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# Harnack inequalities for quasi-minima of variational integrals

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ABSTRACT. — In his fundamental work on linear elliptic equations, De Giorgi established local bounds and Hölder estimates for functions satisfying certain integral inequalities. The main result of this paper is that the Harnack inequality can be proved directly for functions in the De Giorgi classes. This implies that every non-negative Q-minimum (in the terminology of Giaquinta and Giusti) satisfies a Harnack inequality.

RÉSUMÉ. — Dans son travail fondamental sur les équations linéaires elliptiques, De Giorgi a donné des estimations locales et hölderiennes pour des fonctions satisfaisant certaines inégalités intégrales. Le résultat principal de cet article est que l'inégalité de Harnack peut être démontrée directement pour les fonctions appartenant aux classes de De Giorgi. Ceci implique que tout Q-minimum (au sens de Giaquinta et Giusti) non-négatif vérifie une inégalité de Harnack.

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## 1. INTRODUCTION

In his fundamental work on linear elliptic equations, De Giorgi [*1*] established local bounds and Hölder estimates for functions satisfying certain integral inequalities. His analysis was further developed by Ladyzhenskaya and Ural'tseva [*5*] and applied to a wide range of quasilinear elliptic and parabolic equations.

Through a different approach, Moser [*9*] established a Harnack inequality for linear elliptic equations which was extended to quasilinear equations by Serrin [*10*] and Trudinger [*11*].

The main result of this paper is that Harnack inequality can be proved directly for functions in the De Giorgi classes.

Let $\Omega$ be an open set in $\mathbb{R}^N$ and $m > 1$. The De Giorgi classes $DG_m^{\pm}(\Omega)$ are defined to consist of functions $u$ in the Sobolev space $W_{loc}^{1,m}(\Omega)$, which satisfy for any ball $B_R = B_R(y) \subset \Omega$, $\sigma \in (0, 1)$, $k \geq 0$, inequalities of the form

$$(1.1)\quad \int_{B_{\sigma R}} |\nabla(u-k)^{\pm}|^m \leq \gamma\left\{\frac{1}{(1-\sigma)^m R^m}\int_{B_R} |(u-k)^{\pm}|^m + (\chi^m + (R^{-\alpha}k)^m)\,|A_{k,R}^{\pm}|^{1-\frac{m}{N}+\varepsilon}\right.$$

where $\gamma$, $\chi$ and $\varepsilon$ are non-negative constants, $0 < \varepsilon \leq m/N$, $\alpha = N\varepsilon/m$ and

$$A_{k,R}^{\pm} \equiv \{\, x \in B_R \mid (u-k)^{\pm} > 0 \,\},$$

and $|\Sigma|$ denotes the Lebesgue measure of the set $\Sigma$.

We further define the De Giorgi classes $DG_m(\Omega)$ by

$$DG_m(\Omega) = DG_m^+(\Omega) \cap DG_m^-(\Omega)$$

and refer to these classes as homogeneous when $\chi = 0$.

We can now assert the following Harnack type inequalities.

THEOREM 1. — *Let* $u \in DG_m^{\pm}(\Omega)$, $B_E = B_E(y) \subset \Omega$. *Then for any* $\sigma \in (0, 1)$, $p > 0$

$$\sup_{B_{\sigma R}} u^{\pm} \leq C(1-\sigma)^{-m/p\varepsilon}\left\{\left(\fint_{B_R} (u^{\pm})^p\right)^{1/p} + \chi R^{\alpha}\right\}$$

*where* C *depends only on* $m$, N, $\gamma$, $\varepsilon$ *and* $p$.

Here we have set

$$\fint_{B_R} v^p = |B_R|^{-1}\int_{B_R} v^p .$$

THEOREM 2. — *Let* $u \geq 0$ *and* $u \in DG_m^-(\Omega)$, $B_R = B_R(y) \subset \Omega$. *Then there* *exists a positive constant p depending only on m,* N, $\gamma$, $\varepsilon$ *such that for any* $\sigma, \tau \in (0, 1)$ *we have*

$$\left( \fint_{B_{\sigma R}} u^p \right)^{1/p} \leq C(\inf_{B_{\tau R}} u + \chi R^\alpha)$$

*where* C *depends only on m,* N, $\gamma$, $\varepsilon$, $\sigma$, $\tau$.

Combining Theorems 1 and 2 we have the full Harnack inequality.

