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Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents

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by

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ABSTRACT. — In this paper we study the existence of nontrivial solutions for the boundary value problem

$$\begin{cases} -\Delta u - \lambda u - u\,|\,u\,|^{2^*-2} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

when $\Omega \subset \mathbf{R}^n$ is a bounded domain, $n \geqslant 3$, $2^* = \dfrac{2n}{n-2}$ is the critical exponent for the Sobolev embedding $H_0^1(\Omega) \subset L^p(\Omega)$, λ is a real parameter.

We prove that there is bifurcation from any eigenvalue λ_j of $-\Delta$ and we give an estimate of the left neighbourhoods $]\lambda_j{}^*, \lambda_j]$ of λ_j, $j \in \mathbf{N}$, in which the bifurcation branch can be extended. Moreover we prove that, if $\lambda \in \,]\lambda_j{}^*, \lambda_j[$, the number of nontrivial solutions is at least twice the multiplicity of λ_j.

The same kind of results holds also when Ω is a compact Riemannian manifold of dimension $n \geqslant 3$, without boundary and Δ is the relative Laplace-Beltrami operator.

RÉSUMÉ. — Dans cet article, nous étudions l'existence de solutions non triviales pour le problème aux limites

$$\begin{cases} -\Delta u - \lambda u - u \mid u \mid^{2^{*}-2} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

où $\Omega \subset \mathbb{R}^{n}$ est un domaine borné, $n \geq 3$, $2^{*} = \dfrac{2n}{n-2}$ est l'exposant critique pour le plongement de Sobolev $H_{0}^{1}(\Omega) \subset L^{p}(\Omega)$, λ est un paramètre réel.

Nous démontrons que toute valeur propre λ_{j} de $-\Delta$ est une valeur de bifurcation, et nous donnons une estimation des voisinages $[\lambda_{j}^{*}, \lambda_{j}]$ de λ_{j} où existent des solutions non triviales. Nous montrons en outre que le nombre de celles-ci est au moins le double de la multiplicité de λ_{j}.

On a les mêmes résultats quand Ω est une variété riemannienne compacte de dimension $n \geq 3$, et Δ l'opérateur de Laplace-Beltrami.

INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^{n}$, $n \geqslant 3$, $2^{*} = \dfrac{2n}{n-2}$ the critical exponent for the Sobolev embedding $H_{0}^{1}(\Omega) \to L^{q}(\Omega)$. For a real parameter $\lambda \in \mathbb{R}$ consider the boundary value problem

$$(0.1) \qquad \begin{cases} -\Delta u - \lambda u - u \mid u \mid^{2^{*}-2} = 0 & \text{in } \Omega \\ u \mid_{\partial\Omega} = 0 \end{cases}$$

corresponding to the functional $f_{\lambda} : H_{0}^{1}(\Omega) \to \mathbb{R}$ given by

$$(0.2) \qquad f_{\lambda}(u) = 1/2 \int_{\Omega} (\mid \nabla u \mid^{2} - \lambda \mid u \mid^{2}) dx - 1/2^{*} \int_{\Omega} \mid u \mid^{2^{*}} dx .$$

Since the embedding $H_{0}^{1}(\Omega) \to L^{2^{*}}(\Omega)$ is not compact the functional f_{λ} in general will not satisfy the Palais-Smale condition.

However, recently Brezis and Nirenberg [5] were able to establish

the existence of positive solutions of (0.1) for any λ in a certain range $]\lambda^*, \lambda_1[$, where $\lambda_j, j \in \mathbb{N}$ ($\lambda_1 < \lambda_2 < \ldots$), denote the eigenvalues of the operator $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega) = (H_0^1(\Omega))^*$, and $\lambda^* \geqslant 0$ is some constant depending on n and Ω.

In this paper we study the existence of nontrivial solutions for (0.1) also for $\lambda > \lambda_j$ to obtain bifurcation from any eigenvalue λ_j. We give an estimate of the left neighbourhoods $]\lambda_j^*, \lambda_j]$ of λ_j, in which the bifurcation branch « can be extended »; moreover we prove that, if $\lambda \in]\lambda_j^*, \lambda_j[$, the number of nontrivial solutions of (0.1) is at least twice the multiplicity of λ_j (cp. Theorem 1.1).

