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Normal modes of a Lagrangian system constrained in a potential well

by

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ABSTRACT. — Let a, $U \in C^2(\Omega)$ where Ω is a bounded set in $\mathbb{R}^n$ and let

$$(*) \qquad L(x, \xi) = \frac{1}{2} a(x) | \xi |^2 - U(x), \qquad x \in \Omega ; \ \xi \in \mathbb{R}^n .$$

We suppose that a, $U > 0$ for $x \in \Omega$ and that

$$\lim_{x \to \partial\Omega} U(x) = + \infty .$$

Under some smoothness assumptions, we prove that the Lagrangian system associated with the above Lagrangian L has infinitely many periodic solutions of any period T.

Key-words: Lagrangian system, periodic solutions, minimax principle, Palais-Smale condition.

RÉSUMÉ. — Soit a, $U \in C^2(\Omega)$ où Ω est un borné de $\mathbb{R}^n$, on pose

$$(*) \qquad L(x, \xi) = \frac{1}{2} a(x) | \xi |^2 - U(x), \qquad x \in \Omega ; \ \xi \in \mathbb{R}^n .$$

Nous supposons que a, $U > 0$ pour $x \in \Omega$ et que

$$\lim_{x \to \partial\Omega} U(x) = + \infty .$$

Moyennant quelques hypothèses de différentiabilité, nous démontrons que le système Lagrangien associé à L a une infinité de solutions T-périodiques, quel que soit T.

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1. INTRODUCTION AND MAIN RESULTS

Let a, U $\to$ R where Ω is an open set in R^n. We make the following assumption

(L_0) Ω is bounded and its boundary is C^2.
(L_1) $U \in C^2(\Omega)$
(L_2) $\lim_{x \to \partial\Omega} U(x) = +\infty$
(L_3) $\lim_{x \to \partial\Omega} \frac{\nabla U(x) \cdot \nu(x)}{U(x)} = +\infty$ where $\nu(x) = -\nabla \operatorname{dist}(x, \partial\Omega)$
(L_4) $a \in C^2(\overline{\Omega})$
(L_5) $a(x) > 0$ for every $x \in \Omega$
(L_6) for every $x \in \partial\Omega$ such that $a(x) = 0$, $\nabla a(x) \neq 0$.

We consider the Lagrangian

$$(1.1)\quad L(x, \xi) = \frac{1}{2} a(x) |\xi|^2 - U(x),$$
$x \in \Omega$, $\xi \in T_x\Omega = R^n$, and $|\cdot|$ denotes the norm in R^n,

and we look for normal modes of the dynamical system associated to this Lagrangian; i. e. periodic solutions of the following systems of ordinary differential equations :

$$(1.2)\quad \begin{cases} \gamma \in C^2(R, \Omega) \\ a(\gamma)\ddot{\gamma} = \frac{1}{2} |\dot{\gamma}|^2 \nabla a(\gamma) - (\nabla a(\gamma) \cdot \dot{\gamma})\dot{\gamma} - \nabla U(\gamma) \end{cases}$$

where « $\cdot$ » denotes $\frac{d}{dt}$ and « $\cdot$ » denoted the dot product in R^n. We restrict our attention to periodic solution of a given period T, and in order to simplify the notation we suppose T = 1. Also it is not restrictive to suppose that

(L_7) $\quad U(x) \geqslant 0 \quad$ for $\quad x \in \Omega, \quad \min_{x \in \Omega} U(x) = 0 \quad$ and $\quad a(x) \leqslant 1$.

The main result of this paper is the following theorem:

THEOREM 1.1. — *If* (L_0)-(L_7) *hold then the equation* (1.2) *has infinitely many periodic distinct solutions of period* 1. *More exactly there exists a positive integer* N_0 *and two positive constants* E^+ *and* E^- *such that for every* $N \geqslant N_0$ *there exists* $\gamma_N \in C^2(R, \Omega)$ *such that*

i) γ_N *is solution of* (1.2)

ii) γ_N *has period* $\frac{1}{N}$

iii) $E^- N^2 \leqslant E(\gamma_N) \leqslant E^+ N^2$

where $E(\gamma) = \frac{1}{2} a(\dot\gamma(t)) \,|\, \gamma(t) \,|^2 + U(\gamma(t))$ *is the « energy » of* γ.

iv) $\alpha N^2 \leqslant J(\gamma_N) \leqslant \beta N^2$

where $J(\gamma_N) = \int \alpha(\gamma_N) \,|\, \dot\gamma_N \,|^2 - U(\gamma_N) dt$ *and* α *and* β *are constant which depend only on* Ω *(but not on* U *and* N*). Moreover if* $U(x_M) = 0$ *(i. e.* x_M *is a minimum point) and*

$$U(x) = o(\,|\, x - x_M \,|^2) \quad \textit{for} \quad x \to x_M,$$

then we can choose $N_0 = 1$.

Remarks I. — Notice that (*ii*) does not say that $\frac{1}{N}$ is the minimal period of γ_N. It might happen that γ_N has a smaller period. Thus it may happen that $\gamma_N = \gamma_M$ for some $M \neq N$. However (*iii*) implies that if $M \gg N$ then $\gamma_M \neq \gamma_N$.

II. — As easy one-dimensional examples show it is possible that equation 1.2 has no periodic solution with minimal period 1.

III. — Assumption (L_3) which may appear as the less natural one, describes the behaviour of $U(x)$ as $x \to \partial\Omega$. It says that $U(x)$ cannot « oscillate » too badly near the boundary.

IV. — We have decided to consider Lagrangian of the form (1.1) (i. e. with $a(x)$ not identically 1 and in particular with $a(x)$ which may degenerate on $\partial\Omega$) because in this way theorem 1.1 can be applied to the study of closed geodesic for the Jacobi metric (which degenerates for $x \to \partial\Omega)_i$ cf. [B_2].

V. — By the proof of the theorem it will be clear that the same result hold for a Lagrangian of the form

$$L(x, \xi) = \Sigma_{ij} a_{ij}(x) \xi_i \xi_j - U(x)$$

with $\Sigma_{ij} a_{ij}(x) \xi_i \xi_j \geqslant a(x) \,|\, \xi \,|^2$ and satisfies (L_4-L_6). More in general, the same method apply also to the case in which Ω is a Riemann manifold with a C^2-boundary. We have decided to consider a simpler case in order to not make the notation and the uninteresting technicalities too heavy.

VI. — The results of theorem 1.1 holds also if a and U are of class C^1 (cf. remark II after theorem 2.3). However, in order to not get involved in technicalities which will obscure the main ideas we have preferred to treat the C^2-case.

The study of normal modes of nonlinear Hamiltonian or Lagrangian systems is an old problem which in the last years has attracted new interest. We refer to [R_1] for recent references on this subject. However, as far as I know there are no results of the nature of theorem 1.1, i. e. periodic solutions in a potential well. The more similar situations to the one considered in this paper are the following ones.

a) Ω is a compact manifold without boundary

b) $\Omega = R^n$ but U grows more than quadratically for $|x| \to +\infty$ or, more precisely

$$0 < U(x) \leqslant \theta U(x) \cdot x \quad \text{for} \quad x \text{ large} \quad \text{and} \quad \vartheta < \frac{1}{2}$$

(notice that the above condition is the analogous of (L_3) when $\Omega = R^n$).

In both cases (*a*) and (*b*) we have a result similar to theorem 1.1 (cf. [B_2] for (*a*); [R_4], [BF] or [G] for (*b*); also the case (*b*) has been considered in [R_2], [BR] and [BCF] in the context of Hamiltonian systems).

What we want to remark here is the similarity of these three situations. In case (*a*), the existence of infinitely many periodic orbits can be proved by virtue of the compactness of Ω (provided that Ω satisfy some suitable geometric assumption as having the fundamental group finite). In (*b*) and in theorem 1.1, the lack compactness is replaced by the growth of U.

A last remark about the technique used to prove theorem 1.1. We have used variational arguments reducing our problem to the proof of existence of critical points of a functional defined on an open set in a Hilbert space. In proving the existence of critical points for functional in infinite dimensional manifold the well known condition (*c*) of Palais and Smale (P. S.) has been used. However in our situation (since we deal with a non-closed manifold) (P.S.) is not sufficient. For this reason we have used a variant of (P.S.), which fit our case, obtaining an abstract theorem (theorem 2.3) which might have some interest in itself as a further step in understanding the critical point theory in infinite dimensional manifolds.