THEOREM 3. — *Let* $u \geq 0$ *and* $u \in DG_m(\Omega)$, $B_R = B_R(y) \subset \Omega$. *Then for any* $\sigma \in (0, 1)$

$$\sup_{B_{\sigma R}} u \leq C(\inf_{B_{\sigma R}} u + \chi R^\alpha)$$

*where* C *depends only on* N, *m*, $\gamma$, $\varepsilon$, $\sigma$.

It is well known that weak solutions of quasilinear elliptic equations in divergence form, under appropriate structure conditions, belong to $DG_m(\Omega)$; [*5*]. Therefore our work provides alternative proofs of the Harnack inequalities in [*9*] [*10*] and [*11*]. However our main motivation comes from quasi-minima in the calculus of variations. Consider the functional

$$J(u, \Omega) = \int_\Omega f(x, u, \nabla u) \tag{1.2}$$

for $f$ satisfying the usual Caratheodory conditions on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ and the structure conditions

$$|p|^m - b|z|^m - g(x) \leq f(x, z, p) \leq \mu |p|^m + b|z|^m + g(x) \tag{1.3}$$

for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, where $m$, $\mu$, $b$ are non-negative constants and $g$ a non-negative function, $m > 1$. In the terminology of Giaquinta and Giusti [*3*], $u$ is a Q-minimum for J if $Q \geq 1$ and

$$J(u, K) \leq QJ(u + \phi, K) \tag{1.4}$$

for every $\phi \in W^{1,m}(\Omega)$ with supp $\phi \subset K$. In [*2*] it was demonstrated that if $u \in W^{1,m}_{loc}(\Omega)$ is a Q-minimum, then $u$ satisfies inequalities like (1.1) and therefore is locally bounded and Hölder continuous. Our results imply that every non-negative Q-minimum satisfies a Harnack inequality (under appropriate integrability conditions on $g$), which is homogeneous when $g \equiv 0$.

The main tools in our proof consist of a suitable modification of the De Giorgi estimates, as presented in [*5*], and a fundamental covering lemma due to Krylov and Safonov [*6*] and used by them in their treatment of equations in non-divergence form.

Theorems 1 and 2 are proved in Sections 2 and 3. In the last section we consider the application of these results to quasi-minima.

## 2. PROOF OF THEOREM 1

The proof of the following lemma closely follows [5], except that we are more careful about constant dependence.

LEMMA 2.1. — *Let* $u \in \mathrm{DG}_m^{\pm}(\Omega)$, $B_R = B_R(y) \subset \Omega$. *Then for every* $\sigma \in (0, 1)$, *we have*

$$\sup_{B_{\sigma R}} u^{\pm} \le \frac{C}{(1-\sigma)^{1/\varepsilon}} \left\{ \left( \int_{B_R} (u^{\pm})^m \right)^{1/m} + \chi R^{\alpha} \right\}, \tag{2.1}$$

*where* C *depends only upon* $m$, N, $\gamma$, $\varepsilon$ *and* $\alpha = \mathrm{N}\varepsilon/m$.

*Proof.* — We normalize so that $R = 1$; this has the effect of replacing $\chi$ by $\chi R^{\alpha}$ in the final result. Taking some $k > 0$, to be chosen later, we set

$$k_n = k(1 - 2^{-n}), \qquad n = 0, 1, 2, \ldots,$$

and for fixed $\sigma > 0$, consider the sequence of radii

$$R_n = \sigma + 2^{-n}(1-\sigma), \bar{R}_n = \frac{1}{2}(R_n + R_{n+1}) = \sigma + \frac{3}{4} 2^{-n}(1-\sigma), \quad n = 0, 1, 2, \ldots,$$

and the corresponding balls, $B_n = B_{R_n}$, $\tilde{B}_n = B_{\bar{R}_n}$. Observing that

$$(R_n - R_{n+1})^{-1} = \frac{2^{n+1}}{1-\sigma}, \qquad (R_n - \bar{R}_n)^{-1} = \frac{2^{n+2}}{1-\sigma},$$

we let $\zeta_n$ be a cut-off function in $\tilde{B}_n$ such that $\zeta_n = 1$ on $B_{n+1}$ and $|\nabla \zeta_n| \le 2^{n+2}/(1-\sigma)$. Let us consider the case $m < \mathrm{N}$; the case $m = \mathrm{N}$ follows by minor modification while the case $m > \mathrm{N}$ can be deduced directly from the Sobolev imbedding theorem. Applying now the Sobolev imbedding theorem, (Theorem 7.10 of [4]), to (1.1), we obtain, for $u \in \mathrm{DG}_m^{+}(\Omega)$,

