

ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 1, n° 5 (1984), p. 341-350

http://www.numdam.org/item?id=AIHPC_1984__1_5_341_0

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Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents

by

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ABSTRACT. — In this paper we study the existence of nontrivial solutions for the boundary value problem

$$\begin{cases} -\Delta u - \lambda u - |u|^{2^*-2}u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

when $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \geq 3$, $2^* = \frac{2n}{n-2}$ is the critical exponent for the Sobolev embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, λ is a real parameter.

We prove that there is bifurcation from any eigenvalue λ_j of $-\Delta$ and we give an estimate of the left neighbourhoods $]\lambda_j^*, \lambda_j]$ of λ_j , $j \in \mathbb{N}$, in which the bifurcation branch can be extended. Moreover we prove that, if $\lambda \in]\lambda_j^*, \lambda_j]$, the number of nontrivial solutions is at least twice the multiplicity of λ_j .

⁽¹⁾ ⁽²⁾ Supported by Ministero P. I. (Italy) and by G. N. A. F. A. (CNR).

⁽³⁾ Supported by SFB72 of the Deutsche Forschungsgemeinschaft. The third author is indebted to the University of Bari for its kind hospitality.

The same kind of results holds also when Ω is a compact Riemannian manifold of dimension $n \geq 3$, without boundary and Δ is the relative Laplace-Beltrami operator.

Key-words: Boundary value problem, critical Sobolev exponent, bifurcation, critical points, eigenvalue, variational problem, Riemannian manifold.

RÉSUMÉ. — Dans cet article, nous étudions l'existence de solutions non triviales pour le problème aux limites

$$\begin{cases} -\Delta u - \lambda u - u|u|^{2^*-2} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

où $\Omega \subset \mathbb{R}^n$ est un domaine borné, $n \geq 3$, $2^* = \frac{2n}{n-2}$ est l'exposant critique pour le plongement de Sobolev $H_0^1(\Omega) \subset L^p(\Omega)$, λ est un paramètre réel.

Nous démontrons que toute valeur propre λ_j de $-\Delta$ est une valeur de bifurcation, et nous donnons une estimation des voisinages $[\lambda_j^*, \lambda_j]$ de λ_j où existent des solutions non triviales. Nous montrons en outre que le nombre de celles-ci est au moins le double de la multiplicité de λ_j .

On a les mêmes résultats quand Ω est une variété riemannienne compacte de dimension $n \geq 3$, et Δ l'opérateur de Laplace-Beltrami.

AMS (MOS) Subject Classifications: 35 A 15, 35 J 20, 58 E 99.

INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, $2^* = \frac{2n}{n-2}$ the critical exponent for the Sobolev embedding $H_0^1(\Omega) \rightarrow L^q(\Omega)$. For a real parameter $\lambda \in \mathbb{R}$ consider the boundary value problem

$$(0.1) \quad \begin{cases} -\Delta u - \lambda u - u|u|^{2^*-2} = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

corresponding to the functional $f_\lambda : H_0^1(\Omega) \rightarrow \mathbb{R}$ given by

$$(0.2) \quad f_\lambda(u) = 1/2 \int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) dx - 1/2^* \int_{\Omega} |u|^{2^*} dx.$$

Since the embedding $H_0^1(\Omega) \rightarrow L^{2^*}(\Omega)$ is not compact the functional f_λ in general will not satisfy the Palais-Smale condition.

However, recently Brezis and Nirenberg [5] were able to establish

the existence of positive solutions of (0.1) for any λ in a certain range $]\lambda^*, \lambda_1[$, where $\lambda_j, j \in \mathbb{N}$ ($\lambda_1 < \lambda_2 < \dots$), denote the eigenvalues of the operator $-\Delta : H_0^1(\Omega) \rightarrow H^{-1}(\Omega) = (H_0^1(\Omega))^*$, and $\lambda^* \geq 0$ is some constant depending on n and Ω .

