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Some remarks on isoperimetry of gaussian type


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by

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ABSTRACT. – We give a martingale proof of Gaussian isoperimetry,
which also contains Bobkov’s inequality on the two-point space and
its extension to non symmetric Bernoulli measures. We derive the
equivalence of different forms of Gaussian type isoperimetry. This allows
us to prove a sharp form of Bobkov’s inequality for the sphere and to get
isoperimetric estimates for the unit cube. © 2000 Éditions scientifiques et
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RÉSUMÉ. – Nous donnons une démonstration de l’isopérimétrie gauss-
ienne par une technique de martingale. Cette démonstration fournit
aussi l’inégalité de Bobkov sur l’espace à deux points et une gé-
néralisation aux mesures de Bernoulli non symétriques. Nous mon-
trons l’équivalence de plusieurs formes d’inégalités isopérimétriques
de type gaussien. Ceci nous permet de prouver une inégalité de Bob-
kov précise pour la sphère et d’obtenir des estimations isopérimétriques

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1. INTRODUCTION

It is well known that among subsets of the sphere with prescribed volume, spherical caps have minimal boundary measure. Using Poincaré’s limit argument, which gives a representation of the Gaussian measure on \( \mathbb{R}^n \) as a limit of projections on \( \mathbb{R}^n \) of the invariant probability measures on high dimensional spheres, Borell [9] and Sudakov-Tsirel’son [18] proved that half-spaces are solutions to the isoperimetric problem in Gauss space; later, Ehrhard [11] obtained this result by means of symmetrization techniques. In this article, we shall be mainly interested by a different approach due to Bobkov [4], who emphasized a functional version of this Gaussian isoperimetric problem (inequality (2) below); we begin with some notation.

The Euclidean norm of a vector \( x \in \mathbb{R}^n \) is denoted by \( |x| \); if \( \mu \) is a Borel probability measure on \( \mathbb{R}^n \) and if \( A \) is an arbitrary Borel subset of \( \mathbb{R}^n \), the boundary \( \mu \)-measure of \( A \) is denoted by

\[
\mu^+(A) = \liminf_{h \to 0^+} \frac{\mu(A_h) - \mu(A)}{h},
\]

where \( A_h = \{ x \in \mathbb{R}^n ; d(x, A) \leq h \} \) is the \( h \)-enlargement of \( A \) for the Euclidean distance. Let \( \gamma_1 \) be the standard Gaussian probability measure on \( \mathbb{R} \), with density \( \varphi(x) = \exp(-x^2/2)/\sqrt{2\pi}, \quad x \in \mathbb{R}, \) and let \( \Phi(x) = \int_{-\infty}^{x} d\gamma_1 \). The Gaussian isoperimetric function \( I \) is defined for every \( t \in [0, 1] \) by

\[
I(t) = \varphi \circ \Phi^{-1}(t);
\]

this value \( I(t) \) represents the minimal Gaussian boundary measure of an interval of Gaussian measure \( t \), which is achieved for sets of the form \((-\infty, a) \) or \((b, +\infty) \) with \( a = \Phi^{-1}(t) \) or \( b = \Phi^{-1}(1 - t) \). We have thus \( I(t) = e^{-t^2/2}/\sqrt{2\pi} \) exactly when \( t = \int_{-\infty}^{a} d\gamma_1 \). Notice that \( I(0) = 0, \) and \( I(t) = I(1 - t) \) for \( 0 \leq t \leq 1 \); the reader can check that this function \( I \) satisfies on \((0, 1)\) the differential relation \( I'' = -1 \), which will play an important role in the next section (proof of Proposition 1).
The standard estimate for the tail of the Gaussian distribution gives that 
\( I(t) \sim t\sqrt{2\log(1/t)} \) as \( t \to 0 \).

The isoperimetric inequality for the standard Gaussian measure \( \gamma_n \) on \( \mathbb{R}^n \) with density \( \exp(-|x|^2/2)/\sqrt{2\pi} \), \( x \in \mathbb{R}^n \) can be stated as follows: for every measurable set \( A \subset \mathbb{R}^n \), we have

\[
\gamma_n^+(A) \geq I(\gamma_n(A)).
\]

(1)

In other words, if we define \( a \in \mathbb{R} \) by the equation \( \gamma_n(A) = \int_{-\infty}^a d\gamma_1 \), then \( \gamma_n^+(A) \geq \exp(-a^2/2)/\sqrt{2\pi} \). Clearly, this inequality is an equality for affine half-spaces in \( \mathbb{R}^n \).

