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On the dynamics of Bose-Einstein condensation

by

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ABSTRACT. — We study a non-linear integro-differential equation describing the evolution of a gas of Bosons interacting with a heat bath. The solution is shown to exist globally. A family of Liapunov functionals is constructed and used to prove convergence to equilibrium. The linearized equation determines a semi-group of contractions. The nature of approach to equilibrium (exponential at low density, but not in the two-phase region) is studied in relation to the spectral properties of the generator of the semi-group.

RÉSUMÉ. — Nous étudions une équation intégro-différentielle non linéaire décrivant l'évolution d'un gaz de Bosons en interaction avec un bain thermique. Nous montrons l'existence globale de la solution. On construit une famille de fonctionnelles de Liapounov, et l'on s'en sert pour démontrer la convergence vers l'équilibre. L'équation linéarisée définit un semi-groupe de contractions. La vitesse de convergence vers l'équilibre (exponentielle en basse densité, mais non dans la région diphasique) est reliée aux propriétés spectrales du générateur de ce semi-groupe.

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§ 1. INTRODUCTION

In this article, we study the kinetic equation which governs the evolution of a gas of Bosons interacting with a heat bath. This is one of the very few models where the dynamical aspect of a phase transition in a continuous system can be analysed rigorously. This provides a testing ground for the predictions of phenomenological theories concerning the occurence and the nature of *critical slowing down.*

We concentrate here on the analysis of the mathematical properties of the equation, and the physical consequences of these. The actual derivation of the equation from first principles is given in a related article [*1*] (although for a slightly simplified model involving an energy cut-off). An outline of this derivation is given in section 2 together with the relevant physical background.

Our kinetic equation is a non-linear integro-differential equation for the function which gives the population of the various energy levels of the Bose gas at time t (see (16)). Since the integral of this function is related to the density of the system, the natural mathematical framework for the problem is a Banach space of L^1-type. The first question is the existence and uniqueness of the solution, and we treat it in section 3. As the right-hand side of the equation is given by an unbounded non-linear operator, even the existence of a *local solution* is a non-trivial matter. To deal with this problem, we note that the operator can be split into a bounded non-linear part and a linear part which, although unbounded, generates a contraction semi-group (Proposition 1). This remark, together with further regularity properties of the non-linear part (Lemma 1), yields local existence and uniqueness of the solution (Proposition 2).

Moreover, the linear semi-group is *positivity-preserving* (Proposition 1), and we can use this property to show that the local solution of the full equation is positive for positive initial conditions (Theorem 1). This feature does not only ensure that if the initial condition is physically acceptable so is the corresponding local solution ; it yields also (using the conservation of the overall density) an *a priori bound* on the local solution, and this implies that the solution exists *globally* (Theorem 1). The method of proof is similar to that used by Arkeryd in his study of the classical Boltzmann equation [*2*] [*3*].

The next step consists in investigating the *asymptotic behaviour* of our solution (section 4). As in most comparable problems (see [*2*] [*3*] [*4*] for instance), physical considerations suggest both a fixed point and a Liapunov functional for our equation. The fixed point is of course the Bose distribution at the temperature of the bath, whereas the Liapunov functional φ is related to the free energy density of the Bose gas (see (83) and remark (*i*) at the end of section 4). One expects that all physical initial conditions will be attracted by the fixed point. However, it turns out that the « natural » Liapunov functional φ does not suffice to prove this fact. This is essentially because φ fails to be continuous on the whole of the physical domain (i. e. the positive cone of L^1). However, φ is continuous on a restricted physical domain, consisting of those positive functions in L^1 which have finite first and second moments (i. e. the mean and variance of the energy density are finite). But this does not solve the difficulty: even if one is willing to restrict one's attention to such initial conditions, there is no *a priori* guarantee that the finiteness of the second moment will be preserved by the evolution.

In order to prove that the restricted physical domain is invariant under the evolution (Proposition 5), we construct a new Liapunov functional H_2 (Proposition 4). We can then use the continuity of φ (Lemma 6) and various other technical results (Lemmas 5, 7) to prove that every initial condition in the restricted physical domain is attracted by the physical fixed point (Theorem 2). We can also generalize our Liapunov functional to obtain the convergence of *arbitrary moments*. The result is of the following type: assume that the moment of order $n + 1$ is finite for the initial condition ; then the time-dependent moments of order $1, 2, \ldots, n$ converge to the corresponding equilibrium moments (see Corollary 2 and remark (*ii*) in section 4).

We turn finally to the linearized equation, which we analyse in a Hilbert space framework (section 5). The phase transition manifests itself through a change in the spectral properties of a certain self-adjoint operator (see Theorem 3). At low density, there is a gap in the spectrum of this operator, and this ensures that the approach to equilibrium is exponentially fast (Theorem 4). But the gap disappears in the two-phase region, so that the relaxation cannot be exponential in that regime (Theorem 5), and our model exhibits critical slowing down. Moreover, we can control the way in which the gap closes up as the parameters of the system tend to their critical value (Corollary 4). This gives a rate of divergence of the relaxation time of the form $\tau = O((T - T_c)^{-2})$ which is compatible with the predictions of the Ginzburg-Landau theory (see remark (*ii*) at the end of section 5).

Our proof of the existence of a spectral gap in the low density regime is based on an argument previously used by W. G. Sullivan in a probabilistic context [*5*].

As a final comment, we mention that our full non-linear equation fits in the scheme recently put forward by Alicki and Messer [*6*].

§ 2. A HEURISTIC DERIVATION OF THE KINETIC EQUATION

In this section, we outline a programme leading to the deduction of our dynamical equation. In a related article [*1*], this programme is carried out in full detail (though for a technically simpler model).

A convenient framework for obtaining kinetic equations is provided by the well developed theory of master equations (see [7] [8]). In this scheme, a finite system (here the Bose gas) is coupled to an infinite heat bath. By tracing out the variables of the bath, the unitary evolution of the total system is turned into a closed equation of motion for the reduced state of the Bose gas. This is the so-called *generalized master equation*, which contains memory effects. The leading Markovian behaviour is then extracted by going to the *weak coupling limit*, in which the strength λ of the interaction with the bath tends to zero, whereas the time variable is rescaled by a factor λ^{-2}. After the weak coupling limit, the evolution of the state of the Bose gas is governed by a semi-group type law. The explicit form of the generator of this semi-group is known once the interaction between the system and the bath is specified (see [7]).

In our case, the main requirement is that the coupling should allow for energy (but not for particles) to be exchanged between the Bose gas and the reservoir; for this we need an interaction which is quadratic in the Bose creation and annihilation operators. Moreover, separate *independent* baths are attached to the different energy levels of the (finite volume) Bose gas. The interaction mechanism is then idealized as follows: particles are created and destroyed in the n-th energy level of the Bose system while elementary excitations are destroyed and created in the corresponding bath. We take (for simplicity) these elementary excitations to be all described by a fixed (but arbitrary) wave function f.

Once the above framework has been agreed upon, the corresponding Hamiltonian is fairly unique (see [1]); inserting this in Davies' formula, we get an explicit form for the dynamics of the open Bose gas in the weak coupling limit: the average of an arbitrary gauge-invariant observable X^V in the state ω_t^V obeys the equation (*see* [1])

$$\frac{d}{dt}\omega_t^V(X^V) = V^{-1}\sum_{\substack{i,j\\ i\neq j}} \hat{C}(E_j^V - E_i^V)\omega_t^V(a_i^* a_j[X^V, a_j^* a_i]) \tag{1}$$

In (1) we introduce the following notation:

E_i^V, $i = 0, 1, 2, \ldots$ denote the increasing energy levels of the free Bose gas and a_i^*, a_j the creation and annihilation operators for Bosons in the corresponding eigenstates. The function $\hat{C}(x)$ is the Fourier transform of the bath correlation function; there is a certain degree of arbitrariness in $\hat{C}(x)$ because of the unspecified wave function f, but one can show (see appendix A in [1]) that for a wide class of functions f, $\hat{C}$ is continuous and satisfies:

$$\hat{C}(x) > 0 \tag{2}$$

$$\hat{C}(-x) = e^{\beta x}\hat{C}(x) \tag{3}$$

$$\hat{C}(x)e^{\beta x} \leqslant C < \infty \qquad \text{when} \quad x \geqslant 0 \tag{4}$$

$$\hat{C}(x) \leqslant C < \infty \qquad \text{when} \quad x \leqslant 0 \tag{5}$$

where β is the inverse temperature of the baths. Formula (3) is merely the K. M. S. relation.

If one puts for the observable X^V in (1) an occupation number $N_k^V = a_k^* a_k$ one finds that the right hand side involves $\omega_t^V(N_k^V N_i^V)$. Thus we do not have yet a closed dynamical description of the average occupation numbers, except if the state ω_t^V has no correlation:

$$\omega_t^V\left(\frac{N_k^V}{V}\frac{N_i^V}{V}\right) = \omega_t^V\left(\frac{N_k^V}{V}\right)\omega_t^V\left(\frac{N_i^V}{V}\right). \tag{6}$$

Property (6) is extremely restrictive as such, but it becomes a reasonable condition if it is only required to hold *asymptotically* as $V \to \infty$ (one has then to be more careful about the formulation of this *decorrelation property* because the energy spectrum becomes continuous; see (36) in [*1*] for a proper statement). The decorrelation property becomes then merely the statement that the occupation numbers do not have abnormally large fluctuations and cross-fluctuations; this is known to hold for instance for canonical Gibbs states of the free Bose gas, although it fails for grand canonical ones (see [*9*]). This should not be confused with the much stronger property of quasi-freeness of the state. Of course, the decorrelation property is useful only if one can prove that it is *preserved* by the evolution equation (1). The proof of this fact given in [*1*] for a related model is of combinatorial nature (see theorem 1 and lemma 1); here we assume that the same holds. The closed equation that we obtain from (1) reads

$$\begin{aligned}\frac{d}{dt}\int_{[0,\infty)} G_t(dx)h(x) = \int_{[0,\infty)} G_t(dx)\int_{[0,\infty)} G_t(dy)\hat{C}(x-y)[h(x)-h(y)] \\ + \int_{[0,\infty)} F(dx)\int_{[0,\infty)} G_t(dy)\hat{C}(x-y)[h(x)-h(y)] \end{aligned}\tag{7}$$

with the following notation:

$h(x)$ is an arbitrary bounded continuous function, the fixed measure $F(dx)$ is given by (see remark (*i*) at the end of the section)

$$F(dx) = \frac{x^{1/2}dx}{\sqrt{2}\pi^2} \tag{8}$$

and the time-dependent measure $G_t(dx)$ describes the energy distribution of the Bose gas at time t; more precisely, if we introduce the distribution function

$$G_t^V(x) = V^{-1} \sum_{E_i^V \leqslant x} \omega_t^V(N_i^V) \tag{9}$$

and its associated measure $G_t^V(dx)$, then $G_t(dx)$ is defined to be the limit of $G_t^V(dx)$ as $V \to \infty$ (we assume the existence of this limit, weakly with respect to the class of bounded continuous functions).

There are two properties that eq. (7) ought to display if it is to provide a consistent dynamical description of the infinite free Bose gas: first the overall density should be a conserved quantity, and second the Bose distribution at the temperature of the bath should be a fixed point. The first property is readily checked by putting $h(x) = 1$ in (7). For the second, recall that the measure $G_{eq.}(dx)$ on $[0, \infty)$ describing the population of the energy levels of an infinite Bose gas at inverse temperature β and overall density ρ is given by

$$G_{eq.}(dx) = \begin{cases} \dfrac{F(dx)}{\exp \beta(x-\mu) - 1} & \rho < \rho_c \qquad (10a) \\[2ex] \dfrac{F(dx)}{\exp \beta x - 1} + (\rho - \rho_c)\delta_0(dx) & \rho \geqslant \rho_c \qquad (10b) \end{cases}$$

where $F(dx)$ is as in (8), $\delta_0(dx)$ is the Dirac measure at the origin, and μ and ρ_c are defined respectively by (note that $\mu \leqslant 0$):

$$\int_0^\infty \frac{F(dx)}{\exp \beta(x-\mu) - 1} = \rho \tag{11}$$

$$\rho_c = \int_0^\infty \frac{F(dx)}{\exp \beta x - 1}. \tag{12}$$

It is straightforward to check that the right hand side of (7) vanishes when $G_{eq.}(dx)$ is substituted for $G_t(dx)$.