$$\begin{aligned} (2.2) \int_{B_{n+1}} & |(u - k_{n+1})^+|^m \\ & \le \int_{\tilde{B}_n} |(u - k_{n+1})^+ \zeta|^m \\ & \le \left( \int_{\tilde{B}_n} |(u - k)_{n+1})^+ \zeta|^{m^*} \right)^{m/m^*} |A_n|^{1 - m/m^*} \\ & \le C \left\{ \int_{\tilde{B}_n} |\nabla (u - k_{n+1})^+|^m + \int_{\tilde{B}_n} |(u - k_{n+1})^+ |D\zeta||^m \right\} |A_n|^{m/\mathrm{N}} \\ & \le \frac{C(1+\gamma)2^{mn}}{(1-\sigma)^m} \left\{ \int_{B_n} |(u - k_n)^+|^m + (\chi^m + k^m)|A_n|^{1 - \frac{m}{\mathrm{N}} + \varepsilon} \right\} |A_n|^{m/\mathrm{N}}, \end{aligned}$$

where $A_n = A^{+}_{k_{n+1}, R_n}$ and as usual $m^* = \mathrm{N}m/(\mathrm{N} - m)$.

Next we set

$$Y_n = k^{-m}\int_{B_n} |(u - k_n)^+|^m$$

and observe that

$$|A_n| \le C2^{mn}Y_n .$$

Therefore, setting

$$b = 2^{\frac{m}{N}(m+N)},$$

we deduce from (2.2), for $k \ge \| u \|_{m,B_0}$,

$$Y_{n+1} \le \frac{Cb^n}{(1-\sigma)^m} Y_n^{1+\varepsilon}\left(\frac{\chi^m + k^m}{k^m}\right). \tag{2.3}$$

Hence if $k \ge \chi$, we have

$$Y_{n+1} \le \frac{Cb^n}{(1-\sigma)^m} Y_n^{1+\varepsilon} \tag{2.4}$$

and consequently, by Lemma 4.7, [5], page 66, $Y_n \to 0$ as $n \to \infty$ provided

$$Y_0 \le C(m, n)(1-\sigma)^{m/\varepsilon},$$

that is for

$$k \ge \frac{C}{(1-\sigma)^{1/\varepsilon}}\left(\int_{B_0}(u^+)^m\right)^{1/m}.$$

The estimate (2.1) follows immediately.

Theorem 1 may now be concluded by means of an interpolation argument. For, setting $v = u^\pm$, $M_\sigma = \sup\limits_{B_{\sigma R}} v$, $\sigma \in (0, 1)$ and

$$\phi(p) = \sup_{0<\sigma<1} (1-\sigma)^{\tilde p}\left(\fint_{B_{\sigma R}} v^m\right)^{1/m}, \qquad 0 < p \le m, \tag{2.5}$$

where $\tilde p = (m-p)/p\varepsilon$, we have for fixed $\eta > 0$,

$$\phi(p) \le (1-\sigma')^{\tilde p}\left(\fint_{B_{\sigma' R}} v^m\right)^{1/m} + \eta$$

for some $\sigma'$ depending on $p$ and $\eta$. But, by Young's inequality,

$$\begin{aligned}\phi(p) &\le (1-\sigma')^{\tilde p} M_{\sigma'}^{1-p/m}\left(\fint_{B_{\sigma' R}} v^p\right)^{1/m} + \eta \\ &< C_\delta\left(\fint_{B_{\sigma' R}} v^p\right)^{1/p} + \delta M_{\sigma'}(1-\sigma')^{m/p\varepsilon} + \eta .\end{aligned} \tag{2.6}$$

By (2.1) applied over $B_{\sigma'' R}$, $\sigma'' = (1+\sigma')/2$, we then obtain,

$$M_{\sigma'}(1-\sigma')^{m/p\varepsilon} \le C(1-\sigma'')^{\tilde p}\left\{\left(\fint_{B_{\sigma'' R}} v^m\right)^{1/m} + \chi R^\alpha\right\}, \tag{2.7}$$

so that letting $\eta \to 0$ and taking $\delta$ sufficiently small, we deduce from (2.6) and (2.7)

$$\phi(p) \leq C\left\{\left(\fint_{B_R} v^p\right)^{1/p} + \chi R^\alpha\right\},$$

and hence, for arbitrary $\sigma \in (0, 1)$,

$$(2.8) \qquad \frac{1}{(1-\sigma)^{1/\varepsilon}}\left(\fint_{B_{\sigma R}} v^m\right)^{1/m} \leq \frac{C}{(1-\sigma)^{m/p\varepsilon}}\left\{\left(\int_{B_R} v^p\right)^{1/p} + \chi R^\alpha\right\}.$$

Theorem 1 now follows by combining (2.1) and (2.8).

