq C\eta(t - s) \tag{37}$$

which proves (38). The other estimates can be shown in a similar manner.

COROLLARY 1. — *Under the hypotheses of proposition 1, there exists a constant* $\eta_0 > 0$ *such that, when the vector field* $E(x, t)$ *satisfies the estimate*

$$\|\nabla_x E(., t)\|_\infty \leqq \eta/(1 + t)^{5/2} \qquad \text{with} \qquad \eta < \eta_0 \tag{40}$$

the following facts are true:

i) The determinant of the matrix $\frac{\partial X}{\partial v}(s, t, x, v)$ *satisfies the estimate, valid for any* (x, v, t) *in* $\mathbb{R}^3_x \times \mathbb{R}^3_v \times \mathbb{R}^+_t$ *and any* s *such that* $0 \leqq s \leqq t$:

$$\left|\det\left(\frac{\partial X}{\partial v}(s, t, x, v)\right)\right| \geqq \frac{(t-s)^3}{2} \tag{41}$$

ii) For any fixed (s, t, x) *in* $\mathbb{R}^+_t \times \mathbb{R}^+_t \times \mathbb{R}^3_x$, *such that* $0 \leqq s \leqq t$, *the mapping:*

$$\theta : v \to X(s, t, x, v) \tag{42}$$

is one to one.

Proof. — *i)* We use the notations of Proposition 1 and we write:

$$\begin{aligned}\left|\det\left(\frac{\partial X}{\partial v}(s)\right)\right| &= |\det((s-t)\,\mathrm{Id} + R(s))| \\ &= (t-s)^3\left|\det\left(\mathrm{Id} + \frac{R(s)}{s-t}\right)\right|.\end{aligned} \tag{43}$$

Now, thanks to (28), the norm of $R(s)/(s-t)$ goes to zero when η goes to zero, uniformly with respect to (s, t, x, v), and by the continuity of the mapping $M \to |\det M|$, we have (41) for η small. This proves (*i*).

ii) Now we shall keep (x, t) fixed, and we denote by θ_s the mapping (42) for a given s. We first notice that for $s \in [t_*, t]$, with

$$|t - t_*| \leqq |||\nabla_x E|||^{-1/2} \tag{44}$$

the mapping θ_s is one to one. Indeed, suppose that there exists an s_* in $[t_*, t]$ and two distinct values v_1, v_2 such that

$$X(s_*, t, x, v_1) = X(s_*, t, x, v_2). \tag{45}$$

Then we will show that the functions $X_i(s) = X(s, t, x, v_i)$ $(i=1, 2)$, which are two solutions of the problem.

$$\ddot{X}_i(s) = E(X_i(s), s), \qquad X_i(t) = x \tag{46}$$

coïncide on the interval $[s_*, t]$. This will prove that $v_1 = v_2$:

We denote by $z(s)$ the function

$$z(s) = X_1(s) - X_2(s). \tag{47}$$

With an integration by parts, we obtain

$$\begin{aligned}\int_{s_*}^{t} |\dot{z}(s)|^2 ds &\leqq |||\nabla_x E||| \int_{s_*}^{t} |z(s)|^2 ds \\ &\leqq |||\nabla_x E|||\,(t - s_*) \sup_{s\in[s_*,t]} |z(s)|^2\end{aligned} \tag{48}$$

As we have

$$(49) \qquad |z(s)|^2 \leqq \left(\int_{s_*}^{s} \dot{z}(u)du\right)^2 \leqq (s - s_*) \int_{s_*}^{s} |\dot{z}(u)|^2 du$$

we deduce that

$$(50) \qquad \int_{s_*}^{t} |\dot{z}(s)|^2 ds \leqq |||\nabla_x \mathrm{E}||| \cdot (t - s_*)^2 \int_{s_*}^{t} |\dot{z}(s)|^2 ds .$$

This shows that the function $\dot{z}$ is identically zero on the interval $[s_*, t]$.