Recently, Bobkov [4] proved a functional version of the Gaussian isoperimetry: for every locally Lipschitz function \( f: \mathbb{R}^n \to [0, 1] \), one has

\[
I\left( \int_{\mathbb{R}^n} f \, d\gamma_n \right) \leq \int_{\mathbb{R}^n} \sqrt{I^2(f) + |\nabla f|^2} \, d\gamma_n.
\]

(2)

It is easy to see that this inequality implies (1). Bobkov deduces (2) from the following “two-point” isoperimetric inequality: for all \( a, b \in [0, 1] \),

\[
I\left( \frac{a+b}{2} \right) \leq \frac{1}{2} \sqrt{I^2(a) + \left( \frac{b-a}{2} \right)^2} + \frac{1}{2} \sqrt{I^2(b) + \left( \frac{b-a}{2} \right)^2}.
\]

(3)

Using the remarkable tensorisation properties of this inequality and the central limit theorem, Bobkov shows that (3) implies (2). As it is noticed in [4], inequality (2) for \( \mathbb{R}^n \) can also be proved from (1) for \( \mathbb{R}^{n+1} \) by choosing \( A \subset \mathbb{R}^n \times \mathbb{R} \) to be the subgraph of \( \Phi^{-1} \circ f \); actually, this reasoning already appears in Ehrhard’s paper [12]: the relation 2.2.1, p. 323 of [12] contains Bobkov’s inequality (2); of course the striking point about the paper [4] is that this inequality (2) is obtained there as a consequence of the simple two-point inequality (3).

In Section 2, we extend a Brownian approach to (2) due to Capitaine, Hsu and Ledoux [10]. We get a unified proof of (3) and (2), and an extension of (3) to an isoperimetric inequality for non symmetric Bernoulli measures. The third section contains a proof of the equivalence of different forms of isoperimetry on the Gaussian model. It follows from works by Wang [19] and by Bakry and Ledoux (Theorem 4.1 of [1]) that for any probability measure \( d\mu(x) = e^{-V(x)} \, dx \) on \( \mathbb{R}^n \), with \( V'' \geq \alpha I d\mu \) for some \( \alpha \in \mathbb{R} \), and such that \( \int \int \exp(\varepsilon |x-y|^2) \, d\mu(x) \, d\mu(y) < \infty \) for
some $\varepsilon > \sup(0, -\alpha)$, there exists $c > 0$ such that for every Borel set $A \subset \mathbb{R}^n$

$$\mu^+(A) \geq c I(\mu(A)).$$  \hfill (4)

A simple proof of this fact for log-concave probability measures is given by Bobkov [5]: (4) is equivalent to the existence of a number $\varepsilon > 0$ such that $\int \exp(\varepsilon|x|^2) \, d\mu(x) < \infty$ (Herbst condition). Moreover he proves that (4) implies that for all locally Lipschitz functions $f : \mathbb{R}^n \to [0, 1]$,

$$I \left( \int_{\mathbb{R}^n} f \, d\mu \right) \leq \int_{\mathbb{R}^n} \left( I(f) + \frac{1}{c} |\nabla f| \right) \, d\mu. \hfill (5)$$

The constants provided by these results are not very good. Sharp ones are given by Bakry and Ledoux [1]: under the hypothesis $V'' \geq c^2 I_{\mathbb{R}^n}$, one has for $d\mu(x) = e^{-V(x)} \, dx$ and every $f$ as above

$$I \left( \int_{\mathbb{R}^n} f \, d\mu \right) \leq \int_{\mathbb{R}^n} \sqrt{I^2(f) + \frac{1}{c^2} |\nabla f|^2} \, d\mu. \hfill (6)$$

Notice that the case $\mu = \gamma_n$ and $c = 1$ gives (2).

It is clear that (6) implies (5), which implies (4). We will show that they are equivalent, with the same constant $c$. The proof strongly relies on the Gaussian model (2). Then, we give a sharp form of Bobkov’s inequality for spheres, using the Gaussian isoperimetric function $I$. Finally, we improve the isoperimetric estimates of Hadwiger [15] for the unit cube in $\mathbb{R}^n$. In particular, we recover the following result of Hadwiger: among subsets of measure 1/2 of the unit cube, half-cubes have the smallest boundary measure.

### 2. BROWNIAN PROOF OF BOBKOV’S INEQUALITIES

In order to simplify the notation we work with real-valued processes and functions. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion on $\mathbb{R}$ with natural filtration $(\mathcal{F}_t)_{t \geq 0}$ and such that $B_0 = 0$. We assume that all the processes appearing below are adapted with respect to this filtration.

