Ann. I. H. Poincaré – PR 38, 4 (2002) 437–464

S0246-0203(01)01102-5/FLA

A CENTRAL LIMIT THEOREM FOR THE ASYMMETRIC SIMPLE EXCLUSION PROCESS ☆

Fraydoun REZAKHANLOU

Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA

Received 28 September 2000, revised 14 April 2001

ABSTRACT. – We establish a central limit theorem for the density fluctuations of a one dimensional particle system known as the totally asymmetric simple exclusion process (TASEP). Because of our method in this article, it is more convenient to regard TASEP as a growth model. Let the configuration space Γ consists of functions $h:\mathbb{Z}\to\mathbb{Z}$ such that $0\leqslant h(i+1)-h(i)\leqslant 1$ for all $i\in\mathbb{Z}$. With rate one, each $h(i)$ increases by one unit provided that the resulting configuration does not leave the configuration space; otherwise the growth is suppressed. We establish a central limit theorem for the rescaled height function $u^\varepsilon(x,t)=\varepsilon h([\frac{x}{\varepsilon}],\frac{t}{\varepsilon})$ where $x\in\mathbb{R}$, $[\frac{x}{\varepsilon}]$ denotes the integer part of $\frac{x}{\varepsilon}$, and $h(\cdot,t)$ denotes the configuration after t seconds. We assume that initially, the probability law of $u^\varepsilon(x,0)$ is the same as $g(x)+\sqrt{\varepsilon}B(x)+\mathrm{o}(\sqrt{\varepsilon})$ for a continuous function g and a continuous random process $B(\cdot)$. It is expected that at later times, the rescaled process $u^\varepsilon(x,t)$ can be stochastically represented as $\bar{u}(x,t)+\sqrt{\varepsilon}Z(x,t)+\mathrm{o}(\sqrt{\varepsilon})$ where $\bar{u}$ is the unique solution of the Hamilton–Jacobi equation $\bar{u}_t=\bar{u}_x(1-\bar{u}_x)$ with the initial condition $\bar{u}(\cdot,0)=g(\cdot)$, and $Z(x,t)$ is a random process that is given by a variational expression involving $B(\cdot)$. This will be established if g is piecewise convex. We also define a random lattice curve as a microscopic backward characteristic curve and prove a law of large numbers for it. © 2002 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Un théorème limite central est établi pour les fluctuations de la densité d'un système de particules unidimensionnel connu comme le processus d'exclusion simple totalement asymétrique. La méthode employée conduit à la considérer comme un modèle de croissance. © 2002 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Various phenomena such as the formation of crystals and the spread of infections are modeled by stochastic growth models. To simplify the geometry, we regard a crystal as a collection of cubes of small size with their centers lying on some n-dimensional lattice (in practice n is 2 or 3), and assume that the growth can only occur in the direction of the last coordinate axis. It is customary to take cubes of side length one in our microscopic

☆ Research supported in part by NSF Grant DMS-0072666.

description, and then multiply the side lengths by a small factor ε that will go to zero at the end. If initially the center of cubes lie in a set of the form

$$A(0)=\{(i,k)\in\mathbb{Z}^{d+1}\colon\ k\leqslant h(i)\}$$

for some $h:\mathbb{Z}^d\to\mathbb{Z}$, then at later times our crystal is of the same form and the centers of their cubes lie in the set

$$A(t)=\{(i,k)\in\mathbb{Z}^{d+1}\colon\ k\leqslant h(i,t)\}$$

for a function $h:\mathbb{Z}^d\times[0,\infty)\to\mathbb{Z}$. For the *macroscopic* description of our crystal we rescale h and study

$$u^{\varepsilon}(x,t)=\varepsilon h\left(\left[\frac{x}{\varepsilon}\right],\frac{t}{\varepsilon}\right), \tag{1.1}$$

where $(x,t)\in\mathbb{R}^d\times[0,\infty)$ and $[a]$ denotes the integer part of a. We normally assume that the growth is random, and the rate at which $h(i)$ increases to $h(i)+1$ depends on the height differences $(h(i)-h(j)\colon\ j\in\mathbb{Z}^d)$. When $d=1$ and if we assume that h is always nondecreasing, then a different interpretation of our model is available. One may interpret the height difference $\eta(i)=h(i+1)-h(i)$ as the number of particles that are sitting at the site i. With such interpretation the increase of $h(i)$ by one unit is equivalent to the jump of a particle from the site $i+1$ to the site i, modeling a one-dimensional fluid. For a class of such models, it was shown in Rezakhanlou [9] that the limit of $u^{\varepsilon}(x,t)$ as $\varepsilon\to 0$ exists and the limiting function $\bar{u}$ solves a *Hamilton–Jacobi* equation of the form

$$\bar{u}_t+H(\bar{u}_x)=0, \tag{1.2}$$

for a suitable function H. In [9] the macroscopic density $\bar{\rho}(x,t)=\lim_{\varepsilon\to 0}\eta([\frac{x}{\varepsilon}],\frac{t}{\varepsilon})$ was studied in the context of hydrodynamic limit and it was shown that the function $\bar{\rho}=\bar{u}_x$ satisfies a conservation law of the form

$$\bar{\rho}_t+H(\bar{\rho})_x=0. \tag{1.3}$$

It is well-known that Eq. (1.3) enjoys the following *monotonicity* property: If ρ_1 and ρ_2 are two solutions of (1.3) and if $\rho_1(x,0)\leqslant\rho_2(x,0)$, then $\rho_1(x,t)\leqslant\rho_2(x,t)$ for all t. A refined version of this principle leads to the so-called *entropy inequalities.* In general (1.3) does not possess classical solutions, and (1.3) has infinitely many nonclassical (weak) solutions that share the same initial data. If we require that for a solution, the entropy inequalities hold, then for a given initial data there exists a unique solution. Such a solution is physically relevant because the entropy inequalities are closely related to the second law of thermodynamics. To apply the method of [9], one needs to have two properties for the underlying particle system. First, one needs to assume that for each density ρ there exists an ergodic invariant measure for the η-process that has the average density equal to ρ. Secondly, one needs to assume that the jump rates (or the growth rates for the h-process) satisfy certain monotonicity so that the aforementioned monotononicity for the solutions of (1.3) are also true microscopically.

The two properties we just described are true for the so-called *totally asymmetic simple exclusion process* (TASEP). In the TASEP, one assumes that there exists at most one particle per site, and the jump from $i+1$ to i is suppressed if the site i is occupied. In this case, $H(\rho) = -\rho(1-\rho)$, which is nothing other than the average of the jump rate with respect to the unique ergodic invariant measure with density ρ. In [14], Seppäläinen was able to derive (1.3) for the generalized exclusion processes. He calls a particle system K-exclusion if each site can have at most K particles and a jump is suppressed if such restriction is violated. The important aspect of his work is that it does not rely on the existence of the ergodic invariant measures. In [14] however, only the existence of H is shown and no simple expression for the function H is given. In [12], Rezakhanlou generalizes the work of [14] to a class of growth models that are defined in all dimensions. It was observed in [12] that the key property in [14] that was used for the derivation of (1.2) is some type of *strong monotonicity*. To motivate the definition of strong monotonicity, first recall that if H is convex, then by Hopf–Lax–Oleinik formula, $\bar{u}$ can be expressed by a variational formula of the form

$$\bar{u}(x,t) = \inf_y \left\{ \bar{u}(y,0) + tL\left(\frac{x-y}{t}\right) \right\}, \tag{1.4}$$

where L is the convex conjugate of H:

$$L(q) = \sup_p \big(pq - H(p)\big).$$

Let us write $g(y)$ for $\bar{u}(y,0)$ and let us denote $u(\cdot,t)$ by $T_t g$. Then T_t is a semigroup, and more importantly, a consequence of (1.4) is the following strong monotonicity property of T_t:

$$T_t(\inf_\alpha g_\alpha) = \inf_\alpha T_t g_\alpha. \tag{1.5}$$

In [12] we showed that a microscopic version of such a strong monotonicity is valid for a class of growth models that includes the K-exclusion processes. For a given nonnegative function $v:\mathbb{Z}^d \to \mathbb{Z}$ with $v(0)=0$, we define $\Gamma = \Gamma_v$ to be the set of functions $h:\mathbb{Z}^d \to \mathbb{Z}$ such that $h(i) - h(j) \leqslant v(i-j)$ for every $i, j \in \mathbb{Z}^d$. Now $h(i)$ increases to $h(i)+1$ with rate one, but this increase is suppressed if the resulting configuration h^i does not belong to Γ. If we choose $v(x) = Kx^+$ with $x \in \mathbb{Z}$ and $K \in \mathbb{Z}^+$, then our model coincides with a K-exclusion process. Let us call our model a v-exclusion process.It turns out that a microscopic analog of (1.5) is true for all v-exclusion processes (see (2.3) of Section 2). A consequence of such a strong monotonicity property is the following microscopic version of (1.4):

$$u^\varepsilon(x,t) = \inf_y \big\{ u^\varepsilon(y,0) + w^\varepsilon(x,y,t) \big\}, \tag{1.6}$$

where $w^\varepsilon(x,y,t) = \varepsilon w([\frac{x}{\varepsilon}], [\frac{y}{\varepsilon}], \frac{t}{\varepsilon})$, a nd $w(i,j,t)$ denotes the height function at time t that initially starts from $w(i,j,0) = v(i-j)$. See (2.7) of Section 2 for a proof of (1.6). The formula (1.6) for the first time appeared in [14]. The analog of (1.6) for a closely related particle system known as Hammersley Process was derived by Aldous and Diaconis in [1].

Apparently, the only υ-exclusion process with an explicit simple formula for its invariant measures is TASEP.

Our goal in this article is to establish a central limit theorem for the convergence of u^ε to $\bar{u}$ in the case of TASEP. To motivate the statement of the main results of this article, let us start with formulating some conjectures for the υ-exclusion processes or even more general growth models. Let us pretend that $h(x,t)$ is defined for all $x \in \mathbb{R}^d$, $t \geqslant 0$, and h is a sufficiently nice function. It is expected that h satisfies

$$h_t + H(h_x) = \operatorname{div} A(h_x) + \xi + R \tag{1.7}$$

where A is a suitable vector-valued function, ξ is a space-time white noise, and R is the remainder. Of course R may involve differential operators of higher orders and more complicated randomness. The main aspect of the almost meaningless formula (1.7) is that various terms in (1.7) are scaled differently, and after a rescaling of h, the remainder becomes smaller than the other terms and can be ignored. More precisely, we expect for the function u^ε to satisfy,

$$u_t^\varepsilon + H\left(u_x^\varepsilon\right) = \varepsilon \operatorname{div} A\left(u_x^\varepsilon\right) + \varepsilon^{\frac{d+1}{2}} \xi(x,t) + \mathrm{o}(\varepsilon). \tag{1.8}$$

The derivation of (1.8) even for TASEP seems to be a difficult problem. To have a more mathematically tractable problem, we address two consequences of (1.8). When $d \geqslant 3$, we may ignor the random term on the right–hand side to write

$$u_t^\varepsilon + H\left(u_x^\varepsilon\right) = \varepsilon \operatorname{div} A\left(u_x^\varepsilon\right) + \mathrm{o}(\varepsilon). \tag{1.9}$$

What we will establish in this article is a consequence of (1.8). Assume $d = 1$. The expression (1.8) certainly implies

$$u_t^\varepsilon + H(u_x^\varepsilon) = \mathrm{o}\left(\sqrt{\varepsilon}\right). \tag{1.10}$$

If at time $t = 0$, we have a central limit theorem of the form

$$u^\varepsilon(x,0) = g(x) + \sqrt{\varepsilon} B(x) + \mathrm{o}\left(\sqrt{\varepsilon}\right), \tag{1.11}$$

then we expect to have

$$u^\varepsilon(x,t) = \bar{u}(x,t) + \sqrt{\varepsilon} Z(x,t) + \mathrm{o}\left(\sqrt{\varepsilon}\right), \tag{1.12}$$

for a suitable random process $Z(x,t)$. In fact, $Z(x,t)$ can be calculated by replacing $\mathrm{o}(\sqrt{\varepsilon})$ with zero in (1.10). The result is

$$Z(x,t) = \inf_{y \in I(x,t)} B(y) \tag{1.13}$$

where $I(x,t)$ is the set of y at which the infimum in (1.4) is attained. (For calculating L, we should define $H(\rho) = -\rho(1-\rho)$ for $\rho \in [0,1]$ and set $H(\rho) = +\infty$ if $\rho \notin [0,1]$.)

In fact we can prove more, namely, if we replace x with $x + \sqrt{\varepsilon}e$ on the left-hand side of (1.12), then (1.12) is still valid provided that the process Z is replaced with the

$$Z^e(x,t) = \inf_{y \in I(x,t)} \left\{ B(y) + eL'\left(\frac{x-y}{t}\right) \right\}. \tag{1.14}$$

We establish (1.12) in the case of TASEP, provided that the set $I(x,t)$ is finite and for every $y \in I(x,t)$, the function g is convex in a neighborhood of y.