2. AN ABSTRACT THEOREM

Let X be a Hilbert space with norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$ and let Λ be an open set in X (or more in general a Riemannian manifold embedded in X). $C^n(\Lambda, R)$ will denote the set of *n*-times Frechét differentiable functions from X to R.

If $f \in C^n(\Lambda, R)$, f' will denote its Frechét derivative which can be identified, by virtue of $\langle \cdot, \cdot \rangle$, with a function from Λ to X.

DEFINITION 2.1. — A function $\rho : \Lambda \to X$ is called a weight function for Λ if it satisfies the following assumptions:

i) $\rho \in C^1(\Lambda, R)$

ii) $\rho(x) > 0$ for every $x \in \Lambda$
iii) if $x_n \to \bar{x} \in \partial\Omega$ then $\rho(x_n) \to +\infty$ for $n \to +\infty$.

DEFINITION 2.2. — We say that a functional $J \in C^1(\Lambda, R)$ satisfies the weighted Palais-Smale condition (abbreviated W. P. S.) if there exists a weight function ρ such that given any sequence $x_n \in \Lambda$ the following happens:
(WPS 1) if $\rho(x_n)$ and $J(x_n)$ are bounded and $J'(x_n) \to 0$ then x_n has a subsequence converging to $\bar{x} \in \Lambda$
(WPS 2) if $J(x_n)$ is convergent and $\rho(x_n) \to +\infty$, then there exists $v > 0$ such that

$$\| J'(x_n) \| \geqslant v \, \| \rho'(x_n) \| \quad \text{for} \quad n \text{ large enough}.$$

Remarks I. — We say that a functional satisfy the Palais-Smale assumption on a Hilbert (or Banach) manifold Λ if every sequence x_n such that $J(x_n)$ is bounded and $J'(x_n) \to 0$ has a converging subsequence. Most results in critical point theory have been obtained using the (P. S.) assumption. However, as easy examples show (P. S.) is not sufficient to obtain existence results when Λ is an open set in a Hilbert space, or to be more precise, when Λ is not complete with respect to the Riemannian structure which we want to use.

II. — If Λ is a closed Hilbert (or Banach) manifold then (P. S.) implies (W. P. S.) (it is enough to take $\rho \equiv 1$). Moreover if $\Lambda = X$, choosing $\rho(x) = \log(1 + \| x \|^2)$, then (W. P. S.) reduces to a generalization of (P. S.) introduced by G. Cerami [C] (cf. also [B. B. F.] and [B. C. F.]).

III. — If a functional J satisfy (P. S.) then the set

$$K_c = \{ \gamma \in \Lambda \mid J(\gamma) = c, \, J'(\gamma) = 0 \}$$

is compact. If J satisfies (W. P. S.) we can only conclude that

$$K_c \cap \{ \gamma \in \Lambda \mid \rho(\gamma) \leqslant M \}$$

is compact for every $M > 0$. Thus (W. P. S.) might be an useful tool for analyzing situations in which we do not expect to find a compact set of critical points at a given value c. (However if $\rho'(x) \neq 0$ when $\rho(x)$ is large, then K_c is compact).

DEFINITION 2.2′. — Let X be an Hilbert (or Banach) space. Let S be a closed set in X, and let Q be an Hilbert manifold with boundary ∂Q. We say that S and ∂Q link if

a) $S \cap \partial Q = \phi$
b) if $h : \bar{Q} \to \Lambda$ is a continuous map such that $h(u) = u$ for every $u \in \partial Q$, then $h(Q) \cap S \neq \phi$.

THEOREM 2.3. — *Let* Λ *be a Riemannian manifold embedded in a Hilbert space* X *and let* $J \in C^2(\Lambda, R)$. We suppose that

(J_1) J *satisfy* (*W. P. S.*)

(J_2) *there exists a closed subset* $S \subset \Lambda$ *and an Hilbert manifold* $Q \subset \overline{\Lambda}$ *with boundary* ∂Q, *and two constants* $0 < \alpha < \beta$ *such that*

a) $J(\gamma) \leqslant \beta$ *for* $\gamma \in Q$ *and* $\max \lim_{\gamma \to \partial Q} J(\gamma) \leqslant 0$

b) $J(\gamma) \geqslant \alpha$ *for every* $\gamma \in S$

c) S *and* ∂Q *link.*

We set $H = \{ h : \overline{Q} \to \overline{\Lambda} \mid h(\gamma) = \gamma \text{ if } J(\gamma) \leqslant 0 \}$ *and*

$$c = \inf_{h \in H} \sup_{\gamma \in Q} J \circ h(\gamma)$$

Then $c \in [\alpha, \beta]$ *and it is either a critical value of* J *or an accumulation point of critical values of* J.

Remarks I. — Theorem 2.3 is a variant of similar results (see [BBF] theorem 2.3, [BR] or [R_3]). The novelty lies in the fact that Λ might be an open set, therefore (P. S.) is not sufficient to guarantee that c is a critical value of J. Therefore we have to require (W. P. S.). A consequence of this fact is that we do not know that c is a critical value of J; it might be an accumulation point of critical values of J (unless (P. S.) is also satisfied).

II. — The assumption $J \in C^2(\Lambda, R)$ is not necessary. It would be enough to assume $J \in C^1(\Lambda, R)$. With the latter assumption the proof of lemma 2.4 would be more technical. However if the reader is interested to prove theorem 2.3 under the less restrictive assumption, he has to « interpolate » between the proof of lemma 2.4 and theorem 1.3 in [BBF]. Since in our application, it is sufficient to assume $J \in C^2(\Lambda, R)$, we did not bother to be as general as possible.

To prove theorem 2.3, we need the following lemma

LEMMA 2.4. — *Let* $J \in C^2(\Lambda, R)$ *satisfy* (*W. P. S.*). *Suppose that* c *is not a critical value of* J *nor an accumulation point of critical values of* J. *Then there exist constants* $\bar{\varepsilon} > \varepsilon > 0$ *and a function* $\eta : [0, 1] \times \Lambda \to \Lambda$ *such that*

a) $\eta(0, x) = x$ *for every* $x \in \Lambda$

b) $\eta(1, x) = x$ *for every* x *such that* $J(x) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}]$ *and every* $t \in R$

c) $\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon}$

where $A_b = \{ x \in \Lambda \mid J(x) \leqslant b \}$. *Moreover* $\bar{\varepsilon}$ *can be chosen arbitrarily small.*

Proof. — We set

$$S_\varepsilon = \{ x \in \Lambda \mid c - \varepsilon \leqslant J(x) \leqslant c + \varepsilon \}$$
$$\Delta_M = \{ x \in \Lambda \mid \rho(x) \leqslant M \}.$$

We claim that there exists $\bar{\varepsilon}$, $\overline{\mathrm{M}}$ and b such that

$$\text{(2.1)} \qquad \| J'(x) \| \geqslant b \| \rho'(x) \| \quad \text{for every} \quad x \in S_{\bar{\varepsilon}} - \Delta_{\overline{M}} .$$

In order to prove (2.1) we argue indirectly. Suppose that (2.1) does not hold. Then there exists a sequence x_n such that

$$\text{(2.2)} \qquad \begin{aligned} &a) \quad J(x_n) \to c \\ &b) \quad \rho(x_n) \to +\infty \\ &c) \quad \| J'(x_n) \| \leqslant b_n \| \rho'(x_n) \| \quad \text{with} \quad b_n \to 0 . \end{aligned}$$

Then we have

$$0 < \nu \leqslant \min \lim \frac{\| J'(x_n) \|}{\| \rho'(x_n) \|} \qquad \text{(by (2.2)}(a)\ (b) \text{ and (W. P. S) } (ii)\text{)}$$

$$\leqslant \min \lim b_n = 0 \qquad \text{by (2.2) } (c) .$$

This is a contradiction which proves (2.1). It is not restrictive to suppose that $\bar{\varepsilon}$ is so small that $[c - \bar{\varepsilon}, c + \bar{\varepsilon}]$ does not contain critical values of J; this is possible because we have supposed that c is not an accumulation point of critical values.