Now, we define a number T by

$$(51) \qquad \mathrm{T} = \mathrm{Sup}\ \{ s \in [0, t]/\theta_s \ \text{ is not one to one} \} .$$

From what precedes, we have $\mathrm{T} < t$. We will show that such a number does not exist, which implies that the set of s for which the mapping θ_s is not one to one, is empty.

We denote by $\mathrm{X}(v, s)$, the vector $\mathrm{X}(s, t, x, v)$, for any fixed (t, x). Let $v_1, v_2, \underline{s}$ be such that.

$$(52) \qquad \mathrm{X}(v_1, \underline{s}) = \mathrm{X}(v_2, \underline{s}) .$$

Then, thanks to (41), we can apply the implicit function theorem to the function

$$\varphi(w, s) = \mathrm{X}(w, s) - \mathrm{X}(v_1, s)$$

and prove that there exists a function $\mathrm{W}(s)$ defined for $|s - \underline{s}| < \varepsilon$, such that one has

$$(53) \qquad \mathrm{X}(v_1, s) = \mathrm{X}(\mathrm{W}(s), s) \quad \text{for} \quad |s - \underline{s}| < \varepsilon, \quad \text{and} \quad \mathrm{W}(\underline{s}) = v_2 .$$

Furthermore, the estimates (28) on $\dfrac{\partial \mathrm{X}}{\partial v}$ being independant on (v, s) in $\mathbb{R}^3_v \times [0, t_*]$, the number ε itself is independant on v_1, v_2 and $\underline{s}$.

Then according to the definition of T, there exists $\underline{s}^\varepsilon$ in $]\mathrm{T} - \varepsilon/2, \mathrm{T}[$, and two different values $v_1^\varepsilon, v_2^\varepsilon$ such that

$$(54) \qquad \mathrm{X}(v_1^\varepsilon, \underline{s}^\varepsilon) = \mathrm{X}(v_2^\varepsilon, \underline{s}^\varepsilon) .$$

Thanks to what precedes, there exists a function W defined for $|s - \underline{s}^\varepsilon| < \varepsilon$, verifying (53). Now, for s in the interval $]\mathrm{T} - \varepsilon/2, \mathrm{T} + \varepsilon/2[$, the points v_1^ε and $\mathrm{W}(s)$ never coincide otherwise this would violate point (i). We have therefore proved that T is not an upper bound, which ends the proof of corollary 1.

REMARK 1. — This result seems very classical, though the authors have not been able to find it in the literature. The Hamiltonian system (24), (25), is interpreted as describing the trajectories of the particules starting from the same point x, with velocity v ranging in $\mathbb{R}^3$. In geometrical optics (or in hyperbolic partial differential equations), these curves never intersect

in the space $\mathbb{R}^3_x \times \mathbb{R}^3_v$. However, their projection on the subspace $\mathbb{R}^3_x$, which describe the rays, or trajectories of the particules may intersect. This corresponds to the appearance of caustics in optics. The turbulence, in the Vlasov equation may actually be related to the appearance of caustics; however, in our approach (small initial data, space dimension equal to 3), we rule out these phenomenas.

IV. CONSTRUCTION OF THE SMOOTH SOLUTION

THEOREM 1. — *We assume that the functions $u_{0,\alpha}(x, v)$ for $\alpha = 1, \ldots, N$, are twice continuously differentiable, and that they satisfy, the following estimates, for any pair (x, v) in $\mathbb{R}^3_x \times \mathbb{R}^3_v$, and any α:*

$$0 \leqq u_{\alpha,0}(x, v) \leqq \varepsilon/(1 + |x|)^4(1 + |v|)^4 \tag{55}$$

$$|\nabla_{(x,v)} u_{0,\alpha}(x, v)| \leqq \varepsilon/(1 + |x|)^4(1 + |v|)^5 \tag{56}$$

$$|\nabla^2_{(x,v)} u_{0,\alpha}(x, v)| \leqq \varepsilon/(1 + |x|)^4(1 + |v|)^4 . \tag{57}$$