In some sense, the process Z is a solution to the linear equation

$$Z_t + H'(\bar{u}_x)Z_x = 0, \tag{1.15}$$

with the random initial condition $Z(x,0) = B(x)$. Similarly, if the initial condition g is differntiable, then the process Z^e satisfies the same equation but now with the initial condition $B(x) + g'(x)e$. Since in general g is not differentiable and the coefficient $H'(\bar{u}_x)$ is multivalued at the nondifferentiability points of the function $\bar{u}$, Eq. (1.15) does not possess classical solutions and the formulas (1.13)–(1.14) offer some type of generalized solutions to (1.15).

To describe our next result, let us assume that if $x_1 \neq x_2$, then $B(x_1) \neq B(x_2)$ with probability one. Such an assumption implies that there exists a unique minimizer $\bar{y}(x,t)$ such that $Z(x,t) = B(\bar{y}(x,t))$. Let $y^\varepsilon(x,t)$ be any random process such that for each (x,t) and $\varepsilon > 0$, the point $y^\varepsilon(x,t)$ is a minimizer in the variational problem (1.5). In the last section we show that the finite dimensional marginals of $y^\varepsilon(x,t)$ converge to the finite dimensional marginals of the process $\bar{y}(x,t)$. When $\bar{u}$ is differentable at (x,t), then the set $I(x,t)$ consists of a single point and $y^\varepsilon(x,t)$ converges to the only element of $I(x,t)$. When $\bar{u}$ is not differentiable at (x,t), then $I(x,t)$ consists of more that one point and the limit of $y^\varepsilon(x,t)$ is a suitable random point in $I(x,t)$. Such a point (x,t) lies on a *discontinuity shock* and if $y \in I(x,t)$, then a characteristic line emenating from y at time 0, is involved in the formation of such a shock.

The central limit theorem (1.12) for the simple exclusion process was established by Ferrari and Fonte [5] in two cases, either when the η-process is in equilibrium, or when $g(x)$ is the infimum of two linear functions. The latter case in the language of conservation laws, corresponds to a Riemann solution of (1.3). In Ferrari et al. [6] the work of [5] is generalized to the case of an initial data g that is the infimum of finitely many linear functions. In comparison with [6] our result is stronger because we allow more general initial data g. The work of [5] however applies to simple exclusion processes for which $h(i)$ can decrease as well.

The proof of (1.12) is naturally divided into parts:

$$u^\varepsilon(x,t) \leqslant \bar{u}(x,t) + \sqrt{\varepsilon}Z(x,t) + \mathrm{o}\big(\sqrt{\varepsilon}\,\big), \tag{1.16}$$

$$u^\varepsilon(x,t) \geqslant \bar{u}(x,t) + \sqrt{\varepsilon}Z(x,t) + \mathrm{o}\big(\sqrt{\varepsilon}\,\big). \tag{1.17}$$

It turns out that the proof of (1.16) is a straightforward consequence of the work of Johansson [7] and holds for arbitrary initial data g. In fact Johansson shows that if the initial height function is of the form $h(i,0)=i^+$, then

$$u^\varepsilon(x,t)=tL\left(\frac{x}{t}\right)+\mathrm{O}\left(\varepsilon^{2/3}\right). \tag{1.18}$$

See Section 4 for more details on how (1.18) implies (1.16). Our main contribution is (1.17) and for this we need to assume that the initial data is piecewise convex. Our approach can been used to obtain (1.16) provided that we assume the initial data is piecewise concave. Such an assumption can be avoided if we appeal to (1.18).

In spirit our method is close to the method of Rezakhanlou–Tarver [13] and Rezakhanlou [11]. In these papers a central limit theorem for the convergence of u^ε is established where u^ε satisfies a Hamilton–Jacobi equation

$$u_t^\varepsilon+H\left(\frac{x}{\varepsilon},u_x^\varepsilon,\omega\right)=0,$$

where the random Hamiltonian $H(\frac{x}{\varepsilon},p,\omega)$ is stationary and ergodic in the spatial variable. If we assume $H(y,\rho,\omega)$ is convex in ρ, then formula (1.6) is true where w^ε is now given by

$$w^\varepsilon(x,y,t)=\inf\left\{\int_0^t L\left(\varepsilon^{-1}\gamma(\theta),\gamma'(\theta),\omega\right)\mathrm{d}\theta\right\}.$$

Here the infimum is over smooth curves $\gamma:[0,t]\to\mathbb{R}$ with $\gamma(0)=y$ and $\gamma(t)=x$, and $L(y,q,\omega)$ denotes the convex conjugate of $H(y,\rho,\omega)$ in the ρ-variable. The sequence u^ε converges to a function $\bar{u}$ that solves a *homogenized* Hamilton–Jacobi equation of the form (1.2). The main result of [11] asserts that (1.12) is valid for the sequence u^ε provided that we have a central limit theorem for the solutions of the form

$$u_\rho^\varepsilon(x,t)=q^\varepsilon(x,t,\omega)-t\bar{H}(\rho)=\varepsilon q\left(\frac{x}{\varepsilon},\omega\right)-t\bar{H}(\rho),$$

where q satisfies $\lim_{\varepsilon\to 0}q^\varepsilon(x,\omega)=x\rho$. In the case of TASEP, the role of u_ρ^ε are played by random height functions for which the height differences are distributed according to an equilibrium measure with density ρ. As we mentioned earlier, a result of Ferrari and Fontes [4] establishes a central limit theorem for the convergence of u_ρ^ε. To apply the arguments of [11], we need a stronger version of [4] result, namely, the family of processes

$$J^\varepsilon(x,t,\rho):=\varepsilon^{-1/2}\left(u_\rho^\varepsilon(x,t)-x\rho\right)$$

is convergent as ε goes to zero. (See condition (iv) of Theorem 2.8 of [11].) Unfortunately we have not been able to prove this for our model. In fact such a strong central limit theorem would allow us to have (3.2) of Section 3 with Y^ε in place of R^ε. The piecewise convexity assumption on the initial data can be dropped if we can prove (3.2) with Y^ε. An interested reader should compare (3.2) with Assumption 2.3 of [11].

The organization of this paper is as follows. In the next section the main results are stated. A suitable bound on the fluctuations of u^ε is given in Section 3 when the initial height function is $h(i,0)=i^+$. The statement (1.12) is established in Section 4. The last section is devoted to a law of large numbers for the process y^ε.

2. Notations and main results

The space of configurations Γ consists of $k:\mathbb{Z}\to\mathbb{Z}$ such that $0\leqslant k(i+1)-k(i)\leqslant 1$ for all $i\in\mathbb{Z}$. We also write $\bar\Gamma$ for the set of functions $g:\mathbb{R}\to\mathbb{R}$ with

$$g(x)-g(y)\leqslant (x-y)^+,$$

for every $x,y\in\mathbb{R}$. The process $h(i,t)$ is a Markov process with the infinitesimal generator

$$(\mathcal{A}F)(k)=\sum_i \mathbb{1}\big(k^i\in\Gamma\big)\big(F\big(k^i\big)-F(k)\big) \tag{2.1}$$

where $F:\mathbb{Z}^{\mathbb{Z}}\to\mathbb{R}$ is any cylindrical function ($F(k)$ depends on finitely many $k(i)$'s) and k^i is defined by

$$k^i(j)=\begin{cases} k(i)+1 & \text{if } j=i,\\ k(j) & \text{if } j\neq i.\end{cases} \tag{2.2}$$

When necessary, we write $h(i,t;k)$ for the process with the initial configuration k, i.e., $h(i,0;k)=k(i)$. The function $v(i)=i^+\in\Gamma$ plays a key role in our arguments. In fact, one can easily see that if $k\in\Gamma$ and $k(j)=a$, then $k(i)\leqslant a+v(i-j)$. It was shown in [12] that we always have

$$h\big(i,t;\inf_\alpha k_\alpha\big)=\inf_\alpha h(i,t;k_\alpha) \tag{2.3}$$

where $\{k_\alpha\}$ is a family of configurations of Γ with $\inf_\alpha k_\alpha=k$ finite. From

$$k(i)=\inf_j\big\{k(j)+v(i-j)\big\} \tag{2.4}$$

and (2.3) we deduce

$$h(i,t;k)=\inf_j h\big(i,t;v\big(\cdot,j,k(j)\big)\big), \tag{2.5}$$

where $v(r,j,k(j))=k(j)+v(r-j)$.

The proof of (2.3) follows from a suitable construction of the process $h(i,t)$ in terms of a sequence of independent rate one Poisson processes $(\ell_i(\cdot)\colon i\in\mathbb{Z})$. Let $\mathcal{D}$ denote the set of step functions $\ell:[0,\infty)\to\mathbb{Z}^+$ such that for an increasing sequence of numbers $\sigma_0(\ell)=0,\sigma_1(\ell),\ldots,$ we have $\ell(t)=k$ for $t\in[\sigma_k(\ell),\sigma_{k+1}(\ell))$. We set $\Omega_{dy}=\mathcal{D}^{\mathbb{Z}}$ and let P_{dy} denote the law of a sequence of independent rate one Poisson processes. Given a realization

$$\omega_1=\big(\ell_i(\cdot)\colon i\in\mathbb{Z}\big)\in\Omega_{dy},$$

we can define a sequence $(\sigma_r(\ell_i)\colon r,i\in\mathbb{Z})$ where $\sigma_r(\ell_i)$ is the rth time the process ℓ_i has increased by one unit. We set $\hat\Omega_{dy}$ to be the set of realizations ω_1 for which all $\sigma_r(\ell_i)$ are distinct. It is not hard to show that $P_{dy}(\hat{\Omega}_{dy}) = 1$. For every $\omega_1 \in \hat{\Omega}_{dy}$ we can construct the process $h(i,t;k) = h(i,t;k,\omega_1)$ by adding one to $h(i)$ at each time $\sigma_r(\ell_i)$ provided that the resulting configuration h^i stays in Γ. From our construction we see that if for some $a \in \mathbb{Z}$, we have $\hat{k}(i) = k(i) + a$ for every $i \in \mathbb{Z}$, then

$$h(i,t;\hat{k}) = h(i,t;k) + a. \tag{2.6}$$

In particular

$$h\big(i,t;v\big(\cdot,j,k(j)\big)\big) = k(j) + h\big(i,t;v^j\big),$$

where $v^j(i) = v(i-j)$.

To ease the notation, we write $w(i,j,t) = w(i,j,t;\omega_1)$ for $h(i,t;v^j)$. As a result, (2.5) becomes

$$h(i,t;k) = \inf_j \big\{k(j) + w(i,j,t)\big\}. \tag{2.7}$$

This is (4.9) of [14]. See also [1] where a similar formula is derived for the Hammersly Process. (The Hammersly process is another example of a strongly monotone particle system.)

Throughout the paper we write $\omega_0 \in \Omega_0$ for the randomness of the initial data, $\omega_1 \in \Omega_{dy}$ for the randomness of the dynamics, and ω for the pair (ω_0, ω_1). The space of such pairs will be denoted by Ω. Recall that the probability distribution of ω_1 is denoted by P_{dy}. The corresponding expectation is denoted by E_{dy}. In the nonequilibrium case we write $p^\varepsilon(\mathrm{d}\omega_0)$ for the probability measure at time zero (that may depend on ε) and this combined with the probability measure coming from dynamics will be denoted by $P^\varepsilon(\mathrm{d}\omega_0, \mathrm{d}\omega_1)$. The correspondind expectation is denoted by E^ε.