*Remarks.* — *i)* The proof of Theorem 1 extends to the case $m = 1$.

*ii)* Lemma 1 may be alternatively derived by Moser iteration. To see this, in the case $\varepsilon = m/N$, we set $R = 1$ and $\bar{u} = u^\pm + \chi$. Multiplying (1.1) by $k^{\beta-2}$ for $\beta > 1$ and integrating over $k$, we thus obtain, with the aid of Fubini's theorem,

$$(2.9) \qquad \int_{B_\sigma} (\bar{u})^{\beta-1} |\nabla u|^m \leq \gamma\left(\frac{1}{(1-\sigma)^m} + 2^m\right)\int_{B_1} (\bar{u})^{m+\beta-1}.$$

Clearly, by (1.1) again, (2.9) continues to hold for $\beta \geq 1$. By applying the Moser iteration method [*8*] as described for example in [*4*] or [*11*], we arrive at (2.1).

## 3. PROOF OF THEOREM 2

The proof is based on the following proposition which is closely related to the strong maximum principle. For non-negative supersolutions of divergence structure equations, the corresponding result, obtained using the logarithm function, was a cornerstone in Moser's approach to Holder estimates; (see [*8*] [*7*]. Theorem 5.3.2, or [*4*] Problem 8.6).

PROPOSITION 3.1. — *Let* $u \geq 0$, $u \in DG_m^-(\Omega)$, $B_{4R} = B_{4R}(y) \subset \Omega$. *Then, if for some* $\delta \in (0, 1)$,

$$|\{x \in B_R \mid u(x) \geq 1\}| \geq \delta |B_R|,$$

*we have*

$$\inf_{B_R} u \geq \lambda - \chi R^\alpha$$

*where* $\lambda$ *is a positive constant depending only on* $m$, N, $\varepsilon$, $\gamma$ *and* $\delta$.

*Proof.* — By replacing $u$ with $u + \chi R^\alpha$, it suffices to take $\chi = 0$ in (1.1). Again we normalize $R = 1$, and consider (1.1) over the balls $B_2$ and $B_4$ for the levels,

$$k_s = \mu + 2^{-s}, \; s = 1, 2, \ldots$$

where

$$\mu = \inf_{B_4} u.$$

We obtain thus

$$(3.1) \qquad \int_{B_2} |\nabla(u - k_s)^-|^m \leq C|A^-_{k_s,4}|^{1-m/N}(2^{-ms} + k_s^m),$$

where C depends on $\gamma$ and N. We recall now the following lemma due to De Giorgi [1].

LEMMA 3.2. — *Let $u \in W^{1,1}(B_r)$ and $l > k$. Then*

$$(l-k)|[u<k] \cap B_r|^{1-1/N} \leq \frac{\beta r^N}{|B_r \backslash [u<l]|} \int_{\Delta_r} |\nabla u|$$

*where $\beta$ depends only on* N *and*

$$\Delta_r = [k < u < l] \cap B_r.$$

Using Lemma 3.2 we shall derive,

LEMMA 3.3. — *Let $\theta \in (0, 1)$ be fixed. Then there exists a positive integer $s^*$ such that*

$$|\{x \in B_2 \mid u(x) < \mu + 2^{-s^*}\}| < \theta |B_2|,$$

*with $s^*$ depending only on m,* N, *$\gamma$, $\varepsilon$, $\delta$ and $\theta$.*

*Proof of Lemma* 3.3. — Taking a particular $s^*$ to be fixed later, we may assume that

$$(3.2) \qquad \mu < 2^{-s} \quad \text{for} \quad s^* > s > 1.$$

By the hypotheses of Lemma 3.1, we have

$$(3.3) \qquad |\{x \in B_2 \mid u(x) \geq 1\}| \geq \frac{\delta}{2^N}|B_2|,$$

and hence, by (3.2),

$$(3.4) \qquad |B_2 \backslash [u < \mu + 2^{-s}]| \geq \frac{\delta}{2^N}|B_2|, \qquad \forall s \geq 1.$$

We now apply Lemma 3.2 over the ball $B_2$ for the levels $l = \mu + 2^{-s}$, $k = \mu + 2^{-s-1}$, $s = 1, 2, \ldots$ Using (3.4) and writing $A_s = A^-_{k_s,2}$, we thus obtain

$$2^{-s}|A_{s+1}|^{1-\frac{1}{N}} \leq C \int_{A_s \backslash A_{s+1}} |\nabla u|.$$