Then, there exists $\varepsilon_0 > 0$, such that for $\varepsilon < \varepsilon_0$ the Vlasov-Poisson equations

$$\left.\begin{aligned} &\frac{\partial u_\alpha}{\partial t} + (v.\nabla_x)u_\alpha + \frac{q_\alpha}{m_\alpha}(E.\nabla_v)u_\alpha = 0 && (58)\\ &u_\alpha(x, v, 0) = u_{\alpha,0}(x, v) && (59)\end{aligned}\right\} \quad 1 \leqq \alpha \leqq N$$

$$\rho(x, t) = \sum_\alpha q_\alpha \int u_\alpha(x, v, t)dv \tag{60}$$

$$E(x, t) = 4\pi \int \frac{x - y}{|x - y|^3} \rho(y, t)dy \tag{61}$$

have a global in time solution such that $u_\alpha(x, v, t)$ is continuously differentiable in $\mathbb{R}^3_x \times \mathbb{R}^3_v \times \mathbb{R}^+_t$, and such that $E(x, t)$ and $\rho(x, t)$ are continuous in $\mathbb{R}^3_x \times \mathbb{R}^+_t$. Furthermore, this solution satisfies the following uniform estimates.

$$\|(u_\alpha(.,.,t)\|_\infty \leqq \varepsilon, \qquad \|u_\alpha(.,.,t)\|_1 \leqq C\varepsilon \tag{62}$$

$$\sup_{x\in\mathbb{R}^3} \int u_\alpha(x, v, t)dv \leqq C\varepsilon/(1 + t)^3 \tag{63}$$

$$\iint |\nabla_v u_\alpha(x, v, t)|\, dxdv \leqq C\varepsilon t \tag{64}$$

$$\sup_{x\in\mathbb{R}^3} \int |\nabla_v u_\alpha(x, v, t)|\, dv \leqq C\varepsilon/(1 + t)^2 \tag{65}$$

$$\|E(.,t)\|_\infty \leqq C/(1 + t)^2 \tag{66}$$

$$\|\nabla_x E(.,t)\|_\infty + \|\nabla^2_x E(.,t)\|_\infty \leqq C/(1 + t)^{5/2} . \tag{67}$$

Proof. — We proceed by iteration and introduce a sequence of functions

$$u^n(x, v) = \left(u_1^n(x, v), \ldots, u_\alpha^n(x, v), \ldots, u_N^n(x, v)\right)$$

in the following manner:

We first define $u_\alpha^1(x, v, t)$ by the formula

$$\frac{\partial u_\alpha^1}{\partial t} + (v.\nabla_x)u_\alpha^1 = 0\,; \qquad u_\alpha^1(x, v, 0) = u_{\alpha,0}(x, v) \tag{68}$$

and the charge ρ_1 and the Electric field E_1 by the equations

$$\rho_1(x, t) = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} u_\alpha^1(x, v, t)dv \tag{69}$$

$$E_1(x, t) = 4\pi \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3}\, \rho_1(y, t)dy\,. \tag{70}$$

Now, suppose that $u_\alpha^n(x, v, t)$ is defined. Then ρ_n and E_n are defined by the equations

$$\rho_n(x, t) = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} u_\alpha^n(x, v, t)dv \tag{71}$$

$$E_n(x, t) = 4\pi \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3}\, \rho_n(y, t)dy \tag{72}$$

u_α^{n+1} is now supposed to satisfy the linear transport equation

$$\begin{cases} \dfrac{\partial u_\alpha^{n+1}}{\partial t} + (v.\nabla_x)u_\alpha^{n+1} + \dfrac{q_\alpha}{m_\alpha}(E_n.\nabla_v)u_\alpha^{n+1} = 0 \\ u_\alpha^{n+1}(x, v, 0) = u_{\alpha,0}(x, v) \end{cases} \tag{73}$$

