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Conservation laws
for the nonlinear Schrödinger equation

by

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ABSTRACT. — We propose a method of calculating the operator densities $\hat{h}_n \ n = 0, 1, \ldots$ of the conservation laws for the quantum nonlinear Schrödinger equation. It follows from the method that $\hat{h}_n$ are polynomials in fields and their derivatives and in the coupling constant. The densities $\hat{h}_n \ n \leq 4$ are explicitly calculated. Comparison with the integral densities $b_n \ n = 0, 1, \ldots$ for the classical nonlinear Schrödinger equation shows that the correspondence between $\hat{h}_n$ and $b_n$ breaks down after $n = 3$.

1. INTRODUCTION

We consider the quantum nonlinear Schrödinger equation (NLSE) in 1 + 1 space-time dimensions

$$i \Psi_t = - \Psi_{xx} + 2c \Psi \Psi^\dagger \Psi^2 . \tag{1.1}$$

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Its Hamiltonian

\[ \hat{H}_2 = -\int dx (\Psi^\dagger \Psi_{xx} - c\Psi^\dagger \Psi^2) \]  

(1.2)

is the second quantized form of the many body Hamiltonian

\[ H_2^{(N)} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + c \sum_{i \neq j} \delta(x_i - x_j) \]  

(1.3)

Hamiltonian (1.3) describes the interaction of \( N \) identical particles on the line via elastic collisions and \( c \) is the strength of interaction. The famous « Bethe Ansatz » \cite{1} \cite{2} exhibits the system of generalized eigenstates \( |\Psi_N(k_1, \ldots, k_N)\rangle = |\Psi_N(k)\rangle \) of \( H_2^{(N)} \) which is complete if \( c \geq 0 \). We have

\[ H_2^{(N)} |\Psi_N(k)\rangle > = \left( \sum_{i=1}^{N} k_i^2 \right) |\Psi_N(k)\rangle > . \]  

(1.4)

Since Bethe Ansatz eigenstates depend on \( N \) quantum numbers \( k_1, \ldots, k_N \) the Hamiltonian (1.3) must be completely integrable. This means that there are \( N \) independent operators \( H_n^{(N)} \) \( n = 1, \ldots, N \) such that

\[ H_n^{(N)} |\Psi_N(k)\rangle > = \left( \sum_{i=1}^{N} k_i^n \right) |\Psi_N(k)\rangle > \]  

(1.5)

\[ H_1^{(N)} = (-i) \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \] is of course the total momentum and \( H_2^{(N)} \) is the Hamiltonian (1.3). Existence of \( H_n^{(N)} \) should imply the infinite sequence of independent conservation laws \( \hat{H}_n \) \( n = 1, 2, \ldots \) for the NLSE given by their operator densities \( \hat{h}_n \)

\[ H_n = \int dx \hat{h}_n(x). \]  

(1.6)

We have

\[ \hat{h}_1 = (-i)\Psi^\dagger \Psi_x \]  

(1.7)

\[ \hat{h}_2 = (-i)^2 (\Psi^\dagger \Psi_{xx} - c\Psi^\dagger \Psi^2). \]  

(1.8)

Operators \( \hat{H}_n \) are completely characterized by the property that for any \( N \)

\[ \hat{H}_n |\Psi_N(k)\rangle > = \left( \sum_{i=1}^{N} k_i^n \right) |\Psi_N(k)\rangle > . \]  

(1.9)
It is desirable to have explicit expressions for the operator densities $\hat{h}_n$. In this paper I suggest a method for calculating $\hat{h}_n$ for any $n$. Using this method I calculate $\hat{h}_3$ and $\hat{h}_4$. In section 4 I compare $\hat{h}_n$ with the functional densities $b_n$ of the integrals of motion for the classical NLSE

$$i\varphi_t = -\varphi_{xx} + 2c |\varphi|^2\varphi$$  \hspace{1cm}(1.10)

Thacker [3] has obtained $\hat{h}_3$ using a completely different approach. Kulish and Sklyanin [4] and Thacker [4] have integrated (1.1) using the quantum inverse scattering method. Their method however does not yield explicit formulas for $\hat{h}_n$ in terms of the fields (*).