Our results are based on the observation that although the Palais-Smale condition does not hold globally for f_λ (cp. Remark 2.3) it is satisfied locally in a certain energy range (cp. Lemma 2.1 or [5, Remark 2.2]).

We observe that the tools used in proving the above results do not depend on the shape of Ω and on the dimension n.

With suitable modifications the existence and bifurcation results also apply to problem (0.1) posed on a compact Riemannian manifold without boundary of dimension $n \geqslant 3$ (cp. Theorem 1.3).

We thank Prof. H. Brezis for his useful comments.

1. RESULTS

Let $\| u \| = \left(\int_\Omega | \nabla u |^2 dx \right)^{1/2}$, $| u |_p = \left(\int_\Omega | u |^p dx \right)^{1/p}$ denote the norms in $H_0^1(\Omega)$, $L^p(\Omega)$, respectively, and let

$$S = \inf \{ \| u \|^2 / | u |_{2*}^2 : u \in H_0^1(\Omega) \backslash \{ 0 \} \}$$

denote the best constant for the embedding $H_0^1(\Omega) \to L^{2*}(\Omega)$.

THEOREM 1.1. — *For $\lambda > 0$ let $\lambda_+ = \min \{ \lambda_j | \lambda < \lambda_j \}$, and suppose*

$$\lambda_+ - \lambda < S [\text{meas} (\Omega)]^{-2/n} .$$

Let m be the multiplicity of λ_+. Then problem (0.1) admits at least m pairs of nontrivial solutions

$$\{ u_k(\lambda), - u_k(\lambda) \} \qquad k = 1, \ldots, m$$

such that

$$\| u_k(\lambda) \| \to 0 \qquad as \qquad \lambda \to \lambda_+ .$$ ∎

REMARK 1.2. — If Ω is starshaped, it is well known that (0.1) admits only the trivial solution for $\lambda \leqslant 0$ (cp. [5] [8]).

A result analogous to Theorem 1.1 holds for the problem

$$(1.1) \qquad\qquad -\Delta_M u - \lambda u - u\,|\,u\,|^{2^*-2} = 0$$

on a compact Riemannian manifold M of dimension $\geqslant 3$ and without boundary. Here Δ_M is the Laplace-Beltrami operator on M, $\lambda \geqslant 0$ a parameter and $2^* = \dfrac{2n}{n-2}$ as before. Denote by $H^1(M)$ the closure of $C^\infty(M)$ with respect to the norm

$$\|\,u\,\|_M = \left(\int_M (\,|\,\nabla u\,|^2 + |\,u\,|^2)dM\right)^{1/2}$$

which in local coordinates on a covering $\{\,T_h\,\}$ of M is given by

$$\|\,u\,\|_M = \left(\sum_h \int_{T_h} \left(\sum_{i,j=1}^n g^{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + |\,u\,|^2\right)\sqrt{g}dx\right)^{1/2}$$

g^{ij} denoting the metric tensor, and $g = \det(g^{ij})$. Note that the quadratic form $\displaystyle\int_M |\,\nabla u\,|^2 dM$ is only positive semidefinite in $H^1(M)$, then the operator

$$-\Delta_M : H^1(M) \rightarrow H^{-1}(M) := (H^1(M))'$$

possesses eigenvalues $\mu_1 < \mu_2 < \ldots \mu_k < \ldots$ which are $\geqslant 0$ (cp. Appendix 1 of [4]).