So far our dynamical description of the open Bose gas is in terms of a non-linear differential equation for a measure, and this is not an easy object to handle. On the other hand, we expect on a physical basis that the relevant class of measures $G_t(dx)$ is of the following type: a sum of an absolutely continuous part and of an atom at the origin. Hence we seek a solution of eq. (7) in the form:

$$G_t(dx) = g_t(x)G(dx) + \rho_t^0\delta_0(dx) \tag{13}$$

where $G(dx)$ is the continuous part of $G_{eq.}(dx)$ namely (with μ as in (11)):

$$G(dx) = \frac{F(dx)}{\exp \beta(x-\mu) - 1} \qquad \mu \leqslant 0. \tag{14}$$

If we insert (13) in (7), we obtain two coupled differential equations for the *time-dependent occupation function* $g_t(x)$ and the *time-dependent* *condensate* ρ_t^0. Note that the second equation is in fact superfluous, because by virtue of density conservation we have:

$$\rho_t^0 + \int_0^\infty G(dx) g_t(x) = \rho = \text{constant}. \tag{15}$$

Thus all the relevant information is contained in the equation governing the occupation function $g_t(x)$:

$$\frac{d}{dt} g_t(x) = \int_0^\infty G(dy) D(x, y) [A(y) g_t(x)(g_t(y) - 1) - A(x) g_t(y)(g_t(x) - 1)] + \rho_t^0 \hat{C}(x)(e^{\beta(x-\mu)} - 1) - \rho_t^0 g_t(x) \hat{C}(x)(e^{\beta x} - 1) \tag{16}$$

with

$$A(x) = 1 - e^{-\beta(x-\mu)} \tag{17}$$

$$D(x, y) = \hat{C}(x - y) e^{\beta(x-\mu)} \tag{18}$$

and $\hat{C}(x)$, μ as before.

The non-linear integro-differential equation (16) is our final kinetic equation, and the remaining part of the paper is devoted to studying its properties. Note that the physical fixed point (10) is now given by $g_{\text{eq.}}(x) = 1$. To check this directly on (16), it is enough to notice that one at least of the two terms μ and $\rho_{\text{eq.}}^0 = \rho - \int_0^\infty G(dx)$ vanishes; indeed there are only three possibilities:

$$\rho < \rho_c \Rightarrow \begin{cases} \mu < 0 \\ \rho_{\text{eq.}}^0 = 0 \end{cases} \tag{19a}$$

$$\rho > \rho_c \Rightarrow \begin{cases} \mu = 0 \\ \rho_{\text{eq.}}^0 > 0 \end{cases} \tag{16b}$$

$$\rho = \rho_c \Rightarrow \begin{cases} \mu = 0 \\ \rho_{\text{eq.}}^0 = 0 . \end{cases} \tag{19c}$$

We end this section with some comments.

i) The reason why the measure (8) appears in the problem is that it is the limit of the Stieltjes measures associated to the energy spectrum of the finite Bose gas. In terms of the corresponding distribution function (sometimes called the integrated density of states) one has, see [*10*]:

$$\lim_{V \to \infty} V^{-1} \sum_{E_i^V \leqslant x} 1 = \frac{\sqrt{2}}{3\pi^2} x^{3/2} = F(x). \tag{20}$$

ii) We see from (2), (3) that the function $D(x, y)$ defined in (18) obeys:

$$D(x, y) > 0 \tag{21}$$

$$D(x, y) = D(y, x). \tag{22}$$

iii) At first sight, the occupation function $g_t(x)$ seems to describe only the uncondensed part of the Bose gas: but we stress again that once $g_t(x)$ is known, the condensate density can be deduced using (15):

$$\rho_t^0 = \rho - \int_0^\infty G(dx)g_t(x). \tag{23}$$

iv) Finally the reader should not be misled by the fact that the *same* function $g_{\text{eq.}}(x) = 1$ is a fixed point of (16) for all values of the density; its interpretation *differs* according to $\rho \leqslant \rho_c$ or $\rho > \rho_c$ since the condensate is absent in the first case and present in the second (see (19)).

§ 3. EXISTENCE AND UNIQUENESS OF THE SOLUTION

We write our differential equation (16) in the form

$$\frac{d}{dt} g_t = \mathrm{L}g_t + \mathrm{N}(g_t) \tag{24}$$

where the operators L (linear) and N (non-linear) act on the real Banach space $\mathscr{B} = \mathrm{L}^1(\mathbb{R}^+, \mathrm{G}(dx))$ in the following way:

$$(\mathrm{L}g)(x) = \int \mathrm{G}(dy)\mathrm{D}(x, y)[\mathrm{A}(x)g(y) - \mathrm{A}(y)g(x)] \tag{25}$$

$$\begin{aligned}\mathrm{N}(g)(x) &= \int \mathrm{G}(dy)\mathrm{D}(x, y)[\mathrm{A}(y) - \mathrm{A}(x)]g(x)g(y) \\ &\quad + \left[\rho - \int \mathrm{G}(dy)g(y)\right]\hat{\mathrm{C}}(x)\{ e^{\beta(x-\mu)} - 1 - g(x)(e^{\beta x} - 1)\}\end{aligned} \tag{26}$$

with μ, A(x), D(x, y) as in (11), (17), (18).

We investigate the properties of these operators; the non-linear part is very regular:

LEMMA 1. — *i*) The operator N is bounded: there exists constants $\mathrm{K}_1, \mathrm{K}_2 \geqslant 0$ such that (with C as in (4), (5))

$$\| \mathrm{N}(g) \| \leqslant 4\mathrm{C} \| g \|^2 + \mathrm{K}_1 \| g \| + \mathrm{K}_2 . \tag{27}$$

ii) The operator N satisfies a local Lipschitz condition: for every g_0 in $\mathscr{B}$ and $\varepsilon > 0$ there exists a constant K_3 such that

$$\left.\begin{array}{l}\| f - g_0 \| \leqslant \varepsilon \\ \| h - g_0 \| \leqslant \varepsilon\end{array}\right\} \Rightarrow \| \mathrm{N}(f) - \mathrm{N}(h) \| \leqslant \mathrm{K}_3 \| f - h \| . \tag{28}$$

iii) The operator N is continuously Fréchet-differentiable; namely for every f in $\mathscr{B}$ the operator (DN)(f) is bounded and moreover the mapping $f \to$ (DN)(f) is continuous.

Proof. — All three points of the lemma are simple consequences of the following inequalities (see (4), (5)):

$$| D(x, y)[A(y) - A(x)] | = | \hat{C}(x - y) - \hat{C}(y - x) | \leqslant 2C \tag{29}$$

$$\hat{C}(x)(e^{\beta x} - 1) \leqslant 2C \qquad \text{when} \quad x \geqslant 0 \tag{30}$$

$$\hat{C}(x)(e^{\beta(x-\mu)} - 1) \leqslant C(e^{-\beta\mu} + 1) \qquad \text{when} \quad x \geqslant 0 \tag{31}$$

$$\int G(dx) \leqslant \rho \tag{32}$$

and of the fact that the non-linearity in N is merely quadratic. ■

We turn now to L; this is an unbounded operator (see remark (*i*) at the end of this section) so that the formal expression (25) does not suffice to define an operator. We consider first the two domains

$$\tilde{\mathscr{L}} = \left\{ g \in \mathscr{B} : \int G(dx) \left| \int G(dy) D(x, y) [A(x)g(y) - A(y)g(x)] \right| < \infty \right\} \tag{33}$$

$$\hat{\mathscr{L}} = \left\{ g \in \mathscr{B} : \sup | g(x) | < \infty, \operatorname{supp} g \subset [0, a] \text{ for some } a < \infty \right\} \tag{34}$$

and we denote the corresponding operators by $\tilde{L}$ and $\hat{L}$ respectively. One can check that $\tilde{\mathscr{L}}$ contains $\hat{\mathscr{L}}$ and that the latter is dense in $\mathscr{B}$, so that both $\tilde{L}$ and $\hat{L}$ are densely defined. Moreover, $\tilde{L}$ is a closed extension of $\hat{L}$:

LEMMA 2. — The operator $\tilde{L}$ is closed.

Proof. — Let $\{ g_n \}$ be a sequence in $\tilde{\mathscr{L}}$ such that $g_n \to g$ and $\tilde{L}g_n \to f$, both in L^1-sense. Then (see Theorem 1.12 and Corollary in [*11*]) there exists a subsequence $\{ g_{n_i} \}$ of $\{ g_n \}$ such that

$$g_{n_i}(x) \to g(x) \qquad \text{a. e.} \tag{35}$$

$$(\tilde{L}g_{n_i})(x) \to f(x) \qquad \text{a. e.} \tag{36}$$

$$| g_{n_i}(x) | \leqslant h(x) \qquad \text{for some } h \text{ in } \mathscr{B}. \tag{37}$$

Using (35), (37) together with the inequality

$$D(x, y)A(y) \leqslant C(e^{\beta(x-\mu)} - 1) \tag{38}$$

we deduce, using the dominated convergence theorem:

$$(\tilde{L}g_{n_i})(x) \to (\tilde{L}g)(x) \qquad \text{a. e.} \tag{39}$$

Comparing (36) and (39), we see that g belongs to $\tilde{\mathscr{L}}$ and that $f = \tilde{L}g$ so that $\tilde{L}$ is closed. ■

The domain $\tilde{\mathscr{L}}$ might be too big for $\tilde{L}$ itself to generate a contraction semi-group. However there exists a restriction of $\tilde{\mathrm{L}}$ which has this property:

PROPOSITION 1. — Let L (with domain $\mathscr{L}$) be the closure of $\hat{\mathrm{L}}$; then L generates a semi-group of positivity-preserving contractions on

$$\mathscr{B} = \mathrm{L}^1(\mathbb{R}^+, \mathrm{G}(dx)).$$

Proof. — We use the Lumer-Phillips theorem (see [12] Theorem 2.25) which states that the following set of conditions on $\hat{\mathrm{L}}$ is sufficient for its closure L to generate a contraction semi-group:

a) For every g in $\hat{\mathscr{L}}$ there is an element l of $\mathscr{B}^* = \mathrm{L}^\infty(\mathbb{R}^+, \mathrm{G}(dx))$ such that

$$i) \qquad \| l \|_\infty = 1 \tag{40}$$

$$ii) \qquad \langle l, g \rangle = \| g \| \tag{41}$$

$$iii) \qquad \langle l, \hat{\mathrm{L}}g \rangle \leqslant 0 \tag{42}$$

$$b) \qquad \mathrm{Ran}\,(\mathrm{I} - \hat{\mathrm{L}}) \text{ is dense in } \mathscr{B}. \tag{43}$$

To show that conditions $a(i)$, (iii) are fulfilled, we exhibit an explicit element l of $\mathscr{B}^*$:

$$l(x) = \mathrm{sgn}\ g(x) = \begin{cases} 1 & \text{when} \quad g(x) \geqslant 0 \\ -1 & \text{when} \quad g(x) < 0. \end{cases} \tag{44}$$

Properties (i), (ii) are readily checked. For (iii), note that:

$$\langle l, \hat{\mathrm{L}}g \rangle = \int \mathrm{G}(dx) \int \mathrm{G}(dy)\mathrm{D}(x, y)\,[\mathrm{A}(x)g(y)\,\mathrm{sgn}\,g(x) - \mathrm{A}(y)g(x)\,\mathrm{sgn}\,g(x)] \tag{45}$$

$$= \int \mathrm{G}(dx) \int \mathrm{G}(dy)\mathrm{D}(x, y)\mathrm{A}(y)\,[g(x)\,\mathrm{sgn}\,g(y) - |\,g(x)\,|\,]. \tag{46}$$

The last formula is deduced from the previous one by using Fubini's theorem and the definition (34) of $\hat{\mathscr{L}}$. Now (46) is clearly non-positive because $\mathrm{D}(x, y)$ and $\mathrm{A}(y)$ are non-negative (see (21), (17)).

We prove condition (b) by contradiction. If $\mathrm{Ran}\,(\mathrm{I} - \hat{\mathrm{L}})$ is *not* dense in $\mathscr{B}$ there must exist (by the Hahn-Banach theorem) an element $k \neq 0$ in $\mathscr{B}^* = \mathrm{L}^\infty(\mathbb{R}^+, \mathrm{G}(dx))$ such that

$$\langle k, (\mathrm{I} - \hat{\mathrm{L}})h \rangle = 0 \qquad \text{for all } h \text{ in } \hat{\mathscr{L}}. \tag{47}$$

To see that this is impossible, consider the following family of elements of $\hat{\mathscr{L}}$:

$$h_a(x) = \mathrm{A}(x)\chi_{[0,a]}(x)\ \mathrm{sgn}\ k(x). \tag{48}$$

We have

$$\langle k, h_a \rangle = \int_0^a \mathrm{G}(dx)\mathrm{A}(x)\,|\,k(x)\,| \to \int_0^\infty \mathrm{G}(dx)\mathrm{A}(x)\,|\,k(x)\,| > 0 \quad \text{as} \quad a \to \infty. \tag{49}$$

On the other hand

$$
\begin{aligned}
\langle k, \hat{\mathrm{L}}h_a \rangle = & \int_0^a \mathrm{G}(dx) \int_0^a \mathrm{G}(dy)\mathrm{D}(x, y)\mathrm{A}(x)\mathrm{A}(y)\,[k(x)\,\mathrm{sgn}\,k(y) - |\,k(x)\,|\,] \\
& - \int_0^a \mathrm{G}(dx) \int_a^\infty \mathrm{G}(dy)\mathrm{D}(x, y)\mathrm{A}(x)\mathrm{A}(y)\,|\,k(x)\,| \\
& + \int_a^\infty \mathrm{G}(dx) \int_0^a \mathrm{G}(dy)\mathrm{D}(x, y)\mathrm{A}(x)\mathrm{A}(y)k(x)\,\mathrm{sgn}\,k(y)
\end{aligned}
\tag{50}
$$

The first two terms in the right hand side of (50) are obviously non-positive; moreover one can check that the last term tends to zero as $a \to \infty$, so that

$$
\lim_{a \to \infty} \langle k, \hat{\mathrm{L}}h_a \rangle \leqslant 0 . \tag{51}
$$

Comparing (47), (49) and (51), we obtain the desired contradiction. Thus L generates a contraction semi-group on $\mathscr{B}$.