We claim that for every M, there exists $b_M > 0$ such that

$$\text{(2.3)} \qquad \| J'(x) \| \geqslant b_M \quad \text{for every } x \in S_{\bar{\varepsilon}} \cap \Delta_M .$$

In fact if (2.3) does not hold there exists a sequence x_n such that

$$\text{(2.4)} \qquad \begin{aligned} &a) \quad J(x_n) \in [c - \bar{\varepsilon}, c + \bar{\varepsilon}] \\ &b) \quad \rho(x_n) \leqslant M \\ &c) \quad \| J'(x_n) \| \leqslant b_n \quad \text{for some sequence } b_n \to 0 . \end{aligned}$$

Then, by (WPS 1), it follows that x_n has a subsequence converging to some limit $\bar{x}$. So we have

$$J'(\bar{x}) = 0 \quad \text{and} \quad J(\bar{x}) = \bar{c} \stackrel{\text{def}}{=} \lim_{n \to +\infty} J(x_n) .$$

Thus $\bar{c} \in [c - \bar{\varepsilon}, c + \bar{\varepsilon}]$ is a critical value of J contradicting our choice of $\bar{\varepsilon}$. Let $\phi : \Lambda \to \mathbb{R}$ be a Lipschitz continuous function such that

$$\phi(x) = \begin{cases} 1 & \text{if} \quad x \in S_{\bar{\varepsilon}/2} \\ 0 & \text{if} \quad x \notin S_{\bar{\varepsilon}} \end{cases}$$

and we set

$$\text{(2.5)} \qquad V(x) = \begin{cases} \phi(x) \dfrac{J'(x)}{\| J'(x) \|^2} & \text{for} \quad x \in S_{\bar{\varepsilon}} \\ 0 & \text{for} \quad x \notin S_{\bar{\varepsilon}} \end{cases}$$

By (2.1) and the definition of ϕ, V is well defined and locally Lipschitz continuous. We now consider the following initial value problem

$$\text{(2.6)} \qquad \begin{aligned} \dot{\eta} &= -V(\eta) \\ \eta(0) &= x . \end{aligned}$$

By basic existence theorems for such equations, for each $x \in \Lambda$ there exists a unique solution $\eta(t, x)$ of (2.6) defined for $t \in (t^-(x), t^+(x))$, a maximal interval depending on x. We claim that $t^\pm(x) = \pm\infty$. Let us prove that

$$t^+(x) = +\infty .$$

We argue indirectly and suppose that $t^+(x) < +\infty$.

First of all we can suppose that

$$\eta(t, x) \in S_{\bar{\varepsilon}} \quad \text{for} \quad t \in [0, t^+(x)) \tag{2.6'}$$

otherwise the conclusion follows directly from (2.5). Now we claim that

$$\rho(\eta(t)) \leqslant M \quad \text{for every} \quad t \in [0, t^+(x)) \tag{2.6''}$$

where $M = M_1 + \dfrac{t^+(x)}{b}$ and $M_1 = \max\{\rho(0), \overline{M}\}$. If the above inequality does not hold then there exists t_1, t_2 with $0 \leqslant t_1 < t_2 < t^+(x)$ such that

$$M_1 \leqslant \rho(\eta(t)) \leqslant M \quad \text{for} \quad t \in [t_1, t_2]$$

and

$$\rho(\eta(t_1)) = M_1 ; \qquad \rho(\eta(t_2)) = M ; \tag{2.7}$$

then, for $t \in [t_1, t_2]$

$$\begin{aligned} \left| \frac{d}{dt} \rho(\eta(t, x)) \right| &= | \langle \rho'(\eta(t, x)), V(\eta(t, x) \rangle | && [\text{by } (2.6)] \\ &\leqslant \phi(\eta(t)) \frac{\| \rho'(\eta(t, x) \|}{\| J'(\eta(t, x) \|} && [\text{by } (2.5)] \\ &\leqslant \frac{1}{b} && [\text{by } (2.6'), (2.4') \text{ and } (2.1)] \end{aligned}$$

Then we have

$$\begin{aligned} M - M_1 &= \rho(\eta(t_2, x)) - \rho(\eta(t_1, x)) && [\text{by } (2.7)] \\ &\leqslant \int_{t_1}^{t_2} \left| \frac{d}{dt} \rho(\eta(t, x)) \right| dt \\ &\leqslant (t_2 - t_1) \frac{1}{b} && [\text{by the above inequality}] \\ &< \frac{t^+(x)}{b} = M - M_1 && [\text{by the definition of } M]. \end{aligned}$$

This is a contradiction. Therefore (2.6″) is proved.

Then by (2.3), there exists $b_M > 0$ such that

$$\| J'(\eta(t, x)) \| \geqslant b_M \quad \text{for} \quad t \in [0, t^+(x)) . \tag{2.7'}$$

Now let t_n be a sequence such that $t_n \to t^+(x)$. So we have

$$\begin{aligned} \| \eta(t_{n+k}, x) - \eta(t_n, x) \| &= \left\| \int_{t_n}^{t_{n+k}} V(\eta(t, x))\, dt \right\| && \text{[by (2.6)]} \\ &\leqslant \int_{t_n}^{t_{n+k}} \frac{dt}{\| J'(\eta(t, x)) \|} && \text{[by (2.4) and (2.5)]} \\ &\leqslant b_M^{-1}(t_{n+k} - t_n) && \text{[by (2.7')]}. \end{aligned}$$

This implies that $\eta(t_n, x)$ is a Cauchy sequence converging some $x_0 \in \bar{\Lambda}$ as $t_n \to t^+(x)$. Moreover $\rho(\bar{x}) = \lim_{n \to +\infty} \rho(\eta(x, t_n)) \leqslant M$, therefore, by Definition (2.1) (*iii*), $\bar{x} \in \Lambda$. But the solution of (2.6) with initial condition $\bar{x}$ furnishes a continuation of $\eta(t, x)$ contradicting the maximality of $t^+(x)$. Analogously we can prove that $t^-(x) = -\infty$. Therefore $\eta(t, x)$ is defined for every $t \in R$. Since $\frac{d}{dt} J(\eta(t, x)) = -1$ if $\eta(t, x) \in S_{\bar{\varepsilon}/2}$ by an easy standard argument the conclusion follows. □

Proof of Theorem 2.3. — By virtue of lemma 2.4, the proof of theorem 2.3 is almost a repetition of analogous proofs (cf. e. g. [BR], [R_3] or [BF]). We sketch it for completeness. By the first part of (J_2) (*a*) and since the identity belong H, $c \leqslant \beta$. By the second part of (J_2) (*a*) and (J_2) (*c*), $h(Q) \cap S \neq \phi$; then by (J_2) (*b*), $c \geqslant \alpha$. Then c is well defined and is in $[\alpha, \beta]$. It remains to prove that c is a critical value of J or it is an accumulation point of critical values of J. Suppose that neither possibility holds. Then the assumptions of lemma 2.4 are satisfied. Choose $\bar{\varepsilon} \in (0, \alpha]$, ε and η as in lemma 2.4. By the definition of c, there exists $\bar{h} \in H$ such that

$$\sup_{x \in Q} J \circ \bar{h}(x) \leqslant c + \varepsilon .$$

By lemma (2.4) (*c*) and the above inequality we have

$$\sup_{x \in Q} J \circ \eta \circ \bar{h}(x) \leqslant c - \varepsilon . \tag{2.8}$$

By lemma (2.4) (*b*) and the choice of $\bar{\varepsilon}$, $\eta \circ \bar{h} \in H$; then by the definition of c

$$\sup_{x \in Q} J \circ \eta \circ \bar{h}(x) \geqslant c .$$

The above inequality contradicts (2.8). Thus the theorem is proved. □

3. PROOF OF THEOREM 1.1

We set

$$\Lambda^1\Omega = \{ \gamma \in H^1(S^1, R^n) \mid \gamma(t) \in \Omega \} \qquad (S^1 = [0, 1]/\{0, 1\}) \tag{3.1}$$

where $H^1(S^1, R^n)$ denotes the Sobolev space obtained by the closure of C^∞-functions (periodic of period 1) with respect to the norm

$$\| \gamma \| = \left[\int_0^1 \{ | \dot{\gamma} |^2 + | \gamma |^2 \} \, dt \right]^{1/2}.$$

Since $H^1(S^1, R^n) \subset C^0(S^1, R^n)$, then the set $\Lambda^1\Omega$ is an open set in $H^1(\Omega, R^n)$. The periodic solutions of (1.2) are, at least formally, the critical value of the functional

$$\text{(3.2)} \qquad J(\gamma) = \int \left\{ \frac{1}{2} a(\gamma) | \dot{\gamma} |^2 - U(\gamma) \right\} dt.$$

However the functional (3.2) does not satisfy W. P. S. (nor the condition (J_2) of theorem 2.3) on the set (3.1). Therefore it is necessary to modify the functional (3.2) in a suitable way. Then we shall apply theorem 2.3 to the modified functional and finally we shall prove that the solutions of the modified functional are the solutions of our problem.