THEOREM 1.3. — *For $\lambda > 0$ let $\mu_+ = \min\{\,\mu_j\,|\,\lambda < \mu_j\,\}$ and suppose*

$$\mu_+ - \lambda < S\left(\int_M dM\right)^{-2/n}.$$

Let m be the multiplicity of μ_+. Then problem (1.1) admits at least m pairs of nonconstant solutions

$$\{\,u_k(\lambda),\,-u_k(\lambda)\,\} \qquad k = 1, \ldots, m$$

such that
$$\|\,u_k(\lambda)\,\|_M \rightarrow 0 \qquad as \qquad \lambda \rightarrow \mu_+.$$ ■

2. PROOF OF THEOREMS 1.1, 1.3

The proof of Theorem 1.1 requires some lemmata.

LEMMA 2.1. — *For any $\lambda \in \mathbb{R}$ the functional f_λ (see (0.2)) satisfies the Palais-Smale condition in $\left]-\infty, \dfrac{1}{n}S^{n/2}\right[$ in the following sense:*

(P. S.)   *If* $c < \dfrac{1}{n} S^{n/2}$ *and* $\{u_m\}$ *is a sequence in* $H_0^1(\Omega)$ *such that as* $m \to \infty$

$f_\lambda(u_m) \to c$, $df_\lambda(u_m) \to 0$ *strongly in* $H^{-1}(\Omega)$, *then* $\{u_m\}$ *contains a subsequence converging strongly in* $H_0^1(\Omega)$.

REMARK 2.2. — An analogous result has been proved in [5]. Nevertheless for completeness we give here a proof of lemma 2.1 which is slightly different from that contained in [5].

Proof. — Let $\lambda \in \mathbb{R}$, and suppose $\{u_m\}$ is a sequence in $H_0^1(\Omega)$ such that as $m \to \infty$

(2.1) $f_\lambda(u_m) \to c_1 < \dfrac{1}{n} S^{n/2}$

(2.2) $df_\lambda(u_m) \to 0$ strongly in $H^{-1}(\Omega)$.

As in [5, estimates (2.18)] from (2.1), (2.2) we obtain that

(2.3) $\{ \| u_m \| \}$ is bounded.

Hence we may extract a subsequence $\{ u_m \}$ (relabeled) such that

(2.4) $u_m \to u$ weakly in $H_0^1(\Omega)$

(2.5) $u_m \to u$ strongly in $L^p(\Omega)$ for any $p \in [1, 2^*[$.

Moreover u is a solution of (0.1). Indeed, letting $\phi \in C_0^\infty(\Omega)$, by (2.4), (2.5) and (2.2) we deduce that

$$\langle df_\lambda(u), \phi \rangle = \langle df_\lambda(u_m), \phi \rangle + o(1) = o(1).$$

Hence u weakly solves (0.1). But by regularity results (cp. [5] [6] [7] and [10]) it follows that

(2.6) $u \in L^\infty(\Omega)$

and hence that u is regular and is a solution of (0.1) in the classical sense. To show that $u_m \to u$ strongly in $H_0^1(\Omega)$ as $m \to \infty$, let $v_m = u_m - u$. Testing (2.2) with v_m we obtain

(2.7) $o(1) = \langle df_\lambda(u_m), v_m \rangle$

$$= \int_\Omega (\nabla u \nabla v_m + | \nabla v_m |^2 - \lambda(u + v_m)v_m - | u + v_m |^{2^*-2}(u + v_m)v_m) dx.$$

By (2.4) and (2.5) we have

(2.8) $\int_\Omega (\nabla u \nabla v_m - \lambda(u + v_m)v_m) dx = o(1).$

Whence from (2.7), (2.8) we deduce that

(2.9) $\| v_m \|^2 = \int_\Omega | u + v_m |^{2^*-2}(u + v_m)v_m \, dx + o(1).$

Now we claim that

(2.10) $$\| v_m \|^2 = | v_m |_{2^*}^{2^*} + o(1).$$

In fact, by using (2.5) and (2.6), we have

(2.11) $$\left| \int_\Omega (u + v_m) | u + v_m |^{2^* - 2} v_m dx - \int_\Omega | v_m |^{2^*} dx \right|$$

$$= \left| \int_\Omega \int_0^{u(x)} \frac{\partial}{\partial \xi} [(v_m + \xi) | v_m + \xi |^{2^* - 2}] v_m d\xi dx \right|$$