To see that $e^{t\mathrm{L}}$ preserves positivity, consider for g fixed but arbitrary in $\mathscr{L}$ the function:

$$
\phi(t) = \int \mathrm{G}(dx)\,|\,(e^{t\mathrm{L}}g)(x)\,| - \int \mathrm{G}(dx)(e^{t\mathrm{L}}g)(x) . \tag{52}
$$

This function is non-negative; moreover it is non-increasing: indeed the first term is non-increasing because $e^{t\mathrm{L}}$ is a contraction and the second is constant in time (because the integrand of L is skew-symmetric). Hence if $\phi(0) = 0$ (i. e. if $g(x) \geqslant 0$ a. e.) one has $\phi(t) = 0$ for all $t \geqslant 0$ (i. e. $(e^{t\mathrm{L}}g)(x) \geqslant 0$ a. e. for all $t \geqslant 0$). ■

Because of lemma 1 and proposition 1, the equation (24) falls into the class of semilinear differential equations (see [*13*]). The importance of proposition 1 is apparent if one considers the integral equation associated to (24):

$$
g_t = e^{t\mathrm{L}}g_0 + \int_0^t ds e^{(t-s)\mathrm{L}}\mathrm{N}(g_s) . \tag{53}
$$

Existence and uniqueness of the local solution of (53) follows from lemma 1 (*ii*) by a standard fixed point argument (see [*13*]). To prove that this is also a local solution of (24), one has to check that it remains in $\mathscr{L}$. This follows from the additional regularity of N stated in (*iii*) of lemma 1 (see theorem 3.2 in chapter 8 of [*13*]). We thus have:

PROPOSITION 2. — Equation (24) admits a unique local solution. Namely every initial condition g_0 in $\mathscr{L}$ gives rise to a unique solution g_t, $0 \leqslant t \leqslant \mathrm{T}$, and T depends only on $\|\, g_0 \,\|$.

We note immediately two properties of the local solution.

LEMMA 3. —

i) If $\int G(dx)g_0(x) = \rho$, then $\int G(dx)g_t(x) = \rho$, $0 \leqslant t \leqslant T$.

ii) If $\int G(dx)g_0(x) < \rho$, then $\int G(dx)g_t(x) < \rho$, $0 \leqslant t \leqslant T$.

The proof is a simple application of Gronwall's lemma, and we omit it (see lemma 2 in [*1*] for a hint and [*14*] for further details). The importance of lemma 3 (besides its physical interpretation, see remark (*iv*) at the end of the section) is that if we can prove that g_t remains non-negative we shall have the *a priori bound*

$$\| g_t \| \leqslant \rho \qquad 0 \leqslant t \leqslant T. \tag{54}$$

Such an *a priori* bound on the local solution yields immediately *global* existence because the local solution starting at g_T exists on $T \leqslant t \leqslant 2T$, and so on. Moreover, the physical interpretation of g_t as the *time-dependent occupation function* requires that it remains *non-negative* if the initial condition g_0 is so. To prove this property, we proceed in a way similar to Arkeryd's study of the Boltzmann equation [*2*] [*3*]. The positivity-preserving property of e^{tL} is an essential ingredient of the proof.

THEOREM 1. — If the initial condition g_0 in $\mathscr{L}$ is non-negative, so is the solution g_t of (24) for some time interval $[0, T]$; consequently the solution exists globally in time.

Proof. — As mentioned above, global existence is an immediate consequence of the positivity-preserving property. To prove this last property we write (24) as

$$\frac{d}{dt} g_t = L' g_t + N'(t, g_t) \tag{55}$$

with

$$(L'g)(x) = (Lg)(x) - K\rho g(x) \tag{56}$$

$$\begin{aligned} N'(t, g)(x) = & \int G(dy) \{ K + D(x, y) [A(y) - A(x)] \} g(x)g(y) \\ & + F_{g_0}(t) g(x) \{ K - \hat{C}(x)(e^{\beta x} - 1) \} \\ & + F_{g_0}(t) \hat{C}(x)(e^{\beta(x-\mu)} - 1). \end{aligned} \tag{57}$$

In the above, K is some constant larger than 2C (see (4), (5)), and $F_{g_0}(t)$ stands for:

$$F_{g_0}(t) = \rho - \int G(dx) g_t(x) \tag{58}$$

where g_t is the local solution of (24) (known in principle) corresponding to the initial condition g_0 in $\mathscr{L}$. Note the inequalities (see (29), (30) and lemma 3):

$$K + D(x, y)[A(y) - A(x)] \geqslant 0 \tag{59}$$

$$K - \hat{C}(x)(e^{\beta x} - 1) \geqslant 0 \tag{60}$$

$$F_{g_0}(t) \geqslant 0 . \tag{61}$$

The integral equation corresponding to (55) reads:

$$g_t = e^{tL'} g_0 + \int_0^t ds e^{(t-s)L'} N'(s, g_s) . \tag{62}$$

Clearly, a solution of this equation is also a solution of (53), and thus of (24) (see proposition 2). We prove now that if g_0 is non-negative, so is the solution of (62). For this we construct (as in [2] [3]) a sequence of non-negative functions $g_t^{(n)}$ which converges to the solution g_t of (62); this sequence is obtained by iterating (62):

$$g_t^{(n)} = e^{tL'} g_0 + \int_0^t ds e^{(t-s)L'} N'\big(s, g_s^{(n-1)}\big) \tag{63a}$$

$$g_t^{1)} = 0 \tag{63b}$$

When g_0 is non-negative, the sequence $g_t^{(n)}$ is non-negative, pointwise increasing, and bounded in norm for $t \leqslant T$, T small enough:

$$g_t^{(n)}(x) \geqslant 0 \qquad \text{a. e.} \tag{64a}$$

$$g_t^{(n+1)}(x) \geqslant g_t^{(n)}(x) \qquad \text{a. e.} \tag{64b}$$

$$\| g_t^{(n)} \| \leqslant 2\rho . \tag{64c}$$

These properties are proved by induction. The first two are straightforward because

$$e^{tL'} = e^{-K\rho t} e^{tL} \tag{65}$$

preserves positivity just as e^{tL} does (see proposition 1), and $N'(t, \cdot)$ preserves positivity and pointwise order, namely (see (57)-(61)):

$$g_1(x) \geqslant g_2(x) \geqslant 0 \quad \text{a. e.} \Rightarrow N'(t, g_1)(x) \geqslant N'(t, g_2)(x) \geqslant 0 \quad \text{a. e.} \tag{66}$$

To prove $(64c)$, note that because both L and N have skew-symmetric integrands we have

$$\int G(dx)(e^{tL} f)(x) = \int G(dx) f(x) \tag{67}$$

$$\int G(dx) N(f)(x) = 0 . \tag{68}$$

Using these relations, together with (63 a), (57), (64 a) we obtain

$$\|g_t^{(n)}\| \leqslant e^{-\mathrm{K}\rho t}\|g_0\| + \int_0^t ds e^{-(t-s)\mathrm{K}\rho}\{\mathrm{K}\|g_s^{(n-1)}\|^2 + \mathrm{F}_{g_0}(s)[\mathrm{K}\|g_s^{(n-1)}\| + \|k\|]\} \tag{69}$$

where we have set

$$k(x) = \hat{\mathrm{C}}(x)(e^{\beta(x-\mu)} - 1). \tag{70}$$

Using $\|g_0\| \leqslant \rho$, $\|g_s^{(n-1)}\| \leqslant 2\rho$, $\mathrm{F}_{g_0}(s) \leqslant 2\rho$ we get

$$\|g_t^{(n)}\| \leqslant 8\rho + 2\|k\|\mathrm{K}^{-1} - 7\rho e^{-\mathrm{K}\rho t}. \tag{71}$$

Now, we can always take the constant K large enough to have

$$2\|k\|\mathrm{K}^{-1} \leqslant \rho/2. \tag{72}$$

Moreover, we can also restrict the time interval $[0, \mathrm{T}]$ on which we work in such a way that

$$e^{-\mathrm{K}\rho t} \geqslant 13/14 \qquad 0 \leqslant t \leqslant \mathrm{T}. \tag{73}$$

Gathering (71), (72) and (73) we have

$$\|g_t^{(n)}\| \leqslant \rho\left(\frac{17}{2} - \frac{13}{2}\right) = 2\rho \tag{74}$$

so that the proof of (64 c) is complete.

By the monotone convergence theorem, equations (64 a, b, c) imply that $g_t^{(n)} \to g_t^{(\infty)}$ in L^1-sense, with $g_t^{(\infty)}(x) \geqslant 0$ a. e. As both N′ and $e^{t\mathrm{L}'}$ are continuous operators, $g_t^{(\infty)}$ is the solution of equation (62) and thus of (24). ■

We conclude this section with a few remarks.

i) To see the origin of the unboundedness of the operator L, we can look separately at its multiplicative part and at its integral part. First of all, the multiplicative part is clearly unbounded because, see (8), (14), (18), (25):

$$\int \mathrm{G}(dy)\mathrm{D}(x, y)\mathrm{A}(y) = (\sqrt{2}\pi^2)^{-1}\int dy y^{1/2}\hat{\mathrm{C}}(y - x) \tag{75}$$

$$\geqslant x^{1/2}(\sqrt{2}\pi^2)^{-1}\int dz \hat{\mathrm{C}}(z). \tag{76}$$

The integral part is also unbounded, because if we put

$$g_\alpha(x) = (e^{\beta x} - 1)x^\alpha \tag{77}$$

we have

$$\lim_{x\to\infty} x^{-\alpha-1/2}e^{-\beta x}\int \mathrm{G}(dy)\mathrm{D}(x, y)\mathrm{A}(x)g_\alpha(y) > 0 \tag{78}$$

so that, taking $-2 < \alpha < -3/2$, we see that g_0 belongs to $\mathscr{B}$ but that its image does not.

ii) The point of Lemma 2 is the following: it ensures that the generator L of the semi-group (see Proposition 1) is included into $\tilde{\mathrm{L}}$, and consequently that L is given by formula (25) on the whole of its domain $\mathscr{L}$.

iii) The set of non-negative elements of $\mathscr{L}$ with norm smaller than ρ is invariant under the evolution generated by (24). This shows the consistency of the interpretation of $g_t(x)$ as the time-dependent occupation function and $\rho - \int \mathrm{G}(dx)g_t(x)$ as the time-dependent condensate.

iv) Lemma 3 shows that the qualitative feature of presence or absence of condensate in the initial condition is preserved for all finite times. This does *not* mean that these features are shared by the asymptotic occupation function, but rather it places restrictions on the nature of convergence to equilibrium when $\rho > \rho_c$, see theorem 2.