In order to prove the convergence of the sequences u_α^n, ρ_n and E_n towards smooth functions, we will need the basic following estimates, valid for ε small:

$$\begin{aligned} &\| \rho_n(.,t)\|_\infty + \|\nabla_x \rho_n(.,t)\|_\infty + \|\nabla_x^2 \rho_n(.,t)\|_\infty \leqq A/(1+t)^3, \quad \forall t > 0 \\ &\| \rho_n(.,t)\|_1 + \|\nabla_x \rho_n(.,t)\|_1 \leqq A\varepsilon \end{aligned} \tag{74}$$

where A is a fixed constant which does not depend on n and ε. The estimates on ρ_n and $\nabla_x \rho_n$ will permit us to prove that E_n converges strongly in a suitable space, whereas the estimate on $\nabla_x^2 \rho_n$ will give us the regularity of the solution. We shall prove that, if these relations are true at order n, they are also true at order $n + 1$, provided the initial data are small enough. To this purpose, we will first consider the case t close to zero ($0 \leqq t < 1$) and use the uniform integrability in v (decay of order $(1 + |v|)^{-\alpha}$, $\alpha > 3$); then we consider the case t large, and use the uniform integrability in x

(decay of order $(1 + |x|)^{-\alpha}$, $\alpha > 3$). Of course, the estimates of section III will be the basic ingredients of the proof in the case t large.

First, it is easily shown by iteration, that E_n and ρ_n are continuous in (x, t) and twice continuously differentiable with respect to x. So the solution of (73) is

$$u_\alpha^{n+1}(x, v, t) = u_{\alpha,0}(X_\alpha^n(0, t, x, v), V_\alpha^n(0, t, x, v)) \tag{75}$$

where $X_\alpha^n(s, t, x, v)$, $V_\alpha^n(s, t, x, v)$ is the solution of the Hamiltonian system

$$\begin{aligned} \dot{X}_\alpha^n(s) &= V_\alpha^n(s) \quad ; \quad & X_\alpha^n(t) &= x \\ \dot{V}_\alpha^n(s) &= \frac{q_\alpha}{m_\alpha} E_n(X_\alpha^n(s), s) \; ; \quad & V_\alpha^n(t) &= v \, . \end{aligned} \tag{76}$$

In the sequel, X_α^n, V_α^n will mean $X_\alpha^n(0, t, x, v)$, $V_\alpha^n(0, t, x, v)$ and C will denote any numerical constant, which may eventually depend on q_α and m_α, but not on ε, n, and A. From (75) we deduce the following uniform estimates:

$$0 \leqq u_\alpha^{n+1}(x, v, t) \leqq \varepsilon \; ; \qquad \int_{\mathbb{R}^3} |\rho_{n+1}(x, t)| \, dx \leqq A\varepsilon \, . \tag{77}$$

Now, we suppose that relation (74) is true at order n, and we apply lemma 1 (with $\theta = 3/5$). Thanks to (77), we obtain

$$\| E_n(., t) \|_\infty \leqq C.A.\varepsilon^{1/3}/(1 + t)^2 \tag{78}$$

$$\| \nabla_x E_n(., t) \|_\infty + \| \nabla_x^2 E_n(., t) \|_\infty \leqq C.A.\varepsilon^{1/6}/(1 + t)^{5/2} \, . \tag{79}$$

So proposition 1 applies and gives with $\varepsilon^{1/6} < 1/CA$

$$\left| \frac{\partial X_\alpha^n}{\partial x} \right| + \left| \frac{\partial V_\alpha^n}{\partial x} \right| + \left| \frac{\partial^2 X_\alpha^n}{\partial x^2} \right| + \left| \frac{\partial^2 V_\alpha^n}{\partial x^2} \right| + \frac{1}{t} \left| \frac{\partial V_\alpha^n}{\partial v} \right| + \frac{1}{t} \left| \frac{\partial X_\alpha^n}{\partial v} \right| \leqq C \, . \tag{80}$$

with (75) we may now deduce the following estimates on ρ_{n+1}:

$$|\rho_{n+1}(x, t)| \leqq C \sum_\alpha \int u_{\alpha,0}(X_\alpha^n, V_\alpha^n) dv \, . \tag{81}$$

$$|\nabla_x \rho_{n+1}(x, t)| \leqq C \sum_\alpha \int |\nabla_{(x,v)} u_{\alpha,0}(X_\alpha^n, V_\alpha^n)| \, dv \tag{82}$$

$$|\nabla_x^2 \rho_{n+1}(x, t)| \leqq C \sum_\alpha \int \left(|\nabla_{(x,v)}^2 u_{\alpha,0}(X_\alpha^n, V_\alpha^n)| + |\nabla_{(x,v)} u_{\alpha,0}(X_\alpha^n, V_\alpha^n)| \right) dv \tag{83}$$

