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Percolation on nonamenable products at the uniqueness threshold

by

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ABSTRACT. – Let X and Y be infinite quasi-transitive graphs, such that the automorphism group of X is not amenable. For i.i.d. percolation on the direct product $X \times Y$, we show that the set of retention parameters p where a.s. there is a unique infinite cluster, does not contain its infimum p_u. This extends a result of Schonmann, who considered the direct product of a regular tree and $\mathbb{Z}$. © 2000 Éditions scientifiques et médicales Elsevier SAS

Key words: Percolation, Cayley graphs, Amenability

RÉSUMÉ. – Soit X et Y des graphes infinis quasi-transitifs, tels que le groupe d'automorphismes de X n' est pas moyennable. Pour la percolation i.i.d. sur le produit direct $X \times Y$, nous montrons que l'ensemble des paramètres p pour lesquels p.s. il y a un unique amas infini ne contient pas son infimum p_u. Cela étend un résultat de Schonmann, qui considérait le produit direct d'un arbre régulier avec $\mathbb{Z}$. © 2000 Éditions scientifiques et médicales Elsevier SAS

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1. INTRODUCTION

Let $X = (V_X, E_X)$ be an infinite, locally finite, connected graph. Say that X is *transitive* if its automorphism group $\mathrm{Aut}(X)$ has a single orbit in V_X; more generally, if $\mathrm{Aut}(X)$ has finitely many orbits in V_X, then X is called *quasi-transitive*. In i.i.d. bond percolation with retention parameter $p \in [0, 1]$ on X, each edge is independently assigned the value 1 (open) with probability p, and the value 0 (closed) with probability $1 - p$. We write $\mathbf{P}_p^X$, or simply $\mathbf{P}_p$, for the resulting probability measure on $\{0, 1\}^{E_X}$. A connected component of open edges is called a *cluster*. The critical parameters for percolation on X are

$$p_c(X) = \inf\{p \in [0, 1]\colon\ \mathbf{P}_p^X(\exists \text{ an infinite cluster}) = 1\};$$

$$p_u(X) = \inf\{p \in [0, 1]\colon\ \mathbf{P}_p^X(\exists \text{ a unique infinite cluster}) = 1\}.$$

We now state our result; further background and references will follow.

THEOREM 1.1. – *Let X and Y be infinite, locally finite, connected quasi-transitive graphs and suppose that* $\mathrm{Aut}(X)$ *is not amenable. Then on the direct product graph $X \times Y$,*

$$\mathbf{P}_{p_u}(\exists \textit{ a unique infinite cluster}) = 0.$$

Remarks. –
- For the definition of amenable groups, see, e.g., [14].
- Theorem 1.1 and its proof may be adapted to site percolation as well.
- In the case where X is a regular tree of degree $d \geqslant 3$ and $Y = \mathbb{Z}$, Theorem 1.1 is due to Schonmann [20].
- Given two graphs $X = (V_X, E_X)$ and $Y = (V_Y, E_Y)$, the *direct product graph* $X \times Y$ has vertex set $V_X \times V_Y$; the vertices (x_1, y_1) and (x_2, y_2) in $V_X \times V_Y$ are adjacent in $X \times Y$ iff either $x_1 = x_2$ and $[y_1, y_2] \in E_Y$, or $y_1 = y_2$ and $[x_1, x_2] \in E_X$.
- Our proof of Theorem 1.1 is based on the following ingredients:
  - (i) The characterization of p_u in terms of connection probabilities between large balls, due to Schonmann [19]; see Theorem 2.1.
  - (ii) The principle that for a (possibly dependent) percolation process, that is invariant under a nonamenable automorphism group, *high marginals yield infinite clusters*. This principle was proved by Häggström [8] for regular trees; it was extended to graphs with a nonamenable automorphism group by Ben-

jamini, Lyons, Peres and Schramm [2]. (See Theorems 2.2 and 2.3 below.)