**Proposition 1.** Let $(M_t)_{t \geq 0}$ and $(N_t)_{t \geq 0}$ be $(\mathcal{F}_t)_{t \geq 0}$ real-valued martingales with $M_t = M_0 + \int_0^t m_s \, dB_s$, $N_t = N_0 + \int_0^t n_s \, dB_s$, and let $A_t = A_0 + \int_0^t a_s \, ds$ be an increasing process, such that $A_t$ is bounded for
every $t \geq 0$ and $A_0 \geq 0$. Assume that $a_t |N_t|^2 \geq |m_t|^2$ for every $t \geq 0$, and that for some $\varepsilon \in (0, 1/2)$, we have $M_t \in [\varepsilon, 1 - \varepsilon]$ for every $t \geq 0$. Then

$$\left( \sqrt{\frac{I^2(M_t)}{t} + A_t |N_t|^2} \right)_{t \geq 0}$$

is a submartingale.

The result remains true, with essentially the same proof, when $M_t$ is a real martingale and $N_t$ a vector-valued martingale with respect to the $n$-dimensional Brownian motion $(B_t^{(n)})$. In this case, $(m_t)$ is a vector process, $dM_t = m_t \cdot dB_t^{(n)}$ is the scalar product in $\mathbb{R}^n$ and $(n_t)$ is a matrix-valued process. The condition above remains $a_t |N_t|^2 \geq |m_t|^2$, this time with the Euclidean norm.

Proof. – Let $J : \mathbb{R} \to \mathbb{R}$ be a positive $C^2$ function, constant outside $[0, 1]$ and such that $J(x) = I(x)$ when $\varepsilon \leq x \leq 1 - \varepsilon$. Let $F(x, y, t) = \sqrt{J^2(x) + ty^2}$. Direct computations give

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{F^3} \left( tJ^2(x)y^2 + J^3(x)J''(x) + ty^2J(x)J''(x) \right),$$

$$\frac{\partial^2 F}{\partial y^2} = \frac{tJ^2(x)}{F^3}, \quad \frac{\partial^2 F}{\partial x \partial y} = -\frac{tyJ(x)J'(x)}{F^3}.$$ 

Writing $Q_t$ for the triple $(M_t, N_t, A_t)$, we get by Itô’s formula

$$X_t := F(Q_t) = F(Q_0) + \int_0^t \left( \frac{\partial F}{\partial x}(Q) dM_t + \frac{\partial F}{\partial y}(Q) dN_t \right) + \int_0^t \Delta(s) ds,$$

with

$$\Delta(s) = \frac{\partial F}{\partial t} a_s + \frac{1}{2} \left( \frac{\partial^2 F}{\partial x^2}(Q_s)m_s^2 + 2 \frac{\partial^2 F}{\partial x \partial y}(Q_s)m_s n_s + \frac{\partial^2 F}{\partial y^2}(Q_s)n_s^2 \right).$$

Since the stochastic integral has a bounded integrand, it is a martingale. Hence $(X_t)$ is a martingale plus $\int_0^t \Delta(s) ds$. But since $J(M_t)$ and $I(M_t)$ coincide, we get using the relation $II'' = -1$ and omitting the variables

$$2F^3 \Delta = F^2 N^2 a + (I^2AN^2 - I^2 - AN^2)m^2 - 2(II'AN)mn + I^2 AN^2.$$ 

Since $N^2 a \geq m^2$, we have $F^2 N^2 a \geq (I^2 + AN^2)m^2$, hence

$$2F^3 \Delta \geq A(I^2 N^2 m^2 - 2II' Nmn + I^2 n^2) = A(I'Nm - In)^2 \geq 0,$$

thus $(X_t)$ is a submartingale. □
The preceding computation is not difficult, but does not really explain why the result was intuitively clear. Given a non-negative semimartingale $Z$ such that $Z + dZ = Z + q dB + r dt$, Itô’s formula shows that $\sqrt{Z}$ is a submartingale precisely when the formal second degree polynomial in the $\beta$ variable

$$T = Z + q\beta + r\beta^2$$

has a non-positive discriminant $q^2 - 4Zr \leq 0$, or in other words when $T \geq 0$ for every real value of $\beta$. When $Z = I^2(M) + AN^2$, our formal expression is equal to

$$T = (I^2 + 2II'm\beta + I^2m^2\beta^2) + II''m^2\beta^2 + aN^2\beta^2 + A(N^2 + 2Nn\beta + n^2\beta^2)$$

$$\geq (I + I'm\beta)^2 + A(N + n\beta)^2 \geq 0$$

since $II''m^2 + aN^2 = -m^2 + aN^2 \geq 0$. The trick is simply that the increase of $A_t$ (multiplied by $N_t^2$) must compensate the fact that $II'' < 0$. If we were trying to do the same for a different function $J$, we see that all is needed is that $(JJ')(M_t) m_t^2 + a_t N_t^2 \geq 0$ for every $t$.