### 2. N-PARTICLE SECTOR

In this section we fix $N$ and omit the superscript $N$ in formulas. The Hamiltonian $H_2$ is equal to the Laplacian $-\sum_{i=1}^{N} \partial^2/\partial x_i^2$ with the boundary conditions

$$(\partial/\partial x_j - \partial/\partial x_i)F = cF$$ \hspace{1cm}(2.1)

on hyperplanes $\{ x_i - x_j = 0 \} \ i, j = 1, \ldots, N$.

Because of the symmetry of function $F$ it suffices to restrict it to $R^N_i = \{ x_1 \leq x_2 \leq \ldots \leq x_N \}$ and to impose boundary conditions

$$(\partial/\partial x_{k+1} - \partial/\partial x_k)F = cF$$ \hspace{1cm}(2.2)

on hyperplanes $x_k = x_{k+1} \ \ k = 1, \ldots, N - 1$.

I will use the following fact. There is an operator $P$ on symmetric functions in $R^N$ that intertwines Laplacian with the Neumann boundary conditions

$$(\partial/\partial x_{k+1} - \partial/\partial x_k)F = 0$$ \hspace{1cm}(2.3)

and Laplacian with boundary conditions (2.2) for $c \geq 0$. The operator $P$ constructed as follows. For any $i \neq j$ let $P_{ij}$ be given by

$$(P_{ij}f)(x_1, \ldots, x_N) = \int_{0}^{\infty} dt e^{-ct}f(x_1, \ldots, x_i-t, \ldots, x_j+t, \ldots, x_N).$$ \hspace{1cm}(2.4)

Denote by $S$ the operator from all functions $f$ on $R^N$ into symmetric functions on $R^N$ obtained by restricting $f$ to $R^N_i$ and then extending it to $R^N$ by symmetry. Then [5]

$$P = S \prod_{i<j} (1 - cP_{ij}).$$ \hspace{1cm}(2.5)

(* Added in proofs: in a forthcoming paper I show that the formulas for integrals of the NLSE obtained in [4] via the quantum scattering method are false.

Denoting by $\Delta_2$ the Laplacean with boundary conditions (2.3) we express the intertwining property of $P$ by

$$H_2P = P\Delta_2.$$  \hspace{1cm} (2.6)

Let $\Delta_n$ be given by $(-i)^{n} \sum_{i=1}^{N} \partial^n / \partial x_i^n$ with "higher" Neumann boundary conditions

$$(\partial / \partial x_{k+1} - \partial / \partial x_k)^{2i+1} f = 0$$  \hspace{1cm} (2.7)

for $i = 0, 1, \ldots, [n/2] - 1$ on hyperplanes $\{x_k = x_{k+1}\}$ for $k = 1, \ldots, N-1$. Let $H_n$ be defined from

$$P\Delta_n = H_nP$$  \hspace{1cm} (2.8)

for $n = 1, \ldots$. Since operators $\Delta_n$ commute, $H_n$ also commute. It follows from (2.5) that $P$ takes boundary conditions (2.7) into boundary conditions

$$(\partial / \partial x_{k+1} - \partial / \partial x_k)^{2i+1} f = c(\partial / \partial x_{k+1} - \partial / \partial x_k)^{2i} f$$  \hspace{1cm} (2.9)

So $H_n$ is equal to $(-i)^{n} \sum_{i=1}^{N} \partial^n / \partial x_i^n$ with boundary conditions (2.9) for $i = 0, \ldots, [n/2] - 1$. It remains to obtain formulas for $H_n$ similar to the formula (1.3) for $H_2$.

Let $g(x_1, \ldots, x_N)$ be an infinitely differentiable function and let $f$ satisfy the boundary conditions (2.9). Then

$$\langle g | H_3 | f \rangle = (-i)^{3} \int d^N x \bar{g} \left( \sum_{i=1}^{N} \frac{\partial^3}{\partial x_i^3} f \right).$$  \hspace{1cm} (2.10)

Integrating by parts and taking (2.9) into account we get

$$\langle g | H_3 | f \rangle = -(-i)^{3} \int d^N x \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \bar{g} \frac{\partial^2}{\partial x_i^2} f$$

$$- c(-i)^{3} \sum_{i \neq j} \int d^N x \delta(x_i - x_j) g \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) f.$$  \hspace{1cm} (2.11)
Integrating by parts again

\[ \langle g \mid H_3 \mid f \rangle = (-i)^3 \int d^N x \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} g \frac{\partial}{\partial x_i} f \]

\[ + (-i)^3 c \int d^N x \sum_{i \neq j} \delta(x_i - x_j) \frac{\partial}{\partial x_i} \bar{g} f \]

\[ + (-i)^3 c \int d^N x \sum_{i \neq j} \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) \delta(x_i - x_j) \bar{g} f \]. \tag{2.12} \]