So, thanks to Liouville's theorem and to (56), we have

$$\int_{\mathbb{R}^3} |\nabla_x \rho_{n+1}(x, t)| \, dx \leqq \tilde{C}\varepsilon \tag{84}$$

and eventually replacing A by Sup (A, $\tilde{C}$), we deduce that the second inequality in (74) is true at order $(n + 1)$.

Now we consider the case $0 \leqq t \leqq 1$, and we have

$$(85) \qquad | V_\alpha^n - v | \leqq \frac{| q_\alpha |}{m_\alpha} \int_0^t \| E_n \cdot, s) \|_\infty ds \leqq CA\varepsilon^{1/3} .$$

Therefore we have

$$(86) \qquad | V_\alpha^n | \geqq | v | - CA\varepsilon^{1/3}$$

and, thanks to (81) and (56), we obtain

$$(87) \qquad | \rho_{n+1}(x, t) | \leqq C\varepsilon \sum_\alpha \int \frac{dv}{(1 + | v | - CA\varepsilon^{1/3})^4}$$

So, for ε small, we have:

$$(88) \qquad \| \rho_{n+1}(\cdot, t) \|_\infty \leqq A/(1 + t)^3 \qquad 0 \leqq t \leqq 1 .$$

Similarly, the same estimate is valid for $\nabla_x \rho_{n+1}$ and $\nabla_x^2 \rho_{n+1}$.

We can now consider the case $t \geqq 1$. Estimate (79) and corollary 1 proves that for ε small, the mapping

$$(89) \qquad v \to X_\alpha^n = X_\alpha^n(t, x, v) ,$$

is one to one, and that we have

$$(90) \qquad \left| \det \left(\frac{\partial X_\alpha^n}{\partial v} \right) \right| \geqq \frac{t^3}{2} .$$

Therefore, we can use the change of variables $v \to X_\alpha^n$, in the computation of the integrals which appear in the right hand side of (81), (82) and (83). This gives, with (55):

$$(91) \qquad \begin{aligned} | \rho_{n+1}(x, t) | &\leqq C\varepsilon \sum_\alpha \int \frac{dv}{(1 + | X_\alpha^n |)^4} \\ &\leqq C\varepsilon \sum_\alpha \int \frac{1}{(1 + | X_\alpha^n |)^4} \left| \det \left(\frac{\partial X_\alpha^n}{\partial v} \right) \right|^{-1} dX_\alpha^n \\ &\leqq C\varepsilon/t^3 \\ &\leqq A/(1 + t)^3 \qquad \text{(for } \varepsilon \text{ small)} . \end{aligned}$$

Similar results hold for $\nabla_x \rho_{n+1}$ and $\nabla_x^2 \rho_{n+1}$. Together with the estimates in the case $t \leqq 1$, these relations prove that the first inequality (74) is true at order $(n + 1)$, which completes the iteration.

We notice the following three other estimates, which will be of some interest

$$(92) \qquad \operatorname*{Sup}_x \int | \nabla_v u_\alpha^n(x, v, t) | \, dv \leqq C\varepsilon/(1 + t)^2$$

$$\iint |\nabla_v u_\alpha^n(x, v, t)|\, dx dv \leqq C\varepsilon t \tag{93}$$

$$\left\| \frac{\partial E_n}{\partial t}(., t) \right\|_\infty \leqq C_1 \tag{94}$$

(92) and (93) will be used to prove the convergence of the iterative scheme; (94) will be used to prove $W^{2,\infty}$ regularity of the solution.