Given a realization $\omega_1 = (\ell_i(\cdot)\colon i \in \mathbb{Z}) \in \Omega_{dy}$, we define

$$\tau_j \omega_1 = \big(\ell_{i-j}(\cdot)\colon i \in \mathbb{Z}\big) \in \Omega_{dy}.$$

We also define the shift operator τ_j on Γ by

$$\tau_j k(i) = k(i-j),$$

for every $k \in \Gamma$ and every $j \in \mathbb{Z}$. From our construction, it is not hard to see

$$h(i-j,t;k,\omega_1) = h(i,t;\tau_j k, \tau_j \omega_1). \tag{2.8}$$

The translation invariant equilibrium measures for the TASEP are well-known. To define them, let us consider a random height funcion $h_0^\rho(\cdot;\omega_0)$ and a probability measure $p^\rho(\mathrm{d}\omega_0)$ so that $h_0^\rho(0;\omega_0) = 0$, and the sequence

$$\big(\eta_0^\rho(i;\omega_0) = h_0^\rho(i+1;\omega_0) - h_0^\rho(i;\omega_0)\colon i \in \mathbb{Z}\big)$$

are independent identically distributed random variables with

$$p^\rho\big(\big\{\omega_0\colon \eta_0^\rho(i;\omega_0) = 1\big\}\big) = \rho.$$

The probability measure $p^\rho(\mathrm{d}\omega_0)$ at time zero combined with the probability measure associated with the dynamics is denoted by $P^\rho(\mathrm{d}\omega_0, \mathrm{d}\omega_1)$. The corresponding expectation is denoted by E^ρ. Define

$$h^\rho(i,t) = h^\rho(i,t;\omega) = h^\rho(i,t;\omega_0,\omega_1) := h\big(i,t; h_0^\rho(\cdot;\omega_0),\omega_1\big). \tag{2.9}$$

We may choose $\Omega_0 = \{0,1\}^{\mathbb{Z}}$ so that $\omega_0 \in \Omega_0$ is of the form

$$\omega_0 = \big(\eta_0(i)\colon\, i \in \mathbb{Z}\big).$$

For such a realization ω_0, the shifted realization is defined in an obvious way:

$$\tau_j\omega_0 = \big(\eta_0(i-j)\colon\, i \in \mathbb{Z}\big).$$

Note that we always have

$$h_0^\rho(i;\omega_0) - h_0^\rho(j;\omega_0) = h_0^\rho(i-j;\tau_{-j}\omega_0).$$

From this, (2.6) and (2.8) we deduce

$$\begin{aligned} h^\rho(i,t;\omega_0,\omega_1) - h^\rho(j,0;\omega_0) &= h\big(i,t; h_0^\rho(\cdot;\omega_0) - h_0^\rho(j;\omega_0),\omega_1\big) \\ &= h\big(i,t;\tau_j h_0^\rho(\cdot;\tau_{-j}\omega_0),\omega_1\big) \\ &= h^\rho(i-j,t;\tau_{-j}\omega_0,\tau_{-j}\omega_1). \end{aligned} \tag{2.10}$$

It is shown in Ferrari and Fontes [4] that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} E^\rho\big[u_\rho^\varepsilon(x,t) - u_\rho^\varepsilon\big(x - tH'(\rho),0\big) - tL\big(H'(\rho)\big)\big]^2 = 0, \tag{2.11}$$

where $u_\rho^\varepsilon(x,t) = \varepsilon h^\rho([\frac{x}{\varepsilon}],\frac{t}{\varepsilon})$ and L is the convex conjugate of H. A simple caculation reveals $L(H'(\rho)) = \rho^2$. Note that the translation invariance of P^ρ and (2.10) imply that the left-hand side of (2.11) is independent of x. In particular, (2.11) is equivalent to

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} E^\rho\big[u_\rho^\varepsilon\big(tH'(\rho),t\big) - tL\big(H'(\rho)\big)\big]^2 = 0.$$

Also, in (2.11) we may replace x with $x + \sqrt{\varepsilon}e$. This equivalent variation of (2.11) will be used in Section 3.

We now state a definition for the convergence of processes.

DEFINITION 2.1. – *Let $(\Omega,\mathcal{F},p)$ and $(\bar\Omega,\bar{\mathcal{F}},\bar p)$ be two probability measures and suppose $D \subseteq \mathbb{R}^r$. Let $X^\varepsilon : D\times\Omega \to \mathbb{R}$ be a sequence of measurable functions such that for each $\omega\in\Omega$, the function $X^\varepsilon(\cdot,\omega)$ is continuous. Let $X : D\times\bar\Omega \to \mathbb{R}$ be a measurable function such that for each $\bar\omega\in\bar\Omega$, the function $X(\cdot,\bar\omega)$ is continuous. We may regard X^ε (respectively X) as a function from Ω (respectively $\bar\Omega$) into the space of continuous functions $C(D;\mathbb{R})$. Then we say that the processes X^ε converge to the process X if for every compact set $A\subseteq D$ and every bounded continuous function $F : C(A;\mathbb{R})\to\mathbb{R}$, we have*

$$\lim_{\varepsilon\to 0}\int F\big(X^\varepsilon(\omega)\big)\,p(\mathrm{d}\omega) = \int F\big(X(\bar\omega)\big)\,\bar p(\mathrm{d}\bar\omega). \tag{2.12}$$

We say the finite-dimensional marginals of X^ε *converge to the finite-dimensional marginals of* X *if we require* (2.12) *to hold only for sets* $A \subseteq D$ *that are finite.*

In our second definition, we define several sets of points (x,t) for which different versions of our central limit theorem (1.12) will be established.

DEFINITION 2.2. – *Let* $\bar{u}$ *be a solution of* (1.2) *and let* $I(x,t)$ *consist of points* y *at which the infimum in* (1.4) *is attained. Let* G_1 *be the set of points* (x,t) *for which the set* $I(x,t)$ *is bounded, the set*

$$\hat{I}(x,t) := I(x,t) \cap (x-t, x+t)$$

is finite, and for every $y \in \hat{I}(x,t)$ *the function* $g(\cdot) = \bar{u}(\cdot,0)$ *is convex in a neighborhood of* y*. We also define*

$$\begin{aligned}
&G_2 = \{(x,t)\colon\ g \textit{ is differentiable at every } y \in I(x,t)\},\\
&G_3 = \{(x,t)\colon\ I(x,t) \subseteq [x-t, x+t]\},\\
&G_4 = G_1 \cap G_2, \qquad G_5 = G_4 \cap G_3.
\end{aligned}$$

See Lemmas 4.2–4.4 of Section 4 and Remark 2.7 for some relavant information about the sets $G_1 - G_5$.

To state our main results, first take a function $g \in \bar{\Gamma}$. Let $(\Omega_0, \mathcal{F}, p^\varepsilon)$ be a family of probability measures and $(g^\varepsilon : \mathbb{R} \times \Omega_0 \to \mathbb{R}\colon\ \varepsilon > 0)$ be a family of measurable functions such that $g^\varepsilon(\cdot, \omega_0) \in \bar{\Gamma}$ for every ω_0, and the processes

$$B^\varepsilon(x;\omega_0) = \varepsilon^{-1/2}\big(g^\varepsilon(x,\omega_0) - g(x)\big); \quad \varepsilon > 0 \tag{2.13}$$

converge to a continuous process $B(x,\omega_0)$ as $\varepsilon \to 0$. Set $\hat{g}^\varepsilon(i,\omega_0) = [\varepsilon^{-1} g^\varepsilon(i\varepsilon, \omega_0)]$. Define

$$u^\varepsilon(x,t;\omega) = \varepsilon h\left(\left[\frac{x}{\varepsilon}\right], \frac{t}{\varepsilon}; \hat{g}^\varepsilon(\cdot,\omega_0), \omega_1\right).$$

THEOREM 2.3. – *Let* $e : \mathbb{R} \to \mathbb{R}$ *be a continuous function and set* $x^\varepsilon = x + e(x)\sqrt{\varepsilon}$*. Then the finite-dimensional marginals of the processes*

$$X^\varepsilon\big(x^\varepsilon, t, \omega_1, \omega_0\big) = \frac{1}{\sqrt{\varepsilon}}\big(u^\varepsilon\big(x^\varepsilon, t;\omega\big) - \bar{u}(x,t)\big); \quad (x,t) \in G_4, \tag{2.14}$$

converge to the finite-dimensional marginals of the process

$$Z^e(x,t,\omega_0) = \inf_{y \in I(x,t)} B^e(x,y,t;\omega_0); \quad (x,t) \in G_4, \tag{2.15}$$

where,

$$B^e(x,y,t;\omega_0) = B(y;\omega_0) + L'\left(\frac{x-y}{t}\right) e(x).$$

When $e \equiv 0$*, the set* G_4 *may be replaced with the possibly larger set* G_1*.*

The main ingredient for the proof of Theorem 2.3 is a variant of

$$w^\varepsilon(x,y,t) = tL\left(\frac{x-y}{t}\right) + \mathrm{o}\big(\sqrt{\varepsilon}\,\big). \tag{2.16}$$

(Note that (2.16) is consistent with Theorem 2.3 because the initial data $w(\cdot, j, 0)$ is deterministic.) A variant of (2.16) will be established in Section 3. The proof of Theorem 2.3 will be given in Section 4.

Given a point (x,t) and a realization $\omega = (\omega_0, \omega_1)$, let $I^\varepsilon(x,t) = I^\varepsilon(x,t;\omega)$ denote the set of y at which the infimum in (1.6) is attained. We also write $y_+^\varepsilon(x,t) = y_+^\varepsilon(x,t;\omega)$ (respectively $y_-^\varepsilon(x,t) = y_-^\varepsilon(x,t;\omega)$) for the largest (respectively smallest) number in $I^\varepsilon(x,t)$.

THEOREM 2.4. – *Let x^ε be as in the previous theorem. Suppose that for $\tau > 0$, every (x,t) and every pair of distinct points (y_1, y_2), we have $B^e(x,y_1,t) \neq B^e(x,y_2,t)$ almost surely, and that $|B^e(x,y_1,t) - B^e(x,y_2,t)| \leqslant \tau$ occurs with positive probability. Choose $\bar{y}(x,t) = \bar{y}(x,t;\omega)$ to be the unique point $y \in I(x,t)$ at which the infimum in* (2.15) *is attained*;

$$Z^e(x,t) = B\big(\bar{y}(x,t)\big) + L'\left(\frac{x - \bar{y}(x,t)}{t}\right) e\big(\bar{y}(x,t)\big).$$

Then the finite dimensional marginals of $(y_\pm^\varepsilon(x^\varepsilon, t)\colon\ (x,t) \in G_5)$ converge to the finite dimensional marginals of $(\bar{y}(x,t)\colon\ (x,t) \in G_5)$. Again, when $e \equiv 0$, we may replace the set G_5 with the set $G_1 \cap G_3$.

Note that Lemma 4.4 of Section 4 implies that if $(x,t) \notin G_3$, then $I(x,t)$ contains a nonempty interval. That is why the processes $y^\pm$ in Theorem 2.4 are restricted to the set $G_5 \subseteq G_3$ so that the set $I(x,t)$ is finite whenever $(x,t) \in G_5$.

Example 2.5. – Let $\rho^0 : \mathbb{R} \to [0,1]$ be a function that has first-kind discontinuities and assume ρ^0 is right continuous. Choose the measure p^ε in such a way that the random variables $(\eta(i,0)\colon\ i \in \mathbb{Z})$ are independent and $p^\varepsilon(\eta(i,0) = 1) = \rho^0(\varepsilon i+)$. The configuration $h(\cdot,0)$ is defined uniquely from $h(0,0) = 0$ and $h(i+1,0) - h(i,0) = \eta(i,0)$. But standard arguments, one can show that the processes $B^\varepsilon(x) = \varepsilon^{-1/2}(u^\varepsilon(x,0) - \int_0^x \rho^0(y)\,\mathrm{d}y)$ converge to a continuous Gaussian process $B(x)$ with $B(0) = 0$ and the variance $EB^2(x) = \int_0^x \rho^0(1-\rho^0)\,\mathrm{d}y$ if $x > 0$, and $EB^2(x) = \int_x^0 \rho^0(1-\rho^0)\,\mathrm{d}y$ if $x < 0$. By convergence of B^ε to B, we mean the convergence of the processes $\hat{B}^\varepsilon$ to the process B where $\hat{B}^\varepsilon$ is a continuous process with $|\hat{B}^\varepsilon(x) - B^\varepsilon(x)| \leqslant \varepsilon$ for all x. The process $\hat{B}^\varepsilon$ is defined by linear interpolation between the points $(i\varepsilon, B^\varepsilon(i\varepsilon))$. It is not hard to see that the conditions of Theorem 2.4 are satisfied for the corresponding $B^e(x,y,t)$.

We write $B^\varepsilon(x,\rho;\omega_0)$ for $B^\varepsilon(x;\omega_0)$ when the initial data ρ^0 is identically the constant ρ.

Remark 2.6. – A better rate of convergence is expected for (2.11). It is conjectured that in fact

$$E^\rho\big[u_\rho^\varepsilon\big(tH'(\rho),t\big) - tL\big(H'(\rho)\big)\big]^2 = \mathrm{O}\big(\varepsilon^{4/3}\big).$$

If we assume this, then one can readily check that our results are still valid if $\sqrt{\varepsilon}$ is replaced with ε^α, provided that $\alpha \in (0, 2/3)$.

Remark 2.7. – When the function $\bar{u}$ is differentiable at (x_0, t_0), then $I(x_0, t_0) = \{y(x_0, t_0)\}$ is a singleton and

$$\bar{u}_x(x_0, t_0) = \rho(x_0, t_0) = L'\left(\frac{x_0 - y(x_0, t_0)}{t_0}\right).$$

In general, the set

$$D^*\bar{u}(x_0, t_0) := \left\{ L'\left(\frac{x_0 - y}{t_0}\right):\ y \in I(x_0, t_0)\right\}$$

coincides with the set of the limit points of the set

$$\{\bar{u}_x(x, t):\ u \text{ is differentiable at } (x, t)\}$$

as (x, t) approaches the point (x_0, t_0). (See for example [2].) In particular, if $I(x_0, t_0) \cap (-\infty, x_0 - t_0] \neq \emptyset$, then $1 \in D^*\bar{u}(x_0, t_0)$. Similarly, if $I(x_0, t_0) \cap [x_0 + t_0, \infty) \neq \emptyset$, then $0 \in D^*\bar{u}(x_0, t_0)$.