§ 4. ASYMPTOTIC BEHAVIOUR OF THE SOLUTION

In this section, we show that under mild restrictions every physically meaningful initial condition g_0 is driven by the evolution (24) to the fixed point $g_{\text{eq.}}(x) = 1$. The corresponding problem for a *finite energy range* $[0, b]$ was treated in [*1*]. We were dealing there with the equation

$$\frac{d}{dt} g_t = \mathrm{L}_b g_t + \mathrm{N}_b(g_t) \tag{79}$$

where L_b and N_b are characterized as follows: first define two operators $\tilde{\mathrm{L}}_b$, $\tilde{\mathrm{N}}_b$ are replacing in L and N (see (25), (26)) the measure $\mathrm{G}(dy)$ by

$$\mathrm{G}_b(dy) = \chi_{[0,b]}(y)\mathrm{G}(dy)\,. \tag{80}$$

Put then

$$(\mathrm{L}_b g)(x) = \chi_{[0,b]}(x)(\tilde{\mathrm{L}}_b g)(x) \tag{81}$$

$$\mathrm{N}_b(g)(x) = \chi_{[0,b]}(x)\tilde{\mathrm{N}}_b(g)(x)\,. \tag{82}$$

The main ingredient of the proof of approach to equilibrium for eq. (79) was the fact that the following functional has the *Liapunov property* (namely it is bounded from below and it decreases along the trajectories of (79)):

$$\varphi_b(g) = \beta \int \mathrm{G}_b(dx)xg(x) - \int \mathrm{G}_b(dx)\left\{\frac{1 + b(x)g(x)}{b(x)}\log(1 + b(x)g(x)) - g(x)\log(b(x)g(x))\right\} \tag{83}$$

with

$$b(x) = (\exp(\beta(x - \mu)) - 1)^{-1}. \tag{84}$$

A similar result holds for the infinite energy range: the functional φ, defined as φ_b but with $G(dx)$ in place of $G_b(dx)$, is a Liapunov functional for eq. (24). This can be seen from the result of the cut-off case, with the help of the following property:

LEMMA 4. — Let g_0 be a non-negative element of $\mathscr{L}$, and let g_t be the corresponding solution of eq. (24). Put $g_0^b(x) = \chi_{[0,b]}(x)g_0(x)$, and denote by g_t^b the corresponding solution of (79). Then for any $T < \infty$

$$\sup_{0 \leqslant t \leqslant T} \| g_t - g_t^b \| \to 0 \qquad \text{as} \quad b \to \infty. \tag{85}$$

The proof of Lemma 4 is not particularly enlightening, and we omit it; the techniques involved are similar to those used in [*4*]. The decrease in time of $\varphi(g_t)$ is then obtained as follows; we know that

$$\varphi_b(g_0^b) - \varphi_b(1) \geqslant \varphi_b(g_t^b) - \varphi_b(1). \tag{86}$$

Moreover, one can check (see eq. (103) in [*1*]), that the right hand side of (86) can be written as the integral of a non-negative function. On the other hand, we deduce from Lemma 4 that there exists a sequence $b_i \nearrow \infty$ such that

$$| g_t(x) - g_t^{b_i}(x) | \to 0 \qquad \text{a. e.} \qquad \text{as} \quad i \to \infty. \tag{87}$$

Supposing that g_0 is such that $\varphi(g_0) < \infty$ (see (100), (102) below for sufficient conditions) we can apply Fatou's lemma to obtain from (86)

$$\varphi(g_0) - \varphi(1) \geqslant \varphi(g_t) - \varphi(1) \tag{88}$$

which is the desired result.

But in fact, the decrease in time of $\varphi(g_t)$ is not sufficient to prove approach to equilibrium for equation (24). The main reason is that, because of the infinite energy range, we face the additional difficulty that a fraction of the mass of the measure $g_t(x)G(dx)$ or $xg_t(x)G(dx)$ might flow out to infinity. In order to exclude this *depletion* phenomenon, we have to prove that both $g_t(x)G(dx)$ and $xg_t(x)G(dx)$ are *stochastically bounded*, namely (see [*15*]): for every $\varepsilon > 0$, there exists $a > 0$ (independent of t) such that

$$\int_a^\infty G(dx)g_t(x) < \varepsilon \qquad \text{for all} \quad t \geqslant 0 \tag{89}$$

$$\int_a^\infty G(dx)xg_t(x) < \varepsilon \qquad \text{for all} \quad t \geqslant 0 \tag{90}$$

(note that (90) implies (89)).

In fact, using the decrease in time of $\varphi(g_t)$ one can show the following implication:

$$\int G(dx)xg_0(x) < \infty \Rightarrow \int G(dx)xg_t(x) \leqslant E(g_0) < \infty . \tag{91}$$

The physical interpretation is the following: if the initial energy distribution has a finite mean, this mean remains uniformly bounded in the course of the evolution (even though it need not decrease). This property implies immediately (89), but not (90). This suggests that in order to prove (90), one should construct an alternative Liapunov functional, in which the second moment $\int G(dx)x^2g(x)$ plays a role analogous to that of the first moment $\int G(dx)xg(x)$ in φ. In fact, we can construct a whole family of Liapunov functionals. This is done as follows, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous increasing function such that $f(x) = 0$ iff $x = 0$. Put

$$h(y, x) = -\int_0^y dz f\left(\log \frac{1+z}{z} - \beta(x-\mu)\right) \tag{92}$$

and define (formally) the functional H on the positive cone of

$$\mathscr{B} = L^1(\mathbb{R}^+, G(dx))$$

by

$$H(g) = \int G(dx)\frac{h(b(x)g(x), x)}{b(x)} - \left[\rho - \int G(dx)g(x)\right]f(\beta\mu). \tag{93}$$

We first give a formal argument which indicates why $H(g_t)$ should be a decreasing function of t. For that purpose, it is useful to write the dynamical equation (24) in the form

$$\begin{aligned}\frac{d}{dt}g_t(x) = \int G(dy)D(x, y)[r_t(y) - r_t(x)]g_t(x)g_t(y)\\ + D(x, 0)[A(0) - r_t(x)]g_t(x)\rho_t^0\end{aligned} \tag{94}$$

where

$$r_t(x) = A(x)\frac{g_t(x) - 1}{g_t(x)}. \tag{95}$$

With the same notation, we have

$$\log \frac{1 + b(x)g_t(x)}{b(x)g_t(x)} - \beta(x-\mu) = \log(1 - r_t(x)) \tag{96}$$

so that formally

$$\frac{d}{dt}H(g_t) = -\int G(dx)[f(\log(1 - r_t(x))) - f(\beta\mu)]\frac{d}{dt}g_t(x). \tag{97}$$

Inserting (94) in (97), we obtain

$$\begin{aligned}\frac{d}{dt}\mathrm{H}(g_t) &= -\frac{1}{2}\int \mathrm{G}(dx)\int \mathrm{G}(dy)\mathrm{D}(x,y)g_t(x)g_t(y)\,[f(\log(1-r_t(x))) \\ &\qquad\qquad - f(\log(1-r_t(y)))]\,[r_t(y)-r_t(x)] \\ &\quad -\rho_t^0\int \mathrm{G}(dx)\mathrm{D}(x,0)\,[f(\log(1-r_t(x)))-f(\log e^{\beta\mu})]\,[1-r_t(x)-e^{\beta\mu}] \end{aligned} \tag{98}$$

which is non-positive by inspection (remember that f is increasing).

This argument can be made rigorous (see Proposition 4). From now on we restrict ourselves to the following class of functions f:

$$f_n(u) = u\,|\,u\,|^{n-1} \qquad n \geqslant 1\,. \tag{99}$$

The corresponding $h(y, x)$ and $\mathrm{H}(g)$ see (92), (93) are denoted by $h_n(y, x)$ and $\mathrm{H}_n(g)$ respectively. Note that the functional $\varphi(g)$ (defined as in (83) but with $\mathrm{G}(dx)$ instead of $\mathrm{G}_b(dx)$) coincides with $\mathrm{H}_1(g)$ up to a constant:

$$\mathrm{H}_1(g) = \varphi(g) - \rho\beta\mu\,. \tag{100}$$

Our first task is to find a suitable domain for H_n. Define, with $\mathscr{L}$ as in Proposition 1

$$\mathscr{D}_\rho^n = \left\{ g \in \mathscr{L} : g \geqslant 0,\ \|\, g\,\| \leqslant \rho,\ \int \mathrm{G}(dx)x^n g(x) < \infty \right\}. \tag{101}$$

PROPOSITION 3. — *Let* g *belong to* $\mathscr{D}_\rho^n$; then

$$-\,n!2^{n+1}\rho \leqslant \mathrm{H}_n(1) \leqslant \mathrm{H}_n(g) < \infty\,. \tag{102}$$

Proof. — *a*) First note that since μ is non-positive, so is $f_n(\beta\mu)$ and thus (see (93))

$$\mathrm{H}_n(g) \geqslant \int \mathrm{G}(dx)\frac{h_n(b(x)g(x),\,x)}{b(x)}\,. \tag{103}$$

Moreover, the function $y \to h_n(y, x)$ takes its minimum at $y = b(x)$, so that the right hand side of (103) is minimum when $g = 1$; hence $\mathrm{H}_n(g) \geqslant \mathrm{H}_n(1)$.

b) To get a lower bound on $\mathrm{H}_n(1)$ we note the following inequalities:

$$f_n(u) \leqslant n!2^n \exp(u/2) \qquad \text{when} \quad u \geqslant 0 \tag{104}$$

$$\log\frac{1+z}{z} - \beta(x-\mu) = \log\left(\frac{1+z}{z}\,\frac{b(x)}{1+b(x)}\right) \geqslant 0 \qquad \text{when} \quad z \leqslant b(x) \tag{105}$$

Combining (104) and (105) we have

$$0 \leqslant f_n\left(\log\frac{1+z}{z} - \beta(x-\mu)\right) \leqslant n!2^n\left(\frac{b(x)}{z}\right)^{1/2} \qquad \text{when} \quad z \leqslant b(x) \tag{106}$$

This implies (see (92))

$$0 \geqslant h_n(y, x) \geqslant -n!2^{n+1}b(x) \qquad \text{when} \quad y \leqslant b(x)\,. \tag{107}$$

And thus, using (103)

$$\mathrm{H}_n(1) \geqslant -n!2^{n+1}\int \mathrm{G}(dx) \geqslant -n!2^{n+1}\rho\,. \tag{108}$$

c) To conclude the proof, we show that $\mathrm{H}_n(g)$ is finite. Note that

$$\log\frac{1+z}{z} - \beta(x-\mu) = \log\left(\frac{1+z}{z}\frac{b(x)}{1+b(x)}\right) < 0 \qquad \text{when} \quad z > b(x) \tag{109}$$

Since f_n is increasing, this implies

$$0 > f_n\left(\log\frac{1+z}{z-} - \beta(x-\mu)\right) \geqslant f_n(-\beta(x-\mu)) \qquad \text{when} \quad z > b(x) \tag{110}$$

Using (106) and (110), we obtain (see (92))

$$|\, h_n(y, x)\,| \leqslant n!2^{n+1}b(x) + \beta^n(x-\mu)^n y \qquad \text{for all} \quad x, y \geqslant 0 \tag{111}$$

which yields (see (93))

$$\mathrm{H}_n(g) \leqslant (n!2^{n+1} + \beta^n\,|\,\mu\,|^n)\rho + \beta^n\int \mathrm{G}(dx)(x-\mu)^n g(x)\,. \tag{112}$$

This is clearly finite since g belongs to $\mathscr{D}_\rho^n$ (see (101)); note that

$$\mathscr{D}_\rho^n \subset \mathscr{D}_\rho^m \qquad \text{when} \quad n > m \tag{113}$$

■

Now that the functionals H_n are properly defined, we can consider their time behaviour.

PROPOSITION 4. — Let g_0 be an element of $\mathscr{D}_\rho^n$, and g_t be the corresponding solution of (24). Then

$$\mathrm{H}_n(g_t) - \mathrm{H}_n(g_0) \leqslant -\int_0^t ds\Gamma_n(s) \tag{114}$$

with

$$\Gamma_n(s) = \frac{1}{2}\int_{\Omega_s}\mathrm{G}(dx)\int \mathrm{G}(dy)\mathrm{D}(x, y)g_s(x)g_s(y)\,[r_s(y) - r_s(x)]\,[f_n(\log(1-r_s(x))) - f_n(\log(1-r_s(y)))] \tag{115}$$

where $r_s(x)$ is as in (95) and

$$\Omega_s = \{\,(x, y) : x \geqslant 0,\ y \geqslant 0,\ g_s(x) \neq 0 \quad \text{or} \quad g_s(y) \neq 0\,\}\,. \tag{116}$$

Proof. — We only sketch the proof, because it does not involve any new idea. The first step is to prove the equivalent of (114) for the truncated functionals $\mathrm{H}_n^b(g_t^b)$ defined as H_n but with $\mathrm{G}_b(dx)$ (see (80)) in place of $\mathrm{G}(dx)$,

g_t^b being the solution of the truncated equation (79). The case $n = 1$ is treated in appendix C of [1] (note that $H_1^b = \varphi_b - \rho\beta\mu$) and a similar proof works for arbitrary n. In the second step, one uses lemma 4 to deduce (114) by virtue of Fatou's lemma, as in (87), (88). ∎

The domains $\mathscr{D}_\rho^n$ have a further crucial feature: each of them is *invariant* under the evolution generated by eq. (24). We know already that the first two conditions in (101) are preserved by the evolution (see Theorem 1 and Lemma 3). It turns out that the third condition is also preserved in the following strong sense:

PROPOSITION 5. — Suppose that the initial condition g_0 belongs to $\mathscr{D}_\rho^n$; then there exists a constant $E_n(g_0) < \infty$ such that

$$\int G(dx)x^n g_t(x) \leqslant E_n(g_0) \qquad \text{for all} \quad t \geqslant 0 \tag{117}$$

Proof. — We define the function

$$d(x) = \frac{\exp \beta(x - \mu) - 1}{\exp\,[\beta(x - \mu)/2] - 1}. \tag{118}$$