In order to carry out this program we start defining a function $h \in C^2(\overline{\Omega})$ with the following properties

$$\text{(3.3)} \qquad \begin{array}{rlll} i) & h(x) = d(x, \partial\Omega) & \text{if} & d(x, \partial\Omega) \leqslant d_0 \\ ii) & h(x) > d_0 & \text{whenever} & d(x, \partial\Omega) > d_0 \\ iii) & \nabla h(x) \leqslant 1 & \text{for every} & x \in \overline{\Omega} \\ iv) & h(x) \leqslant 1 & \text{for every} & x \in \overline{\Omega} \end{array}$$

where d_0 is a constant sufficiently small. Such a function h exists since Ω is assumed to have a C^2-boundary. Also we set

$$\text{(3.4)} \qquad h_0 = \sup_{\substack{x \in \overline{\Omega} \\ \delta x \in R^n}} \frac{d^2 h(x) [\delta x]^2}{| \delta x |^2}$$

where $d^2 h$ denotes the second differential of h. Moreover we set

$$\text{(3.5)} \quad v(x) = -\nabla h(x) \quad \text{so that} \quad v(x) \in C^1(\Omega, R^n) \quad \text{and} \quad | v(x) | \leqslant 1.$$

Now let $\phi, \chi \in C^\infty(R)$ be two functions such that

$$\begin{array}{lll} \phi(t) = t & \text{for} & t \geqslant 1 \\ \phi(t) = \dfrac{1}{2} & \text{for} & t \leqslant \dfrac{1}{2} \\ 0 \leqslant \phi'(t)t \leqslant \phi(t) & \text{for} & t \in R \\ \chi(t) = 0 & \text{for} & t \leqslant 1 \\ \chi(t) = 1 & \text{for} & t \geqslant 2 \\ \chi'(t) \geqslant 0 & \text{for} & t \in R \end{array}$$

and set

$$a_\lambda(x) = \frac{1}{\lambda}\phi(\lambda a(x))$$

$$U_{\lambda,N}(x) = \frac{1}{N^2}\left\{\chi(\lambda h(x))U(x) + [1 - \chi(\lambda h(x))]\frac{M_\lambda}{h(x)^2}\right\}$$

where $M_\lambda = \sup\left\{ U(x) \mid x \in h^{-1}\left(\left[\frac{1}{\lambda}, \frac{2}{\lambda}\right]\right)\right\}$. Clearly a_λ and $U_{\lambda,N}$ are C^2-functions and

$$(3.6) \qquad a_\lambda(x) \geqslant \frac{1}{2\lambda} \quad \text{for every} \quad x \in \Omega .$$

Our modified functional will be

$$J_{\lambda,N}(\gamma) = \int_0^1 \left\{\frac{1}{2}a_\lambda(\gamma)\,|\,\dot\gamma\,|^2 - U_{\lambda,N}(\gamma)\right\} dt .$$

It is easy to check that $J_{\lambda,N}(\gamma) \in C^2(\Lambda^1\Omega, R)$ and that

$$J'_{\lambda,N}(\gamma)[\delta\gamma] = \int_0^1 \left\{ a_\lambda(\gamma)\dot\gamma \cdot \delta\dot\gamma + \frac{1}{2}(\nabla a_\lambda(\gamma)\cdot\delta\gamma)\,|\,\dot\gamma\,|^2 - \nabla U_{\lambda,\gamma}(\gamma)\cdot\delta\gamma \right\} dt$$

$$\text{for} \quad \gamma \in \Lambda^1\Omega \quad \text{and} \quad \delta\gamma \in H^1(S^1, R^n) .$$

Now we want to apply theorem 2.3 to the functional $J_{\lambda,N}$. In order to do this some lemmas are necessary.

LEMMA 3.1. — *a) there exists a constant* $b = b(\lambda, N)$ *such that*

$$b\left(\frac{1}{h(x)^2}\right) \leqslant U_{\lambda,N}(x) \leqslant b\left(\frac{1}{h(x)^2} + 1\right)$$

b) there are positive constants β *and* K_1 *(which may depend on* λ *and* N*) such that*

$$\nabla U_{\lambda,N}(x)\cdot\nu(x) \geqslant \beta\frac{1}{h(x)^3} - K_1$$

c) for every $M > 0$ *there are constants* $a(M)$ *and* $\bar\lambda(M)$ *such that*

$U_{\lambda,N}(x) \leqslant \frac{1}{M}\nabla U_{\lambda,N}(x)\cdot\nu(x) + a(M)\cdot$ *for every* $x \in \Omega$ *and every* $\lambda \geqslant \bar\lambda(M)$

d) there exists a function $\lambda \to \theta(\lambda)$ *such that*

i) $\lim_{\lambda\to+\infty}\theta(\lambda) = +\infty$

ii) for every $x \in \Omega$ *such that* $U_{\lambda,N}(x) \leqslant \frac{1}{N^2}\theta(\lambda)$, *we have* $U_{\lambda,N}(x) = \frac{1}{N^2}U(x)$, *and* $a_\lambda(x) = a(x)$

e) there exists a constant K *such that*

$$\nabla a_\lambda(x)\cdot\nu(x) \leqslant K a_\lambda(x) \quad \textit{for every} \quad x \in \Omega \quad \textit{and every} \quad \lambda > 0 .$$

Proof. — a and b follows by the fact that for x sufficiently close to $\partial\Omega$,

$U_{\lambda,N}(x)=\frac{M_\lambda}{N^2}\cdot\frac{1}{h(x)^2}$ (remember that for x sufficiently close to $\partial\Omega$, $|\nabla h(x)|=1$).

Let us prove (c). Since $v(x) = -\nabla h(x)$ we have:

$$(3.6\,a)\quad \nabla U_{\lambda,N}(x)\cdot v(x)=\frac{1}{N^2}\left\{\chi(\lambda h(x))\nabla U(x)\cdot v(x)+[1-\chi(\lambda h(x))]\frac{|v(x)|^2}{h^3(x)}M_\lambda + \lambda\chi'(\lambda h(x))\left[\frac{M_\lambda}{h(x)^2}-U(x)\right]|v(x)|^2\right\}.$$

If $\chi'(\lambda h(x)) \neq 0$, then $x \in h^{-1}\left(\left[\frac{1}{\lambda},\frac{2}{\lambda}\right]\right)$. Thus for such values of x, by (3.3) (iv) and the definition of M_λ we have

$$\frac{M_\lambda}{h(x)^2}-U(x) \geqslant M_\lambda - U(x) \geqslant 0.$$

Thus, since $\chi'(t) \geqslant 0$ for every $t \in \mathbf{R}$,

$$(3.6\,b)\qquad \chi'(\lambda h(x))\left[\frac{M_\lambda}{h(x)^2}-U(x)\right] \geqslant 0 \quad \text{for every} \quad x \in \Omega.$$

By (L_3) and easy computations, for every $M > 0$ there exists a_0 such that

$$(3.6\,c)\qquad \nabla U(x)\cdot v(x) \geqslant MU(x) - a_0 \quad \text{for every} \quad x\in\Omega.$$