We have indeed, thanks to (80):

$$\int |\nabla_v u_\alpha^n(x, v, t)|\, dv \leqq Ct \int |\nabla_{(x,v)} u_{\alpha,0}(X_\alpha^n, V_\alpha^n)|\, dv \tag{95}$$

and the same arguments as above lead to (92) and (93). Now, with (73), (75), (80), we obtain

$$\begin{aligned} \left| \frac{\partial u_\alpha^n}{\partial t} \right| &\leqq C(1 + |v|)\, |\nabla_{(x,v)} u_{\alpha,0}(X_\alpha^n, V_\alpha^n)| \\ &\leqq C\varepsilon \frac{1 + |v|}{(1 + |X_\alpha^n|)^4 (1 + |V_\alpha^n|)^5} \end{aligned} \tag{96}$$

since equality (86) is valid for any time t, we have for small ε:

$$\left| \frac{\partial u_\alpha^n}{\partial t} \right| \leqq C \frac{1 + |v|}{\left(\frac{1}{2} + |v|\right)^5} \leqq C/(1 + |v|)^4 . \tag{97}$$

Thanks to Lebesgue's theorem, we deduce that ρ_n is differentiable with respect to t, and that we have

$$\frac{\partial \rho_n}{\partial t} = \sum_\alpha q_\alpha \int \frac{\partial u_\alpha^n}{\partial t}(x, v, t) dv ; \qquad \left\| \frac{\partial \rho_n}{\partial t}(., t) \right\|_\infty \leqq C \tag{98}$$

Furthermore, from (86) and (96), we obtain:

$$\left| \frac{\partial u_\alpha^n}{\partial t} \right| \leqq C \frac{1 + C + |V_\alpha^n|}{(1 + |X_\alpha^n|)^4 (1 + |V_\alpha^n|)^5} \leqq \frac{C}{(1 + |X_\alpha^n|)^4 (1 + |V_\alpha^n|)^4} . \tag{99}$$

So that Liouville's theorem leads to

$$\left\| \frac{\partial \rho_n}{\partial t}(., t) \right\|_1 \leqq C . \tag{100}$$

Now, with Lemma 1, we deduce that E_n is differentiable with respect to t, and that (94) holds.

Now, we prove the convergence of the sequences u_α^n, ρ_n, E_n, to smooth

solutions. First we fix a time T positive and prove that E_n converges in the strong topology of $L^\infty([0, T] \times \mathbb{R}^3_x)$: we have

$$(101)\quad \frac{\partial}{\partial t}(u_\alpha^{n+1} - u_\alpha^n) + (v.\nabla_x)(u_\alpha^{n+1} - u_\alpha^n) + \frac{q_\alpha}{m_\alpha}(E_n.\nabla_v)(u_\alpha^{n+1} - u_\alpha^n)$$
$$= -\frac{q_\alpha}{m_\alpha}(E_n - E_{n-1}).\nabla_v u_\alpha^n$$

So that $u_\alpha^{n+1} - u_\alpha^n$ is written

$$(102)\quad (u_\alpha^{n+1} - u_\alpha^n)(\tau) = -\frac{q_\alpha}{m_\alpha}\int_0^\tau (E_n(X_\alpha^n, t) - E_{n-1}(X_\alpha^n, t)).\nabla_v u_\alpha^n(X_\alpha^n, V_\alpha^n, t)dt$$

Therefore, thanks to Liouville's theorem, and to formulae (71), (72), we obtain:

$$(103)\quad \iint |u_\alpha^{n+1}(x, v, \tau) - u_\alpha^n(x, v, \tau)|\, dxdv$$
$$\leqq C\int_0^\tau \left(\iint |E_n(x, t) - E_{n-1}(x, t)| \,.\, |\nabla_v u_\alpha^n(x, v, t)|\, dxdv\right)d\tau$$
$$\leqq C\int_0^\tau \left(\iiint \frac{1}{|x-y|^2}|\rho_n(y, t) - \rho_{n-1}(y, t)| \,.\, |\nabla_v u_\alpha^n(x, v, t)|\, dydxdv\right)dt$$
$$\leqq C\int_0^\tau \int |\rho_n(y, t) - \rho_{n-1}(y, t)| \left(\int \frac{dx}{|x-y|^2}\left(\int |\nabla_v u_\alpha^n(x, v, t)|\, dv\right)\right)dydt$$