3. A bound on the fluctuations of w^ε

In this section, we use (2.11) to establish a suitable version of (2.16). Recall that $x^\varepsilon = x + e(x)\sqrt{\varepsilon}$, where e is a continuous function. Set

$$\begin{aligned}
R^\varepsilon(x, y, t) &= R^\varepsilon(x, y, t; \omega_1) = w^\varepsilon(x, y, t; \omega_1) - tL\left(\frac{x - y}{t}\right),\\
Y^\varepsilon(x, y, z, t; \omega_1) &= \begin{cases} \frac{(y-z)^2}{4t} + R^\varepsilon(x^\varepsilon, y, t; \omega_1) & \text{if } \left|\frac{x-y}{t}\right| \leqslant 1,\\ R^\varepsilon(x^\varepsilon, y, t; \omega_1) & \text{if } \left|\frac{x-y}{t}\right| > 1,\end{cases}\\
I_\rho(x, t) &= \begin{cases} \{x - tH'(\rho)\} & \text{if } \rho \in (0, 1),\\ [x + t, \infty) & \text{if } \rho = 0,\\ (-\infty, x - t] & \text{if } \rho = 1.\end{cases}
\end{aligned}\tag{3.1}$$

The set $I_\rho(x, t)$ is simply the set $I(x, t)$ when the initial data is $g(y) = \rho y$. The main result of this section is Lemma 3.1.

LEMMA 3.1. – *Let A be a finite subset of $\mathbb{R} \times (0, \infty)$. Let $\rho \in [0, 1]$ and define $\hat{y}(x, t) = x - tH'(\rho)$. Then there exist a function $\psi_\delta^\rho(\cdot) = \psi_\delta^\rho(\cdot; A)$ with $\lim_{\theta\to 0} \psi_\delta^\rho(\theta) = 0$, and a set $\Omega^\varepsilon(\delta, \rho) = \Omega^\varepsilon(\delta, \rho; A) \subset \Omega_{dy}$ such that*

$$P_{dy}\big(\Omega_{dy} - \Omega^\varepsilon(\delta, \rho)\big) \leqslant \delta,$$

and if $\psi(\cdot)$ is any function with $\lim_{\theta\to 0} \psi(\theta) = 0$, $\psi \geqslant \psi_\delta^\rho$, then

$$\lim_{\varepsilon\to 0} \sup_{(x,t)\in A} \sup_{\omega_1\in\Omega^\varepsilon(\delta,\rho)} \inf_{|y - I_\rho(x,t)|\leqslant\psi(\varepsilon)} \varepsilon^{-1/2} Y^\varepsilon\big(x, y, \hat{y}(x, t), t; \omega_1\big) = 0. \tag{3.2}$$

We state several lemmas that will be needed for the proof of Lemma 3.1. The first lemma appeared as Lemma 4.2 of [11] and its proof is omitted.

LEMMA 3.2. – *Let $\bar{u}$ be as in* (1.4) *and let the set A be as in Lemma* 3.1. *Define*

$$a_\ell(\lambda) = \inf_{(x,t)\in A}\left[\min\left\{g(y) + tL\left(\frac{x-y}{t}\right)\colon |y| \leqslant \ell, |y - I(x,t)| \geqslant \lambda\right\} - \bar{u}(x,t)\right].$$

Then $a_\ell(\lambda) > 0$ if $\lambda > 0$ and $\lim_{\lambda\to 0} a_\ell(\lambda) = 0$ for sufficiently large ℓ.

Next we state and prove a lemma that is related to the fact that the speed of propagation in our model is finite.

LEMMA 3.3. – *For every $T > 0$, there exists a function $\ell_T^\varepsilon(\omega_1)$ such that*

$$u^\varepsilon(x,t) = \inf\{u^\varepsilon(y,0) + w^\varepsilon(x,y,t)\colon |y| \leqslant \ell_T^\varepsilon(\omega_1)\}, \tag{3.3}$$

for every (x,t) with $|x| \leqslant T$, $t \in [0,T]$, and

$$\limsup_{\varepsilon\to 0} \ell_T^\varepsilon(\omega_1) \leqslant 2T, \tag{3.4}$$

in probability.

Proof. – Define random walks $x_i^\pm(t;\omega_1)$ with $x_i^\pm(0;\omega_1) = i$ such that x_i^- jumps to the left and x_i^+ jumps to the right with rate one. We use the realization $\omega_1 = (\ell_j(\cdot)\colon j \in \mathbb{Z})$ of the Poisson processes to decide when to jump $x_i^\pm$. More precisely, if $x_i^\pm(t-,\omega_1) = j$ and $\ell_j(t+) = \ell_j(t-) + 1$, then $x_i^\pm(t+;\omega_1) = j \pm 1$. From the definition of the process h, it is not hard to show that if $k_1(j) = k_2(j)$ for $j \in [i-\ell, i+\ell]$, and if $x_{i-\ell}^+(t) < i < x_{i+\ell}^-(t)$, then

$$h(i,t;k_1) = h(i,t;k_2). \tag{3.5}$$

Given $k \in \Gamma$ and $\ell \in \mathbb{Z}^+$, define

$$k_\ell(r) = \inf_{|j-i|\leqslant\ell}\{k(j) + v(r-j)\}.$$

Evidently $k_\ell(j) = k(j)$ for $j \in [i-\ell, i+\ell]$. From this, (2.3) and (3.5) we deduce that

$$h(i,t;k) = h(i,t;k_\ell) = \inf_{|j-i|\leqslant\ell}\{k(j) + w(i,j,t)\}, \tag{3.6}$$

whenever $x_{i-\ell}^+(t) < i < x_{i+\ell}^-(t)$. We now define

$$\ell_T^\varepsilon(\omega_1) = \varepsilon\ell\left(\left[\frac{T}{\varepsilon}\right], \frac{T}{\varepsilon}; \omega_1\right). \tag{3.7}$$

where

$$\ell(L_1, L_2; \omega_1) = \inf\{\ell\colon x_{-\ell}^+(L_1;\omega_1) < -L_2 < L_2 < x_\ell^-(L_1;\omega_1)\},$$

for every pair of positive numbers (L_1, L_2). Eq. (3.3) is an immediate consequence of (3.6) and the definition ℓ_T^ε.

It remains to show

$$\limsup_{\varepsilon\to 0} \ell_T^\varepsilon(\omega_1) \leqslant 2T$$

in probability. This is a straightforward consequence of

$$P_{dy}\big(\ell(L_1, L_2; \omega_1) > r\big) \leqslant P_{dy}\big(x_{-r}^+(L_1) \geqslant -L_2\big) + P_{dy}\big(x_r^-(L_1) \leqslant L_2\big),$$

and a law of large numbers for the random walks $x^\pm$. □

The next lemma appeared as Theorem 4.1 in [12] and its proof is omitted.

LEMMA 3.4. – *For every $T > 0$,*

$$\lim_{\varepsilon\to 0} E_{dy} \sup_{|x|\leqslant T} \sup_{t\leqslant T} \sup_{|y|\leqslant T} \left| w^\varepsilon(x, y, t) - tL\left(\frac{x-y}{t}\right)\right| = 0. \tag{3.8}$$

To this end let us fix a function $\beta : \mathbb{Z}^+ \to (0, \infty)$ and a sequence of non-decreasing functions $\alpha = (\alpha_r\colon r \in \mathbb{Z}^+)$ with $\lim_{\theta\to 0}\alpha_r(\theta) = 0$ for every $r \in \mathbb{Z}^+$. Let $\mathcal{K}(\alpha, \beta)$ denote the set of functions $b(x)$ such that

$$|b(x_1) - b(x_2)| \leqslant \alpha_r(|x_1 - x_2|), \quad |b(x_1)| \leqslant \beta(r),$$

for every x_1, x_2 with $|x_1|, |x_2| \leqslant r$ and each $r \in \mathbb{Z}^+$. We then define

$$\Omega_0^\varepsilon(\alpha, \beta) = \big\{(\omega_0, \omega_1)\colon\ B^\varepsilon(\cdot; \omega_0) \in \mathcal{K}(\alpha, \beta)\big\}.$$

Note that our assumption on the process B^ε implies that for every $\delta > 0$, there exists $(\alpha^\delta, \beta^\delta)$ and $\varepsilon_0(\delta) > 0$ such that

$$\inf_{0<\varepsilon<\varepsilon_0(\delta)} P^\varepsilon\big(\Omega_0^\varepsilon\big(\alpha^\delta, \beta^\delta\big)\big) \geqslant 1 - \delta. \tag{3.9}$$

We write $\Omega_0^\varepsilon(\alpha, \beta; \rho)$ for $\Omega_0^\varepsilon(\alpha, \beta)$ when the initial distribution is the equilibrium measure p^ρ. Hence

$$\inf_{0<\varepsilon<\varepsilon_0(\delta)} P^\rho\big(\Omega_0^\varepsilon\big(\alpha^\delta, \beta^\delta; \rho\big)\big) \geqslant 1 - \delta. \tag{3.10}$$

Furthermore, given $\delta > 0$, Lemma 3.3 implies that there exist a set of realizations Ω_1^ε and a positive number $\varepsilon_1(\delta)$ such that

$$\begin{gathered}(x, t) \in [-T, T] \times [0, T], \quad \omega_1 \in \Omega_1^\varepsilon(\delta) \Rightarrow \ell_T^\varepsilon(\omega_1) \leqslant 2T + 1, \\ \inf_{0<\varepsilon<\varepsilon_1(\delta)} P_{dy}\big(\Omega_1^\varepsilon(\delta)\big) \geqslant 1 - \delta.\end{gathered} \tag{3.11}$$

As the next step, we show that every minimizer in (1.6) is close to a minimizer in (1.4).

LEMMA 3.5. – *Let the set A be as in Lemma 3.1. For every $\ell, \varepsilon, \delta > 0$, there exist $\varepsilon_2(\delta) = \varepsilon_2(\delta; A) > 0$, two functions $\psi_\delta(\cdot) = \psi_\delta(\cdot; A)$, $\psi_{\delta,\ell}(\cdot) = \psi_{\delta,\ell}(\cdot; A)$ with*

$$\lim_{\theta\to 0} \psi_\delta(\theta) = 0, \qquad \lim_{\theta\to 0} \psi_{\delta,\ell}(\theta) = 0,$$

and a set $\Omega^\varepsilon(\delta) = \Omega^\varepsilon(\delta; A) \subseteq \Omega_0^\varepsilon(\alpha^\delta, \beta^\delta)$ such that

$$P^\varepsilon\big(\Omega^\varepsilon(\delta)\big) > 1 - 3\delta,$$

and if $\varepsilon \in (0, \varepsilon_2(\delta))$, $(x,t) \in A$, $\omega \in \Omega^\varepsilon(\delta)$, $z \in I^\varepsilon(x^\varepsilon, t; \omega) \cap [-\ell, \ell]$, then

$$u^\varepsilon\big(x^\varepsilon, t; \omega\big) = \inf\big\{u^\varepsilon(y, 0; \omega_0) + w^\varepsilon\big(x^\varepsilon, y, t; \omega_1\big)\colon\ |y - I(x,t)| \leqslant \psi_\delta(\varepsilon)\big\}, \tag{3.12}$$

and,

$$|z - I(x,t)| \leqslant \psi_{\delta,\ell}(\varepsilon). \tag{3.13}$$

Proof. – Assume $A \subseteq [-T, T] \times (0, T]$ and set $T_1 = T + 1$. Define

$$\begin{aligned} M_{\varepsilon,\ell}(\omega_1) &= \sup_{|x|\leqslant T_1} \sup_{t\leqslant T} \sup_{|y|\leqslant \ell} \left| w^\varepsilon(x, y, t; \omega_1) - tL\left(\frac{x-y}{t}\right)\right|, \\ \mu(\varepsilon, \ell) &= E_{dy} M_{\varepsilon,\ell}, \\ \Omega_2^\varepsilon &= \big\{\omega_1\colon\ M_{\varepsilon,\ell}(\omega_1) \leqslant \mu(\varepsilon,\ell)^{1/2}\big\}. \end{aligned} \tag{3.14}$$

By Chebychev Inequality,

$$P_{dy}\big(\Omega_{dy} - \Omega_1^\varepsilon\big) \leqslant \frac{\mu(\varepsilon,\ell)}{\mu(\varepsilon,\ell)^{1/2}} = \mu(\varepsilon,\ell)^{1/2}.$$

By Lemma 3.4, we can find $\varepsilon_3(\delta)$ such that for $\varepsilon \in (0, \varepsilon_3(\delta))$,

$$P_{dy}\big(\Omega_{dy} - \Omega_2^\varepsilon\big) \leqslant \delta. \tag{3.15}$$

We then set

$$\Omega^\varepsilon(\delta) = \big\{(\omega_0, \omega_1)\colon\ (\omega_0, \omega_1) \in \Omega_0^\varepsilon\big(\alpha^\delta, \beta^\delta\big), \omega_1 \in \Omega_1^\varepsilon \cap \Omega_2^\varepsilon\big\}. \tag{3.16}$$

From (3.9), (3.11) and (3.15) we deduce,

$$P^\varepsilon\big(\Omega^\varepsilon(\delta)\big) \geqslant 1 - 3\delta, \tag{3.17}$$

for every positive $\varepsilon < \varepsilon_2(\delta) = \min\{\varepsilon_0(\delta), \varepsilon_1(\delta), \varepsilon_3(\delta)\}$.