Then $d(x)$ has the following properties, see (14), (17):

$$d(x) \geqslant 1$$

$$\int G(dx)(x - \mu)^m d(x) \equiv \theta_m < \infty \tag{120}$$

$$\log \frac{1 + d(x)b(x)}{d(x)b(x)} = \beta(x - \mu)/2\,. \tag{121}$$

Consider the function $h_n(y, x)$ in the region where $y > d(x)b(x)$; if we split the range of integration $[0, y]$ into three parts $[0, b(x)]$, $(b(x), d(x)b(x))$ and $[(d(x)b(x), y]$, we obtain using (107), (110), (119), (121):

$$h_n(y, x) \geqslant -n!2^{n+1}b(x) + f_n(\beta(x-\mu)/2)\,[y - d(x)b(x)] \quad \text{when} \quad y > d(x)b(x) \tag{122}$$

This implies

$$\int_{g_t(x)>d(x)} G(dx)\frac{h_n(b(x)g_t(x), x)}{b(x)} \geqslant -\,n!2^{n+1}\rho + (\beta/2)^n \int_{g_t(x)>d(x)} G(dx)(x - \mu)^n(g_t(x) - d(x)) \tag{123}$$

We have thus, using (120)

$$(\beta/2)^n \int_{g_t(x)>d(x)} G(dx)(x - \mu)^n g_t(x) \leqslant n!2^{n+1}\rho + (\beta/2)^n\theta_n + \int_{g_t(x)>d(x)} G(dx)\frac{h_n(b(x)g_t(x), x)}{b(x)} \tag{124}$$

We obtain now a bound for the third term on the right hand side of (124). First of all we obtain from the time-decrease property of $H_n(g_t)$ (see Proposition 4 and (93)):

$$\int G(dx)\frac{h_n(b(x)g_t(x), x)}{b(x)} \leqslant H_n(g_0). \tag{125}$$

This yields, using successively (111) and (120):

$$\begin{aligned}\int_{g_t(x)>d(x)} G(dx)\frac{h_n(b(x)g_t(x), x)}{b(x)} \\ &\leqslant H_n(g_0) + \int_{g_t(x)\leqslant d(x)} G(dx)\frac{|h_n(b(x)g_t(x), x)|}{b(x)} \qquad (126)\\ &\leqslant H_n(g_0) + n!2^{n+1}\rho + \beta^n\theta_n. \qquad (127)\end{aligned}$$

Finally, we treat the region where $g_t(x) \leqslant d(x)$; this is immediate because

$$\int_{g_t(x)\leqslant d(x)} G(dx)(x-\mu)^n g_t(x) \leqslant \theta_n. \tag{128}$$

Putting (124), (127) and (128) together we obtain the stated result (117) ■

COROLLARY 1. — Let the initial condition g_0 belong to $\mathscr{D}_\rho^n$; then for every $\varepsilon > 0$ there is a number α such that for all $t \geqslant 0$

$$\int_\alpha^\infty G(dx)x^m g_t(x) < \varepsilon \qquad m = 0, 1, 2, \ldots, n-1. \tag{129}$$

Proof. — Using (117) we have for $m < n$

$$E_n(g_0) \geqslant \int_0^\infty G(dx)x^n g_t(x) \geqslant \alpha^{n-m}\int_\alpha^\infty G(dx)x^m g_t(x) \tag{130}$$

which gives (129) if we take

$$\alpha = \max_{m\leqslant n-1} \{ (E_n(g)/\varepsilon)^{\frac{1}{n-m}} \}. \qquad ■ \tag{131}$$

The corollary reduces to (89), (90) when $n = 2$; we mentioned earlier that once these stochastic boundedness properties are established, the proof of approach to equilibrium for the solution of eq. (24) is essentially reduced to that given in [*1*]. We explicitate now the various steps involved, but to avoid repetition we give only those proofs which were omitted in [*1*]. We stress again that the functional φ (defined as in (83) but with $G(dx)$ instead of $G_b(dx)$) is essentially identical to H_1 (see (100)). The point of Lemmas 5, 6, 7 is to establish various connections between the convergence of $\varphi(g_t)$ as $t \to \infty$ and that of g_t itself.

LEMMA 5. — Let $\{ t_n \}$ be a sequence tending to $+\infty$, such that $\lim_{n\to\infty} \Gamma_1(t_n) = 0$ with Γ_1 as in (115); then for every $b > 0$

$$\lim_{n\to\infty} \int_{\substack{0 \\ g_{t_n}(x)<1}}^{b} G(dx)(1 - g_{t_n}(x)) = 0 . \tag{132}$$

LEMMA 6. — Let the initial condition g_0 belong to $\mathscr{D}_\rho$. Then either of the following conditions implies that $\varphi(g_t) \to \varphi(1)$ as $t \to \infty$:

i) There exists a sequence $\{ t_n \}$ tending to $+\infty$ such that

$$\int_0^\infty G(dx) \,|\, g_{t_n}(x) - 1 \,| \to 0 \qquad \text{as} \quad n \to \infty .$$

ii) $\rho \geqslant \rho_c$ and for every $\delta > 0$ there exists a sequence $\{ t_n \}$ tending to $+\infty$ such that $\int_\delta^\infty G(dx) \,|\, g_{t_n}(x) - 1 \,| \to 0$ as $n \to \infty$.

Proof. — First notice that since $\varphi(g_t)$ is monotone decreasing in t, it suffices to prove that (*i*) or (*ii*) implies $\varphi(g_{t_n}) \to \varphi(1)$ as $n \to \infty$. Define for u, v positive:

$$k(u, v) = -(1 + u)\log(1 + u) + (1 + u)\log(1 + v) + u\log u - u\log v . \tag{133}$$

One can check the following properties:

$$k(u, v) \geqslant 0, \quad \text{and} \quad k(u, v) = 0 \quad \text{iff} \quad u = v \tag{134}$$
$$k(u, v) < (u - v)\log(1 + 1/v) \quad \text{when} \quad u > v \tag{135}$$
$$k(u, v) \leqslant v - u \quad \text{when} \quad u \leqslant v . \tag{136}$$

Moreover we see from the definition of φ that

$$\varphi(g_{t_n}) - \varphi(1) = \int_0^\infty G(dx) \frac{k(b(x)g_{t_n}(x), b(x))}{b(x)} - \beta\mu \int_0^\infty G(dx)(1 - g_{t_n}(x)). \tag{137}$$

We split the integrals in the right hand side of (137) into two domains: one in which $g_{t_n}(x) \leqslant 1$, and the other where $g_{t_n}(x) > 1$. Using (136) we obtain in the first case the upper bound:

$$(1 - \beta\mu) \,\|\, 1 - g_{t_n} \,\| \tag{138}$$

which tends to zero if assumption (*i*) holds. For the second case, we use (135) to get the upper bound

$$\int_{g_{t_n}(x)>1} G(dx)(g_{t_n}(x) - 1)\beta(x - \mu) . \tag{139}$$

By virtue of corollary 1, it is enough to prove that for every $\alpha > 0$

$$\int_{\substack{0 \\ g_{t_n}(x) > 1}}^{\alpha} G(dx)(g_{t_n}(x) - 1)\beta(x - \mu) \tag{140}$$

tends to zero (indeed the complementary integral can be made arbitrarily small uniformly in time). But (140) has the obvious upper bound

$$\beta(\alpha - \mu) \| g_{t_n} - 1 \| \tag{141}$$

so that the proof of the first part of the lemma is complete.

When $\rho \geqslant \rho_c$, one has $\mu = 0$ and the only additional information needed to prove the second part of the lemma is that

$$\int_0^{\delta} G(dx) \frac{k(b(x) g_{t_n}(x), b(x))}{b(x)} \tag{142}$$

can be made arbitrarily small (uniformly in n) for δ small enough. To prove this result, we use

$$k(u, v) < u \log(1 + 1/v) + \log(1 + v) \tag{143}$$

which implies for (142) the upper bound

$$\int_0^{\delta} G(dx) g_{t_n}(x) \beta x + \int_0^{\delta} G(dx) \frac{-\log(1 - \exp(-\beta x))}{b(x)} \tag{144}$$

$$\leqslant \beta\delta\rho + \int_0^{\delta} F(dx)[-\log(1 - \exp(-\beta x))] \tag{145}$$

which, in view of the form of $F(dx)$ (see (8)), can be made arbitrarily small by choosing δ small enough. ■

LEMMA 7. — *i*) If $\varphi(g_t) \to \varphi(1)$ as $t \to \infty$, then for every $\delta > 0$,

$$\int_{\delta}^{\infty} G(dx) \,|\, g_t(x) - 1 \,| \to 0 \quad \text{as} \quad t \to \infty .$$

ii) If in addition $\rho < \rho_c$, then

$$\int_0^{\infty} G(dx) \,|\, g_t(x) - 1 \,| \to 0 \quad \text{as} \quad t \to \infty .$$

Proof. — *i*) We prove separately that $\int_{\substack{\delta \\ g_t(x) < 1}}^{\infty} G(dx)(1 - g_t(x))$ and $\int_{\substack{\delta \\ g_t(x) > 1}}^{\infty} G(dx)(g_t(x) - 1)$ converge to zero.

Note that the second term in the right hand side of (137) is always non-negative; indeed this term is either zero (when $\mu=0$) or $-\beta\mu(\rho-\|g_t\|)\geqslant 0$ (when $\mu<0$), see Lemma 3. Using this fact and (134), (137) we have:

$$\varphi(g_t)-\varphi(1)\geqslant \int_{\substack{\delta\\ g_t(x)\leqslant 1}}^{\infty} G(dx)\frac{k(b(x)g_t(x),\, b(x))}{b(x)} \tag{146}$$

and

$$\varphi(g_t)-\varphi(1)\geq \int_{\substack{\delta\\ g_t(x)>1}}^{\infty} G(dx)\frac{k(b(x)g_t(x),\, b(x))}{b(x)}. \tag{147}$$

Now the function $k(u, v)$ (see (133)) can be shown to obey

$$k(u,v)\geqslant \frac{(u-v)^2}{2v(v+1)} \qquad \text{when} \quad u\leqslant v \tag{148}$$

so that (146) implies:

$$\varphi(g_t)-\varphi(1)\geqslant \frac{1}{2}\int_{\substack{\delta\\ g_t(x)\leqslant 1}}^{\infty} G(dx)\frac{(g_t(x)-1)^2}{1+b(x)}. \tag{149}$$

Moreover we have for $b(x)$ the bound (see (84))

$$b(x)\leqslant \alpha_1\equiv[\exp(\beta(\delta-\mu))-1]^{-1} \qquad \text{when} \quad x\geqslant\delta. \tag{150}$$

On the other hand we can use the Schwarz inequality to deduce

$$\int_{\substack{\delta\\ g_t(x)<1}}^{\infty} G(dx)(1-g_t(x))\leqslant \rho\int_{\substack{\delta\\ g_t(x)<1}}^{\infty} G(dx)(1-g_t(x))^2. \tag{151}$$

Taking (150) and (151) into account in (149), we have

$$\varphi(g_t)-\varphi(1)\geqslant[2\rho(1+\alpha_1)]^{-1}\int_{\substack{\delta\\ g_t(x)<1}}^{\infty} G(dx)(1-g_t(x)) \tag{152}$$

proving that if the left hand side of (152) converges to zero, so does the right hand side.

To deal with the other term, we use the following inequalities:

$$k(u,v)\geqslant\frac{(u-v)^2}{4v(1+v)} \qquad \text{when} \quad v\leqslant u\leqslant 2v \tag{153}$$

$$k(u,v)\geqslant\frac{u-v}{4(1+v)} \qquad \text{when} \quad u\geqslant 2v. \tag{154}$$

Putting these in (147) we have

$$\varphi(g_t)-\varphi(1)\geqslant \int_{\substack{\delta \\ 1\leqslant g_t(x)\leqslant 2}}^{\infty} G(dx)\frac{(g_t(x)-1)^2}{4(1+b(x))}+\int_{\substack{\delta \\ g_t(x)>2}}^{\infty} G(dx)\frac{g_t(x)-1}{4(1+b(x))} \tag{155}$$

$$\geqslant \alpha_2 \int_{\substack{\delta \\ 1\leqslant g_t(x)\leqslant 2}}^{\infty} G(dx)(g_t(x)-1)^2+\alpha_2\int_{\substack{\delta \\ g_t(x)>2}}^{\infty} G(dx)(g_t(x)-1) \tag{156}$$

with $\alpha_2 = [4(1+\alpha_1)]^{-1}$ (see (150)). But Schwarz's inequality implies

$$\int_{\substack{\delta \\ 1\leqslant g_t(x)\leqslant 2}}^{\infty} G(dx)(g_t(x)-1) \leqslant \rho \int_{\substack{\delta \\ 1\leqslant g_t(x)\leqslant 2}}^{\infty} G(dx)(g_t(x)-1)^2 \tag{157}$$

which together with (156) yields

$$\varphi(g_t)-\varphi(1)\geqslant \alpha_3 \int_{\substack{\delta \\ g_t(x)\geqslant 1}}^{\infty} G(dx)(g_t(x)-1) \tag{158}$$

with $\alpha_3 = \alpha_2 \min\{1, \rho^{-1}\}$. This completes the proof of part (*i*) of the lemma.

ii) Note that all the above statements hold for all values of the density, so that the only additional proof needed is that when $\rho < \rho_c$, δ can be taken to be zero. This is straightforward since then $\mu < 0$, so that α_1 remains finite when we put $\delta = 0$, see (150). ■

We are now in a position to formulate the main result of this section: the fixed point $g_{\text{eq.}}(x) = 1$ is a global attractor (in the appropriate technical sense) for eq. (24).