After one more integration by parts and obvious transformations (2.12) becomes

\[ \langle g \mid H_3 \mid f \rangle = i^3 \int d^N x \sum_{i=1}^{N} \frac{\partial^3}{\partial x_i^3} \bar{g} f \]

\[ + (-i)^3 \frac{3}{2} c \int d^N x \sum_{i \neq j} \delta(x_i - x_j) \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) \bar{g} f \] \tag{2.13} \]

which yields

\[ H_3 = (-i)^3 \left( \sum_{i=1}^{N} \frac{\partial^3}{\partial x_i^3} \right) - \frac{3}{2} c \sum_{i \neq j} \delta(x_i - x_j) \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) \] \tag{2.14} \]

For \( H_4 \) we have

\[ \langle g \mid H_4 \mid f \rangle = \int d^N x \bar{g} \sum_{i=1}^{N} \frac{\partial^4}{\partial x_i^4} f \]. \tag{2.15} \]

Integrating by parts the right hand side

\[ - \int d^N x \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \bar{g} \frac{\partial^3}{\partial x_i^3} f \]

\[ - c \int d^N x \sum_{i \neq j} \delta(x_i - x_j) \bar{g} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2} \right) f \]. \tag{2.16} \]

Integrating by parts the first term in (2.16) we get

\[ \int d^N x \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \bar{g} \frac{\partial^2}{\partial x_i^2} f \]

\[ + \frac{1}{2} c \int d^N x \sum_{i \neq j} \delta(x_i - x_j) \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) \bar{g} \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) f \]. \tag{2.17} \]
Integrating (2.17) by parts again and remembering the second term in (2.16) we have

\[
\langle g \mid H_4 \mid f \rangle = \int d^N x \sum_{i=1}^{N} \frac{\partial^4}{\partial x_i^4} \bar{g} f - \frac{c}{2} \int d^N x \sum_{i \neq j} \delta(x_i - x_j) \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_j^2} \right) \bar{g} f
\]

\[
- \frac{c}{2} \int d^N x \sum_{i \neq j} \delta(x_i - x_j) \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right)^2 \bar{g} f
\]

\[
- c \int d^N x \sum_{i \neq j} \delta(x_i - x_j) \bar{g} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2} \right) f. \tag{2.18}
\]

The last term in (2.18) can again be integrated by parts yielding

\[
- c \int d^N x \sum_{i \neq j} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2} \right) \delta(x_i - x_j) \bar{g} f
\]

\[
+ 3c^2 \int d^N x \sum_{i \neq j \neq k} \delta(x_i - x_j) \delta(x_j - x_k) \bar{g} f. \tag{2.19}
\]

From (2.18) and (2.19) we have

\[
H_4 = \sum_{i=1}^{N} \frac{\partial^4}{\partial x_i^4} - c \sum_{i \neq j} \delta(x_i - x_j) \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2} \right)
\]

\[
- c \sum_{i \neq j} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2} \right) \delta(x_i - x_j) + 3c^2 \sum_{i \neq j \neq k} \delta(x_i - x_j) \delta(x_j - x_k). \tag{2.20}
\]

3. SECOND QUANTIZED FORM OF $H_n$

A standard calculation gives

\[
\hat{H}_3 = (-i)^3 \int d x \Psi^\dagger \Psi_{xxx}
\]

\[
- \frac{3}{2} c (-i)^3 \int d x \int d y \Psi^\dagger(x) \Psi^\dagger(y) \delta(x - y) \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \Psi(x) \Psi(y)
\]

\[
= (-i)^3 \int d x \left[ \Psi^\dagger \Psi_{xxx} - 3c \Psi^\dagger \Psi^\dagger \Psi_x \right]. \tag{3.1}
\]
Thus

\[ \hat{h}_3(x) = (-i)^3(\Psi^t \Psi_{xxx} - 3c \Psi^{t2} \Psi \Psi_x). \]  

(3.2)