Now, with estimates (92) and (93) on $\nabla_v u_\alpha^n$, and with lemma 1, we deduce that we have

$$(104)\quad \int \frac{dx}{|x-y|^2}\cdot\left(\int |\nabla_v u_\alpha^n(x, v, t)|\, dv\right) \leqq C, \qquad \forall y \in \mathbb{R}^3, \qquad \forall t \in [0, T]$$

and adding (103) for α ranging from 1 to N, we obtain

$$(105)\quad \|(\rho_{n+1} - \rho_n)(., t)\|_1 \leqq C\int_0^t \|(\rho_n - \rho_{n-1})(., s)\|_1 ds$$

which by iteration gives

$$(106)\quad \|(\rho_{n+1} - \rho_n)(., t)\|_1 \leqq \frac{C^{n+1}t^n}{n!}$$

Thus ρ_n converges in the strong topology of $L^\infty([0, T], L^1(\mathbb{R}^3_x))$ towards a function ρ. Futhermore, with estimates (74) and lemma 1, we deduce that E_n converges in the strong topology of $L^\infty([0, T] \times \mathbb{R}^3_x)$ towards E, and that the pair (E, ρ) satisfies (61).

Now, with estimates (77), u_α^n converges in the weak star topology of $L^\infty([0, T] \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$, towards a function u_α, which satisfies equation (58)

in the sense of distributions. Furthermore equation (60) is trivially verified. It remains to prove the regularity and the initial condition (59).

First, estimates (79) and (94) show that E is continuous in $[0, T] \times \mathbb{R}^3_x$; similarly, estimates (74) and (98) prove that ρ is continuous. On the other hand, the vector $\nabla_{(x,v)} u_\alpha^{n+1}$ is a solution of the equation

$$\frac{\partial}{\partial t}(\nabla_{(x,v)} u_\alpha^{n+1}) + v.\nabla_x(\nabla_{(x,v)} u_\alpha^{n+1}) + \frac{q_\alpha}{m_\alpha} E_n.\nabla_v(\nabla_{(x,v)} u_\alpha^{n+1}) = A_\alpha^n.\nabla_{(x,v)} u_\alpha^{n+1} \tag{107}$$

where A_α^n is the 6×6 matrix:

$$A_\alpha^n = \begin{pmatrix} 0 & -\frac{q_\alpha}{m_\alpha}\nabla_x E_n \\ -\mathrm{Id} & 0 \end{pmatrix} \tag{108}$$

In particular, thanks to (79), A_α^n is uniformly bounded in $[0, T] \times \mathbb{R}^3_x$, with respect to n. We deduce that

$$\| \nabla_{(x,v)} u_\alpha^{n+1}(., ., t) \|_\infty \leqq \| \nabla_{(x,v)} u_{\alpha,0}(., .) \|_\infty + C \int_0^t \| \nabla_{(x,v)} u_\alpha^{n+1}(., ., s) \|_\infty ds \tag{109}$$

This proves that $\nabla_{(x,v)} u_\alpha^{n+1}$, is a bounded sequence in $L^\infty([0, T] \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$. A similar proof would show that the sequence $\nabla^2_{(x,v)} u_\alpha^{n+1}$ is also a bounded sequence in $L^\infty([0, T] \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$. So $\nabla_{(x,v)} u_\alpha$ and $\nabla^2_{(x,v)} u_\alpha$ belong to $L^\infty([0, T] \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$. Then, using the Vlasov equation (58), we notice that $\frac{\partial u_\alpha}{\partial t}$, $\nabla_x\left(\frac{\partial u_\alpha}{\partial t}\right)$ and $\nabla_v\left(\frac{\partial u_\alpha}{\partial t}\right)$ belong to $L^\infty_{\mathrm{loc}}([0, T] \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$.