$$= \left| (2^* - 1) \int_\Omega \int_0^1 | v_m + tu |^{2^* - 2} v_m u dt dx \right|$$

$$\leqslant \text{const.} \left[\int_\Omega (| u | | v_m |^{2^* - 1} + | v_m | | u |^{2^* - 1}) dx \right] = o(1)$$

and (2.10) easily follows from (2.9) and (2.11).
Since

$$\langle df_\lambda(u_m), u_m \rangle = o(1)$$

we have

$$| u_m |_{2^*}^{2^*} = \int_\Omega (| \nabla u_m |^2 - \lambda | u_m |^2) dx + o(1).$$

Inserting into the expression for $f_\lambda(u_m)$ we obtain

(2.12) $$f_\lambda(u_m) = \frac{1}{n} \int_\Omega (| \nabla u_m |^2 - \lambda | u_m |^2) dx + o(1)$$

$$= \frac{1}{n} \int_\Omega (| \nabla u |^2 - \lambda | u |^2) dx + \frac{1}{n} \int_\Omega | \nabla v_m |^2 dx + o(1).$$

Moreover, since u is a solution of (0.1)

$$\int_\Omega (| \nabla u |^2 - \lambda | u |^2) dx - \int_\Omega | u |^{2^*} dx = \langle df_\lambda(u), u \rangle = 0.$$

Whence in particular

(2.13) $$\int_\Omega (| \nabla u |^2 - \lambda | u |^2) dx \geqslant 0.$$

From (2.12) and (2.13) we now infer

$$\| v_m \|^2 \leqslant nf_\lambda(u_m) + o(1).$$

Then, by (2.1), for m sufficiently large we obtain

(2.14) $$\| v_m \|^2 \leqslant c_2 < S^{n/2}.$$

Now, by (2.10)

$$\| v_m \|^2 \leqslant S^{-2^*/2} \| v_m \|^{2^*} + o(1).$$

Or equivalently

$$\| v_m \|^2 (S^{2^*/2} - \| v_m \|^{2^* - 2}) \leqslant o(1) .$$

Taking account of (2.14) this implies that $v_m \to 0$ strongly in $H_0^1(\Omega)$, concluding the proof. ∎

REMARK 2.3. — Complementing the preceding lemma we have a non-compactness result for energies $\geqslant \dfrac{1}{n} S^{n/2}$. In fact we now show that for any $\lambda \in \mathbb{R}$ there exists a sequence $\{ u_m \} \subset H_0^1(\Omega)$ satisfying the P-S assumptions in $c = \dfrac{1}{n} S^{n/2}$, which is not relatively compact in $H_0^1(\Omega)$.

Let $x_0 \in \Omega$ and choose a function $\phi \in C_0^\infty(\Omega)$ such that $\phi \equiv 1$ in a neighbourhood $\mathcal{N}$ of x_0. The functions $u_\mu : \mathbb{R}^n \to \mathbb{R}$

$$u_\mu(x) = \frac{[n(n - 2)\mu^2]^{\frac{n-2}{4}}}{[\mu^2 + | x - x_0 |^2]^{\frac{n-2}{2}}}$$

solve the equation

(2.15) $$- \Delta u_\mu = u_\mu | u_\mu |^{2^* - 2} \qquad \text{in } \mathbb{R}^n .$$

Let

$$u_m = \phi u_{\mu_m} \qquad \mu_m = \frac{1}{m} .$$

Note that $u_m \in H_0^1(\Omega)$ and moreover

(2.16) $$\{ u_m \} \text{ is uniformly bounded in } H_0^1(\Omega) .$$

Also we easily derive that as $m \to + \infty$

(2.17) $$\nabla u_{\mu_m} \to 0 \qquad \text{in } L^2(\mathbb{R}^n \backslash \mathcal{N})$$

(2.18) $$u_m \to 0 \qquad \text{in } L_{\text{loc}}^\infty(\Omega \backslash \{ x_0 \}) .$$