The previous result appears in the more abstract setting of the Wiener space in [10], for the special case of $M_t = \mathbb{E}[f(B_t) \mid \mathcal{F}_t]$, $A_t = t \wedge 1$ and $N_t = \mathbb{E}[\nabla f(B_t) \mid \mathcal{F}_t]$ (the three processes are constant when $t \geq 1$), where $f$ is any regular function on taking values in $[0, 1]$. As a consequence, Capitaine, Hsu and Ledoux obtain Bobkov’s inequality (2) in the equivalent form:

$$I(\mathbb{E} f(B_1)) = \mathbb{E} F(M_0, N_0, 0) \leq \mathbb{E} F(M_1, N_1, 1)$$

$$= \mathbb{E} \sqrt{I^2(f(B_1)) + |\nabla f(B_1)|^2}.$$

**Remark.** - When $f(s) = 1_{s \geq x}$,

$$F(M_t, N_t, t) = \frac{1}{\sqrt{1-t}} \exp - \frac{(x + B_t)^2}{2(1-t)}$$

is a martingale. This corresponds to the equality case in the isoperimetric inequality.

Our next aim is to recover Bobkov’s two-point inequality by this submartingale approach. We will use the following stopping times, for $d, e \in \mathbb{R}$

$$T_d = \inf\{t \geq 0; B_t = d\} \quad \text{and} \quad T_{d,e} = T_d \wedge T_e.$$
**Proposition 2.** Let $m \in (0, 1)$, $d < 0 < e$ be such that $[m + d, m + e] \subset (0, 1)$. Then

$$ (Y_t)_{t \geq 0} := \left( \sqrt{I^2(m + B_{T_{d,e} \wedge t})} + T_{d,e} \wedge t \right)_{t \geq 0} $$

is a submartingale.

**Proof.** Let $\tau = T_{d,e}$. We apply Proposition 1 with $M_t = m + B_{t \wedge \tau}$, $N_t = 1$, and $\tau = t \wedge \tau$; the conditions on $m, d, e$ imply that $M_t$ stays in some interval $[\varepsilon, 1 - \varepsilon]$; for $t \leq \tau$ we have $m_t = a_t = 1$, and $m_t = a_t = 0$ when $t > \tau$, thus $a_t N_t^2 = a_t \geq m_t^2$ for every $t$. \( \Box \)

We need the following classical facts about the exit time of an interval (see for example [17]); recall that these results are obtained by applying the stopping time theorem to $T_{d,e}$ and to the martingales $(B_t)_{t \geq 0}$, $(B_t^2 - t)_{t \geq 0}$ and $(B_t^3 - 3t B_t)_{t \geq 0}$.

**Lemma 3.** Let $d, e$ be such that $ed < 0$. Then, the hitting times of $d$ and $e$ satisfy

$$ P(T_d < T_e) = \frac{e}{e - d}, \quad \mathbb{E} T_{d,e} = -de, $$

$$ \mathbb{E} [T_d \mid T_d < T_e] = \frac{d(d - 2e)}{3}. $$

Using the preceding results, we may now derive Bobkov’s two-point inequality for non-symmetric measures (if we take $p = 1/2$ in the next result, we get inequality (3), the symmetric case):

**Proposition 4.** Let $a, b, p, q \in [0, 1]$, with $p + q = 1$. Then one has

$$ I(pa + qb) \leq p \sqrt{I^2(a) + \frac{1 - p^2}{3} (a - b)^2} + q \sqrt{I^2(b) + \frac{1 - q^2}{3} (a - b)^2}. $$

**Proof.** As a consequence of Proposition 2, and with the same notation, we have $I(m) = \mathbb{E} Y_0 \leq \mathbb{E} Y_t$. When $t$ tends to infinity, we get letting $p_{d,e} := P(T_d < T_e)$

$$ I(m) \leq \mathbb{E} \left[ \sqrt{I^2(m + B_{T_{d,e}})} + T_{d,e} \wedge t \right] $$

$$ = p_{d,e} \mathbb{E} \left[ \sqrt{I^2(m + e)} + T_e \mid T_e < T_d \right] $$

$$ + p_{d,e} \mathbb{E} \left[ \sqrt{I^2(m + d) + T_d} \mid T_d < T_e \right]. $$
By the concavity of the square root function, and using Lemma 3,

\[ I(m) \leq P(T_e < T_d) \sqrt{I^2(m + e) + \mathbb{E}[T_e | T_e < T_d]} + P(T_d < T_e) \sqrt{I^2(m + d) + \mathbb{E}[T_d | T_d < T_e]} \]

\[ \leq \frac{-d}{e - d} \sqrt{I^2(m + e) + \frac{e(e - 2d)}{3}} + \frac{e}{e - d} \sqrt{I^2(m + d) + \frac{-d(2e - d)}{3}}. \]

Assume that \( a > b \). To prove the result, we set \( m = pa + qb, e = a - m = q(a - b) \) and \( d = b - m = p(b - a) \).