In this paper we study the existence of nontrivial solutions for (0.1) also for $\lambda > \lambda_j$ to obtain bifurcation from any eigenvalue λ_j . We give an estimate of the left neighbourhoods $]\lambda_j^*, \lambda_j]$ of λ_j , in which the bifurcation branch « can be extended »; moreover we prove that, if $\lambda \in]\lambda_j^*, \lambda_j[$, the number of nontrivial solutions of (0.1) is at least twice the multiplicity of λ_j (cp. Theorem 1.1).

Our results are based on the observation that although the Palais-Smale condition does not hold globally for f_λ (cp. Remark 2.3) it is satisfied locally in a certain energy range (cp. Lemma 2.1 or [5, Remark 2.2]).

We observe that the tools used in proving the above results do not depend on the shape of Ω and on the dimension n .

With suitable modifications the existence and bifurcation results also apply to problem (0.1) posed on a compact Riemannian manifold without boundary of dimension $n \geq 3$ (cp. Theorem 1.3).

We thank Prof. H. Brezis for his useful comments.

1. RESULTS

Let $\|u\| = \left(\int_{\Omega} |\nabla u|^2 dx \right)^{1/2}$, $|u|_p = \left(\int_{\Omega} |u|^p dx \right)^{1/p}$ denote the norms in $H_0^1(\Omega)$, $L^p(\Omega)$, respectively, and let

$$S = \inf \{ \|u\|^2 / |u|_{2^*}^2 : u \in H_0^1(\Omega) \setminus \{0\} \}$$

denote the best constant for the embedding $H_0^1(\Omega) \rightarrow L^{2^*}(\Omega)$.

THEOREM 1.1. — *For $\lambda > 0$ let $\lambda_+ = \min \{ \lambda_j | \lambda < \lambda_j \}$, and suppose*

$$\lambda_+ - \lambda < S [\text{meas}(\Omega)]^{-2/n}.$$

Let m be the multiplicity of λ_+ . Then problem (0.1) admits at least m pairs of nontrivial solutions

$$\{ u_k(\lambda), -u_k(\lambda) \} \quad k = 1, \dots, m$$

such that

$$\|u_k(\lambda)\| \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow \lambda_+.$$

■

REMARK 1.2. — If Ω is starshaped, it is well known that (0.1) admits only the trivial solution for $\lambda \leq 0$ (cp. [5] [8]).

A result analogous to Theorem 1.1 holds for the problem

$$(1.1) \quad -\Delta_M u - \lambda u - |u|^{2^*-2} = 0$$

on a compact Riemannian manifold M of dimension ≥ 3 and without boundary. Here Δ_M is the Laplace-Beltrami operator on M , $\lambda \geq 0$ a parameter and $2^* = \frac{2n}{n-2}$ as before. Denote by $H^1(M)$ the closure of $C^\infty(M)$ with respect to the norm

$$\|u\|_M = \left(\int_M (|\nabla u|^2 + |u|^2) dM \right)^{1/2}$$

which in local coordinates on a covering $\{T_h\}$ of M is given by

$$\|u\|_M = \left(\sum_h \int_{T_h} \left(\sum_{i,j=1}^n g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + |u|^2 \right) \sqrt{g} dx \right)^{1/2}$$

g^{ij} denoting the metric tensor, and $g = \det(g^{ij})$. Note that the quadratic form $\int_M |\nabla u|^2 dM$ is only positive semidefinite in $H^1(M)$, then the operator

$$-\Delta_M : H^1(M) \rightarrow H^{-1}(M) := (H^1(M))'$$

possesses eigenvalues $\mu_1 < \mu_2 < \dots \mu_k < \dots$ which are ≥ 0 (cp. Appendix 1 of [4]).