For (3.13), it suffices to find a function $\psi_{\delta,\ell}(\cdot)$ such that $\lim_{\theta\to 0}\psi_{\delta,\ell}(\theta) = 0$, and for every $(\omega_0, \omega_1) \in \Omega^\varepsilon(\delta)$, $(x,t) \in A$, and every y with $|y| \leqslant \ell$, $|y - I(x,t)| > \psi_{\delta,\ell}(\varepsilon)$,

$$u^\varepsilon\big(x^\varepsilon, t; \omega_0, \omega_1\big) < u^\varepsilon(y, 0; \omega_0) + w^\varepsilon\big(x^\varepsilon, y, t; \omega_1\big). \tag{3.18}$$

To construct $\psi_{\delta,\ell}$, let us write a_ℓ^{-1} for the right-continuous inverse of a_ℓ. By Lemma 3.2, we certainly have $a_\ell^{-1}(\lambda) > 0$ if $\lambda > 0$ and $\lim_{\lambda\to 0} a_\ell^{-1}(\lambda) = 0$. We now claim that $\psi_{\delta,\ell}(\varepsilon) = a_\ell^{-1}(c_1\sqrt{\varepsilon} + 2\mu(\varepsilon,\ell)^{1/2})$ will do the job for a suitable constant c_1 to be determined later. To see this, suppose $(x,t) \in A$, $|y - I(x,t)| > \lambda$, $|y| \leqslant \ell$ and let $\bar{y}(y)$ denote the closest point in $I(x,t)$ to y. Take $(\omega_0,\omega_1) \in \Omega^\varepsilon(\delta)$. We certainly have

$$\begin{aligned}
&\left|u^\varepsilon(y,0;\omega_0) - g(y)\right| \leqslant \sqrt{\varepsilon}\beta^\delta(\ell),\\
&\left|w^\varepsilon(x^\varepsilon,y,t;\omega) - tL\left(\frac{x^\varepsilon - y}{t}\right)\right| \leqslant \mu(\varepsilon,\ell)^{1/2}.
\end{aligned}$$

Choose ℓ_0 large enough so that if $\bar{y} \in I(x,t)$ for some (x,t) with $|x| \leqslant T$ and $t \in [0,T]$, then $|\bar{y}| \leqslant \ell_0$. Hence, for $\ell \geqslant \ell_0$,

$$\begin{aligned}
&u^\varepsilon(y,0;\omega_0) + w^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\\
&\quad\geqslant g(y) + tL\left(\frac{x^\varepsilon - y}{t}\right) - \sqrt{\varepsilon}\beta^\delta(\ell) - \mu(\varepsilon,\ell)^{1/2}\\
&\quad\geqslant g(y) + tL\left(\frac{x - y}{t}\right) - \sqrt{\varepsilon}\beta^\delta(\ell) - \mu(\varepsilon,\ell)^{1/2} - c_0\sqrt{\varepsilon}\\
&\quad\geqslant \bar{u}(x,t) + a_\ell(\lambda) - \sqrt{\varepsilon}\beta^\delta(\ell) - \mu(\varepsilon,\ell)^{1/2} - c_0\sqrt{\varepsilon}\\
&\quad= g\big(\bar{y}(y)\big) + tL\left(\frac{x - \bar{y}(y)}{t}\right) + a_\ell(\lambda) - \sqrt{\varepsilon}\beta^\delta(\ell) - \mu(\varepsilon,\ell)^{1/2} - c_0\sqrt{\varepsilon}\\
&\quad= g\big(\bar{y}(y)\big) + tL\left(\frac{x^\varepsilon - \bar{y}(y)}{t}\right) + a_\ell(\lambda) - \sqrt{\varepsilon}\beta^\delta(\ell) - \mu(\varepsilon,\ell)^{1/2} - 2c_0\sqrt{\varepsilon}\\
&\quad\geqslant u^\varepsilon\big(\bar{y}(y),0;\omega\big) + w^\varepsilon\big(x,\bar{y}(y),t;\omega_1\big) + a_\ell(\lambda) - 2\sqrt{\varepsilon}\beta^\delta(\ell)\\
&\qquad - 2\mu(\varepsilon,\ell)^{1/2} - 2c_0\sqrt{\varepsilon}
\end{aligned}$$

for some constant c_0. Set $c_1 = 2\beta^\delta(\ell) + 2c_0$. Then if $a_\ell(\lambda) - c_1\sqrt{\varepsilon} - 2\mu(\varepsilon,\ell)^{1/2} > 0$, the point y can not be in the set $I^\varepsilon(x^\varepsilon,t;\omega)$. Thus, if $|y - I(x,t)| > \psi_{\delta,\ell}(\varepsilon)$, then $y \notin I^\varepsilon(x^\varepsilon,t;\omega)$, proving (3.13).

Define $\psi_\delta = \psi_{\delta,\ell_1}$, where $\ell_1 = \max(2T+1,\ell_0)$. Now (3.12) follows from Lemma 3.3, (3.11) and (3.13). □

Remark 3.6. – Evidently (3.12) is also true if we replace ψ_δ with any $\psi \geqslant \psi_\delta$.

Proof of Lemma 3.1. – By definition

$$w^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big) = tL\left(\frac{x^\varepsilon - y}{t}\right) + R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big). \tag{3.19}$$

Fix $\rho \in (0,1)$ and define

$$S^\varepsilon(x,t,\rho;\omega_0,\omega_1) = u^\varepsilon_\rho(x,t;\omega_0,\omega_1) - u^\varepsilon_\rho\big(x - tH'(\rho),0;\omega_0\big) - tL\big(H'(\rho)\big). \tag{3.20}$$

Note that (2.11) implies

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}E^{\rho}\sup_{(x,t)\in A}\left[S^{\varepsilon}(x,t,\rho;\omega_0,\omega_1)\right]^2=0.$$

From this and (2.10) we deduce

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}E^{\rho}\sup_{(x,t)\in A}\left[S^{\varepsilon}\left(x^{\varepsilon},t,\rho;\omega_0,\omega_1\right)\right]^2=0. \tag{3.21}$$

Define

$$\Omega_1^{\varepsilon}(\rho)=\left\{\omega\colon \sup_{(x,t)\in A}\varepsilon^{-1/2}S^{\varepsilon}\left(x^{\varepsilon},t,\rho;\omega\right)<\phi(\varepsilon)\right\}$$

where

$$\phi(\varepsilon)=\varepsilon^{-1/4}E^{\rho}\Big\{\sup_{(x,t)\in A}\left[S^{\varepsilon}\left(x^{\varepsilon},t,\rho;\omega_0,\omega_1\right)\right]^2\Big\}^{1/4}.$$

By Chebyshev inequality,

$$P^{\rho}\left(\Omega-\Omega_1^{\varepsilon}(\rho)\right)\leqslant\frac{\varepsilon^{-1}E^{\rho}(\sup S^{\varepsilon})^2}{(\phi(\varepsilon))^2}=\left(\varepsilon^{-1}E^{\rho}\left(\sup S^{\varepsilon}\right)^2\right)^{1/2},$$

where the supremum is over the set A. From this and (3.21) we learn that there exists a positive $\varepsilon_4(\delta)$ such that if $\varepsilon\in(0,\varepsilon_4(\delta))$, then

$$P^{\rho}\left(\Omega_1^{\varepsilon}(\delta,\rho)\right)\geqslant 1-\delta. \tag{3.22}$$

We apply Lemma 3.5 where the initial distribution is the equilibrium measure p^{ρ}. As a result, for every $\delta>0$, there exist $\varepsilon_5(\delta)$, $\psi_{\delta}^{\rho}(\cdot)$ and a set $\Omega_2^{\varepsilon}(\delta,\rho)$ such that

$$\lim_{\theta\to 0}\psi_{\delta}^{\rho}(\theta)=0,\quad P^{\rho}\left(\Omega_2^{\varepsilon}(\delta,\rho)\right)\geqslant 1-\delta, \tag{3.23}$$

for every positive $\varepsilon<\varepsilon_5(\delta)$, every $\omega\in\Omega_2^{\varepsilon}(\delta,\rho)$, and every $\psi(\cdot)$ with $\lim_{\theta\to 0}\psi(\theta)=0$, $\psi\geqslant\psi_{\delta}^{\rho}$,

$$u_{\rho}^{\varepsilon}\left(x^{\varepsilon},t;\omega_0,\omega_1\right)=\inf_{|y-I_{\rho}|\leqslant\psi(\varepsilon)}\left\{u_{\rho}^{\varepsilon}(y,0;\omega_0)+tL\left(\frac{x^{\varepsilon}-y}{t}\right)+R^{\varepsilon}\left(x^{\varepsilon},y,t;\omega_1\right)\right\},$$

where $I_{\rho}=I_{\rho}(x,t)$ was defined in (3.1). We next define

$$\Omega_3^{\varepsilon}(\delta,\rho)=\Omega_1^{\varepsilon}(\rho)\cap\Omega_2^{\varepsilon}(\delta,\rho)\cap\Omega_0^{\varepsilon}\left(\alpha^{\delta},\beta^{\delta};\rho\right).$$

By (3.10), (3.22) and (3.23),

$$P^{\rho}\left(\Omega_3^{\varepsilon}(\delta,\rho)\right)\geqslant 1-3\delta. \tag{3.24}$$

To ease the notation, let us write $\hat{y}$ for $\hat{y}(x,t)=x-tH'(\rho)$. For every positive $\varepsilon<\varepsilon_6(\delta)=\min(\varepsilon_4(\delta),\varepsilon_5(\delta))$, $(x,t)\in A$, and every $(\omega_0,\omega_1)\in\Omega_3^\varepsilon(\delta,\rho)$,

$$\begin{aligned}
&u_\rho^\varepsilon\big(\hat{y}+\sqrt{\varepsilon}e(x),0;\omega_0\big)+tL\big(H'(\rho)\big)+S^\varepsilon\big(x^\varepsilon,t,\rho;\omega_0,\omega_1\big)\\
&\quad=u_\rho^\varepsilon\big(x^\varepsilon,t;\omega_0,\omega_1\big)\\
&\quad=\inf_{|y-\hat{y}|\leqslant\psi(\varepsilon)}\left\{u_\rho^\varepsilon(y,0;\omega_0)+tL\left(\frac{x^\varepsilon-y}{t}\right)+R^\varepsilon(x^\varepsilon,y,t;\omega_1)\right\}\\
&\quad=\inf_{|y-\hat{y}|\leqslant\psi(\varepsilon)}\left\{\rho y+\sqrt{\varepsilon}B^\varepsilon(y,\rho;\omega_0)+\sqrt{\varepsilon}L'\left(\frac{x-y}{t}\right)e(x)\right.\\
&\qquad\left.+tL\left(\frac{x-y}{t}\right)+R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\right\}+\mathrm{o}\big(\sqrt{\varepsilon}\big)\\
&\quad=\sqrt{\varepsilon}B^\varepsilon(\hat{y},\rho;\omega_0)+\sqrt{\varepsilon}L'\left(\frac{x-\hat{y}}{t}\right)e(x)\\
&\qquad+\inf_{|y-\hat{y}|\leqslant\psi(\varepsilon)}\left\{\rho y+tL\left(\frac{x-y}{t}\right)+R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\right\}+\mathrm{o}\big(\sqrt{\varepsilon}\big)
\end{aligned}\tag{3.25}$$

where for the second equality we used Lemma 3.5, and for the last identity we used the the fact that $\Omega_3^\varepsilon(\delta,\rho)\subset\Omega_0^\varepsilon(\alpha^\delta,\beta^\delta;\rho)$. Here and below, by $\mathrm{o}(\sqrt{\varepsilon})$ we mean an error term $r_1^\varepsilon(\omega)$ for which there exists a function $\psi_1(\varepsilon)$ such that

$$\lim_{\varepsilon\to0}\frac{\psi_1(\varepsilon)}{\sqrt{\varepsilon}}=0,\qquad\big|r_1^\varepsilon(\omega)\big|\leqslant\psi_1(\varepsilon),$$

for every $\omega\in\Omega_3^\varepsilon(\delta,\rho)$. Observe that for $\omega\in\Omega_0^\varepsilon(\alpha^\delta,\beta^\delta;\rho)$,

$$\begin{aligned}
u_\rho^\varepsilon\big(\hat{y}+\sqrt{\varepsilon}e(x),0;\omega_0\big)&=\big(\hat{y}+\sqrt{\varepsilon}e(x)\big)\rho+\sqrt{\varepsilon}B^\varepsilon\big(\hat{y}+\sqrt{\varepsilon}e(x),\rho;\omega_0\big)\\
&=\big(\hat{y}+\sqrt{\varepsilon}e(x)\big)\rho+\sqrt{\varepsilon}B^\varepsilon(\hat{y},\rho;\omega_0)+\mathrm{o}\big(\sqrt{\varepsilon}\big).
\end{aligned}$$

From this, (3.25) and the fact that $\Omega_3^\varepsilon(\delta,\rho)\subset\Omega_1^\varepsilon(\rho)$ we deduce that if $\omega\in\Omega_3^\varepsilon(\delta,\rho)$, then

$$\begin{aligned}
\big(\hat{y}+\sqrt{\varepsilon}e(x)\big)\rho+tL\big(H'(\rho)\big)&=\inf_{|y-\hat{y}|\leqslant\psi(\varepsilon)}\left\{\rho y+tL\left(\frac{x-y}{t}\right)+R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\right\}\\
&\quad+\sqrt{\varepsilon}L'\left(\frac{x-\hat{y}}{t}\right)e(x)+\mathrm{o}\big(\sqrt{\varepsilon}\big).
\end{aligned}\tag{3.26}$$