We majorize the right hand side of (3.5), by making use of inequality (3.1), as follows

$$\int_{A_s \backslash A_{s+1}} |\nabla u| \leq \left(\int_{B_1} |\nabla(u - (\mu + 2^{-s}))^-|^m\right)^{1/m} |A_s \backslash A_{s+1}|^{1-1/m}$$
$$\leq C2^{-s}|A_s \backslash A_{s+1}|^{\frac{m-1}{m}} |A^-_{k_s,4}|^{\frac{N-m}{mN}}$$

provided $s < s^*$. Substituting in (3.5) we therefore obtain

$$|A_{s+1}|^{\frac{m}{m-1}} \leq C\,|A_s \backslash A_{s+1}|, \tag{3.6}$$

so that, by summation from $s = 1$ to $s = s^* - 1$, we have

$$|A_{s^*}|^{\frac{m}{m-1}} \leq \frac{C}{s^* - 2}\,|B_2|^{\frac{m}{m-1}}. \tag{3.7}$$

Lemma 3.3 now follows by choosing $s^*$ sufficiently large, for example

$$s^* = 3 + \left[C\theta^{-\frac{m}{m-1}}\right] \tag{3.8}$$

where $[a]$ denotes the largest integer less than $a$.

*Proof of Proposition* 3.1 (concluded). — Consider the sequence of balls $B_n = B_{\rho_n}$ where

$$\rho_n = 1 + 2^{-n}, \qquad n = 0, 1, 2, \ldots,$$

and the sequence of levels

$$k_n = \mu + 2^{-s^*-1}(1 + 2^{-n}), \qquad n = 0, 1, 2, \ldots$$

Obviously $B_0 = B_2$ and $k_0 = \mu + 2^{-s^*}$. We use inequalities (1.1) over the balls $B_{n+1}$ and $B_n$ for the levels $k_n$. We observe that

$(\rho_n - \rho_{n+1})^{-m} = 2^{(n+1)m}$, and since $\mu = \inf_{B_4} u$, $B_n \subset B_4$,

$$\sup_{B_n} (u - (\mu + 2^{-s^*-1} + 2^{-s^*-n-1}))^- \leq 2^{-s^*}.$$

Using these remarks we rewrite (1.1) as follows,

$$\int_{B_{n+1}} |\nabla(u-k_n)^-|^m \leq \gamma\left\{2^{(n+1-s^*)m}|A_n| + (\mu+2^{-s^*})^m |A_n|^{1-\frac{m}{N}+\varepsilon}\right\}. \tag{3.9}$$

where $A_n = A^-_{k_n,\rho_n}$.

Consider now Lemma 3.2 applied over the ball $B_{n+1}$ for the levels $k_n > k_{n+1}$. We thus have

$$2^{-s^*-n-2}\,|A_{n+1}|^{1-\frac{1}{N}} \leqslant \frac{\beta\,|B_{n+1}|}{|B_{n+1}\backslash A^-_{k_n,\rho_{n+1}}|}\int_{\Delta_n} |\nabla u| \tag{3.10}$$

where $\Delta_n = A^-_{k_n,\rho_{n+1}} \backslash A^-_{k_{n+1},\rho_{n+1}}$. As before

$$|B_{n+1}\backslash A^-_{k_n,\rho_{n+1}}| \geq \frac{\delta}{2^N}\,|B_{n+1}|$$

and

$$\int_{\Delta_n} |\nabla u| \leq \left(\int_{B_{n+1}} |\nabla(u - k_n)^-|^m\right)^{1/m} |A_n|^{1-1/m}.$$

Substituting these estimates into (3.10), we thus obtain

(3.11) $$|A_{n+1}|^{1-1/N} \leq C2^{2n}\{|A_n| + |A_n|^{1-\frac{1}{N}+\frac{\varepsilon}{m}}\}.$$

Setting

$$Y_n = \frac{|A_n|}{|B_2|},$$

we therefore have

$$Y_{n+1} \leq Cb^n Y_n^{1+\eta}\,;\ \eta = \frac{\varepsilon N}{m(N-1)}\,;\ b = 4^{\frac{N}{N-1}}.$$

From [5], Lemma 4.7, page 66, we conclude $Y_n \to 0$ as $n \to \infty$, provided

(3.12) $$Y_0 \leq C^{-1/\eta} b^{-1/\eta^2} \equiv \theta.$$

Fixing $\theta$ by (3.12) and choosing $s^*$ by Lemma 3.3 we thus have

$$Y_0 = \frac{1}{|B_2|}|\{u < \mu + 2^{-s^*}\} \cap B_2| < \theta,$$

whence

$$u(x) > \mu + 2^{-s^*-1} \qquad x \in B_1.$$

Proposition 3.1 is thus proved with $\lambda = 2^{-s^*-1}$.