Moreover, for x sufficiently close to $\partial\Omega$

$$\frac{|v(x)|^2}{h(x)^3} \geqslant \frac{M}{h(x)^2}.$$

Then there exists $\bar\lambda(M)$ such that, for $\lambda \geqslant \bar\lambda(M)$

$$[1-\chi(\lambda h(x))]\frac{|v(x)|^2}{h(x)^3} \geqslant [1-\chi(\lambda h(x))]\frac{M}{h(x)^2}.$$

So by (3.6 a), (3.6 b), (3.6 c) and the above inequality we get

$$\nabla U_{\lambda,N}(x)\cdot v(x) \geqslant \frac{1}{N^2}\left\{\chi(\lambda h(x))MU(x) - a_0 + [1-\chi(\lambda h(x))]\frac{MM_\lambda}{h(x)^2}\right\} \geqslant MU_{\lambda,N}(x) - a_0.$$

From the above inequality (c) follows. Now let us prove (d). We set

$$\Omega_\lambda = \{\, x\in\Omega \mid \lambda h(x) \geqslant 2 \quad \text{and} \quad \lambda a(x) \geqslant 1 \,\}$$

and

$$\theta(\lambda) = \inf\left\{ \chi(\lambda h(x))U(x) + [1-\chi(\lambda h(x))]\frac{M_\lambda}{h(x)^2} \,\middle|\, x\in\Omega-\Omega_\lambda \right\}$$

by (3.3), (L_5) and (L_2), $\theta(\lambda) \to +\infty$ for $\lambda \to +\infty$. Moreover, if $U_{\lambda,N}(x) \leqslant \frac{\theta(\lambda)}{N^2}$, by the definition of $\theta(\lambda)$, $x \in \Omega_\lambda$. Then $\lambda h(x) \geqslant 2$ and $\lambda a(x) \geqslant 1$. Therefore $\chi(\lambda h(x)) = 1$ and $\frac{1}{\lambda}\phi(\lambda a(x)) = a(x)$. This proves (*d*). In order to prove (*e*), we set

$$\Gamma = \{ x \in \partial\Omega \mid a(x) = 0 \}.$$

Since $a(x) > 0$ for $x \in \Omega$ it follows that $\nabla a(x) \cdot \nu(x) < 0$ for every $x \in \Gamma$. By virtue of (L_6) and the compactness of Γ, there exists a constant $\delta > 0$ such that

$$(3.6\,d) \qquad \nabla a(x) \cdot \nu(x) \leqslant -\delta \quad \text{for every} \quad x \in \Gamma .$$

Let

$$B = \{ x \in \Omega \mid \nabla a(x) \cdot \nu(x) < 0 \}.$$

By (3.6 *d*), B is an open neighbourhood of Γ relative to $\overline{\Omega}$. Then, since $\overline{\Omega} - B$ is compact, by (L_5), there exists a constant c_1 such that

$$a(x) \geqslant c_1 \quad \text{for every} \quad x \in \overline{\Omega} - B .$$

Using again the compactness of $\overline{\Omega} - B$, there exists a constant c_2 such that

$$\nabla a(x) \cdot \nu(x) \leqslant c_2 \quad \text{for every} \quad x \in \overline{\Omega} - B .$$

So, choosing $K = c_2/c_1$, it follows that

$$(3.6\,e) \qquad \nabla a(x) \cdot \nu(x) \leqslant K a(x) \quad \text{for every} \quad x \in \Omega .$$

Then we have

$$\begin{aligned} \nabla a_\lambda(x) \cdot \nu(x) &= \phi'(\lambda x) \nabla a(x) \cdot \nu(x) \\ &\geqslant K\phi'(\lambda x) a(x) \qquad \text{by } (3.6\,e) \\ &\geqslant K \frac{1}{\lambda} \phi(\lambda x) = K a_\lambda(x) \text{ by the definition of } \phi \text{ and } a_\lambda . \end{aligned}$$ □

In order to apply theorem 2.3 to the functional $J_{\lambda,N}$ it is necessary to choose an appropriate weight function; we make the following choice

$$(3.7) \qquad \rho(\gamma) = \left[\int_0^1 \frac{1}{h(\gamma)^2} dt \right]^{1/2}$$

LEMMA 3.2. — *The function* ρ *defined by* (3.7) *satisfies the assumptions of definition* 2.1.

Proof. — (*i*) *and* (*ii*) are trivial. Let us prove (*iii*). Let $\gamma_k \in \Lambda^1\Omega$ be a sequence approaching $\partial\Lambda^1\Omega$ and let t_k be such that $\text{dist}(\gamma_k(t_k), \partial\Omega) \leqslant \text{dist}(\gamma_k(t), \partial\Omega)$ for every $t \in (0, 1)$. We want to prove that $\rho(\gamma_k) \to +\infty$. Since ρ is invariant for « time translations », we can suppose that $t_k = 0$ for every k.

By the Schwartz inequality we have

$$|\gamma_k(t) - \gamma_k(0)| \leqslant \int_0^t |\dot\gamma_k(t)| \leqslant t^{1/2}\left[\int_0^t |\dot\gamma_k(t)|^2\right]^{1/2} \leqslant \|\gamma_k\| \cdot t^{1/2}.$$

Then, by (3.3) (*iii*),

$$(3.8)\quad |h(\gamma_k(t)) - h(\dot\gamma_k(0))| \leqslant \max_{x\in\Omega} |\nabla h(x)| \cdot |\gamma_k(t) - \gamma_k(0)| \leqslant \|\gamma_k\| \, t^{1/2}.$$

If we set

$$m(\gamma_k) = h(\gamma_k(0))$$

by (3.8) we get

$$h(\gamma_k(t)) \leqslant m(\gamma_k) + \|\gamma_k\| \, t^{1/2}.$$

We can assume that $\|\gamma_k\| \geqslant \alpha > 0$ for k large and some positive constant α. Then we have

$$\frac{1}{h(\gamma_k(t))^2} \geqslant \frac{1}{(m(\gamma_k) + \|\gamma_k\| \, t^{1/2})^2} \geqslant \frac{1}{2}\,\frac{1}{m(\gamma_k)^2 + \|\gamma_k\|^2 t}$$

$$\rho(\gamma_k)^2 = \int_0^1 \frac{1}{h(\gamma_k(t))^2}\,dt \geqslant \frac{1}{2}\int_0^1 \frac{dt}{m(\gamma_k)^2 + \|\gamma_k\|^2 t} = \frac{1}{\|\gamma_k\|^2}\log\left(1 + \frac{\|\gamma_k\|^2}{m(\gamma_k)^2}\right).$$

From the above inequality and since $\|\gamma_k\| \geqslant \alpha > 0$ and $m(\gamma_k) \to 0$ for $k \to +\infty$ the conclusion follows.

LEMMA 3.3. — *For every* $N \geqslant 1$ *and* $\lambda > 0$, *the functional* $J_{\lambda,N}$ *satisfy W. P. S.*

Proof. — To simplify the notation, in this proof we shall write J, U and a instead of $J_{\lambda,N}$, $U_{\lambda,N}$ and a_λ. Let us start to prove WPS 1. In the following $a_1, a_2, \ldots$ will denote suitable positive constants. Since $\rho(\gamma_n)$ is bounded, then by lemma 3.1

$$(3.9)\qquad \int U(\gamma_n(t))dt \text{ is bounded}.$$

Since $J(\gamma_n)$ is bounded it follows that

$$\int \frac{1}{2} a(\gamma_n)\dot\gamma_n^2 dt \text{ is bounded}.$$

Then $\|\gamma_n\|_{H^1}$ is bounded, therefore (may be taking a subsequence) we have that

$$(3.10)\qquad \gamma_n \to \bar\gamma \text{ weakly in } H^1(S^1, R^n) \text{ and uniformly}.$$