Note that since $\rho\in(0,1)$, we have

$$\left|\frac{x-\hat{y}}{t}\right|<1.$$

Moreover, since $|y-\hat{y}|\leqslant\psi(\varepsilon)$, we also have

$$\left|\frac{x-y}{t}\right|<1,\tag{3.27}$$

provided that ε is sufficiently small. An elementary calculation yields

$$\rho y + tL\left(\frac{x-y}{t}\right) - \rho\hat{y} - tL\big(H'(\rho)\big) = \frac{(y-\hat{y})^2}{4t}, \quad \rho = L'\left(\frac{x-\hat{y}}{t}\right). \tag{3.28}$$

From this and (3.26) we deduce (3.2) in the case of $\rho \in (0,1)$, provided that we choose

$$\Omega^\varepsilon(\delta,\rho) = \big\{\omega_1\colon\ (\omega_0,\omega_1) \in \Omega_3^\varepsilon(\delta,\rho) \text{ for some } \omega_0\big\}.$$

We now turn to the case $\rho \in \{0,1\}$. We only treat the case $\rho = 1$ because the case $\rho = 0$ can be treated in the same way. Note that in this case

$$u_1^\varepsilon(x,t) = x + \mathrm{O}(\varepsilon). \tag{3.29}$$

Also, the set

$$A^\varepsilon := \big\{y\colon\ |y - I_1(x,t)| \leqslant \psi(\varepsilon)\big\}$$

can be written as the union of two sets;

$$A^\varepsilon = A_1 \cup A_2^\varepsilon := (-\infty, x-t] \cup \big\{y\colon\ 0 < y - \hat{y} \leqslant \psi(\varepsilon)\big\},$$

where $\hat{y} = x - t$. We certainly have

$$\begin{aligned}
&\inf_{y\in A_2^\varepsilon}\left\{u_1^\varepsilon(y,0;\omega_0) + tL\left(\frac{x^\varepsilon - y}{t}\right) + R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\right\} \\
&\quad= \inf_{y\in A_2^\varepsilon}\left\{y + tL\left(\frac{x-y}{t}\right) + \sqrt{\varepsilon}L'\left(\frac{x-y}{t}\right)e(x) + R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\right\} + \mathrm{o}\big(\sqrt{\varepsilon}\,\big) \\
&\quad= \inf_{y\in A_2^\varepsilon}\left\{y + tL\left(\frac{x-y}{t}\right) + R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\right\} + \sqrt{\varepsilon}L'\left(\frac{x-\hat{y}}{t}\right)e(x) + \mathrm{o}\big(\sqrt{\varepsilon}\,\big) \\
&\quad= \inf_{y\in A_2^\varepsilon}\left\{\frac{(y-\hat{y})^2}{4t} + R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\right\} + x + \sqrt{\varepsilon}e(x) + \mathrm{o}\big(\sqrt{\varepsilon}\,\big),
\end{aligned} \tag{3.30}$$

where for the last equality, we used $L'(1) = 1$ and the elementary identity

$$y + tL\left(\frac{x-y}{t}\right) = x + \frac{(y-\hat{y})^2}{4t},$$

for $y \in (\hat{y}, x+t)$, which follows from the fact that $L(q) = (q+1)^2/4$ for $q \in [-1,1]$. On the other hand,

$$\begin{aligned}
&\inf_{y\in A_1}\left\{u_1^\varepsilon(y,0;\omega_0) + tL\left(\frac{x^\varepsilon - y}{t}\right) + R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\right\} \\
&\quad= \inf_{y\in A_1}\left\{y + tL\left(\frac{x^\varepsilon - y}{t}\right) + R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\right\} + \mathrm{O}(\varepsilon) \\
&\quad= x + \sqrt{\varepsilon}e(x) + \inf_{y\in A_1} R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big) + \mathrm{O}(\varepsilon).
\end{aligned} \tag{3.31}$$

This is because if $\frac{x^\varepsilon - y}{t} > 1$, then

$$tL\left(\frac{x^\varepsilon - y}{t}\right) = x^\varepsilon - y,$$

and if

$$\frac{x^\varepsilon - y}{t} \leqslant 1, \qquad \frac{x - y}{t} \geqslant 1,$$

then,

$$tL\left(\frac{x^\varepsilon - y}{t}\right) = x^\varepsilon - y + \mathrm{O}(\varepsilon).$$

Finally observe that (3.29)–(3.31) imply

$$\begin{aligned}
x + \sqrt{\varepsilon}e(x) &= u_1^\varepsilon\bigl(x^\varepsilon, t; \omega_0, \omega_1\bigr) \\
&= \inf_{y \in A^\varepsilon}\left\{u_\rho^\varepsilon(y, 0; \omega_0) + tL\left(\frac{x^\varepsilon - y}{t}\right) + R^\varepsilon\bigl(x^\varepsilon, y, t; \omega_1\bigr)\right\} \\
&= \min\left\{\inf_{y \in A_1} R^\varepsilon\bigl(x^\varepsilon, y, t; \omega_1\bigr), \inf_{y \in A_2^\varepsilon}\left[\frac{(y - \hat{y})^2}{4t} + R^\varepsilon\bigl(x^\varepsilon, y, t; \omega_1\bigr)\right]\right\} \\
&\quad + x + \sqrt{\varepsilon}e(x) + \mathrm{o}\bigl(\sqrt{\varepsilon}\bigr).
\end{aligned}$$

This evidently implies (3.2) when $\rho = 1$. $\square$

4. Proof of Theorem 2.3

This section is devoted to the proof of Theorem 2.3. We start with four lemmas. The first lemma is an immediate consequence of (1.18) and the translation invariance of the measure P_{dy}. Lemma 4.2 is a trivial consequence of the definition of the set $I(x, t)$. The proofs of Lemmas 4.1 and 4.2 are omitted.

LEMMA 4.1. – *Suppose $x^\varepsilon = x + \sqrt{\varepsilon}e$ and $y^\varepsilon = y + \sqrt{\varepsilon}e$. Then for every $(x, t) \in \mathbb{R} \times (0, \infty)$,*

$$w^\varepsilon\bigl(x^\varepsilon, y^\varepsilon, t\bigr) = tL\left(\frac{x - y}{t}\right) + \mathrm{O}\bigl(\varepsilon^{2/3}\bigr),$$

in probability.

LEMMA 4.2. – *Suppose $\bar{y} \in I(x, t)$ and g is differentiable at $\bar{y}$. Then*

$$g'(\bar{y}) = L'\left(\frac{x - \bar{y}}{t}\right). \tag{4.1}$$

LEMMA 4.3. – *Suppose $\bar{y} \in I(x, t)$ and g is convex in the set $[\bar{y} - r, \bar{y} + r]$. Then*

$$g(y) - g(\bar{y}) \geqslant L'\left(\frac{x - \bar{y}}{t}\right)(y - \bar{y}), \tag{4.2}$$

for every $y \in [\bar{y} - r, \bar{y} + r]$.

Proof. – If $\bar{y} \in I(x,t)$, then the fact that $\bar{y}$ is a minimizer implies that

$$-L'\left(\frac{x-\bar{y}}{t}\right) + g'(\bar{y}-) \leqslant 0 \leqslant -L'\left(\frac{x-\bar{y}}{t}\right) + g'(\bar{y}+).$$

This and the convexity of g imply,

$$g'(\bar{y}-) \leqslant L'\left(\frac{x-\bar{y}}{t}\right) \leqslant g'(\bar{y}+), \qquad g(y) - g(\bar{y}) \geqslant g'(\bar{y}\pm)(y-\bar{y}).$$

This evidently implies (4.2). □

LEMMA 4.4. – *If $\bar{y} \in I(x,t) \cap (-\infty, x-t)$, then $[\bar{y}, x-t] \subseteq I(x,t)$ and $g(y) = g(\bar{y}) + v(y-\bar{y})$ for every $y \in [\bar{y}, x-t]$. Similarly, if $\bar{y} \in I(x,t) \cap (x+t, \infty)$, then $[x+t, \bar{y}] \subseteq I(x,t)$ and $g(y) = g(\bar{y}) + v(y-\bar{y})$ for every $y \in [x+t, \bar{y}]$.*

Proof. – We only establish the first claim because the proof of the second claim is similar. Recall $v(z) = z^+$ and that $L(z) = v(z)$ for every z with $|z| \geqslant 1$. Suppose $\bar{y} \in I(x,t) \cap (-\infty, x-t)$ and $y \in [\bar{y}, x-t]$. Then

$$\begin{aligned} g(y) + tL\left(\frac{x-y}{t}\right) &= g(y) + v(x-y) \leqslant g(\bar{y}) + v(x-\bar{y}) \\ &= g(\bar{y}) + tL\left(\frac{x-\bar{y}}{t}\right), \end{aligned} \tag{4.3}$$

because

$$g(y) - g(\bar{y}) \leqslant v(y-\bar{y}) = y - \bar{y} = (x-\bar{y}) - (x-y) = v(x-\bar{y}) - v(x-y).$$

From (4.3) we deduce that $y \in I(x,t)$. This implies that in fact the inequality in (4.3) is an equality. Hence

$$g(y) - g(\bar{y}) = v(x-\bar{y}) - v(x-y) = v(y-\bar{y}). \qquad \square$$

The rest of this section is devoted to the statement and the proof of Lemma 4.5 which is the main ingredient for the proof of Theorem 2.3. We omit the straightforward proof of the fact that Lemma 4.5 implies Theorem 2.3 and refer the reader to Section 5 of [13].

To this end, let us define

$$Z^{e,\varepsilon}(x,y,t;\omega_0) = B^{\varepsilon}(y;\omega_0) + L'\left(\frac{x-y}{t}\right)e(x),$$
$$Z^{e,\varepsilon}(x,t;\omega_0) = \inf_{\bar{y} \in I(x,t)} Z^{e,\varepsilon}(x,\bar{y},t;\omega_0).$$

LEMMA 4.5. – *Let A be a finite subset of G_4. For every $\varepsilon, \delta, \eta > 0$, there exists a set $\bar{\Omega}^{\varepsilon}(\delta,\eta) = \bar{\Omega}^{\varepsilon}(\delta,\eta;A) \subset \Omega$ such that*

$$P^{\varepsilon}\big(\Omega - \bar{\Omega}^{\varepsilon}(\delta,\eta)\big) < 7\delta,$$

and

$$\limsup_{\varepsilon\to 0}\sup_{(x,t)\in A}\sup_{\omega\in\bar\Omega^\varepsilon(\delta)}\varepsilon^{-1/2}\bigl|u^\varepsilon\bigl(x^\varepsilon,t;\omega\bigr)-\bar u(x,t)-\sqrt{\varepsilon}\,Z^{e,\varepsilon}(x,t;\omega_0)\bigr|\leqslant\eta. \tag{4.4}$$

Proof. – Given $\delta>0$, let $\Omega^\varepsilon(\delta)$ and ψ_δ be as in Lemma 3.5. Recall

$$P^\varepsilon\bigl(\Omega^\varepsilon(\delta)\bigr)\geqslant 1-3\delta,\quad \Omega^\varepsilon(\delta)\subset\Omega_0^\varepsilon\bigl(\alpha^\delta,\beta^\delta\bigr), \tag{4.5}$$

and $\lim_{\theta\to 0}\psi_\delta(\theta)=0$. For every function ψ with $\lim_{\theta\to 0}\psi(\theta)=0$, $\psi\geqslant\psi_\delta$, and every $(\omega_0,\omega_1)\in\Omega^\varepsilon(\delta)$,

$$\begin{aligned}
&u^\varepsilon\bigl(x^\varepsilon,t;\omega_0,\omega\bigr)\\
&\quad=\inf_{|y-I(x,t)|\leqslant\psi(\varepsilon)}\bigl\{u^\varepsilon(y,0;\omega_0)+w^\varepsilon\bigl(x^\varepsilon,y,t;\omega_1\bigr)\bigr\}\\
&\quad=\inf_{|y-I(x,t)|\leqslant\psi(\varepsilon)}\bigl\{g(y)+\sqrt{\varepsilon}B^\varepsilon(y;\omega_0)+w^\varepsilon\bigl(x^\varepsilon,y,t;\omega_1\bigr)\bigr\}\\
&\quad=\inf_{\bar y\in I(x,t)}\inf_{|y-\bar y|\leqslant\psi(\varepsilon)}\bigl\{g(y)+\sqrt{\varepsilon}B^\varepsilon(y;\omega_0)+w^\varepsilon\bigl(x^\varepsilon,y,t;\omega_1\bigr)\bigr\}\\
&\quad=\inf_{\bar y\in I(x,t)}\bigl\{\sqrt{\varepsilon}B^\varepsilon(\bar y;\omega_0)+\inf_{|y-\bar y|\leqslant\psi(\varepsilon)}\bigl[g(y)+w^\varepsilon\bigl(x^\varepsilon,y,t;\omega_1\bigr)\bigr]\bigr\}+\mathrm{o}\bigl(\sqrt{\varepsilon}\bigr)\\
&\quad=\inf_{\bar y\in I(x,t)}\biggl\{\sqrt{\varepsilon}B^\varepsilon(\bar y;\omega_0)+\inf_{|y-\bar y|\leqslant\psi(\varepsilon)}\biggl[g(y)+tL\biggl(\frac{x^\varepsilon-y}{t}\biggr)-\bar u(x,t)\\
&\qquad+R^\varepsilon\bigl(x^\varepsilon,y,t;\omega_1\bigr)\biggr]\biggr\}+\bar u(x,t)+\mathrm{o}\bigl(\sqrt{\varepsilon}\bigr)\\
&\quad=\inf_{\bar y\in I(x,t)}\biggl\{\sqrt{\varepsilon}B^\varepsilon(\bar y;\omega_0)+\inf_{|y-\bar y|\leqslant\psi(\varepsilon)}\biggl[g(y)+tL\biggl(\frac{x-y}{t}\biggr)-\bar u(x,t)\\
&\qquad+\sqrt{\varepsilon}L'\biggl(\frac{x-\bar y}{t}\biggr)e(x)+R^\varepsilon\bigl(x^\varepsilon,y,t;\omega_1\bigr)\biggr]+\bar u(x,t)\biggr\}+\mathrm{o}\bigl(\sqrt{\varepsilon}\bigr).
\end{aligned}\tag{4.6}$$