The proof of Theorem 2 may now be completed by means of the procedure of Krylov and Safonov [6], as adapted by Trudinger [4] [12]. For the sake of completeness we repeat some details. First we reformulate Proposition 3.1 in terms of cubes, by setting, for $y \in \Omega$, $R > 0$,

$$K_R(y) = \{x \in \mathbb{R}^N \mid \max_{1 \leq i \leq N} |x_i - y_i| < R\}$$

and assume that $B_{12\sqrt{N}R}(y) \subset \Omega$. Writing $\bar{u} = u + \chi R^\alpha$ and replacing $\bar{u}$ by $\bar{u}/t$ for $t > 0$, we deduce from Proposition 3.1, that if $\delta \in (0, 1)$ and

$$|\{x \in K_R \mid \bar{u}(x) > t\}| > \delta |K_R|,$$

then

(3.13) $$\bar{u} \geq \lambda t, \qquad \forall x \in K_{3R}$$

where $\lambda$ is a positive constant depending only on $m$, N, $\gamma$, $\varepsilon$, $\delta$.

Defining

$$\Gamma_t = \{x \in K_R \mid \bar{u}(x) > t\},$$
$$\Gamma_t^s = \{x \in K_R \mid \bar{u}(x) > t\lambda^s\}, \qquad s = 1, 2, \ldots,$$

we extend this assertion as follows.

LEMMA 3.4. — *Suppose that for fixed $\delta \in (0, 1)$, we have $|\Gamma_t| > \delta^s |K_R|$. Then*

$$\bar{u} \geq \lambda^s t, \qquad \forall x \in K_{3R}.$$

The proof is based on the following covering argument of Krylov and Safonov [*6*], (see [*4*] [*12*]).

LEMMA 3.5. — *Let* $K_R$ *be any cube in* $\mathbb{R}^N$, $\mathscr{E}$ *a measurable subset of* $K_R$, $\delta \in (0, 1)$ *and consider*

$$\mathscr{E}_\delta = \cup \{ K_{3\rho}(x) \cap K_R \mid x \in K_R,\ \rho > 0,\ | \mathscr{E} \cap K_\rho(x) | \geq \delta | K_\rho(x) | \}. \tag{3.14}$$

*Then either* $\mathscr{E}_\delta = K_R$ *or*

$$| \mathscr{E}_\delta | \geq \delta^{-1} | \mathscr{E} |.$$

*Remark.* — The same conclusion holds if in (3.14) we require the elements in the collection defining $\mathscr{E}_\delta$ to be cubes $K_{3\rho}$ with $\rho$ small, say $\rho \leq \rho_0$.

*Proof of Lemma* 3.4. — Let us apply Lemma 3.6 with $\mathscr{E} = \Gamma_t^{n-1}$. Obviously we have $\Gamma^{n-1} \subset \Gamma^n$, $n = 1, 2, \ldots$ If for some $z \in K_R$ and $\rho > 0$, we have

$$| \Gamma_t^{n-1} \cap K_\rho(z) | \geq \delta | K_\rho(z) |,$$

then by (3.13), $\bar{u}(x) > t\lambda^n$ $\forall x \in K_{3\rho}(z)$. Therefore, by virtue of Lemma 3.5,

$$| \Gamma_t^n | \geq \frac{1}{\delta} | \Gamma_t^{n-1} |, \qquad n = 1, 2, 3, \ldots .$$

Suppose now that $| \Gamma_t | > \delta^s | K_R |$. Then

$$| \Gamma_t^{s-1} | \geq \delta^{-1} | \Gamma_t^{s-2} | \geq \ldots \delta^{-s+1} | \Gamma_t | \geq \delta | K_R |,$$

and hence by (3.13), again we have

$$\bar{u}(x) \geq t\lambda^s \qquad \forall x \in K_{3R}.$$

*Proof of Theorem* 2. — For each $t > 0$, choose $s$ so that

$$\delta^s \leq | \Gamma_t | / | K_R |, \qquad \text{i. e. } s \geq \frac{ln | \Gamma_t | / | K_R |}{ln \delta}.$$

By Lemma 3.4,

$$\inf_{K_{3R}} \bar{u} \geq C_1 t \frac{| \Gamma_t |}{| K_R |}\Big)^{C_0} \tag{3.15}$$

for (small) $C_1$ and (large) $C_0$ depending on $\delta$ and $\lambda$. Setting

$$\xi = \inf_{K_{3\rho}} \bar{u}$$

we have from (3.15)

$$| \Gamma_t | / | K_R | \leq \frac{1}{C_1} t^{-\frac{1}{C_0}} \xi^{\frac{1}{C_0}}.$$