THEOREM 2. — Let the initial condition g_0 belong to $\mathcal{D}_\rho^2$, (see (101)). Then for every $\delta > 0$

$$\lim_{t\to\infty}\int_\delta^\infty G(dx)\,|\,g_t(x)-1\,| = 0\,. \tag{159}$$

If in addition $\rho \leqslant \rho_c$, then

$$\lim_{t\to\infty}\int_0^\infty G(dx)\,|\,g_t(x)-1\,| = 0\,. \tag{160}$$

Proof. — We only sketch the proof, because it is substantially the same as that of theorem 3 in [1]. Let g_t be the solution of (24) determined by the initial condition g_0, and $\{ m_i \}$ be a divergent sequence of distinct integers. Using proposition 4 we see that for every integer k

$$\varphi(1) - \varphi(g_0) \leqslant \varphi(g_{m_k}) - \varphi(g_0) \leqslant - \sum_{i=0}^{k-1} \int_{m_i}^{1+m_i} ds \Gamma_1(s) . \tag{161}$$

As the left hand side of (161) is finite, this implies that for some subsequence $\{ p_i \}$ of $\{ m_i \}$

$$\lim_{i \to \infty} \Gamma_1(t + p_i) = 0 \qquad \text{for a. e.} \quad t \text{ in } [0, 1] \tag{162}$$

so that by lemma 5 we have for every $b > 0$

$$\lim_{i \to \infty} \int_{\substack{0 \\ g_{t+p_i}(x) < 1}}^{b} G(dx)(1 - g_{t+p_i}(x)) = 0 \qquad \text{for a. e.} \quad t \text{ in } [0, 1] \tag{163}$$

and by corollary 1 this yields

$$\lim_{i \to \infty} \int_{\substack{0 \\ g_{t+p_i}(x) < 1}}^{\infty} G(dx)(1 - g_{t+p_i}(x)) = 0 \qquad \text{for a. e.} \quad t \text{ in } [0, 1] . \tag{164}$$

On the other hand we have when $\rho \leqslant \rho_c$ (see lemma 4 and formulas (15), (19))

$$\int_0^\infty G(dx) g_t(x) \leqslant \rho = \int_0^\infty G(dx) \tag{165}$$

which can be put in the form

$$\int_{\substack{0 \\ g_t(x) > 1}}^{\infty} G(dx)(g_t(x) - 1) \leqslant \int_{\substack{0 \\ g_t(x) < 1}}^{\infty} G(dx)(1 - g_t(x)) \tag{166}$$

Using lemmas 6 (*i*) and 7 (*ii*), we see that formulas (164), (166) yield the desired conclusion (160).

To complete the proof we have to show that when $\rho > \rho_c$, for every δ, $b > 0$ one can find a subsequence $\{ q_i \}$ of $\{ p_i \}$ such that

$$\lim_{i \to \infty} \int_{\substack{\delta \\ g_{t+q_i}(x) > 1}}^{b} G(dx)(g_{t+q_i}(x) - 1) = 0 . \tag{167}$$

Using (24) one can prove that for every δ, σ, $b > 0$

$$\begin{aligned} \rho &\geqslant \int_0^\sigma \mathrm{G}(dx) g_{t_2}(x) \\ &\geqslant \left[\int_0^\sigma \mathrm{G}(dx) g_{t_1}(x) \right] \exp \left\{ \int_{t_1}^{t_2} dt(-\mathrm{C}_1\mathrm{A}(\sigma) - \mathrm{C}_2\mathrm{S}(t) + \mathrm{C}_3\mathrm{T}(t)) \right\} \end{aligned} \tag{168}$$

where $t_1, t_2, (t_2 > t_1)$ are arbitrary times, $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3$ are positive constants and

$$\mathrm{S}(t) = \int_{\substack{0 \\ g_t(x) < 1}}^{b} \mathrm{G}(dx)(1 - g_t(x)) \tag{169}$$

$$\mathrm{T}(t) = \int_{\substack{\delta \\ g_t(x) > 1}}^{b} \mathrm{G}(dx)(g_t(x) - 1). \tag{170}$$

One can then use the facts that $\mathrm{S}(t + p_i)$ converges to zero (see (163)) and that $\mathrm{A}(\sigma)$ can be made arbitrarily small (for σ small enough), to prove that the uniform bound (168) implies that $\mathrm{T}(t + q_i)$ converges to zero for some subsequence $\{ q_i \}$ of $\{ p_i \}$. We refer to theorem 3 in [*1*] for further details. ■

The assumption of Theorem 2 guarantees also the convergence of the first moment; more generally we have, with $\mathscr{D}_\rho^n$ as in (101):

COROLLARY 2. — Let the initial condition g_0 belong to $\mathscr{D}_\rho^n$; then for all values of the density the moments of order $1, 2, \ldots, n - 1$ converge to their equilibrium value, namely:

$$\lim_{t \to \infty} \left| \int \mathrm{G}(dx) x^m (g_t(x) - 1) \right| = 0 \qquad m = 1, 2, \ldots, n - 1. \tag{171}$$

Proof. — For every δ, $b > 0$

$$\left| \int_\delta^b \mathrm{G}(dx) x^m (g_t(x) - 1) \right| \leqslant b^m \int_\delta^\infty \mathrm{G}(dx) \, | \, g_t(x) - 1 \, | \tag{172}$$

converges to zero, see (159). On the other hand the integrals over $[b, \infty)$ and $[0, \delta]$ can be made arbitrarily small if b is chosen large enough and δ small enough (using Corollary 1 and $m > 0$). ■

We conclude this section with some remarks.

i) The form of the functionals φ_b, φ (see (83)) was dictated by the fact that $\beta^{-1}\varphi$ is the expression representing the free energy density for the class of quasi-free states of the Bose gas [*16*] [*17*]. On the other hand, our generalized Liapunov functionals H_n have no direct physical interpretation.

ii) Corollary 2 can be improved in several ways. First it is clear that in order to obtain the convergence of the moment of order $n > 0$ it is enough to assume that the moment of order $n + \varepsilon$ is finite initially, for some $\varepsilon > 0$. Next, by choosing the function f in (92) in an appropriate manner, we can obtain the following convergence result (valid for any $\delta > 0$):

$$\lim_{t \to \infty} \left| \int G(dx) e^{\beta(1-\delta)x} (g_t(x) - 1) \right| = 0. \tag{173}$$

iii) We see from (160) that g_t converges to equilibrium in L^1-sense when $\rho \leqslant \rho_c$. When $\rho > \rho_c$ we have the somewhat weaker result (159), which is however optimal, see [*1*] section 3.2, remarks (*iii*) and (*iv*).

§ 5. THE LINEARIZED PROBLEM

In this section we obtain finer details on the process of approach to equilibrium. For this purpose, we linearize the kinetic equation (24) about its fixed point $g_{\text{eq.}}(x) = 1$. We obtain in this way the following equation for the deviation from equilibrium $h_t(x)$:

$$\begin{aligned} \frac{d}{dt} h_t(x) = & \int G(dy) D(x, y) [A(y) h_t(y) - A(x) h_t(x)] \\ & - h_t(x) \hat{C}(x)(e^{\beta x} - 1) \left[\rho - \int G(dy) \right] \\ & - \hat{C}(x) e^{\beta x} (e^{-\beta\mu} - 1) \int G(dy) h_t(y) \end{aligned} \tag{174}$$

where all integrals are on $(0, \infty)$.

As a first remark, we note that when $\rho \leqslant \rho_c$ the second term in (174) vanishes (see (19)) and that the equation of motion for $\int G(dx) h_t(x)$ can be solved explicitly. This gives us the linearized motion of the condensate density (see (23)):

$$\rho_t^0 = - \int G(dx) h_t(x) = \rho_0^0 e^{-\gamma t} \tag{175}$$

$$\gamma = (e^{-\beta\mu} - 1) \int G(dx) \hat{C}(x) e^{\beta x}. \tag{176}$$

The fact that ρ_t^0 tends to zero when $\rho < \rho_c$ is no surprise, but it is interesting to not that the condensate relaxation time γ^{-1} blows up when ρ approaches ρ_c (because then μ goes to zero, see (11), (12)). This is the germ of the phenomenon of *critical slowing down* that we shall discuss further in the sequel (see Theorem 5, Corollary 4 and remark (*ii*) at the end of the section).

As our subsequent investigations rely mainly on spectral analysis, we find it more convenient to use a Hilbert space framework. We introduce the complex Hilbert space

$$\mathcal{H} = \mathrm{L}^2(\mathbb{R}^+, \mathrm{M}(dx)) \tag{177}$$

where the finite measure $\mathrm{M}(dx)$ is defined by (see (8), (14), (17)):

$$\mathrm{M}(dx) = \mathrm{A}(x)\mathrm{G}(dx) = e^{-\beta(x-\mu)}\mathrm{F}(dx) = \frac{e^{-\beta(x-\mu)}x^{1/2}dx}{\sqrt{2}\pi^2} \tag{178}$$

The key object is the operator R defined formally by

$$(\mathrm{R}g)(x) = \int \mathrm{M}(dy)\mathrm{D}(x,y)\left[g(y) - \frac{\mathrm{A}(x)}{\mathrm{A}(y)}g(x)\right]. \tag{179}$$

A formal calculation gives

$$(f, \mathrm{R}g) = -\frac{1}{2}\int \mathrm{M}(dx)\int \mathrm{M}(dy)\mathrm{D}(x,y)\left[\left(\frac{\mathrm{A}(x)}{\mathrm{A}(y)}\right)^{1/2}\bar{f}(x) - \left(\frac{\mathrm{A}(y)}{\mathrm{A}(x)}\right)^{1/2}\bar{f}(y)\right]\left[\left(\frac{\mathrm{A}(x)}{\mathrm{A}(y)}\right)^{1/2}g(x) - \left(\frac{\mathrm{A}(y)}{\mathrm{A}(x)}\right)^{1/2}g(y)\right] \tag{180}$$

so that there is some hope to associate with (179) a negative self-adjoint operator.

Consider the two domains

$$\hat{\mathcal{D}} = \{ h \in \mathcal{H} : \operatorname{supp} h \subset [0, a] \quad \text{for some} \quad a < \infty \} \tag{181}$$

$$\tilde{\mathcal{D}} = \left\{ h \in \mathcal{H} : \int \mathrm{M}(dx)\int \mathrm{M}(dy)\mathrm{D}(x,y)\left|\left(\frac{\mathrm{A}(x)}{\mathrm{A}(y)}\right)^{1/2}h(x) - \left(\frac{\mathrm{A}(y)}{\mathrm{A}(x)}\right)^{1/2}h(y)\right|^2 < \infty \right\} \tag{182}$$

and define $\hat{\mathrm{Q}}$ and $\tilde{\mathrm{Q}}$ to be the quadratic forms determined by the right hand side of (180) on $\hat{\mathcal{D}}$ and $\tilde{\mathcal{D}}$ respectively. One can check that $\hat{\mathcal{D}} \subset \tilde{\mathcal{D}}$ and that $\hat{\mathcal{D}}$ is dense in $\mathcal{H}$. Moreover $\hat{\mathrm{Q}}$ is closable; in fact:

PROPOSITION 6. — The quadratic form $\tilde{\mathrm{Q}}$ is closed.