Let \( \mu_p \) be the measure on \( \{-1, 1\} \) defined by \( \mu_p(\{1\}) = p, \mu_p(\{-1\}) = q \), and let \( \mu_p^n = (\mu_p)^\otimes n \). Set \( c(1) = (1 - p^2)/3 \) and \( c(-1) = (1 - q^2)/3 \). For \( f: \{-1, 1\}^n \to [0, 1] \) we consider the following “modulus of gradient”:

\[ D(f)(x) = \sqrt{\sum_{i=1}^n c(x_i) [f(x) - f(s_i(x))]^2}, \]

where \( x = (x_i)_{i=1}^n \in \{-1, 1\}^n \) and \( s_i(x) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n) \). By a classical tensorisation argument (which we recall in the remark after Proposition 5, in a slightly different setting), the previous proposition yields the following isoperimetric inequality on \( \{-1, 1\}^n, \mu_p^n \):

\[ I(\mathbb{E}_{\mu_p^n} f) \leq \mathbb{E}_{\mu_p^n} \sqrt{I^2(f) + D(f)^2}. \]

This extends Corollary 1 in [4]. Similar isoperimetric inequalities for the Bernoulli measure, involving different moduli of gradient, are derived by Bobkov and Götze in [6].

### 3. Equivalent Forms of Gaussian Type Isoperimetry

In order to deal with spheres, we need to generalize slightly our setting. Let \((M, g)\) be a Riemannian manifold; for every subset \( A \) of \( M \), we may define as before \( A_h = \{x \in M; d(x, A) \leq h\} \), the \( h \)-enlargement of \( A \) for
the geodesic distance. For any Borel probability measure $\mu$ on $M$, the boundary $\mu$-measure of a Borel subset $A$ of $M$ is again defined by

$$\mu^+(A) = \liminf_{h \to 0^+} \frac{\mu(A_h) - \mu(A)}{h}.$$ 

**Proposition 5.** Let $c > 0$ and let $\mu$ be a Borel probability measure on $M$, absolutely continuous with respect to the Riemannian volume. Then the following properties are equivalent:

(i) For every measurable $A \subseteq M$, $\mu^+(A) \geq c I(\mu(A))$.

(ii) For every locally Lipschitz function $f : M \to [0, 1]$,

$$I \left( \int_M f \, d\mu \right) \leq \int_M \left( I(f) + \frac{1}{c} |\nabla f| \right) \, d\mu.$$ 

(iii) For every locally Lipschitz function $f : M \to [0, 1]$,

$$I \left( \int_M f \, d\mu \right) \leq \int_M \sqrt{I^2(f) + \frac{1}{c^2} |\nabla f|^2} \, d\mu.$$ 

**Proof.** It is well known that (iii) $\Rightarrow$ (i) (or that (ii) $\Rightarrow$ (i), for the same reason: take $f_\varepsilon(x) = (1 - d(x, A)/\varepsilon)_+$ and let $\varepsilon \to 0$). The implication (i) $\Rightarrow$ (ii) was done in [3] for the Gaussian measure and $c = 1$ but the proof extends to the general case [5]. However, we give here a shorter proof. Assuming (i), the co-area formula yields:

$$\int_M |\nabla f| \, d\mu = \int_0^1 \mu^+([f \leq t]) \, dt \geq c \int_0^1 I(\mu([f \leq t])) \, dt$$

(for this formula, see [13], Theorem 3.2.12, which deals with the Lebesgue measure on $\mathbb{R}^n$ and uses the $n - 1$ dimensional Hausdorff measure of the set $\{f = t\}$ instead of $\mu^+([f \leq t])$; the manifold case follows from 3.2.12 with the usual partition of unity argument; for more general situations of this co-area formula, see also [7]). Let $\nu$ be the distribution of $f$ with respect to $\mu$. Following [1] we may assume that $\nu$ is absolutely continuous with respect to Lebesgue’s measure on $\mathbb{R}$ and has a positive density on its support $[a, b] \subset [0, 1]$. For $t \in [0, 1]$, set
\[ N(t) = \nu([0, t]) = \mu([f \leq t]). \]  
We have to show that
\[ \int_0^1 I(N(t)) \, dt \geq I\left( \int_0^1 t \, d\nu(t) \right) - \int_0^1 I(t) \, d\nu(t). \]