THEOREM 1.3. — *For $\lambda > 0$ let $\mu_+ = \min \{ \mu_j | \lambda < \mu_j \}$ and suppose*

$$\mu_+ - \lambda < S \left(\int_M dM \right)^{-2/n}.$$

Let m be the multiplicity of μ_+ . Then problem (1.1) admits at least m pairs of nonconstant solutions

$$\{ u_k(\lambda), -u_k(\lambda) \} \quad k = 1, \dots, m$$

such that

$$\|u_k(\lambda)\|_M \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow \mu_+.$$

■

2. PROOF OF THEOREMS 1.1, 1.3

The proof of Theorem 1.1 requires some lemmata.

LEMMA 2.1. — *For any $\lambda \in \mathbb{R}$ the functional f_λ (see (0.2)) satisfies the Palais-Smale condition in $\left] -\infty, \frac{1}{n} S^{n/2} \right[$ in the following sense:*

(P. S.) If $c < \frac{1}{n} S^{n/2}$ and $\{u_m\}$ is a sequence in $H_0^1(\Omega)$ such that as $m \rightarrow \infty$ $f_\lambda(u_m) \rightarrow c$, $df_\lambda(u_m) \rightarrow 0$ strongly in $H^{-1}(\Omega)$, then $\{u_m\}$ contains a subsequence converging strongly in $H_0^1(\Omega)$.

REMARK 2.2. — An analogous result has been proved in [5]. Nevertheless for completeness we give here a proof of lemma 2.1 which is slightly different from that contained in [5].

Proof. — Let $\lambda \in \mathbb{R}$, and suppose $\{u_m\}$ is a sequence in $H_0^1(\Omega)$ such that as $m \rightarrow \infty$

$$(2.1) \quad f_\lambda(u_m) \rightarrow c_1 < \frac{1}{n} S^{n/2}$$

$$(2.2) \quad df_\lambda(u_m) \rightarrow 0 \text{ strongly in } H^{-1}(\Omega).$$

As in [5, estimates (2.18)] from (2.1), (2.2) we obtain that

$$(2.3) \quad \{\|u_m\|\} \text{ is bounded.}$$

Hence we may extract a subsequence $\{u_m\}$ (relabelled) such that

$$(2.4) \quad u_m \rightarrow u \text{ weakly in } H_0^1(\Omega)$$

$$(2.5) \quad u_m \rightarrow u \text{ strongly in } L^p(\Omega) \text{ for any } p \in [1, 2^*].$$

Moreover u is a solution of (0.1). Indeed, letting $\phi \in C_0^\infty(\Omega)$, by (2.4), (2.5) and (2.2) we deduce that

$$\langle df_\lambda(u), \phi \rangle = \langle df_\lambda(u_m), \phi \rangle + o(1) = o(1).$$

Hence u weakly solves (0.1). But by regularity results (cp. [5] [6] [7] and [10]) it follows that

$$(2.6) \quad u \in L^\infty(\Omega)$$

and hence that u is regular and is a solution of (0.1) in the classical sense. To show that $u_m \rightarrow u$ strongly in $H_0^1(\Omega)$ as $m \rightarrow \infty$, let $v_m = u_m - u$. Testing (2.2) with v_m we obtain

$$(2.7) \quad o(1) = \langle df_\lambda(u_m), v_m \rangle \\ = \int_{\Omega} (\nabla u \nabla v_m + |\nabla v_m|^2 - \lambda(u + v_m)v_m - |u + v_m|^{2^*-2}(u + v_m)v_m) dx.$$

By (2.4) and (2.5) we have

$$(2.8) \quad \int_{\Omega} (\nabla u \nabla v_m - \lambda(u + v_m)v_m) dx = o(1).$$

Whence from (2.7), (2.8) we deduce that

$$(2.9) \quad \|v_m\|^2 = \int_{\Omega} |u + v_m|^{2^*-2}(u + v_m)v_m dx + o(1).$$