Let us write $M_\varepsilon(\bar y;\omega)$ for the expression inside the curely brackets in the last line of (4.6) and set

$$\hat M_\varepsilon(\omega)=\inf_{\bar y\in\hat I(x,t)}M_\varepsilon(\bar y;\omega),\qquad \tilde M_\varepsilon^\pm(\omega)=\inf_{\bar y\in\tilde I^\pm(x,t)}M_\varepsilon(\bar y;\omega),$$
$$M_\varepsilon^\pm(\omega)=\inf_{\bar y\in I^\pm(x,t)}M_\varepsilon(\bar y;\omega),$$

where

$$\hat I(x,t)=I(x,t)\cap(x-t,x+t),\qquad \tilde I^\pm(x,t)=I(x,t)\cap\{x\pm t\},$$
$$I^-(x,t)=I(x,t)\cap(-\infty,x-t),\qquad I^+(x,t)=I(x,t)\cap(x+t,\infty).$$

Now (4.6) can be written as

$$u^\varepsilon(x^\varepsilon,t;\omega)=\min\bigl\{\hat M_\varepsilon(\omega),M_\varepsilon^-(\omega),M_\varepsilon^+(\omega),\tilde M_\varepsilon^-(\omega),\tilde M_\varepsilon^+(\omega)\bigr\}+\mathrm{o}\bigl(\sqrt{\varepsilon}\bigr). \tag{4.7}$$

On the other hand, if $\bar y\in I(x,t)$, $-t\leqslant x-y\leqslant t$, $-t\leqslant x-\bar y\leqslant t$, and y is sufficiently close to $\bar y$, then

$$\begin{aligned}g(y)+tL\left(\frac{x-y}{t}\right)-\bar{u}(x,t)&=g(y)+tL\left(\frac{x-y}{t}\right)-g(\bar{y})-tL\left(\frac{x-\bar{y}}{t}\right)\\&=g(y)-g(\bar{y})-L'\left(\frac{x-\bar{y}}{t}\right)(y-\bar{y})+\frac{(y-\bar{y})^2}{4t}\\&\geqslant\frac{(y-\bar{y})^2}{4t},\end{aligned}\tag{4.8}$$

where for the last inequality, we used Lemma 4.3. For every $\bar{y}\in\hat{I}(x,t)$, set $\hat{\Omega}^\varepsilon(\delta,\bar{y}):=\Omega^\varepsilon(\delta/r,\rho(\bar{y}))$, where $\rho(\bar{y})$ is the unique number $\rho\in(0,1)$ that satisfies $x-tH'(\rho)=\bar{y}$, the number r is the cardinality of the set $\hat{I}(x,t)$, and the set $\Omega^\varepsilon(\cdot,\cdot)$ is as in Lemma 3.1. We then set

$$\hat{\Omega}^\varepsilon(\delta)=\bigcap_{\bar{y}\in\hat{I}(x,t)}\hat{\Omega}^\varepsilon(\delta,\bar{y})\cap\Omega^\varepsilon(\delta).$$

From Lemma 3.1 and (4.5) we deduce

$$P^\varepsilon\big(\hat{\Omega}^\varepsilon(\delta)\big)\geqslant 1-4\delta.\tag{4.9}$$

Let us assume

$$\psi\geqslant\max_{\bar{y}\in\hat{I}(x,t)}\psi_\delta^{\rho(\bar{y})}.\tag{4.10}$$

From (4.8) we deduce that for $\omega\in\hat{\Omega}^\varepsilon(\delta)$,

$$\begin{aligned}M^\varepsilon(\omega)&\geqslant\inf_{\bar{y}\in\hat{I}(x,t)}\bigg\{\sqrt{\varepsilon}B^\varepsilon(\bar{y};\omega_0)+\sqrt{\varepsilon}L'\left(\frac{x-\bar{y}}{t}\right)e(x)\\&\qquad+\inf_{|y-\bar{y}|\leqslant\psi(\varepsilon)}Y^\varepsilon(x,y,\bar{y},t;\omega_1)\bigg\}+\bar{u}(x,t)+\mathrm{o}\big(\sqrt{\varepsilon}\big)\\&=\bar{u}(x,t)+\sqrt{\varepsilon}\inf_{\bar{y}\in\hat{I}(x,t)}\left\{B^\varepsilon(\bar{y};\omega_0)+L'\left(\frac{x-\bar{y}}{t}\right)e(x)\right\}+\mathrm{o}\big(\sqrt{\varepsilon}\big),\end{aligned}\tag{4.11}$$

where for the equality we used (4.10) and Lemma 3.1.

We now turn to the term $\tilde{M}_\varepsilon^\pm$. If the set $\tilde{I}^\pm(x,t)$ is not empty, we have $I^\pm(x,t)=\{\bar{y}_\pm\}$ for $\bar{y}_\pm=x\pm t$. We claim

$$\begin{aligned}\tilde{M}_\varepsilon^-(\omega)&\geqslant\sqrt{\varepsilon}B^\varepsilon(\bar{y_-};\omega_0)+\sqrt{\varepsilon}L'\left(\frac{x-\bar{y}_-}{t}\right)e(x)\\&\qquad+\inf_{|y-\bar{y_-}|\leqslant\psi(\varepsilon)}Y^\varepsilon(x,y,\bar{y}_-,t;\omega_1)+\bar{u}(x,t)+\mathrm{o}\big(\sqrt{\varepsilon}\big)\\&\geqslant\sqrt{\varepsilon}B^\varepsilon(\bar{y_-};\omega_0)+\sqrt{\varepsilon}L'\left(\frac{x-\bar{y}_-}{t}\right)e(x)\\&\qquad+\inf_{|y-I_1(x,t)|\leqslant\psi(\varepsilon)}Y^\varepsilon(x,y,\bar{y}_-,t;\omega_1)+\bar{u}(x,t)+\mathrm{o}\big(\sqrt{\varepsilon}\big).\end{aligned}\tag{4.12}$$

The second inequality is obvious because $\bar{y}_-\in I_1(x,t)=(\infty,x-t]$, and for the first inequality in (4.12), we simply apply (4.8) with $\bar{y}=\bar{y}_-$ when

$$\left|\frac{x-y}{t}\right|\leqslant 1,\tag{4.13}$$

and use

$$g(y)+tL\left(\frac{x-y}{t}\right)-\bar{u}(x,t)\geqslant 0, \tag{4.14}$$

when

$$\left|\frac{x-y}{t}\right|>1. \tag{4.15}$$

From (4.12) and Lemma 3.1 we deduce

$$\tilde{M}_\varepsilon^-(\omega)\geqslant \bar{u}(x,t)+\sqrt{\varepsilon}\left\{B^\varepsilon(\bar{y}_-;\omega_0)+L'\left(\frac{x-\bar{y}_-}{t}\right)e(x)\right\}+\mathrm{o}\big(\sqrt{\varepsilon}\,\big), \tag{4.16}$$

provided that $\omega\in\Omega^\varepsilon(\delta)\cap\Omega^\varepsilon(\delta,1)$ and $\psi\geqslant\psi_\delta^1$. In the same way we show that if $\omega\in\Omega^\varepsilon(\delta)\cap\Omega^\varepsilon(\delta,0)$ and $\psi\geqslant\psi_\delta^0$, then

$$\tilde{M}_\varepsilon^+(\omega)\geqslant \bar{u}(x,t)+\sqrt{\varepsilon}\left\{B^\varepsilon(\bar{y}_+;\omega_0)+L'\left(\frac{x-\bar{y}_+}{t}\right)e(x)\right\}+\mathrm{o}\big(\sqrt{\varepsilon}\,\big). \tag{4.17}$$

We now turn to the terms $M_\varepsilon^\pm$. The numbers $\bar{y}_\pm=x\pm t$ are defined as before. Again we use (4.8) with $\bar{y}=\bar{y}_-$ when (4.13) holds and apply (4.14) when (4.15) holds. We obtain

$$\begin{aligned}
M_\varepsilon^-(\omega)\geqslant{}& \inf_{\bar{y}\in I^-(x,t)}\bigg\{\sqrt{\varepsilon}B^\varepsilon(\bar{y};\omega_0)+\sqrt{\varepsilon}L'\left(\frac{x-\bar{y}}{t}\right)e(x)\\
&\quad+\inf_{|y-\bar{y}|\leqslant\psi(\varepsilon)}Y^\varepsilon(x,y,\bar{y}_-,t;\omega_1)\bigg\}+\bar{u}(x,t)+\mathrm{o}\big(\sqrt{\varepsilon}\,\big)\\
\geqslant{}& \inf_{\bar{y}\in I^-(x,t)}\left\{\sqrt{\varepsilon}B^\varepsilon(\bar{y};\omega_0)+\sqrt{\varepsilon}L'\left(\frac{x-\bar{y}}{t}\right)e(x)\right\}\\
&\quad+\inf_{|y-I_1(x,t)|\leqslant\psi(\varepsilon)}Y^\varepsilon(x,y,\bar{y}_-,t;\omega_1)+\bar{u}(x,t)+\mathrm{o}\big(\sqrt{\varepsilon}\,\big)\\
={}&\bar{u}(x,t)+\sqrt{\varepsilon}\inf_{\bar{y}\in I^-(x,t)}\left\{B^\varepsilon(\bar{y};\omega_0)+L'\left(\frac{x-\bar{y}}{t}\right)e(x)\right\}+\mathrm{o}\big(\sqrt{\varepsilon}\,\big),
\end{aligned} \tag{4.18}$$

where for the second inequality we used $I^-(x,t)\subseteq I_1(x,t)$ and for the last equality we used Lemma 3.1. The term M_ε^+ can be treated likewise.

Set $\tilde{\Omega}^\varepsilon(\delta)=\hat{\Omega}^\varepsilon(\delta)\cap\Omega^\varepsilon(\delta,1)\cap\Omega^\varepsilon(\delta,0)$. Evidently (4.9) implies

$$P^\varepsilon\big(\tilde{\Omega}^\varepsilon(\delta)\big)\geqslant 1-6\delta. \tag{4.19}$$

Additional to (4.10), we assume

$$\psi\geqslant\max\big(\psi_\delta^0,\psi_\delta^1\big).$$

From (4.7), (4.11) and (4.16)–(4.18) we deduce that if $\omega=(\omega_0,\omega_1)\in\Omega^\varepsilon(\delta)$, then

$$u^\varepsilon\big(x^\varepsilon,t;\omega\big)\geqslant\bar{u}(x,t)+\sqrt{\varepsilon}Z^{e,\varepsilon}(x,t;\omega)+\mathrm{o}\big(\sqrt{\varepsilon}\,\big). \tag{4.20}$$

If $\bar{y}\in I(x,t)$ and $\bar{y}^\varepsilon=\bar{y}+\sqrt{\varepsilon}e(x)$, then by (1.6) and Lemma 4.1,

$$u^\varepsilon\big(x^\varepsilon, t; \omega_0, \omega_1\big) \leqslant u^\varepsilon\big(\bar y^\varepsilon, 0; \omega_0\big) + w^\varepsilon\big(x^\varepsilon, \bar y^\varepsilon, t; \omega_1\big) \\ = u^\varepsilon\big(\bar y^\varepsilon, 0; \omega_0\big) + tL\left(\frac{x-\bar y}{t}\right) + \mathrm{O}\big(\varepsilon^{2/3}\big),$$

in probability. Define

$$\Omega_0^\varepsilon(\bar y) = \left\{\omega\colon \left|w^\varepsilon\big(x^\varepsilon, \bar y^\varepsilon, t; \omega_1\big) - tL\left(\frac{x-\bar y}{t}\right)\right| \leqslant \varepsilon^{7/12}\right\}.$$

Choose a finite set

$$J(x,t) := \bar y_1, \bar y_2, \dots, \bar y_s\} \subseteq I(x,t)$$

such that for every $\omega \in \Omega_0^\varepsilon(\alpha^\delta, \beta^\delta)$,

$$\Big|\inf_{\bar y \in I(x,t)} Z^{e,\varepsilon}(x, \bar y, t; \omega_0) - \inf_{\bar y \in J(x,t)} Z^{e,\varepsilon}(x, \bar y, t; \omega_0)\Big| \leqslant \eta.$$