On the other hand, for any $p < 1/C_0$,

$$\frac{1}{| K_R |} \int_\zeta^\infty t^{p-1} | \Gamma_t | \, dt \leq C_2 \zeta^p$$

and hence

$$\fint_{K_R} (\bar{u})^p = \frac{1}{|K_R|}\int_\zeta^\infty t^{p-1}\,|\Gamma_t|\,dt + p\zeta^p \leq C_3\zeta^p .$$

Returning to balls, we thus have

$$\left(\frac{1}{B_R}\int_{B_R}(\bar{u})^p\right)^{1/p} \leq C \inf_{B_{3R}} \bar{u}\,,$$

provided $B_{12\sqrt{N}R}(y) \subset \Omega$, where C depends on $m$, N, $\gamma$, $\varepsilon$. The conclusion of Theorem 2 now follows by means of a standard covering and chaining argument.

## 4. APPLICATION TO QUASI-MINIMA

We consider functionals of the form

$$\text{(4.1)} \qquad J(u, \Omega) = \int_\Omega f(x, u, \nabla u)$$

where $f(x, z, p)$ is a Caratheodory function, namely measurable in $x$ for every $(z, p)$ and continuous in $(z, p)$ for almost all $x \in \Omega$. The function $f$ is further restricted through structural inequalities:

$$\text{(4.2)} \qquad |p|^m - b\,|z|^m - g(x) \leq f(x, z, p) \leq \mu\,|p|^m + b\,|z|^m + g(x)$$

where $m, \mu \geq 1$ are constants and $b, g$ are non-negative functions satisfying $b, g \in L^q(\Omega)$ for $q > N/m$ if $m \leq N$, and $q = 1$ for $m > N$. We call a function $u \in W^{1,m}_{loc}(\Omega)$, a *sub* Q-*minimum* (*super* Q-*minimum*) for J if $Q \geq 1$ and

$$\text{(4.3)} \qquad J(u, K) \leq QJ(u + \phi, K)$$

for every $\phi \leq 0$, $(\geq 0)$, $\in W^{1,m}(\Omega)$ with supp $\phi \subset K$. A Q-minimum for J is thus both a sub and super Q-minimum. The following lemma, adapted from [2] and [3], provides a connection between Q-minima and De Giorgi classes. We assume for simplicity that $\Omega$ is bounded.

LEMMA 4.1. — *Let* $u \in W^{1,m}_{loc}(\Omega)$ *be a sub*(*super*) Q-*minimum for* J. *Then* $u \in DG^+_m(\Omega)(DG^-_m(\Omega))$, *with constants* $\varepsilon = \dfrac{m}{N} - \dfrac{1}{q}$, $\chi_m = \|g\|_{L^q(\Omega)}$ *and* $\gamma$ *depending on* Q, $\mu$, *and* $(\text{diam}\,\Omega)^{m-\frac{N}{q}}\|b\|_{L^q(\Omega)}$.

*Proof*. — Let $u$ be a sub Q-minimum for J and fix a ball $B_R(y) \subset \Omega$. Normalizing $R = 1$ we take, for $k \geq 0$,

$$\phi = -\eta(u - k)^+$$

where $0 \leq \eta \leq 1$, supp $\eta \subset B_s$, $\eta = 1$ in $B_t$, $|\nabla\eta| \leq 2(s-t)^{-1}$ and $0 < t < s \leq 1$.

Using (4.2), (4.3), we obtain

$$(4.4)\quad \int_{A_{k,s}^+} |\nabla u|^m \leq \mu Q \int_{A_{k,s}^+} \{(1-\eta)^m |\nabla u|^m + |\nabla\eta|^m (u-k)^m\} + (1+Q)\int_{A_{k,s}^+} (b|u|^m + g).$$

Now, for $m < N$, and arbitrary $s > 0$,

$$\begin{aligned}\int_{A_{k,s}^+} b|u|^m &\leq 2^m \int_{A_{k,s}^+} \{b(u-k)^m + bk^m\} \\ &\leq 2^m \left\{ \|b\|_{L^q(\Omega)} \| |(u-k)^+|^m \|_{L^{q'}(\Omega)} + k^m \int_{A_{k,s}^+} b \right\} \\ &\leq 2^m \{ \|b\|_q (\delta \| |(u-k)^+|^m \|_{L^{m^*/m}(B_s)} \\ &\quad + C(N,q)\delta^{N/(N-mq)} \|(u-k)^+\|_{m,B_s}^m) + k^m \int_{A_{k,s}^+} b \}.\end{aligned}$$