Let \( k = N^{-1} \circ \Phi : \mathbb{R} \to [a, b] \). We apply the weak functional form of Gaussian isoperimetry to \( k \) and we get, since the distribution of \( k \) with respect to \( \gamma_1 \) is \( \nu \), that
\[ \int |k'| \, d\gamma_1 \geq I\left( \int k \, d\gamma_1 \right) - \int I(k) \, d\gamma_1 \]

\[ = I\left( \int t \, d\nu(t) \right) - \int I(t) \, d\nu(t). \]

We know by definition that \( \int_{-\infty}^{+\infty} d\gamma_1 = \int_{0}^{k(x)} d\nu = N(k(x)) \) for every \( x \) real, hence \( e^{-x^2/2}/\sqrt{2\pi} = I(N(k(x))) \) and we obtain by the change of variables \( t = k(x) \)
\[ \int_{-\infty}^{+\infty} |k'| \, d\gamma_1 = \int_{-\infty}^{+\infty} k'(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx = \int_0^1 I(N(t)) \, dt. \]

This finishes the proof that \( (i) \Rightarrow (ii) \). We show now that \( (ii) \) implies \( (iii) \). The beginning is similar to [1]. Let \( \nu \) be the distribution of \( f \), and let \( N, k \) be as before. We apply \( (ii) \) to \( \psi_\varepsilon(f) \), where \( \psi_\varepsilon(t) = 1 \) when \( t \leq r \), \( \psi_\varepsilon(t) = 0 \) when \( t \geq r + \varepsilon \) and \( \psi_\varepsilon \) linear (continuous) in between. Letting \( \varepsilon \to 0 \) one gets
\[ cI(N(r)) \leq \theta(r)N'(r), \]
where \( \theta(s) = \mathbb{E}[|\nabla f|] \). Differentiating the relation that defines \( k \), we get \( N'(k(x))k'(x) = e^{-x^2/2}/\sqrt{2\pi} = I(N(k(x))) \). Setting \( r = k(x) \), the previous inequality becomes \( c \, k'(x) \leq \theta(k(x)) \). We apply (2) to \( k \):
\[ I\left( \int k \, d\gamma_1 \right) \leq \int \sqrt{I^2(k) + |k'|^2} \, d\gamma_1 \leq \int \sqrt{I^2(k) + \frac{1}{c^2} \theta(k)^2} \, d\gamma_1. \]

Since the distribution of \( k \) with respect to \( \gamma_1 \) is \( \nu \), we may translate the preceding line as
The Minkowski inequality \( (\mathbb{E} a)^2 + (\mathbb{E} b)^2 \leq \mathbb{E} a^2 + b^2 \) for the conditional expectation gives

\[
I \left( \int_0^1 t \, d\nu(t) \right) \leq \int_0^1 \sqrt{I^2(t) + \frac{1}{c^2} \theta^2(t)} \, d\nu(t)
\]

\[
= \int_0^1 \left( \mathbb{E} \left[ I(f) \mid f = t \right] \right)^2 + \left( \mathbb{E} \left[ \frac{\nabla f}{c} \mid f = t \right] \right)^2 \, d\nu(t).
\]

The Minkowski inequality \( \sqrt{(\mathbb{E} a)^2 + (\mathbb{E} b)^2} \leq \mathbb{E} \sqrt{a^2 + b^2} \) for the conditional expectation gives

\[
I \left( \int_M f \, d\mu \right) \leq \int_0^1 \mathbb{E} \left[ \sqrt{I^2(f) + \frac{1}{c^2} |\nabla f|^2} \mid f = t \right] \, d\nu(t)
\]

\[
= \int_M \sqrt{I^2(f) + \frac{1}{c^2} |\nabla f|^2} \, d\mu.
\]

**Remark.** Among the three equivalent forms of Gaussian isoperimetry, (iii) is formally stronger and can be tensorized: we assume given two probability measures \(\mu_1\) and \(\mu_2\) on \(M_1\) and \(M_2\); the product space \(M_1 \times M_2\) is equipped with the natural Riemannian product metric, for which \(|\nabla|^2 = |\nabla_1|^2 + |\nabla_2|^2\); if \(\mu_1\) and \(\mu_2\) satisfy (iii) with the same constant \(c\), so does \(\mu_1 \otimes \mu_2\). Hence, (i) and (ii) can also be tensorized (this does not seem obvious without using (iii)): if \(\mu_1\) and \(\mu_2\) satisfy (i) with the same constant \(c\), so does \(\mu_1 \otimes \mu_2\).