Proof. — Let $\{ f_n \}$ be a sequence of elements of $\tilde{\mathcal{D}}$ such that for some f in $\mathcal{H}$:

$$\| f_n - f \| \to 0 \quad \text{as} \quad n \to \infty \tag{183}$$

$$\tilde{\mathrm{Q}}(f_n - f_m, f_n - f_m) \to 0 \quad \text{as} \quad n, m \to \infty .- \tag{184}$$

We have to prove (see [*11*] section VIII.6)

$$f \in \tilde{\mathcal{D}} \tag{185}$$

$$\tilde{\mathrm{Q}}(f_n - f, f_n - f) \to 0 \quad \text{as} \quad n \to \infty . \tag{186}$$

If to every function $g : \mathbb{R}^+ \to \mathbb{C}$ we associate $\tilde{g} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ defined by

$$\tilde{g}(x, y) = g(x) - g(y) \tag{187}$$

and if we put

$$\tilde{\mathcal{H}} = L^2(\mathbb{R}^+ \times \mathbb{R}^+, D(x, y)M(dx)M(dy)) \tag{188}$$

we see that (184) reads

$$\| \tilde{f}_n - \tilde{f}_m \|_\sim \to 0 \quad \text{as} \quad n, m \to \infty \tag{189}$$

where $\| \cdot \|_\sim$ is the norm of $\tilde{\mathcal{H}}$. Hence there exists a function $h(x, y)$ such that

$$h \in \tilde{\mathcal{H}} \tag{190}$$

$$\| \tilde{f}_n - h \|_\sim \to 0 \quad \text{as} \quad n \to \infty . \tag{191}$$

Now, (183), (191) imply that for some subsequence $\{ f_{n_i} \}$ of $\{ f_n \}$ we have (see [*11*] Corollary in section 1.3):

$$f_{n_i}(x) \to f(x) \qquad \text{a. e. in } \mathbb{R}^+ \tag{192}$$

$$\tilde{f}_{n_i}(x, y) \to h(x, y) \qquad \text{a. e. in } \mathbb{R}^+ \times \mathbb{R}^+ . \tag{193}$$

This implies

$$h(x, y) = f(x) - f(y) \qquad \text{a. e. in } \mathbb{R}^+ \times \mathbb{R}^+ \tag{194}$$

so that the desired results (185), (186) are just reformulations of (190), (191). ■

Proposition 6 ensures that $\hat{Q}$ is closable, and that its closure Q is given by the right hand side of (180) on the whole of its domain; we do not have an explicit characterization of this domain, but we can prove the following:

LEMMA 8. — Let Q, (with domain $\mathcal{D}$), be the closure of $\hat{Q}$. Then the function $1/A(x)$ belongs to $\mathcal{D}$ when $\rho < \rho_c$.

Proof. — The restriction to $\rho < \rho_c$ comes from the fact that when $\mu = 0$ the function $1/A(x)$ does not belong to $\mathcal{H}$ (see (17), (178)).

Putting

$$\varphi_n(x) = \frac{\chi_{[0,n]}(x)}{A(x)} \tag{195}$$

$$\varphi(x) = 1/A(x) \tag{196}$$

we have

$$Q(\varphi_n, \varphi_n) = - \int_n^\infty M(dx) \int_0^n M(dy) \frac{D(x, y)}{A(x)A(y)} . \tag{197}$$

Using (3), (4), (17), (18) we can prove

$$0 \geqslant Q(\varphi_n, \varphi_n) \geqslant - n^{-1/2} \frac{C\sqrt{2}}{3\pi^2} \int_0^\infty M(dx)x^2 . \tag{198}$$

On the other hand

$$\| \varphi_n - \varphi \| \to 0 \quad \text{as} \quad n \to \infty \tag{199}$$

so that (198), (199) imply

$$\| \varphi_n - \varphi_m \| - Q(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \to 0 \quad \text{as} \quad n, m \to \infty . \tag{200}$$

But since Q is closed this implies that there exists Ψ in $\mathcal{Q}$ such that

$$\| \varphi_n - \Psi \| - Q(\varphi_n - \Psi, \varphi_n - \Psi) \to 0 \quad \text{as} \quad n \to \infty . \tag{201}$$

As each term in (201) must go to zero separately, we see by (199) that $\Psi = \varphi$. ■

The closed negative quadratic form Q defines a unique self-adjoint negative operator R with domain $\mathcal{R} \subset \mathcal{Q}$ (see [*11*] Theorem VIII.15) which is thus the generator of a contraction semi-group. Some properties of R are easily extracted from those of Q:

COROLLARY 3. — Let R (with domain $\mathcal{R}$), be the self-adjoint operator defined by Q. Then:

i) every element of $\mathcal{H}$ which has compact support belongs to $\mathcal{R}$.
ii) when $\rho < \rho_c$ the function $\varphi(x) = 1/A(x)$ belongs to $\mathcal{R}$ and is a zero-eigenvector of R.
iii) the operator R is given by formula (179) on the whole of $\mathcal{R}$.

Proof. — The domain $\mathcal{R}$ is related to $\mathcal{Q}$ by (see formula (5.23) in [*18*])

$$\mathcal{R} = \{ h \in \mathcal{Q} : \exists h' \in \mathcal{H} : Q(g, h) = (g, h') \quad \forall g \in \hat{\mathcal{Q}} \} \tag{202}$$

and one has, with the same notation

$$Rh = h' . \tag{203}$$

All three points of the corollary follow from (202), (203) because, when g has compact support, the exchange of integrals needed to go from (179) to (180) is justified by Fubini's theorem. ■

Now that the self-adjoint negative operators R is defined in an unambiguous manner, we can proceed and obtain more details on its spectral properties. It turns out that the nature of the spectrum of R in the neighbourhood of the origin changes drastically at the critical point:

THEOREM 3. — *i*) When $\rho < \rho_c$, zero is a non-degenerate eigenvalue and an isolated point of the spectrum of R.
ii) When $\rho \geqslant \rho_c$, zero is not an eigenvalue of R, but it belongs to its essential spectrum. In fact, the spectrum of R consists of the whole of $\mathbb{R}^-$.

Proof. — *i*) The fact that zero is an eigenvalue of R was proved in Corollary 3 (*ii*). The non-degeneracy is immediate because, since $\mathscr{R} \subset \mathscr{Q} \subset \tilde{\mathscr{Q}}$

$$(h, \mathrm{R}h) = -\frac{1}{2}\int \mathrm{M}(dx)\int \mathrm{M}(dy)\mathrm{D}(x, y)\left|\left(\frac{\mathrm{A}(x)}{\mathrm{A}(y)}\right)^{1/2} h(x) - \left(\frac{\mathrm{A}(y)}{\mathrm{A}(x)}\right)^{1/2} h(y)\right|^2 \quad \text{for all} \quad h \text{ in } \mathscr{R}. \tag{204}$$

Next we prove that zero does not belong to the essential spectrum of R. In preparation for this, we note that R is formally the difference of the integral operator with kernel $\mathrm{D}(x, y)$ and of the multiplication operator by the non-negative continuous function (see (179)):

$$m(x) = \int \mathrm{M}(dy)\mathrm{D}(x, y)\frac{\mathrm{A}(x)}{\mathrm{A}(y)}. \tag{205}$$

When $\rho \geqslant \rho_c$, we have $\mu = 0$ and thus $m(0) = \mathrm{A}(0) = 0$ (see (17)), whereas when $\rho < \rho_c$ the function $m(x)$ is strictly positive with ($\mu < 0$):

$$m(x) \geqslant \frac{1 - e^{\beta\mu}}{\sqrt{2}\pi^2}\int dz z^{1/2}\hat{\mathrm{C}}(z) \equiv \Delta. \tag{206}$$

The following comparison function plays an important role in our argument:

$$s(x) = \int \mathrm{M}(dy)\mathrm{D}(x, y)\exp\left[\beta(y - x)/2\right]. \tag{207}$$

One can show that $m(x)$ is ultimately larger than $s(x)$; more precisely, there exists $b < \infty$ such that, with Δ as in (206)

$$m(x) - s(x) \geqslant \Delta \qquad \text{when} \quad x \geqslant b. \tag{208}$$

This is because

$$\lim_{x\to\infty} x^{-1/2}s(x) = \frac{\sqrt{2}}{\pi^2}\int dz\hat{\mathrm{C}}(z)\exp(\beta z/2) \tag{209}$$

whereas

$$\lim_{x\to\infty} x^{-1/2}m(x) = \frac{\sqrt{2}}{\pi^2}\int dz\hat{\mathrm{C}}(z)\cosh(\beta z/2)\exp(\beta z/2) \tag{210}$$

and the right hand side of (210), (which may be infinite), is largen than that of (209).

Next we introduce the operator K defined by

$$(\mathrm{K}h)(x) = \int \mathrm{M}(dy)\mathrm{D}(x, y)(1 - \chi(x)\chi(y)h(y) \tag{211}$$

where

$$\chi(z) = \chi_{[b,\infty)}(z) \tag{212}$$

with b as in (208). This is a Hilbert-Schmidt operator with

$$\|\mathrm{K}\|_{\mathrm{H.S.}} \leqslant \sqrt{2}\mathrm{C}\exp\left[\beta(b - \mu)\right]\int \mathrm{M}(dx) \tag{213}$$

where C is as in (4), (5). Moreover we have, for every h in $\hat{\mathscr{Q}}$ (see (174)):

$$(h, (\mathrm{R}-\mathrm{K})h)$$

$$= \int \mathrm{M}(dx) \int \mathrm{M}(dy)\mathrm{D}(x, y)\chi(x)\chi(y)\bar{h}(x)h(y) - \int \mathrm{M}(dx)m(x) \mid h(x) \mid^2 \quad (214)$$

$$= -\frac{1}{2} \int \mathrm{M}(dx) \int \mathrm{M}(dy)\mathrm{D}(x, y) \exp\left[\beta(x+y)/2\right] \mid \chi(x)h(x) \exp(-\beta x/2) - \chi(y)h(y) \exp(-\beta y/2) \mid^2 - \int \mathrm{M}(dx)\left[m(x) - s(x)\chi(x)\right] \mid h(x) \mid^2 \quad (215)$$

$$\leqslant -\Delta \| h \|^2 . \quad (216)$$

As $\hat{\mathscr{Q}}$ is a core for Q, this proves that the spectrum of the self-adjoint operator R $-$ K is located on the left hand side of $-\Delta$. In particular, since $\Delta > 0$, zero does not belong to the spectrum of R $-$ K. But since K is compact, Weyl's theorem implies that zero cannot belong to the essential spectrum of R (see [*10*]).

ii) The fact that zero is not an eigenvalue of R follows again from (204), and from the fact that $1/\mathrm{A}(x)$ does not belong to $\mathscr{H}$ when $\rho \geqslant \rho_c$ (see (17), (178)). The final statement is a special case of our proposition 7, see below. ■

We now use theorem 3 to discuss the behaviour of the solution of the linearized kinetic equation. We treat first the regime $\rho < \rho_c$; in that case (174) becomes

$$\frac{d}{dt} h_t = \mathrm{R}h_t - (k, h_t)\xi \quad (217)$$

where we have set

$$k(x) = \left[\mathrm{A}(x) \| 1/\mathrm{A} \| \right]^{-1} \quad (218)$$

$$\xi(x) = \| 1/\mathrm{A} \| \hat{\mathrm{C}}(x)e^{\beta x}(e^{-\beta\mu} - 1) . \quad (219)$$

The corresponding evolution has the exponential decay property:

THEOREM 4. — There exists a number $\kappa > 0$ such that for every initial condition in $\mathscr{R}$ the solution of (217) obeys

$$\| h_t \| \leqslant e^{-\kappa t}\mathrm{K}(h_0) \quad (220)$$

where $\mathrm{K}(h_0)$ depends only on the initial condition h_0.

Proof. — *a*) We discuss first the case where the initial condition h_0 is orthogonal to k; this property corresponds physically to the absence of condensate, and it is preserved in time because (175), (176) read

$$(k, h_t) = (k, h_0)e^{-\gamma t} \quad (221)$$

$$\gamma = (k, \xi) . \quad (222)$$

Moreover, it follows from theorem 3 (*i*) that there exists $\delta > 0$ with

$$(g, \mathrm{R}g) \leqslant -\delta \| g \|^2 \qquad \text{when} \quad g \in \mathscr{R}, (k, g) = 0. \tag{223}$$

This implies

$$\| e^{t\mathrm{R}} h_0 \| \leqslant e^{-\delta t} \| h_0 \| \qquad \text{if} \quad h_0 \in \mathscr{R}, (k, h_0) = 0. \tag{224}$$

b) For the general case, we note that using (221) we can write (217) as

$$\frac{d}{dt} h_t = \mathrm{R} h_t - e^{-\gamma t}(k, h_0)\xi \tag{225}$$

which has for solution

$$h_t = e^{t\mathrm{R}} h_0 - (k, h_0) \int_0^t ds e^{(t-s)\mathrm{R}} e^{-\gamma s} \xi. \tag{226}$$

But, because k is a fixed point of $e^{t\mathrm{R}}$, this can be put in the form

$$h_t = e^{t\mathrm{R}} [h_0 - (k, h_0)k] + e^{-\gamma t}(k, h_0)k - (k, h_0) \int_0^t ds e^{(t-s)\mathrm{R}} e^{-\gamma s}(\xi - \gamma k). \tag{227}$$

Now, because of (222), we see that in (227) both functions on which the semi-group $e^{t\mathrm{R}}$ acts are orthogonal to k, so that we can use (224) to get:

$$\begin{aligned} \| h_t \| \leqslant e^{-\delta t} \| h_0 - (k, h_0)k \| + e^{-\gamma t} | (k, h_0) | \\ + | (k, h_0) | \frac{e^{-\gamma t} - e^{-\delta t}}{\delta - \gamma} \| \xi - (k, \xi)k \|. \end{aligned} \tag{228}$$

This gives the stated result (220) if we put

$$\kappa = \min (\delta, \gamma). \tag{229}$$

■

Before we treat the regime $\rho \geqslant \rho_c$, we note an interesting property of the class of operators of the form (179): even though their integral part is not compact (not even bounded), it cannot decrease the essential spectrum of the multiplicative part. More precisely we have:

PROPOSITION 7. — Let $m(x)$ be as in (205) and $u(x)$ be a bounded real continuous function; define

$$p(x) = u(x) - m(x) \tag{230}$$

$$(\mathrm{U}h)(x) = u(x)h(x) \qquad \text{for every} \quad h \text{ in } \mathscr{H}. \tag{231}$$

Then every point in the range of the function $p(x)$ belongs to the essential spectrum of the self-adjoint operator R + U, where R is as in Corollary 3.