Now we claim that

$$(2.10) \quad \|v_m\|^2 = \|v_m\|_{2^*}^{2^*} + o(1).$$

In fact, by using (2.5) and (2.6), we have

$$\begin{aligned} (2.11) \quad & \left| \int_{\Omega} (u + v_m) |u + v_m|^{2^*-2} v_m dx - \int_{\Omega} |v_m|^{2^*} dx \right| \\ &= \left| \int_{\Omega} \int_0^{u(x)} \frac{\partial}{\partial \xi} [(v_m + \xi) |v_m + \xi|^{2^*-2}] v_m d\xi dx \right| \\ &= \left| (2^* - 1) \int_{\Omega} \int_0^1 |v_m + tu|^{2^*-2} v_m u dt dx \right| \\ &\leq \text{const.} \left[\int_{\Omega} (|u| |v_m|^{2^*-1} + |v_m| |u|^{2^*-1}) dx \right] = o(1) \end{aligned}$$

and (2.10) easily follows from (2.9) and (2.11).

Since

$$\langle df_{\lambda}(u_m), u_m \rangle = o(1)$$

we have

$$\|u_m\|_{2^*}^{2^*} = \int_{\Omega} (|\nabla u_m|^2 - \lambda |u_m|^2) dx + o(1).$$

Inserting into the expression for $f_{\lambda}(u_m)$ we obtain

$$\begin{aligned} (2.12) \quad f_{\lambda}(u_m) &= \frac{1}{n} \int_{\Omega} (|\nabla u_m|^2 - \lambda |u_m|^2) dx + o(1) \\ &= \frac{1}{n} \int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) dx + \frac{1}{n} \int_{\Omega} |\nabla v_m|^2 dx + o(1). \end{aligned}$$

Moreover, since u is a solution of (0.1)

$$\int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) dx - \int_{\Omega} |u|^{2^*} dx = \langle df_{\lambda}(u), u \rangle = 0.$$

Whence in particular

$$(2.13) \quad \int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) dx \geq 0.$$

From (2.12) and (2.13) we now infer

$$\|v_m\|^2 \leq n f_{\lambda}(u_m) + o(1).$$

Then, by (2.1), for m sufficiently large we obtain

$$(2.14) \quad \|v_m\|^2 \leq c_2 < S^{n/2}.$$

Now, by (2.10)

$$\|v_m\|^2 \leq S^{-2^*/2} \|v_m\|^{2^*} + o(1).$$

Or equivalently

$$\|v_m\|^2(S^{2^*/2} - \|v_m\|^{2^*-2}) \leq o(1).$$

Taking account of (2.14) this implies that $v_m \rightarrow 0$ strongly in $H_0^1(\Omega)$, concluding the proof. ■

REMARK 2.3. — Complementing the preceding lemma we have a non-compactness result for energies $\geq \frac{1}{n}S^{n/2}$. In fact we now show that for any $\lambda \in \mathbb{R}$ there exists a sequence $\{u_m\} \subset H_0^1(\Omega)$ satisfying the P-S assumptions in $c = \frac{1}{n}S^{n/2}$, which is not relatively compact in $H_0^1(\Omega)$.

Let $x_0 \in \Omega$ and choose a function $\phi \in C_0^\infty(\Omega)$ such that $\phi \equiv 1$ in a neighbourhood $\mathcal{N}$ of x_0 . The functions $u_\mu: \mathbb{R}^n \rightarrow \mathbb{R}$

$$u_\mu(x) = \frac{[n(n-2)\mu^2]^{\frac{n-2}{4}}}{[\mu^2 + |x - x_0|^2]^{\frac{n-2}{2}}}$$

solve the equation

$$(2.15) \quad -\Delta u_\mu = u_\mu |u_\mu|^{2^*-2} \quad \text{in } \mathbb{R}^n.$$

Let

$$u_m = \phi u_{\mu_m} \quad \mu_m = \frac{1}{m}.$$

Note that $u_m \in H_0^1(\Omega)$ and moreover

$$(2.16) \quad \{u_m\} \text{ is uniformly bounded in } H_0^1(\Omega).$$

Also we easily derive that as $m \rightarrow +\infty$

$$(2.17) \quad \nabla u_{\mu_m} \rightarrow 0 \quad \text{in } L^2(\mathbb{R}^n \setminus \mathcal{N})$$

$$(2.18) \quad u_m \rightarrow 0 \quad \text{in } L_{loc}^\infty(\Omega \setminus \{x_0\}).$$