(iii) The *shadowing method* used in Pemantle and Peres [16] to prove that there is no automorphism-invariant measure on spanning trees in any nonamenable direct product $X \times Y$ of the type considered in Theorem 1.1.

The first two ingredients are explained in the next section; (ii) was used in [3] to prove that percolation at level p_c on any nonamenable Cayley graph has no infinite clusters. Section 3 contains the proof of Theorem 1.1, and we will point out there where the shadowing method is used.

2. BACKGROUND

In an infinite tree, clearly $p_u = 1$, and in quasi-transitive amenable graphs, the arguments of Burton and Keane [5] yield that $p_u = p_c$ (see [6]). Examples of transitive graphs where $p_c < p_u < 1$ were provided by Grimmett and Newman [7], Benjamini and Schramm [4] and Lalley [11]. The conjecture stated in [4] that $p_c < p_u$ on any nonamenable Cayley graph, is still open. Benjamini and Schramm also conjectured that on any quasi transitive graph, for all $p > p_u$ there is a unique infinite cluster $\mathbf{P}_p$-a.s. This was established by Häggström and Peres [9] under a unimodularity assumption, and by Schonmann [19] in general. The latter paper also contains the following useful expression for p_u:

THEOREM 2.1 ([19]). – *Let X be any quasi-transitive graph. Then*

$$p_u(X) = \inf\Big\{p\colon \lim_{R\to\infty} \inf_{x,z\in V_X} \mathbf{P}_p\big(B_R(x) \leftrightarrow B_R(z)\big) = 1\Big\}, \tag{2.1}$$

Notation. – Let (V, E) be a locally finite graph.

- For $K_1, K_2 \subset V$, we write $K_1 \leftrightarrow K_2$ for the event that there is an open path from some vertex in K_1 to some vertex in K_2.
- For $x, z \in V$ and $F \subset E$, denote by $\operatorname{dist}(x, z; F)$ the minimal length of a path in F from x to z.
- For $x \in V$ and $R > 0$, let $B_R(x) := \{z \in V\colon \operatorname{dist}(x, z; E) \leqslant R\}$.

In [10], Theorem 2.1 is used to show that $p_u(\Gamma) \leqslant p_c(\mathbb{Z}^d)$ for any graph Γ which is a direct product of d infinite connected graphs of bounded degree.

Next, we discuss the relation between nonamenability and invariant percolation. Let X be a locally finite graph, and endow the automorphism group $\mathrm{Aut}(X)$ with the topology of pointwise convergence. Then any closed subgroup G of $\mathrm{Aut}(X)$ is locally compact, and the *stabilizer* $S(x) = S_G(x) := \{g \in G\colon\ gx = x\}$ of any vertex x is compact. We start with a qualitative statement.

THEOREM 2.2 ([2, Theorem 5.1]). – *Let X be a locally finite graph and let G be a closed subgroup of* $\mathrm{Aut}(X)$. *Then G is nonamenable iff there exists a threshold $\eta_G > 0$, such that if a G-invariant site percolation Λ on X satisfies $\mathbf{P}[x \notin \Lambda] < \eta_G$ for all $x \in V_X$, then Λ has infinite clusters with positive probability.*

The proof of this result in [2] uses a method of Adams and Lyons [1], that does not yield any estimate for the threshold η_G. Although Theorem 2.2 suffices for the proof of Theorem 1.1, we take this opportunity to complete the discussion of quantitative thresholds from Section 4 of [2]. This avoids the nonconstructive definition of amenability via invariant means, and will also allow us to obtain quantitative bounds on the intrinsic graph metric within the unique percolation cluster for $p > p_u$. (See the second remark in Section 4.)