Note that the set J depends on δ, η and can be chosen to be independent of ε . In fact we may choose the set J in such a way that every $\bar y \in I(x,t)$ satisfies $|\bar y - J(x,t)| \leqslant \tau$. First choose r large enought so that $I(x,t) \subseteq [-r, r]$. Then use (4.5) and choose τ small enough so that $\alpha_r(\tau) + c_0\tau \leqslant \eta$ where c_0 is a bound on the Lipschitz constant of $L'(\frac{x-y}{t})e(x)$ as a function of y. We then set

$$\bar\Omega_0^\varepsilon(\delta, \eta) = \bigcap_{(x,t)\in A} \bigcap_{\bar y \in J(x,t)} \Omega_0^\varepsilon(\bar y),$$

$$\bar\Omega^\varepsilon(\delta, \eta) = \tilde\Omega^\varepsilon(\delta) \cap \bar\Omega_0^\varepsilon(\delta, \eta).$$

From Lemma 4.1 and (4.19) we know that

$$P^\varepsilon\big(\hat\Omega^\varepsilon(\delta, \eta)\big) \geqslant 1 - 7\delta,$$

for sufficiently small ε. For $\omega \in \bar\Omega^\varepsilon(\delta, \eta)$ and $\bar y \in J(x,t)$,

$$\begin{aligned}
&u^\varepsilon\big(x^\varepsilon, t; \omega_0, \omega\big) \\
&\quad\leqslant g\big(\bar y^\varepsilon\big) + tL\left(\frac{x-\bar y}{t}\right) + \sqrt{\varepsilon}B^\varepsilon\big(\bar y^\varepsilon; \omega_0\big) + \varepsilon^{7/12} \\
&\quad= g\big(\bar y^\varepsilon\big) + tL\left(\frac{x-\bar y}{t}\right) + \sqrt{\varepsilon}B^\varepsilon(\bar y; \omega_0) + \mathrm{o}\big(\sqrt{\varepsilon}\big) \\
&\quad= g(\bar y) + \sqrt{\varepsilon}g'(\bar y)e(x) + tL\left(\frac{x-\bar y}{t}\right) + \sqrt{\varepsilon}B^\varepsilon(\bar y; \omega_0) + \mathrm{o}\big(\sqrt{\varepsilon}\big) \\
&\quad= g(\bar y) + \sqrt{\varepsilon}L'\left(\frac{x-\bar y}{t}\right)e(x) + tL\left(\frac{x-\bar y}{t}\right) + \sqrt{\varepsilon}B^\varepsilon(\bar y; \omega_0) + \mathrm{o}\big(\sqrt{\varepsilon}\big) \\
&\quad= \bar u(x,t) + \sqrt{\varepsilon}Z^{e,\varepsilon}(x, \bar y, t; \omega_0) + \mathrm{o}\big(\sqrt{\varepsilon}\big),
\end{aligned}$$

where for the third equality we used Lemma 4.2. As a result,

$$\begin{aligned}
u^\varepsilon\big(x^\varepsilon, t; \omega_0, \omega_1\big) &\leqslant \bar u(x,t) + \sqrt{\varepsilon}\inf_{\bar y \in J(x,t)} Z^{e,\varepsilon}\big(x, \bar y, t; \omega_0\big) + \mathrm{o}\big(\sqrt{\varepsilon}\big) \\
&\leqslant \bar u(x,t) + \sqrt{\varepsilon}\inf_{\bar y \in I(x,t)} Z^{e,\varepsilon}\big(x, \bar y, t; \omega_0\big) + \eta\sqrt{\varepsilon} + \mathrm{o}\big(\sqrt{\varepsilon}\big).
\end{aligned} \tag{4.21}$$

This and (4.20) complete the proof of (4.4). □

Remark 4.6. – Note that for the proof of (4.20), we only need to assume $A \subset G_1$. It is only for (4.21) that we need the differentiability of g at the points $\bar{y}$ in $I(x,t)$.

5. Proof of Theorem 2.4

The main ingredient for the proof of Theorem 2.4 is Lemma 5.1. Once we have this lemma, we can repeat the proof of Theorem 2.5 of [11] to conclude Theorem 2.4.

Write $I_\ell^\varepsilon(x,t,\omega)$ for $I^\varepsilon(x,t,\omega) \cap [-\ell,\ell]$.

LEMMA 5.1. – *Let $\bar{\Omega}^\varepsilon(\delta,\eta)$ be as in Lemma* 4.5 *and assume that A is a finite subset of G_5. For every positive ℓ,*

$$\limsup_{\varepsilon\to 0} \sup_{(x,t)\in A} \sup_{\omega\in\bar{\Omega}^\varepsilon(\delta,\eta)} \sup_{z\in I_\ell^\varepsilon(x^\varepsilon,t,\omega)} \left| Z^{e,\varepsilon}(x,z,t;\omega_0) - \inf_{\bar{y}\in I(x,t)} Z^{e,\varepsilon}\big(x,\bar{y},t;\omega_0\big)\right| \leqslant \eta.$$

Proof. – Let $z \in I_\ell^\varepsilon(x^\varepsilon,t,\omega)$ and $\omega = (\omega_0,\omega_1) \in \bar{\Omega}^\varepsilon(\delta,\eta)$. Let ψ be any function with $\lim_{\theta\to 0}\psi(\theta) = 0$. We certainly have,

$$\begin{aligned}
&u^\varepsilon\big(x^\varepsilon,t;\omega_0,\omega_1\big)\\
&\quad= \inf_{|y-z|\leqslant\psi(\varepsilon)} \big\{u^\varepsilon(y,0;\omega_0) + w^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\big\}\\
&\quad= \inf_{|y-z|\leqslant\psi(\varepsilon)} \left\{g(y) + \sqrt{\varepsilon}B^\varepsilon(y;\omega_0) + tL\left(\frac{x^\varepsilon-y}{t}\right) + R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\right\}\\
&\quad= \inf_{|y-z|\leqslant\psi(\varepsilon)} \left\{g(y) + tL\left(\frac{x-y}{t}\right) + R^\varepsilon\big(x^\varepsilon,y,t;\omega_1\big)\right\}\\
&\qquad+ \sqrt{\varepsilon}L'\left(\frac{x-z}{t}\right)e(x) + \sqrt{\varepsilon}B^\varepsilon(z;\omega_0) + \mathrm{o}\big(\sqrt{\varepsilon}\,\big).
\end{aligned} \tag{5.1}$$

Let us write $\bar{y}(z)$ for the closest point in the set $I(x,t)$ to the point z. As in (4.8) we can write,

$$\begin{aligned}
&g(y) + tL\left(\frac{x-y}{t}\right) - \bar{u}(x,t)\\
&\quad= g(y) + tL\left(\frac{x-y}{t}\right) - g(\bar{y}(z)) - tL\left(\frac{x-\bar{y}(z)}{t}\right)\\
&\quad= g(y) - g\big(\bar{y}(z)\big) - L'\left(\frac{x-\bar{y}(z)}{t}\right)\big(y-\bar{y}(z)\big) + \frac{(y-\bar{y}(z))^2}{4t}\\
&\quad\geqslant \frac{(y-\bar{y}(z))^2}{4t},
\end{aligned} \tag{5.2}$$

provided that

$$\left|\frac{x-y}{t}\right| \leqslant 1.$$

Otherwise we use

$$g(y)+tL\left(\frac{x-y}{t}\right)-\bar{u}(x,t)\geqslant 0.$$

From this, (5.2) and (5.1) we deduce,

$$u^{\varepsilon}\left(x^{\varepsilon},t;\omega_0,\omega_1\right)\geqslant \bar{u}(x,t)+\inf_{|y-z|\leqslant\psi(\varepsilon)}Y\left(x,y,\bar{y}(z);t\right)+\sqrt{\varepsilon}Z^{e,\varepsilon}(x,z,t;\omega_0)+\mathrm{o}\left(\sqrt{\varepsilon}\right). \tag{5.3}$$

Note that since $A\subseteq G_5$, we have $I(x,t)\subseteq[x-t,x+t]$. Also note that $|y-z|\leqslant\psi(\varepsilon)$ implies $|y-\bar{y}|\leqslant\bar{\psi}(\varepsilon)$ where $\bar{\psi}=\psi+\psi_{\delta,\ell}$ and $\psi_{\delta,\ell}$ is as in Lemma 3.5. Define

$$J^{\varepsilon}(\bar{y})=\begin{cases}\{y\colon\ |y-\bar{y}|\leqslant\bar{\psi}(\varepsilon)\} & \text{if } \bar{y}\in(x-t,x+t),\\ \{y\colon\ y-\bar{y}\leqslant\bar{\psi}(\varepsilon)\} & \text{if } \bar{y}=x-t,\\ \{y\colon\ \bar{y}-y\leqslant\bar{\psi}(\varepsilon)\} & \text{if } \bar{y}=x+t,\end{cases}$$

Evidently (5.3) implies

$$u^{\varepsilon}\left(x^{\varepsilon},t;\omega_0,\omega_1\right)\geqslant\bar{u}(x,t)+\inf_{y\in J^{\varepsilon}(\bar{y}(z))}Y\left(x,y,\bar{y}(z);t\right)+\sqrt{\varepsilon}Z^{e,\varepsilon}(x,z,t;\omega_0)+\mathrm{o}\left(\sqrt{\varepsilon}\right),$$

where $\mathrm{o}(\sqrt{\varepsilon})$ is uniform in $\omega\in\bar{\Omega}^{\varepsilon}(\delta,\eta)$ and $z\in I_{\ell}^{\varepsilon}(x^{\varepsilon},t,\omega)$. We then apply Lemma 3.1 to deduce,

$$u^{\varepsilon}\left(x^{\varepsilon},t;\omega_0,\omega_1\right)\geqslant\bar{u}(x,t)+\sqrt{\varepsilon}Z^{e,\varepsilon}(x,z,t;\omega_0)+\mathrm{o}\left(\sqrt{\varepsilon}\right).$$

This and Lemma 4.5 imply

$$Z^{e,\varepsilon}(x,z,t;\omega_0)-\inf_{\bar{y}\in I(x,t)}Z^{e,\varepsilon}(x,\bar{y},t;\omega_0)\leqslant\mathrm{o}(1)+\eta, \tag{5.4}$$

where $\mathrm{o}(1)$ goes to zero uniformly in $\omega\in\bar{\Omega}^{\varepsilon}(\delta,\eta)$ and $z\in I_{\ell}^{\varepsilon}(x^{\varepsilon},t,\omega)$. On the other hand, by (3.13)

$$\begin{aligned}&Z^{e,\varepsilon}(x,z,t;\omega_0)-Z^{e,\varepsilon}(x,t;\omega_0)\\&\quad=Z^{e,\varepsilon}\left(x,y(z),t;\omega_0\right)-\inf_{\bar{y}\in I(x,t)}Z^{e,\varepsilon}(x,\bar{y},t;\omega_0)+\mathrm{o}(1)\geqslant\mathrm{o}(1).\end{aligned}$$

This and (5.4) complete the proof of lemma. □

REFERENCES

[1] Aldous D., Diaconis P., Hammersley's interacting particle process and longest increasing sequences, Probab. Theory Related Fields 103 (1995) 199–213.

[2] Cannarsa P., Mennucci A., Sinestrari C., Regularity results for solutions of a class of Hamilton–Jacobi equations, Arch. Rational Mech. Anal. 140 (1997) 197–223.

[3] Evans L.C., Partial Differential Equations, American Mathematical Society, 1998.

[4] Ferrari P.A., Fontes L.R.G., Current fluctuations for the asymmetric exclusion process, Ann. Probab. 22 (1994) 319–820.

[5] Ferrari P.A., Fontes L.R.G., Shock fluctuations in the asymmetric simple exclusion process, Probab. Theory Related Fields 99 (1994) 305–319.
[6] Ferrari P.A., Fontes L.R.G., Vares M.E., The asymmetric simple exclusion model with multiple shocks, Ann. Inst. Henri Poincare 36 (2000) 109–126.
[7] Johansson K., Shape fluctuations and random matrices, Comm. Math. Phys. 209 (2000) 437–476.
[8] Liggett T.M., Interacting Particle Systems, Springer, 1985.
[9] Rezakhanlou F., Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$, Comm. Math. Phys. 140 (1991) 417–448.
[10] Rezakhanlou F., Microscopic location of shocks in one conservation laws, Annals de l'IHP, Analyse nonlineaire 12 (1995) 119–153.
[11] Rezakhanlou F., Central limit theorem for stochastic Hamilton–Jacobi equations, Comm. Math. Phys. 211 (2000) 413–438.
[12] Rezakhanlou F., Continuum limit for some growth models, preprint.
[13] Rezakhanlou F., Tarver (III) J.E., Homogenization for stochastic Hamilton–Jacobi equations, Arch. Rat. Mech. Analysis 151 (2000) 277–309.
[14] Seppäläinen T., Existence of hydrodynamics for totally asymmetric simple K-exclusion process, Ann. Probab. 27 (1999) 361–415.