Hence also

(2.19) $$u_m \to 0 \text{ weakly in } H_0^1(\Omega) \qquad (m \to \infty) .$$

Using (2.17) and (2.18) we deduce that

(2.20) $$f_\lambda(u_m) = 1/2 \int_{\mathbb{R}^n} | \nabla u_{\mu_m} |^2 dx - 1/2^* \int_{\mathbb{R}^n} | u_{\mu_m} |^{2^*} dx + o(1)$$

$$= \frac{1}{n} S^{n/2} + o(1) \qquad\qquad (\text{cp. } [1] [9]) .$$

Also using (2.15)-(2.18) we obtain

$$\| df_\lambda(u_m) \|_{H^{-1}(\Omega)} = \sup_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{H_0^1} = 1}} \int_{\mathbb{R}^n} (\nabla u_{\mu_m} \nabla v - u_{\mu_m} | u_{\mu_m} |^{2^* - 2} v) dx + o(1) = o(1)$$

Hence $\{ u_m \}$ satisfies the (P-S) assumptions with $c = \dfrac{1}{n} S^{n/2}$, however, by (2.19) and (2.20), $\{ u_m \}$ cannot be relatively compact in $H^1_0(\Omega)$.

LEMMA 2.4. — *For* $\lambda > 0$ *let* $\lambda_+ = \inf \{ \lambda_j \,|\, \lambda < \lambda_j \}$ *and set*

$$M_+ = \overline{\underset{\lambda_j \geqslant \lambda_+}{\oplus} M(\lambda_j)} \qquad \text{(the closure is taken in } H^1_0(\Omega))$$

$$M_- = \underset{\lambda_j \leqslant \lambda_+}{\oplus} M(\lambda_j)$$

where $M(\lambda_j)$ *denotes the eigenspace of* $-\Delta$ *corresponding to* λ_j. *Then*

$$\beta_\lambda := \sup_{u \in M_-} f_\lambda(u) \leqslant (\lambda_+ - \lambda)^{n/2} \frac{\text{meas}\,(\Omega)}{n} \ ,$$

moreover, there exist constants $\rho_\lambda > 0$, $\delta_\lambda \in \,]0, \beta_\lambda[$ *such that*

$$f_\lambda(u) \geqslant \delta_\lambda \qquad \text{for any } u \in M_+, \ \| u \| = \rho_\lambda .$$

Proof. — For any $u \in M_-$ we have

$$f_\lambda(u) = 1/2 \int_\Omega (\,|\,\nabla u\,|^2 - \lambda\,|\,u\,|^2)dx - 1/2^* \int_\Omega |\,u\,|^{2^*}dx$$

$$\leqslant 1/2(\lambda_+ - \lambda) \int_\Omega |\,u\,|^2 dx - 1/2^* \int_\Omega |\,u\,|^{2^*}dx$$

$$\leqslant 1/2(\lambda_+ - \lambda) \,\text{meas}\,(\Omega)^{2/n} \left\{ \int_\Omega |\,u\,|^{2^*}dx \right\}^{2/2^*} - 1/2^* \int_\Omega |\,u\,|^{2^*}dx .$$

Let

$$g(\rho) = 1/2(\lambda_+ - \lambda) \,\text{meas}\,(\Omega)^{2/n}\rho^2 - 1/2^*\rho^{2^*} .$$

Then

$$\sup_{u \in M_-} f_\lambda(u) \leqslant \sup_{\rho \geqslant 0} g(\rho) = \frac{1}{n}(\lambda_+ - \lambda)^{n/2} \,\text{meas}\,(\Omega)$$

proving the first part of the lemma.
Since for $u \in M_+$ we obtain

$$\int_\Omega (\,|\,\nabla u\,|^2 - \lambda\,|\,u\,|^2)dx \geqslant \left(1 - \frac{\lambda}{\lambda_+} \right) \| u \|^2$$

while

$$|\,u\,|_{2^*}^{2^*} \leqslant \text{const}\, \| u \|^{2^*} .$$

The second part of the claim is immediate.   ■

By lemmata 2.1, 2.4, Theorem 1.1 can be deduced by the following result of Bartolo, Benci, Fortunato (cp. Theorem 2.4 of [3]), which is a variant of some results contained in [0].