Hence also

$$(2.19) \quad u_m \rightarrow 0 \text{ weakly in } H_0^1(\Omega) \quad (m \rightarrow \infty).$$

Using (2.17) and (2.18) we deduce that

$$(2.20) \quad \begin{aligned} f_\lambda(u_m) &= 1/2 \int_{\mathbb{R}^n} |\nabla u_{\mu_m}|^2 dx - 1/2^* \int_{\mathbb{R}^n} |u_{\mu_m}|^{2^*} dx + o(1) \\ &= \frac{1}{n} S^{n/2} + o(1) \end{aligned} \quad (\text{cp. [1] [9]}).$$

Also using (2.15)-(2.18) we obtain

$$\|df_\lambda(u_m)\|_{H^{-1}(\Omega)} = \sup_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{H_0^1} = 1}} \int_{\mathbb{R}^n} (\nabla u_{\mu_m} \nabla v - u_{\mu_m} |u_{\mu_m}|^{2^*-2} v) dx + o(1) = o(1)$$

Hence $\{u_m\}$ satisfies the (P-S) assumptions with $c = \frac{1}{n} S^{n/2}$, however, by (2.19) and (2.20), $\{u_m\}$ cannot be relatively compact in $H_0^1(\Omega)$.

LEMMA 2.4. — For $\lambda > 0$ let $\lambda_+ = \inf \{ \lambda_j \mid \lambda < \lambda_j \}$ and set

$$\begin{aligned} M_+ &= \overline{\bigoplus_{\lambda_j \geq \lambda_+} M(\lambda_j)} \quad (\text{the closure is taken in } H_0^1(\Omega)) \\ M_- &= \bigoplus_{\lambda_j \leq \lambda_+} M(\lambda_j) \end{aligned}$$

where $M(\lambda_j)$ denotes the eigenspace of $-\Delta$ corresponding to λ_j . Then

$$\beta_\lambda := \sup_{u \in M_-} f_\lambda(u) \leq (\lambda_+ - \lambda)^{n/2} \frac{\text{meas}(\Omega)}{n},$$

moreover, there exist constants $\rho_\lambda > 0$, $\delta_\lambda \in]0, \beta_\lambda[$ such that

$$f_\lambda(u) \geq \delta_\lambda \quad \text{for any } u \in M_+, \|u\| = \rho_\lambda.$$

Proof. — For any $u \in M_-$ we have

$$\begin{aligned} f_\lambda(u) &= 1/2 \int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) dx - 1/2^* \int_{\Omega} |u|^{2^*} dx \\ &\leq 1/2(\lambda_+ - \lambda) \int_{\Omega} |u|^2 dx - 1/2^* \int_{\Omega} |u|^{2^*} dx \\ &\leq 1/2(\lambda_+ - \lambda) \text{meas}(\Omega)^{2/n} \left\{ \int_{\Omega} |u|^{2^*} dx \right\}^{2/2^*} - 1/2^* \int_{\Omega} |u|^{2^*} dx. \end{aligned}$$

Let

$$g(\rho) = 1/2(\lambda_+ - \lambda) \text{meas}(\Omega)^{2/n} \rho^2 - 1/2^* \rho^{2^*}.$$

Then

$$\sup_{u \in M_-} f_\lambda(u) \leq \sup_{\rho \geq 0} g(\rho) = \frac{1}{n} (\lambda_+ - \lambda)^{n/2} \text{meas}(\Omega)$$

proving the first part of the lemma.

Since for $u \in M_+$ we obtain

$$\int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) dx \geq \left(1 - \frac{\lambda}{\lambda_+}\right) \|u\|^2$$

while

$$\|u\|_{2^*}^{2^*} \leq \text{const} \|u\|^{2^*}.$$

The second part of the claim is immediate. ■

By lemmata 2.1, 2.4, Theorem 1.1 can be deduced by the following result of Bartolo, Benci, Fortunato (cp. Theorem 2.4 of [3]), which is a variant of some results contained in [0].