Let us sketch the argument, due to Bobkov, which shows that the inequality in (iii) can be tensorized. Let \(f(x_1, x_2)\) be a regular function on \(M_1 \times M_2\) and let \(F(x_1) = \int_{M_2} f(x_1, x_2) \, d\mu_2(x_2)\). For each fixed \(x_1 \in M_1\), we may write using (iii) for \(\mu_2\) and for the function \(x_2 \mapsto f(x_1, x_2)\)

\[
I(F(x_1)) \leq \int_{M_2} \sqrt{I^2(f(x_1, x_2)) + \frac{1}{c^2} |\nabla_2 f(x_1, x_2)|^2} \, d\mu_2(x_2) =: B_1(x_1).
\]

Next, using (iii) for \(\mu_1\) and \(F\),

\[
I \left( \int_{M_1 \times M_2} f \, d\mu_1 \, d\mu_2 \right) = I \left( \int_{M_1} F(x_1) \, d\mu_1(x_1) \right)
\]

\[
\leq \int_{M_1} \sqrt{I^2(F(x_1)) + \frac{1}{c^2} |\nabla_1 F(x_1)|^2} \, d\mu_1(x_1)
\]
We bound $I(F(x_1))$ by $B_1(x_1)$, and expressing $\nabla_1 F(x_1)$ as an integral of $\nabla_1 f$ we get

$$\frac{1}{c} |\nabla_1 F(x_1)| \leq \int_{M_2} \frac{1}{c} |\nabla_1 f(x_1, x_2)| d\mu_2(x_2) =: B_2(x_1);$$

both expressions $B_1(x_1)$ and $B_2(x_1)$ are integrals in the $x_2$ variable of some functions $b_1(x_1, x_2)$ and $b_2(x_1, x_2)$; by Minkowski, we have for every $x_1 \in M_1$

$$A(x_1) \leq \sqrt{B_1^2(x_1) + B_2^2(x_1)} \leq \int_{M_2} \sqrt{b_1^2(x_1, x_2) + b_2^2(x_1, x_2)} d\mu_2(x_2),$$

and the result follows after integrating on $M_1$, since $|\nabla f|^2 = |\nabla_1 f|^2 + |\nabla_2 f|^2$.

We shall give two applications of Proposition 5. For the first one, let $rS^n \subset \mathbb{R}^{n+1}$ be the Euclidean sphere of radius $r > 0$, with the Riemannian metric induced by that of $\mathbb{R}^{n+1}$. Let $\sigma_{rS^n}$ be the uniform probability measure on $rS^n$. The spherical isoperimetric function is for $a \in [0, 1]$:

$$I_{rS^n}(a) = \inf \{\sigma_{rS^n}^+(A), \sigma_{rS^n}(A) = a\}.$$  

Since the infimum is achieved for caps, one can give an analytic expression of it. Notice that with our notations $I_{rS^n} = I_{S^n}/r$. When $a$ is close to 0, it is clear that

$$I_{rS^n}(a) \sim \frac{\alpha_n}{r} a^{\frac{n-1}{n}},$$

for some $\alpha_n$ depending only on the dimension. Furthermore, the Gaussian isoperimetric function satisfies $I(a) \sim a \sqrt{2 \log(1/a)}$ as $a \to 0$, so the constant $c_{rS^n} = \inf_{a \in (0, 1)} I_{rS^n}(a)/I(a)$ is positive (in fact, the infimum is achieved for $a = 1/2$ [2]). Let $\nabla_\circ$ be the spherical gradient. Using the implication (i) $\Rightarrow$ (iii) of Proposition 5, we obtain:
THEOREM 6. — Let $f : rS^n \to [0, 1]$ be a locally Lipschitz function. The following inequality holds:

$$I \left( \int f \, d\sigma_{r,S^n} \right) \leq \int \sqrt{I^2(f) + c_{r,S^n}^{-2} |\nabla f|^2} \, d\sigma_{r,S^n}.$$ 

By construction, this is optimal for caps. Previous versions of this inequality existed, involving the Ricci curvature of $S^n$, $n \geq 2$, instead of $c_{r,S^n}$ [1].

We turn now to our second application, the isoperimetry for the unit cube $[0, 1]^n$ in $\mathbb{R}^n$. Let $d\lambda_n(x) = 1_{[0,1]^n}(x) \, dx$ be the Lebesgue probability measure on the unit cube. Notice that the definition of $\lambda_n^+$ assigns the value 0 to the boundary of the cube: we will be measuring only boundaries inside the open cube.