Also

\[ \hat{H}_4 = \int dx \Psi^t \Psi_{xxxx} \]
\[- c \int dx \int dy \Psi^t(x) \Psi^t(y) \delta(x - y) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \right) \Psi(x) \Psi(y) \]
\[- c \int dx \int dy \Psi^t(x) \Psi^t(y) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \right) \delta(x - y) \Psi(x) \Psi(y) \]
\[+ 3c^2 \int dx \int dy \int dz \Psi^t(x) \Psi^t(y) \Psi^t(z) \delta(x - y) \delta(y - z) \Psi(x) \Psi(y) \Psi(z). \]  

(3.3)

After obvious integrations by parts we have

\[ \hat{H}_4 = \int dx \left[ \Psi^t \Psi_{xxxx} - 2c \Psi^{t2} \Psi_{xx} - c \Psi^{t12} \Psi_x^2 - 2c \Psi^{t1} \Psi_{xx} \Psi^2 \right. \]
\[- \left. c \Psi^{t12} \Psi_x^2 + 3c^2 \Psi^{t13} \Psi^3 \right]. \]  

(3.4)

Thus

\[ \hat{h}_4(x) = \Psi^t \Psi_{xxxx} - 2c \Psi^{t2} \Psi_{xx} - c \Psi^{t12} \Psi_x^2 - 2c \Psi^{t1} \Psi_{xx} \Psi^2 \]
\[- c \Psi^{t12} \Psi_x^2 + 3c^2 \Psi^{t13} \Psi^3. \]  

(3.5)

4. COMPARISON WITH CLASSICAL INTEGRALS

The classical NLSE (or Zakharov-Shabat equation [6])

\[ i \varphi_t = - \varphi_{xx} + 2c | \varphi |^2 \varphi \]  

(4.1)

is a completely integrable Hamiltonian system with infinitely many degrees of freedom [7]. In particular (4.1) has an infinite number of integrals of motion \( B_n(\varphi, \varphi) \). The functionals \( B_n \) are determined by the local densities \( b_n \)

\[ B_n(\varphi, \varphi) = \int_{-\infty}^{\infty} dx b_n(\varphi(x), \varphi(x)). \]  

(4.2)

The densities \( b_n \) are found from the recurrence relation

\[ b_{n+1} = \overline{\varphi} \frac{d}{dx} \left( \frac{b_n}{\varphi} \right) - c \sum_{i+j=n-1} b_i b_j \]  

(4.3)

and

\[ b_0 = \overline{\varphi} \varphi. \]  

(4.4)

From (4.3) and (4.4) we get

\[ b_1 = \overline{\varphi} \varphi_x \]  

(4.5)

\[ b_2 = \overline{\varphi} \varphi_{xx} - c | \varphi |^4 \]  

(4.6)
The local densities $h$ and $g$ that differ by a total derivative are equivalent $h \simeq g$ because they define the same functional $\int_{-\infty}^{\infty} dx h(x) = \int_{-\infty}^{\infty} dx g(x)$.

We have

$$\begin{align*}
b_3 &= \overline{\phi} \phi_{xxx} - 2c\overline{\phi}^2(\phi^2)_x - c\overline{\phi} \phi_x \phi^2 \\
b_4 &= \overline{\phi} \phi_{xxxx} - 2c\overline{\phi}^2(\phi^2)_{xx} - 2c\overline{\phi}^2 \phi_{xx} - c\overline{\phi}^2 \phi_x^2 \\
&\quad - 3c\overline{\phi} \phi_x (\phi^2)_x - c\overline{\phi} \phi_x \phi^2 + 2c^2 |\phi|^6.
\end{align*}$$

The difference corresponds to $c^2 |\phi|^6$ which is a nontrivial density.

5. CONCLUSION

The nonlinear Schrödinger equation (1.1) has an infinite sequence of conservation laws $\hat{H}_n$ given by the operator densities $\hat{h}_n(\Psi^*(x), \Psi(x))$. The densities $\hat{h}_n$ can be found using the method of sections 2 and 3. It is clear from the method that $\hat{h}_n$ are polynomials in the fields and their derivatives. Besides $\hat{h}_n$ are polynomials in the coupling constant $c$. The degree of $\hat{h}_n$ in $c$ is $[n/2]$.

Correspondence between $\hat{h}_n$ and the integral densities $b_n$ of the classical NLSE (4.1) breaks down at $n = 4$.

REFERENCES

[5] Another intertwining operator $\hat{P}$ was constructed in E. GUTKIN, Duke Mat. J., t. 49, 1, 1982 on the space of all functions not regarding the symmetry.

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