Proof. — We begin with two simple remarks.

i) The set of functions in $\mathscr{H}$ with support in $[a, a + \varepsilon]$ can be identified with the Hilbert space

$$\mathscr{H}_{a,\varepsilon} = \mathrm{L}^2([a, a + \varepsilon], \mathrm{M}(dx)). \tag{232}$$

ii) The operators S and P defined by

$$(\mathrm{S}h)(x) = \int \mathrm{M}(dy)\mathrm{D}(x, y)h(y) \tag{233}$$

$$(\mathrm{P}h)(x) = p(x)h(x) \tag{234}$$

are bounded when considered as operators from $\mathscr{H}_{a,\varepsilon}$ to $\mathscr{H}$.

In fact S is even compact, because it satisfies the Hilbert-Schmidt condition with

$$\| \mathrm{S} \|_{\mathrm{H.S.}} \leqslant \mathrm{C} \exp [\beta(a + \varepsilon - \mu)] \int_0^\infty \mathrm{M}(dx). \tag{235}$$

Now let $p(a)$ be a point in the range of the continuous function $p(x)$. One can construct a sequence $\{ f_n \}$ such that:

$$f_n \in \mathscr{H}_{a,\varepsilon} \tag{236}$$

$$\| f_n \| = 1 \tag{237}$$

$$(\mathrm{P} - p(a)) f_n \xrightarrow{s} 0 \tag{238}$$

$$f_n \xrightarrow{w} 0. \tag{239}$$

But because of (236) we have

$$(\mathrm{R} + \mathrm{U} - p(a)) f_n = (\mathrm{S} + \mathrm{P} - p(a)) f_n \tag{240}$$

and, since S is compact, we obtain from (238), (239), (240):

$$(\mathrm{R} + \mathrm{U} - p(a) f_n \xrightarrow{s} 0. \tag{241}$$

Gathering (237), (239), (241), we see that Weyl's criterion for $p(a)$ to be in the essential spectrum of the self-adjoint operator R + U is satisfied, [*18*].

■

We turn finally to the regime $\rho \geqslant \rho_c$. In that case, $\mu = 0$ and the linearized kinetic equation (174) reads:

$$\frac{d}{dt} h_t = (\mathrm{R} - \mathrm{N}) h_t \tag{242}$$

where N is the operator of multiplication by the bounded non-negative function

$$n(x) = \hat{\mathrm{C}}(x)(e^{\beta x} - 1)\left(\rho - \int \frac{\mathrm{M}(dy)}{\mathrm{A}(y)} \right). \tag{243}$$

The operator R − N, defined on $\mathscr{R}$, is again self-adjoint and negative and generates accordingly a contraction semi-group. But in contrast with the low density regime, there is no exponential relaxation in this case (see statement (*ii*) below).

THEOREM 5. — *i*) For every element h of $\mathscr{R}$

$$\lim_{t\to\infty} \| e^{t(\mathrm{R}-\mathrm{N})}h \| = 0 . \tag{244}$$

ii) For every $\lambda > 0$, there exists an infinite dimensional subspace $\mathscr{C}_\lambda$ of $\mathscr{R}$ such that

$$\lim_{t\to\infty} e^{\lambda t} \| e^{t(\mathrm{R}-\mathrm{N})}g \| = \infty \qquad \text{for all} \quad g \text{ in } \mathscr{C}_\lambda . \tag{245}$$

Proof. — *i*) Let

$$-(\mathrm{R}-\mathrm{N}) = \int_0^\infty x\mathrm{E}(dx) \tag{246}$$

be the spectral resolution of the self-adjoint positive operator $-(\mathrm{R}-\mathrm{N})$. We have for all h in $\mathscr{R}$:

$$\| e^{t(\mathrm{R}-\mathrm{N})}h \|^2 = \int_0^\infty e^{-2xt}(h, \mathrm{E}(dx)h) . \tag{247}$$

On the other hand, since $-\mathrm{N}$ is negative, it follows again from (204) that zero is not an eigenvalue of $\mathrm{R}-\mathrm{N}$. Hence the measure $(h, \mathrm{E}(dx)h)$ does not have an atom at the origin, and (244) follows from (247) by virtue of the dominated convergence theorem.

ii) To prove (245), we note that since $\rho \geqslant \rho_c$ we have $\mu = 0$, and thus $m(0) = n(0) = 0$ (see (205), (17), (243)). Consequently, zero belongs to the essential spectrum of $\mathrm{R}-\mathrm{N}$ (see Proposition 7). Thus for every $\lambda > 0$, the subspace

$$\mathscr{C}_\lambda = \mathrm{E}(\lambda/2)\mathscr{R} \tag{248}$$

is infinite dimensional. Moreover, for every g in $\mathscr{C}_\lambda$

$$\| e^{\lambda t}e^{t(\mathrm{R}-\mathrm{N})}g \|^2 = \int_0^\infty e^{2(\lambda-x)t}(g, \mathrm{E}(dx)g) \tag{249}$$

$$\geqslant \int_0^{\lambda/2} e^{2(\lambda-x)t}(g, \mathrm{E}(dx)g) . \tag{250}$$

$$\geqslant e^{\lambda t} \| g \|^2 . \tag{251}$$

■

A comparison between the results of theorems 4 and 5 reveals the qualitative difference between the processes of approach to equilibrium below and above the critical density. In order to understand the transition between the two regimes, it is interesting to know the rate at which the relaxation time κ^{-1} blows up as $\rho \to \rho_c-$ (or $\mu \to 0-$). The following result provides a lower bound on the relaxation time (see also remark (*ii*) below).

COROLLARY 4. — For every number λ such that

$$\lambda > m(0) = (1 - e^{\beta\mu}) \int M(dy) \frac{D(0, y)}{A(y)} \tag{252}$$

there exists infinitely many linearly independent initial conditions in $\mathcal{R}$ such that the corresponding solution h_t of (217) obeys

$$\lim_{t \to \infty} e^{\lambda t} \| h_t \| = \infty . \tag{253}$$

Proof. — If we choose an initial condition orthogonal to k, we see by (221) that the equation (217) reduces to

$$\frac{d}{dt} h_t = R h_t . \tag{254}$$

The result follows then, as in Theorem 5, from the fact that $m(0)$ belongs to the essential spectrum of R (see Proposition 7). ■

We conclude this section with some remarks.

i) When $\rho = \rho_c$, μ vanishes so that (175), (176) give $\rho_t^0 = \rho_0^0 =$ constant, in contradistinction with the result of the exact non-linear theory (160).

ii) There is an extensive literature dealing with the dynamical properties of physical systems near the critical region (see [*19*] [*20*] for reviews); the approach is generally of phenomenological nature, and the connection with first principles rather loose. In particular, the dynamics is obtained by an *ad hoc* method (the time-dependent Ginzburg-Landau theory) rather than derived from the underlying Hamiltonian formalism. One of the results of the theory is that the relaxation time of the system should blow up at the critical point (this is called the phenomenon of *critical slowing down*); moreover the rate at which it blows up as $T \to T_c$ (the so-called *dynamical critical exponent*) is predicted by these methods. The result of the time-dependent Ginzburg-Landau theory, (as it can be extracted from [*20*]), is the following: the rate of divergence of the relaxation time of the order parameter coincides with that of the static susceptibility.

In our model, the result of non-exponential relaxation in the two-phase region is obtained on a purely microscopic basis. We also have an explicit expression for the relaxation time γ^{-1} of the condensate density (see (176)) and a lower bound for the global relaxation time κ^{-1} (see (252)). If we specialize these to the region $\rho \to \rho_c -$ (i. e. $\mu \to 0 -$) we obtain

$$\gamma^{-1} = O(|\mu|^{-1}) \quad \text{as} \quad \rho \to \rho_c - \tag{255}$$

$$\kappa^{-1} \geqslant (m(0))^{-1} = O(|\mu|^{-1}) \quad \text{as} \quad \rho \to \rho_c - \tag{256}$$

Moreover, it is not difficult to see that if we fix the density and use the temperature as a variable we have

$$\mu = O((T - T_c)^2) \quad \text{as} \quad T \to T_c + \tag{257}$$

so that the right hand sides of (255), (256) behave like $O((T - T_c)^{-2})$ as $T \to T_c +$. This supports the Ginzburg-Landau prediction because the « susceptibility » of the free Bose gas is (see [21])

$$\chi(\beta, \mu) = (\beta \mid \mu \mid)^{-1} = O((T - T_c)^{-2}) \quad \text{as} \quad T \to T_c + . \tag{258}$$

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REFERENCES

[1] E. Buffet, Ph. de Smedt and J. V. Pule, On the dynamics of the open Bose gas. *Ann. Phys. (N. Y.)*, t. **155**, 1984, p. 269.
[2] L. Arkeryd, On the Boltzmann equation ; Part I: existence. *Arch. Rat. Mechs. Anal.*, t. **45**, 1972, p. 1.
[3] L. Arkeryd, On the Boltzmann equation ; Part II: the full initial value problem. *Arch. Rat. Mechs. Anal.*, t. **45**, 1972, p. 17.
[4] M. Aizenman and T. A. Bak, Convergence to equilibrium in a system of reacting polymers. *Comm. Math. Phys.*, t. **65**, 1979, p. 203.
[5] W. G. Sullivan, The L^2-spectral gap of certain positive recurrent Markov chains and jump processes. *Z. Warscheinlichkeitstheorie* (in press).
[6] R. Alicki and J. Messer, Nonlinear quantum dynamical semi-groups for many-body open systems. *J. Stat. Phys.*, t. **32**, 1983, p. 299.
[7] E. B. Davies, Markovian master equations. *Comm. Math. Phys.*, t. **39**, 1974, p. 91.
[8] E. B. Davies, Markovian master equations II. *Math. Ann.*, t. **219**, 1976, p. 147.
[9] E. Buffet and J. V. Pulè, Fluctuation properties of the imperfect Bose gas. *J. Math. Phys.*, t. **24**, 1983, p. 1608.
[10] M. Reed and B. Simon, *Analysis of operators*, Methods of modern mathematical physics, Vol. 4, Academic Press, New York and London, 1978.
[11] M. Reed and B. Simon, *Functional analysis*, Methods of modern mathematical physics, Vol. 1, Academic Press, New York and London 1972.
[12] E. B. Davies, *One-parameter semigroups*, London Mathematical Society Monographs, Vol. 15, Academic Press, London and New York, 1980.
[13] R. H. Martin, *Nonlinear operators and differential equations in Banach spaces.* John Wiley, New York and London, 1976.
[14] Ph. de Smedt, *Ph. D. thesis*, Leuven University, 1983.
[15] W. Feller, *An introduction to probability theory and its applications*, Vol. 2. John Wiley, New York, 1971.
[16] M. Fannes, The entropy density of quasi-free states for a continuous boson system. *Ann. Inst. Henri Poincaré*, t. **A 28**, 1978, p. 187.
[17] R. H. Critchley and J. T. Lewis, *The entropy density of quasi-free states.* DIAS, TP, 75, 22, unpublished.

[*18*] J. WEIDMANN, *Linear operators in Hilbert spaces*. Graduate texts in mathematics. Vol. 68. Springer Verlag, New York and Berlin, 1980.
[*19*] P. C. HOHENBERG and B. I. HALPERIN, Theory of dynamic critical phenomena. *Rev. Mod. Phys.*, t. **49**, 1977, p. 435.
[*20*] M. LUBAN, Generalized Landau theories, in *Phase transitions and critical phenomena*, Vol. 5A. C. Domb and M. S. Green editors. Academic Press, London and New York, 1976.
[*21*] J. D. GUNTON and M. J. BUCKINGHAM, Condensation of the ideal Bose gas as a co-operative transition. *Phys. Rev.*, t. **166**, 1968, p. 152.

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