THEOREM 7. — Let $A$ be a Borel subset of $[0, 1]^n$, then

$$\lambda_n^+(A) \geq \sqrt{2\pi} \, I(\lambda_n(A)).$$

Proof. — It is clear that for any non-empty set $A \subset \mathbb{R}$ and for every $\varepsilon > 0$, one has

$$\lambda_1(A_\varepsilon) \geq \inf(1, \lambda_1(A) + \varepsilon).$$

So $\lambda_1^+(A) \geq 1$ if $\lambda_1(A) \in (0, 1)$. Since $\max I = 1/\sqrt{2\pi}$, one has for every measurable subset $A$

$$\lambda_1^+(A) \geq \sqrt{2\pi} \, I(\lambda_1(A)).$$

By the remark following Proposition 5, this property also holds for the product measure $\lambda_n$. □

This yields that among subsets of measure 1/2, half-cubes of the form $H = [0, 1/2] \times [0, 1]^{n-1}$ have the smallest boundary measure. Indeed, for $\lambda_n(A) = 1/2$, one has $\lambda_n^+(A) \geq \sqrt{2\pi} \, I(1/2) = 1 = \lambda_n^+(H)$. This was already implied by [15], where it is proved that $\lambda_n^+(A) \geq 4\lambda_n(A)(1 - \lambda_n(A))$. Nevertheless, our estimate is always better when $\lambda_n(A) \neq 1/2$ (and of course $0 < \lambda_n(A) < 1$). In fact, up to a multiplicative constant, it is the optimal estimate valid for all $n \geq 1$. To see this, consider the sets

$$A_{n,t} = \left\{ x \in [0, 1]^n; \sum_{i=1}^n \left( x_i - \frac{1}{2} \right) \leq \frac{\sqrt{n}}{2\sqrt{3}} \, t \right\}.$$
It is clear that the enlargement \((A_{n,t})_e\) is equal to \(A_{n,t+2,\sqrt{3}e}\). By the central limit theorem, \(\lambda_n(A_{n,t})\) tends to \(\Phi(t)\) when \(n\) tends to infinity. So for every \(t\), \(\lambda_n^+(A_{n,t}) \sim 2\sqrt{3} I(\lambda_n(A_{n,t}))\) for \(n\) large.

**Remark.** – A similar result holds for the flat torus \(\mathbb{T}_n = (\mathbb{R}/\mathbb{Z})^n = (\frac{1}{2\pi} S^1)^n\), where \(\frac{1}{2\pi} S^1\) denotes a circle of length one. The isoperimetric function of \(\mathbb{T}_1\) is also constant on \((0, 1)\), hence by the same method, the product of a half circle and \(\mathbb{T}_{n-1}\) is solution of the isoperimetric problem among sets of measure 1/2. Extensions of these results will appear in [2].

Actually, our result for \([0, 1]^n\) is easy to obtain by a known and simple transportation argument: the Gaussian distribution function \(\Phi\) sends \(\gamma_1\) to the uniform probability measure \(\lambda_1\) on \([0, 1]\), and \(\Phi\) is a Lipschitz map with constant \(1/\sqrt{2\pi}\); tensoring this one-dimensional information, we obtain a Lipschitz map with constant \(1/\sqrt{2\pi}\) from \(\mathbb{R}^n\) to \([0, 1]^n\) that sends \(\gamma_n\) to \(\lambda_n\), and it is then easy to transfer the relevant Gaussian estimates to \([0, 1]^n\).

### 4. LOGARITHMIC SOBOLEV INEQUALITIES

We learned from W. Beckner (see [16]) that the Gaussian logarithmic Sobolev inequality of Gross [14] is a limit case of Bobkov’s inequality (2): taking \(f = \varepsilon g\) in (2) for a bounded function \(g\) and letting \(\varepsilon\) tend to 0 yields

\[
\int g \log g \, d\gamma_n - \left( \int g \, d\gamma_n \right) \log \left( \int g \, d\gamma_n \right) \leq \frac{1}{2} \int \frac{|\nabla g|^2}{g} \, d\gamma_n,
\]

because \(I(\varepsilon) \sim \varepsilon \sqrt{2 \log(1/\varepsilon)}\). All the statements of this note have a log-Sobolev version which can be proved by this limit process (or directly by taking \(F(x, y, t) = -x \log x + ty^2/2\); see [10] for the log-Sobolev versions of the results of Section 2). The two-point inequality is, for \(a, b, p, q > 0\) such that \(p + q = 1\)

\[
p a \log a + q b \log b - (pa + qb) \log(pa + qb)
\leq \frac{pq}{6} \left( \frac{1 + p}{a} + \frac{1 + q}{b} \right)(a - b)^2.
\]

This inequality is rather good when \(p\) is close to 1/2, but for small \(p\) the inequality of [8] is better.
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