THEOREM 2.5. — Let H be a real Hilbert space with norm $\|\cdot\|$ and suppose $I \in C^1(H, \mathbb{R})$ is a functional on H satisfying the following conditions:

$$I_1) \quad I(u) = I(-u), \quad I(0) = 0;$$

I₂) There exists a constant $\beta > 0$ such that the Palais-Smale condition (P-S) holds in $]0, \beta[$;

I₃) There exist two closed subspaces $V, W \subset H$ and positive constants ρ, δ, β' , with $\delta < \beta' < \beta$ such that

- i) $I(u) \leq \beta'$ for any $u \in W$
- ii) $I(u) \geq \delta$ for any $u \in V, \|u\| = \rho$
- iii) $\text{codim } V < +\infty$ and $\dim W \geq \text{codim } V$.

Then there exists at least

$$\dim W - \text{codim } V$$

pairs of critical points of I with critical values belonging to the interval $[\delta, \beta']$.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. — Let $H = H_0^1(\Omega)$, $I = f_\lambda$, $V = M_+$, $W = M_-$, $\beta = \frac{1}{n} S^{n/2}$, $\beta' = \beta_\lambda$, $\delta = \delta_\lambda$, $\rho = \rho_\lambda$ and apply Theorem 2.5 together with lemmata 2.1, 2.4. ■

For the proof of Theorem 1.3 the following result from [2] is needed.

LEMMA 2.6. — If $\{v_m\}$ is a sequence in $H^1(M)$ such that $v_m \rightarrow 0$ weakly in $H^1(M)$ as $m \rightarrow \infty$, then

$$\left(\int_M |v_m|^{2^*} dM \right)^{2/2^*} \leq S^{-1} \|v_m\|_M^2 + o(1).$$

Proof. — By [2, Theorem 2.21] for all $\phi \in H^1(M)$, $\varepsilon > 0$

$$\left(\int_M |\phi|^{2^*} dM \right)^{2/2^*} \leq (S^{-1} + \varepsilon) \int_M |\nabla \phi|^2 dM + A(\varepsilon) \int_M |\phi|^2 dM$$

with a constant $A(\varepsilon)$ independent of ϕ . Applying this inequality with $\phi = v_m$, and noting that by weak convergence $v_m \rightarrow 0$ ($m \rightarrow +\infty$) we have

$$\int_M |v_m|^2 dM \rightarrow 0 \quad m \rightarrow +\infty$$

we deduce that for any $\varepsilon > 0$

$$\left(\int_M |v_m|^{2^*} dM \right)^{2/2^*} \leq (S^{-1} + \varepsilon) \|v_m\|_M^2 + o(1).$$

The lemma follows on letting $\varepsilon \rightarrow 0$. ■

Proof of Theorem 1.3. — Going through the proof of Lemma 2.1 — keeping in mind Lemma 2.6 and the fact that, for any sequence $\{v_m\}$

in $H^1(M)$ tending to 0 weakly in this space, $\|v_m\|_2 = o(1)$ — it is now immediate that also for the functional on $H^1(M)$

$$f_\lambda(u) = 1/2 \int_M (|\nabla u|^2 - \lambda |u|^2) dM - 1/2^* \int_M |u|^{2^*} dM$$

corresponding to problem (1.1) the Palais-Smale condition is satisfied in the interval $\left] -\infty, \frac{1}{n} S^{n/2} \right[$.

Moreover it is easy to see that the same estimates of lemma 2.4 continue to hold (obviously $\lambda_j, \lambda_+, H_0^1(\Omega)$, meas Ω are replaced respectively by $\mu_j, \mu_+, H^1(M), \int_M dM$). Then Theorem 1.3 can be proved by using again the abstract critical point Theorem 2.5. ■

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(Manuscrit reçu le 13 février 1984)