We have to prove that $\gamma_n \to \bar\gamma$ strongly in $H^1(S^1, R^n)$. Since we suppose that $J'(\gamma_n) \to 0$, we have that

$$(3.11)\quad \int\left\{ a(\gamma_n)\dot\gamma_n\delta\dot\gamma + \frac{1}{2}(\nabla a(\gamma_n)\cdot\delta\gamma)\,|\dot\gamma_n|^2 - \nabla U(\gamma_n)\cdot\delta\gamma \right\}dt = \varepsilon_n \|\delta\gamma\|$$

for every $\delta\gamma \in H^1$ (we have identified H^1 with its dual), where ε_n is a sequence conveying to 0. By (3.9) and (3.10) it follows that

$$\int \nabla U(\gamma_n)\delta\gamma \leqslant \| \nabla U(\gamma_n) \|_{L^\infty} \int \delta\gamma \leqslant a_2 \| \delta\gamma \|_{L^\infty} \tag{3.12}$$

Also, using (3.10), we have

$$\frac{1}{2}\int \nabla a(\gamma_n)\cdot\delta\gamma\,|\,\dot\gamma_n\,|^2 \leqslant \frac{1}{2}\| \nabla a(\gamma_n) \|_{L^\infty} \| \delta\gamma \|_{L^\infty} \| \gamma_n \|^2_{H^1} \leqslant a_3 \| \delta\gamma \|_{L^\infty}. \tag{3.13}$$

By (3.11), (3.12) and (3.13) it follows that

$$\int a(\gamma_n)\dot\gamma_n\delta\dot\gamma = \varepsilon_n \| \delta\gamma \| + (a_2 + a_3) \| \delta\gamma \|_{L^\infty}$$

for every $\delta\gamma \in H^1(S^2, R^n)$. In particular, taking $\delta\gamma = \gamma_n - \bar\gamma$ we get

$$\int a(\gamma_n)\dot\gamma_n(\dot\gamma_n - \dot{\bar\gamma}) = \varepsilon_n \| \gamma_n - \bar\gamma \| + o(1) \tag{3.14}$$

since $\| \gamma_n - \bar\gamma \|_{L^\infty} \to 0$ for $n \to +\infty$. So by (3.6) and (3.14) we have

$$\begin{aligned}\frac{1}{2\lambda}\|\gamma_n - \bar\gamma\|^2 &\leqslant \frac{1}{2\lambda}\int |\dot{\bar\gamma} - \dot\gamma_n|^2 + o(1) \leqslant \int a(\gamma_n)|\dot\gamma_n - \dot{\bar\gamma}|^2 + o(1)\\ &= \int a(\gamma_n)\dot\gamma_n(\dot\gamma_n - \dot{\bar\gamma}) - \int a(\gamma_n)\dot{\bar\gamma}(\dot\gamma_n - \dot{\bar\gamma}) + o(1)\\ &\leqslant \varepsilon_n\|\gamma_n - \bar\gamma\| + \|a(\gamma_n)\|_{L^\infty}\int \dot{\bar\gamma}(\dot\gamma_n - \dot{\bar\gamma}) + o(1) = \varepsilon_n\|\gamma_n - \bar\gamma\| + o(1)\end{aligned}$$

from which the conclusion follows. Now we shall prove W. P. S. (*ii*). In the following $b_1, b_2, \ldots$ will denote suitable positive constants. Now let γ_n be a sequence such that

$$\begin{aligned} &a)\quad J(\gamma_n) \text{ is convergent}\\ &b)\quad \rho(\gamma_n) \to +\infty. \end{aligned} \tag{3.15}$$

Then we have

$$\begin{aligned}\frac{1}{2\lambda}\int |\,\dot\gamma_n\,|^2 &\leqslant \int a(\gamma_n)\,|\,\dot\gamma_n\,|^2 && \text{(by (3.6))}\\ &\leqslant \int U(\gamma_n) + b_1 && \text{(by (3.15) (}b\text{))}\\ &\leqslant b_2\int \frac{1}{h(\gamma_n)^2} + b_3 && \text{(by lemma 3.1 (}a\text{))}\\ &= b_2\rho(\gamma_n)^2 + b_3 && \text{(by (3.7))}.\end{aligned} \tag{3.16}$$

We now set
$$\delta\gamma_n(t) = v(\gamma_n(t)) = -\nabla h(\gamma_n(t)).$$
Thus $\delta\gamma_n \in H^1(S^1, R^n)$ and
$$(3.17)\qquad \|\delta\gamma_n\|^2 = \int |d^2h(\gamma_n)[\dot\gamma_n]^2| + |\nabla h(\gamma_n)|^2 dt$$
$$\leqslant b_5\int|\dot\gamma_n|^2 + b_6 \qquad \text{(by (3.3) } (iii) \text{ and (3.4))}$$
$$\leqslant b_5\rho(\gamma_n)^2 + b_6 \qquad \text{(by (3.16))}.$$
Then by the above formula we have
$$(3.18)\qquad \|\delta\gamma_n\| \leqslant b_7\rho(\gamma_n) + b_8.$$
We have
$$(3.19)\qquad \|J'(\gamma_n)\|(b_7\rho(\gamma_n) + b_8) \geqslant \|J'(\gamma_n)\|\,\|\delta\gamma_n\| \qquad \text{(by (3.18))}$$
$$\geqslant -J'(\gamma_n)[\delta\gamma_n]$$
$$= \int_0^1 \nabla U(\gamma_n)\cdot\delta\gamma_n - a(\gamma_n)\dot\gamma_n\delta\gamma_n - \frac{1}{2}\nabla a(\gamma_n)\cdot\delta\gamma_n|\dot\gamma_n|^2 \qquad \text{(by the definition of J')}$$
$$= \int_0^1 \nabla U(\gamma_n)\cdot v(\gamma_n) - a(\gamma_n)d^2h[\dot\gamma]^2 - \frac{1}{2}\nabla a(\gamma_n)\cdot v(\gamma_n)|\dot\gamma_n|^2 \qquad \text{(by the definition of } \delta\gamma_n)$$
$$\geqslant \int_0^1 \nabla U(\gamma_n)\cdot v(\gamma_n) - \|a(\gamma_n)\|_{L^\infty}\|d^2h(\gamma_n)[\cdot]\|_{L^\infty}\int|\dot\gamma_n|^2dt$$
$$-\frac{1}{2}\|\nabla a(\gamma_n)\|_{L^\infty}\cdot\|v(\gamma_n)\|_{L^\infty}\int|\dot\gamma_n|^2dt$$
$$\geqslant b_9\int\frac{1}{h(\gamma_n)^3} - b_{10}\int|\dot\gamma_n|^2 - b'_{10} \qquad \text{(by lemma 3.1) } (b)\text{, (3.5) and (3.4))}$$
$$\geqslant b_9\int\frac{1}{h(\gamma_n)^3} - b_{11}\rho(\gamma_n)^2 - b_{12} \qquad \text{(by (3.16))}.$$

Next we shall compute $\|\rho'(\gamma_n)\|$. We have
$$\rho'(\gamma)[\delta\gamma] = -\left(\int\frac{1}{h(\gamma)^2}dt\right)^{-1/2}\int\frac{\nabla h(\gamma)\cdot\delta\gamma}{h(\gamma)^3}dt$$
for $\gamma \in \Lambda^1\Omega$ and $\delta\gamma \in H^1(S^1, R^n)$

then
$$(3.20)\qquad \|\rho'(\gamma)\| = \sup_{\|\delta\gamma\|\neq 0}\frac{\rho'(\gamma)[\delta\gamma]}{\|\delta\gamma\|} = \frac{1}{\rho(\gamma)}\sup_{\|\delta\gamma\|\neq 0}\frac{1}{\|\delta\gamma\|}\int\frac{\nabla h(\gamma)\delta\gamma}{h(\gamma)^3}dt$$
$$\leqslant \frac{1}{\rho(\gamma)}\sup_{\|\delta\gamma\|\neq 0}\frac{\|\delta\gamma\|_{L^\infty}}{\|\delta\gamma\|}\int\frac{|\nabla h(\gamma)|}{h(\gamma)^3}dt \leqslant \frac{1}{\rho(\gamma)}\int\frac{|\nabla h(\gamma)|}{h(\gamma)^3}dt.$$

By the Hölder inequality we have

$$\int_0^1 \frac{1}{h(\gamma)^2} \leqslant \left[\int \frac{1}{h(\gamma)^3}\right]^{2/3}$$

then

$$\int \frac{1}{h(\gamma)^3} \geqslant \left[\int \frac{1}{h(\gamma)^2}\,dt\right]^{3/2} = \rho(\gamma)^3 .$$