Consequently, by the Sobolev imbedding theorem and appropriate choice of $\delta$, we obtain

$$(4.5)\quad \int_{A_{k,s}^+} b|u|^m \leq \frac{1}{2(1+Q)} \int_{A_{k,s}^+} |\nabla u|^m + C\int_{A_{k,s}^+} (u-k)^m + 2^m k^m \int_{A_{k,s}^+} b$$

where C depends on Q, $m$, N, $q$, $\|b\|_q$ and diam $\Omega$. Inequality (4.5) is also readily extended to the cases $m \geq N$. Hence by substitution of (4.5) into (4.4) we have

$$\int_{A_{k,s}^+} |\nabla u|^m \leq C \left\{ \int_{A_{k,s}^+} |\nabla u|^m + \frac{1}{(s-t)^m} \int_{A_{k,1}^+} (u-k)^m + (k^m + \|g\|_q) A_{k,1}^+ |^{1-\frac{1}{q}} \right\}$$

so that

$$\int_{A_{k,t}^+} |\nabla u|^m \leq \frac{C}{1+C} \left\{ \int_{A_{k,1}^+} |\nabla u|^m + \frac{1}{(s-t)^m} \int_{A_{k,1}^+} (u-k)^m + (k^m + \|g\|_q) |A_{k,1}^+|^{1-\frac{1}{q}} \right\}.$$

Applying Lemma 3.2 of [*3*], we thus infer for any $\sigma \in (0,1)$,

$$\int_{B_\sigma} |\nabla (u-k)^+|^m \leq \gamma \left\{ \frac{1}{(1-\sigma)^m} \int_{B_1} |(u-k)^+|^m + (k^m + \|g\|_q) |A_{k,1}^+|^{1-\frac{1}{q}} \right\}$$

and (1.1) follows. The case of a super Q-minimum is proved similarly.

Combining Theorems 1, 2 and 3 with Lemma 4.1 we obtain the corresponding Harnack inequalities for quasi-minima.

COROLLARY 1. — *Let $u$ be a sub Q-minimum for J, $B_R = B_R(y) \subset \Omega$. Then for any $\sigma \in (0, 1)$, $p > 0$, we have*

$$\sup_{B_{\sigma R}} u \leq C(1-\sigma)^{-N/p\alpha}\left\{\left(\fint_{B_R}(u^+)^p\right)^{1/p} + \chi R^\alpha\right\}$$

*where C depends only on $m$, N, Q, $\mu$, $q$, $R^{m-\frac{N}{q}}\| b \|_q$, and $\alpha = 1 - \dfrac{N}{mq}$, $\chi^m = \| g \|_q$.*

COROLLARY 2. — *Let $u \geq 0$ be a super Q-minimum for J, $m > 1$, $B_R = B_R(y) \subset \Omega$. Then there exists a positive constant $p$ depending only on $m$, N, Q, $\mu$, $q$, $R^{m-\frac{N}{q}}\| b \|_q$ such that for any $\sigma, \tau \in (0, 1)$ we have*

$$\left(\int_{B_{\sigma R}} u^p\right)^{1/p} \leq C(\inf_{B_{\tau R}} u + \chi R^\alpha)$$

*where C depends in addition on $\sigma$, $\tau$.*

COROLLARY 3. — *Let $u \geq 0$ be a Q-minimum for J, $m > 1$, $B_R = B_R(y) \subset \Omega$. Then for any $\sigma \in (0, 1)$*

$$\sup_{B_{\sigma R}} u \leq C(\inf_{B_{\sigma R}} u + \chi R^\alpha)$$

*where C depends only on $m$, N, Q, $\mu$, $q$ and $R^{m-\frac{N}{q}}\| b \|_q$.*

When $g \equiv 0$, Corollary 3 reduces to the usual Harnack inequality. Furthermore, when also $b \equiv 0$ we obtain a Liouville theorem.

COROLLARY 4. — *Let $u \in W^{1,m}_{loc}(\mathbb{R}^n)$, $m > 1$, be a quasi-minimum for the functional*

$$J(u, \mathbb{R}^n) = \int_{\mathbb{R}^n} | D^u |^m$$

*and suppose that $u$ is bounded on one side. Then $u$ is a constant.*

Finally we remark that the structure conditions (4.2) can be generalized in various ways; in particular the function $f$ can be divided by certain types of non-negative weight functions.

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