Since $\frac{\partial E}{\partial t}$ is bounded in virtue of estimate (94), $\frac{\partial^2 u_\alpha}{\partial t^2}$ also belongs to $L^\infty_{\mathrm{loc}}([0, T] \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$. So, u_α belongs to $W^{2,\infty}_{\mathrm{loc}}([0, T] \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$, which proves that u_α is continuously differentiable with respect to (t, x, v). Furthermore, u_α^n is bounded in $W^{2,\infty}_{\mathrm{loc}}([0, T] \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$, and if we denote

$$B_R = \{ x \in \mathbb{R}^3 / | x | \leqq R \}$$

the compactness of the imbedding.

$$W^{2,\infty}([0, T] \times B_R \times B_V) \hookrightarrow C^0([0, T] \times B_R \times B_V)$$

shows that the initial condition (59) is satisfied.

Finally, the uniqueness theorem (section V) proves that the functions u_α, E, and ρ are defined for t in $[0, +\infty]$ and that the estimates (62) to (67) are satisfied. This ends the proof of theorem 1.

V. UNIQUENESS

THEOREM 2. — *Let* $\{ u_\alpha^i(x, v, t) \}_{i=1,2}$ *be two solutions of the Vlasov-Poisson equation for t in* $[0, T]$, *and assume that they satisfy the following estimates*

$$0 \leqq u_\alpha^i(x, v, t) \leqq C \tag{110}$$

$$\iint u_\alpha^i(x, v, t) dx dv \leqq C ; \qquad \iint | \nabla_v u_\alpha^i(x, v, t) | dx dv \leqq C \tag{111}$$

$$\operatorname*{Sup}_x \int u_\alpha^i(x, v, t) dv \leqq C ; \qquad \operatorname*{Sup}_x \int | \nabla_v u^i(x, v, t) | dv \leqq C . \tag{112}$$

Then, if they coincide for $t = 0$, *they coincide everywhere for t in* $[0, T]$.

Proof. — We will follow the method which has been used to prove the convergence of ρ_n in theorem 1: we will show that the charges

$$\rho_i(x, t) = \sum_\alpha q_\alpha \int u_\alpha^i(x, v, t) dv \qquad i = 1, 2 . \tag{113}$$

coincide for $t \in [0, T]$. This implies that the electric fields $E_i(x, t)$ coincide so that u_α^1 and u_α^2 are two smooth solutions of the same linear transport equation, which will prove the uniqueness.

We have

$$\begin{aligned} \frac{\partial}{\partial t}(u_\alpha^1 - u_\alpha^2) + v . \nabla_x(u_\alpha^1 - u_\alpha^2) + \frac{q_\alpha}{m_\alpha} E_1 . \nabla_v(u_\alpha^1 - u_\alpha^2) \\ = \frac{q_\alpha}{m_\alpha} (E_2 - E_1) \nabla_v u_\alpha^2 . \end{aligned} \tag{114}$$

With an integration with respect to x and v, and using lemma 1 and estimates (111), (112), we obtain:

$$\begin{aligned} &\iint | u_\alpha^1(x, v, \tau) - u_\alpha^2(x, v, \tau) | dx dv \\ &\leqq C \int_0^\tau \iiint \frac{1}{| x - y |^2} | \rho_1(y, t) - \rho_2(y, t) | \, | \nabla_v u_\alpha^2(x, v, t) | dx dy dv dt \\ &\leqq C \int_0^\tau \int | \rho_1(y, t) - \rho_2(y, t) | \left(\int \frac{dx}{| x - y |^2} \int | \nabla_v u_\alpha^2(x, v, t) | dv \right) dy dt \\ &\leqq C \int_0^\tau || \rho_1(., t) - \rho_2(., t) ||_1 dt \end{aligned} \tag{115}$$

Adding (115) for α ranging from 1 to N gives the Gronwall relation

$$(116) \qquad \| \rho_1(.,t) - \rho_2(.,t) \|_1 \leqq C \int_0^t \| \rho_1(.,s) - \rho_2(.,s) \|_1 ds$$

which proves that $\rho_1 = \rho_2$, and ends the proof of theorem 2.

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(Manuscrit reçu le 20 mars 1984)