By the above inequality and (3.19) we get

$$\| J'_\lambda(\gamma_n) \| (b_7\rho(\gamma_n) + b_8) \geqslant \frac{1}{2} b_9 \int \frac{1}{h(\gamma)^3} + \frac{1}{2} b_9 \rho(\gamma_n)^3 - b_{11}\rho(\gamma_n)^2 - b_{12} .$$

Now, since $\rho(\gamma_n) \to +\infty$, for n large enough we have

$$\| J'_\lambda(\gamma_n) \| \geqslant \frac{b_{13}}{\rho(\gamma_n)} \int \frac{1}{h(\gamma_n)^3} .$$

Since $|\nabla h| \leqslant 1$ (by (3.3) (*iii*)), the above inequality and (3.20) imply that

$$\| J'_\lambda(\gamma_n) \| \geqslant b_{13} \| \rho'(\gamma_n) \|$$

and this proves W. P. S. (*ii*). □

To simplify the notation we shall suppose that

$$(3.21) \qquad 0 \in \Omega .$$

Now let

$$V = \{ \gamma \in H^1(S^1, R^n) \mid \gamma \text{ is a constant} \}$$

and let $V^\perp$ its orthogonal complement in $H^1(S^1, R^n)$. We set

$$(3.22) \quad Q = [V \times \{ re \sin 2\pi t \mid r \geqslant 0 \}] \cap \Lambda^1\Omega, \quad e \in R^n, |e| = 1 .$$

Let R be a constant small enough in order that the ball of center 0 and radius R is contained in Ω. Then there exists an integer number N_0 such that

$$\frac{1}{N} U(x) \leqslant \frac{R^2}{8} \quad \text{for every } x \in B_R(0) \quad \text{and every } N \geqslant N_0 .$$

By the above inequality we get that

$$(3.23) \quad U_{\lambda,N}(x) \leqslant \frac{R^2}{8} \quad \text{for every } x \in B_R(0) \quad \text{and every} \quad N \geqslant N_0 \quad \text{and } \lambda \geqslant \lambda_0$$

where λ_0 is big enough in order that $U_{\lambda,N}(x) = \dfrac{U(x)}{N}$ for every $\lambda \geqslant \lambda_0$ and every $x \in B_R(0)$. Observe that

(2.23′) if $U(x) = o(|x|^2)$ then we can choose $N_0 = 1$ provided that R is small enough.

Now we set

$$(3.24) \qquad S = \{ \gamma \in V^\perp \mid \| \gamma \| = R \} .$$

We have the following lemma

LEMMA 3.4. — *For every* $\lambda \geqslant \lambda_0$ *and* $N \geqslant N_0$, $J_{\lambda,N}$ *satisfy the assumptions* (J_2) *of theorem* 2.3 *where* S *and* Q *are defined by* (3.22) *and* (3.24) *respectively and* α *and* β *are constants which depend only on* Ω *(but not on* U, λ *and* N).

Proof. — *a*) If $\gamma \in Q$ then $\gamma(t) = y_1 + y_2 e \sin(2\pi t)$ with $y_1 \in R^n$ and $y_2 \in R$. Since $Q \subset \Lambda^1\Omega$ then

$$y_1 + y_2 e \sin(2\pi t) \in \Omega \quad \text{for every} \quad t \in [0, 1].$$

Therefore

$$(3.25) \qquad |y_1| < d; \quad |y_2| < 2d \quad \text{where } d = \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega).$$

Thus, by (3.21)

$$J_{\lambda,N}(\gamma) \leqslant \int \frac{1}{2} |y_2|^2 [2\pi \cos(2\pi t)]^2 dt \leqslant 8\pi^2 d^2 \overset{\text{def}}{=} \beta \quad \text{for every} \quad \gamma \in Q.$$

Also β depend only on d i. e. on the geometry of Ω. Now let us prove that

$$(3.26) \qquad \max \lim_{\gamma \to \partial Q} J(\gamma) \leqslant 0.$$

We have

$$\partial Q \subset (V \cap \Lambda^1\Omega) \cup (Q \cap \partial\Lambda^1\Omega).$$

If $\gamma \in V \cap \Lambda^1\Omega$ we have

$$J_{\lambda,N}(\gamma) = \int - U_{\lambda,N}(\gamma) dt \leqslant 0 \qquad \text{(by (3.24) and the definition of V)}.$$

If $\gamma \in Q \cap \Lambda\Omega$ we have

$$\begin{aligned} J_{\lambda,N}(\gamma) &= \int \left\{ \frac{1}{2} |y_2|^2 [2\pi \cos 2\pi t]^2 - U_{\lambda,N}(\gamma) \right\} dt \\ &\leqslant 8\pi^2 d^2 - b \int \frac{1}{h(\gamma)^2} + b \qquad \text{(by lemma (3.1) (}b\text{))} \\ &\leqslant K - b\rho(\gamma)^2 \qquad \left(\text{with} \quad K = 8\pi d_1^2 + b\right). \end{aligned}$$

Then (3.26) follows by the fact that $\lim_{\gamma \to \partial\Lambda^1\Omega} \rho(\gamma) = +\infty$ (cf. lemma 3.2). Now let us prove that assumption (J_2) (*b*) of theorem 2.3 holds. If $\gamma \in S$ then $\|\gamma\| = R$ and $|\gamma(t)| \leqslant R$ for every $t \in [0, 1]$. Then, for $\lambda \geqslant \lambda_0$ and $N \geqslant N_0$, by (3.23) we have

$$(3.27) \quad U_{\lambda,N}(\gamma(t)) \leqslant \frac{R^2}{8} \quad \text{for every} \quad t \in [0, 1] \quad \text{and every} \quad \gamma \in S.$$

Moreover for $\gamma \in S$, by the Poincaré inequality $\int |\gamma|^2 \leqslant \int |\dot{\gamma}|^2$; then

$$\int \frac{1}{2} |\dot{\gamma}|^2 \geqslant \frac{1}{4} \int |\dot{\gamma}|^2 + |\gamma|^2 = \frac{1}{4} \|\gamma\|^2 = \frac{1}{4} R^2 \quad \text{for} \quad \gamma \in S.$$

Thus by the above inequality and (3.27) we get

$$J_{\lambda,N}(\gamma) = \int \left[\frac{1}{2} |\dot{\gamma}|^2 - U_{\lambda,N}(\gamma(t)) \right] dt \geqslant \frac{1}{4} R^2 - \frac{1}{8} R^2 = \frac{1}{8} R^2 \overset{\text{def}}{=} \alpha \quad \text{for every} \quad \gamma \in S.$$

This proves assumption (*b*) of theorem 2.3 with α depending only on R, i. e. on the geometry of Ω. The fact that S and ∂Q link, is proved in proposition 2.2 of [BBF]. Actually there Q is defined in a slightly different way, but this fact does not affect the proof. □

Finally we are able to find solutions of the modified problem.

LEMMA 3.5. — *For every* $N \geqslant N_0$ *and* $\lambda \geqslant \lambda^* \geqslant \lambda_0$ *(where* λ^* *is a suitable constant) there exists* $\gamma_{\lambda,N} \in C^2(S^2, \Omega)$ *such that*

a) $\alpha \leqslant J_{\lambda,N}(\gamma_{\lambda,N}) \leqslant \beta$ where α and β depend only on Ω.

b) $a_\lambda(\gamma_{\lambda,N})\ddot{\gamma}_{\lambda,N} = \frac{1}{2} |\dot{\gamma}_{\lambda,N}|^2 \nabla a_\lambda(\gamma_{\lambda,N}) - (\nabla a_\lambda(\gamma_{\lambda,N}) \cdot \dot{\gamma}_{\lambda,N})\dot{\gamma}_{\lambda,N} - \nabla U_{\lambda,N}(\gamma_{\lambda,N})$

c) $\alpha \leqslant \frac{1}{2} a_\lambda(\gamma_{\lambda,N}(t)) |\dot{\gamma}_{\lambda,N}(t)|^2 + U_{\lambda,N}(\gamma_{\lambda,N}(t)) \overset{\text{def}}{=} E_{\lambda,N} \leqslant \sigma$ for every $t \in (0, 1)$

where σ is independent of λ and N.