THEOREM 2.5. — *Let* H *be a real Hilbert space with norm* $\| \cdot \|$ *and suppose* $I \in C^1(H, \mathbb{R})$ *is a functional on* H *satisfying the following conditions:*

I_1) $I(u) = I(-u)$, $I(0) = 0$;

I_2) *There exists a constant $\beta > 0$ such that the Palais-Smale condition (P-S) holds in* $]0, \beta[$;

I_3) *There exist two closed subspaces* V, W $\subset$ H *and positive constants ρ, δ, β', with $\delta < \beta' < \beta$ such that*

   i) $I(u) \leqslant \beta'$ *for any $u \in$ W*
   ii) $I(u) \geqslant \delta$ *for any $u \in$ V, $\| u \| = \rho$*
   iii) codim V $< + \infty$ *and* dim W $\geqslant$ codim V .

Then there exists at least
$$\text{dim W} - \text{codim V}$$
pairs of critical points of I *with critical values belonging to the interval* $[\delta, \beta']$.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. — Let H $= H_0^1(\Omega)$, I $= f_\lambda$, V $=$ M$_+$, W $=$ M$_-$, $\beta = \dfrac{1}{n} S^{n/2}$, $\beta' = \beta_\lambda$, $\delta = \delta_\lambda$, $\rho = \rho_\lambda$ and apply Theorem 2.5 together with lemmata 2.1, 2.4. ∎

For the proof of Theorem 1.3 the following result from [2] is needed.

LEMMA 2.6. — *If $\{ v_m \}$ is a sequence in $H^1(M)$ such that $v_m \rightharpoonup 0$ weakly in $H^1(M)$ as $m \to \infty$, then*

$$\left(\int_M | v_m |^{2^*} d M \right)^{2/2^*} \leqslant S^{-1} \| v_m \|_M^2 + o(1).$$

Proof. — By [2, Theorem 2.21] for all $\phi \in H^1(M)$, $\varepsilon > 0$

$$\left(\int_M | \phi |^{2^*} d M \right)^{2/2^*} \leqslant (S^{-1} + \varepsilon) \int_M | \nabla \phi |^2 d M + A(\varepsilon) \int_M | \phi |^2 d M$$

with a constant $A(\varepsilon)$ independent of ϕ. Applying this inequality with $\phi = v_m$, and noting that by weak convergence $v_m \to 0$ $(m \to + \infty)$ we have

$$\int_M | v_m |^2 d M \to 0 \qquad m \to + \infty$$

we deduce that for any $\varepsilon > 0$

$$\left(\int_M | v_m |^{2^*} d M \right)^{2/2^*} \leqslant (S^{-1} + \varepsilon) \| v_m \|_M^2 + o(1).$$

The lemma follows on letting $\varepsilon \to 0$. ∎

Proof of Theorem 1.3. — Going through the proof of Lemma 2.1 — keeping in mind Lemma 2.6 and the fact that, for any sequence $\{ v_m \}$

in $H^1(M)$ tending to 0 weakly in this space, $\| v_m \|_2 = o(1)$ — it is now imme-
diate that also for the functional on $H^1(M)$

$$f_\lambda(u) = 1/2 \int_M (|\nabla u|^2 - \lambda |u|^2)dM - 1/2^* \int |u|^{2^*}dM$$

corresponding to problem (1.1) the Palais-Smale condition is satisfied
in the interval $\left] - \infty, \frac{1}{n} S^{n/2} \right[$.

Moreover it is easy to see that the same estimates of lemma 2.4 continue
to hold (obviously λ_j, λ_+, $H_0^1(\Omega)$, meas Ω are replaced respectively by $\mu_j \cdot \mu_+$.
$H^1(M)$, $\int_M dM$). Then Theorem 1.3 can be proved by using again the abstract
critical point Theorem 2.5. ■

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