Proof. — By lemma 3.3 and lemma 3.4 the functional $J_{\lambda,N}$ satisfies the assumptions of theorem 2.3. Then there exists $\gamma = \gamma_{\lambda,N} \in \Lambda^1\Omega$ such that

$$J'(\gamma)[\delta\gamma] = 0 \quad \text{for every} \quad \delta\gamma \in H^1(S^1, R^n) \tag{3.28}$$

and

$$J(\gamma) = c_{\lambda,N} \quad \text{with} \quad \alpha \leqslant c_{\lambda,N} \leqslant \beta. \tag{3.29}$$

The above equation proves (*a*). Moreover by (3.28) it follows that $\gamma_{\lambda,N}(t)$ satisfies the equation (*b*) in a weak sense. By standard regularity arguments it follows that γ is of class C^2. Now let us prove (*c*). It is easy to check that

$$\frac{1}{2} a_\lambda(\gamma(t)) |\dot{\gamma}(t)|^2 + U_{\lambda,N}(\gamma(t)) \tag{3.30}$$

is an integral of the equation (*b*) (in fact it is just the energy). Therefore it is independent of t; we shall call $E_{\lambda,N}$ its value. Integrating (3.30) between 0 and 1 we get

$$E_{\lambda,N} = \int \left\{ \frac{1}{2} a_\lambda(\gamma) |\dot{\gamma}|^2 + U_{\lambda,N}(\gamma) \right\} dt. \tag{3.31}$$

Writing (3.29) explicitely we have

$$(3.32)\qquad \alpha \leqslant \int \left\{ \frac{1}{2} a_\lambda(\gamma) \mid \dot{\gamma} \mid^2 - U_{\lambda,N}(\gamma) \right\} dt \leqslant \beta .$$

By (3.31) and (3.32) we get

$$(3.33)\qquad \alpha \leqslant E_{\lambda,N} \leqslant 2 \int U_{\lambda,N}(\gamma) dt + \beta .$$

The above formula gives the first of the inequalities (*b*). In order to get the second one more work is necessary (and it will be necessary, for the first time, to use the assumption (L_3) which has been used to prove lemma 3.1 (*c*)). Writing (3.28) explicitely with $\delta\gamma = v(\gamma) = -\nabla h(\gamma)$ we get

$$(3.34)\qquad \int \left\{ a_\lambda(\gamma) d^2 h[\dot{\gamma}]^2 + \frac{1}{2} \nabla a(\gamma) \cdot v(\gamma) \mid \dot{\gamma} \mid^2 - \nabla U(\gamma) \cdot v(\gamma) \right\} dt = 0 .$$

Now take $M = 4h_0 + 2K$. Then, for $\lambda \geqslant \bar{\lambda}(M)$, we have

$$\begin{aligned}
&\int U_{\lambda,N}(\gamma(t)) dt \\
&\leqslant \frac{1}{M} \int \nabla U_{\lambda,N}(\gamma) \cdot v(\gamma) dt + a(M) \qquad \text{(by lemma 3.1 } (c)) \\
&= \frac{1}{M} \int \left\{ a_\lambda(\gamma) d^2 h[\dot{\gamma}]^2 + \frac{1}{2} \nabla a_\lambda(\gamma) \cdot v(\gamma) \mid \dot{\gamma} \mid^2 \right\} dt + a(M) \qquad \text{(by (3.34))} \\
&\leqslant \frac{h_0}{M} \int a_\lambda(\gamma) \mid \dot{\gamma} \mid^2 dt + \frac{K}{2M} \int a_\lambda(\gamma) \mid \dot{\gamma} \mid^2 dt + a(M) \\
&\qquad\qquad \text{(by (3.4) and lemma 3.1 } (e)) \\
&= \frac{1}{M} (2h_0 + K) \left[\frac{1}{2} \int a_\lambda(\gamma) \mid \dot{\gamma} \mid^2 dt \right] + a(M) \\
&\leqslant \frac{1}{2} \left[\int U_{\lambda,N}(\gamma) dt + \beta \right] + a(M) \qquad \text{(by our choice of M and (3.23))} .
\end{aligned}$$

Then we get

$$\frac{1}{2} \int U_{\lambda,N}(\gamma) dt \leqslant \frac{1}{2} \beta + a(M) .$$

By the above inequality and (3.33) the last inequality (*c*) holds with $\sigma = 3\beta + 2a(M)$ and $\lambda^* = \max(\lambda(M), \lambda_0)$. □

Finally we can prove theorem 1.1.

Proof of Theorem 1.1. — For any $N \geqslant N_0$ choose $\lambda(N) \geqslant \lambda^*$ large enough such that

$$\frac{\theta(\lambda(N))}{N^2} \geqslant \sigma$$

where $\theta(\lambda)$ is defined in lemma 3.1 (*d*). Then setting $\tilde{\gamma}_N(t) = \gamma_{\lambda(N),N}(t)$, by lemma 3.5 (*c*) we have

$$U_{\lambda(N),N}(\tilde{\gamma}_N(t)) \leqslant \sigma \leqslant \frac{\theta(\lambda(N))}{N^2} \quad \text{for every} \quad t \in [0, 1].$$

Thus by lemma (3.1) (*d*), we have that

$$(3.36) \qquad \begin{aligned} a_{\lambda(N)}(\tilde{\gamma}_N(t)) &= a(\tilde{\gamma}_N(t)) \\ U_{\lambda(N),N}(\tilde{\gamma}_N(t)) &= \frac{1}{N^2} U(\tilde{\gamma}(t)) \qquad \text{for every} \quad t \in [0, 1]. \end{aligned}$$

By the above identity we have that

$$(3.37) \qquad J_{\lambda,N}(\tilde{\gamma}_N) = \int \frac{1}{2} a(\tilde{\gamma}_N) | \dot{\tilde{\gamma}}_N |^2 - \frac{1}{N^2} U(\tilde{\gamma}_N).$$

Moreover, using again (3.36), by lemma 3.5 (*c*), $\tilde{\gamma}_N$ satisfy the following equation

$$a(\tilde{\gamma}_N)\ddot{\tilde{\gamma}}_N = \frac{1}{2} | \dot{\tilde{\gamma}}_N |^2 \nabla a(\tilde{\gamma}_N) - (\nabla a(\tilde{\gamma}_N) \cdot \dot{\tilde{\gamma}}_N)\dot{\tilde{\gamma}}_N - \frac{1}{N^2} \nabla U(\tilde{\gamma}_N).$$

Therefore, setting $\gamma_N(t) = \tilde{\gamma}_N(Nt)$, it follows that $\gamma_N(t)$ satisfy the equation

$$a(\gamma_N)\ddot{\gamma}_N = \frac{1}{2} | \dot{\gamma}_N |^2 \nabla a(\gamma_N) - (\nabla \gamma_N \cdot \dot{\gamma}_N)\dot{\gamma}_N - \nabla U(\gamma_N).$$

Then (*i*) and (*ii*) of Theorem 1.1 are proved. By (3.37) and lemma 3.5 (*a*) we have that

$$\alpha N^2 \leqslant \int \frac{1}{2} a(\gamma_N) | \dot{\gamma}_N |^2 - U(\gamma_N) \leqslant \beta N^2.$$

The above inequalities prove (*iv*) of theorem 1.1. Moreover, using again lemma 3.5 (*c*) and (3.36) we get

$$\alpha \leqslant \frac{1}{2} a(\tilde{\gamma}_N) | \dot{\tilde{\gamma}} |^2 + \frac{U(\tilde{\gamma}_N)}{N^2} \leqslant \sigma \quad \text{for every} \quad t \in (0, 1).$$

Therefore

$$\alpha N^2 \leqslant \frac{1}{2} a(\gamma_N) | \dot{\gamma}_N |^2 + U(\gamma_N) \leqslant \sigma N^2.$$

Thus (*iii*) of theorem 1.1 is obtained with $E^- = \alpha$ and $E^+ = \sigma$. The last remark of theorem 1.1 follows by (3.13'). □

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