Say that a subgroup G of $\mathrm{Aut}(X)$ is quasi-transitive if it has finitely many orbits in V_X. Let μ be the left Haar measure on G, and denote $\mu_*(v) := \mu[S(v)]$ for $v \in V_X$. For any finite set $K \subset V_X$, denote by ∂K the set of vertices in $V_X \setminus K$ adjacent to K, and let $\mu_*(K) := \sum_{x \in K} \mu_*(x)$. Define

$$\kappa_G := \inf\left\{\frac{\mu_*(\partial K)}{\mu_*(K)}\colon\ K \subset V_X \text{ is finite nonempty}\right\}.$$

For $x \in V_X$ and $\omega \subset V_X$, denote by $\mathcal{C}(x, \omega)$ the connected component of x in ω with respect to the edges induced from E_X. (This component is empty if $x \notin \omega$.)

The next theorem combines several results from [2]; we will provide the additional arguments needed below.

THEOREM 2.3. – *Let X be a locally finite graph, and suppose that G is a closed quasi-transitive subgroup of* $\mathrm{Aut}(X)$. *Choose a complete set $\{v_1, \dots, v_L\}$ of representatives in V_X of the orbits of G. Then*

(i) *G is nonamenable iff $\kappa_G > 0$.*

(ii) *Let Λ be a G-invariant site percolation on X. If $\kappa_G > 0$, then*

$$\sum_{i=1}^{L} \mathbf{P}\big[|\mathcal{C}(v_i, \Lambda)| < \infty\big] \leqslant \sum_{i=1}^{L} \frac{\kappa_G + \deg(v_i)}{\kappa_G} \mathbf{P}[v_i \notin \Lambda]. \tag{2.2}$$

Consequently, if

$$\forall x \in V_X, \quad \mathbf{P}[x \notin \Lambda] < \frac{\kappa_G}{\kappa_G + \deg(x)}, \tag{2.3}$$

then Λ has infinite clusters with positive probability.

(The threshold in (2.3) is sharp for regular trees, see Häggström [8, Theorem 8.1].)

To prove Theorem 2.3, we need the following version of the *mass transport principle*, obtained from Corollary 3.7 in [2] by setting $a_i \equiv 1$:

LEMMA 2.4. – *Let X, G and $\{v_1, \ldots, v_L\}$ be as in Theorem 2.3. Suppose that the function $f : V_X \times V_X \to [0, \infty]$ is invariant under the diagonal action of G. Then*

$$\sum_{i=1}^{L} \sum_{z \in V_X} f(v_i, z) = \sum_{j=1}^{L} \sum_{u \in V_X} f(u, v_j) \frac{\mu_*(u)}{\mu_*(v_j)}.$$

Proof of Theorem 2.3. –

(i) This follows from Theorem 3.9 and Lemma 3.10 in [2].

(ii) Let $v, z \in V_X$ and $\omega \subset V_X$. If $v \in \omega$, the component $\mathcal{C}(v, \omega)$ is finite, and $z \in \partial\mathcal{C}(v, \omega)$, then define

$$f_0(v, z, \omega) = \frac{\mu_*(z)}{\mu_*(\partial\mathcal{C}(v, \omega))};$$

otherwise, take $f_0(v, z, \omega) = 0$. For any vertex v, clearly

$$\sum_{z \in V_X} f_0(v, z, \omega) = \mathbf{1}_{\{0 < |\mathcal{C}(v, \omega)| < \infty\}}. \tag{2.4}$$

Since v can be adjacent to at most $\deg(v)$ components of ω,

$$\sum_{u \in V_X} f_0(u, v, \omega) \frac{\mu_*(u)}{\mu_*(v)} = \sum_{u \in V_X} \mathbf{1}_{\{v \in \partial\mathcal{C}(u, \omega)\}} \frac{\mu_*(u)}{\mu_*(\partial\mathcal{C}(u, \omega))}$$
$$\leqslant \frac{\deg(v)}{\kappa_G} \mathbf{1}_{\{v \notin \omega\}}. \tag{2.5}$$

The function $f(v,z) := \mathbf{E} f_0(v,z,\Lambda)$ is invariant under the diagonal action of G. By (2.4) and (2.5), for any $v \in V_X$ we have $\sum_{z\in V_X} f(v,z) = \mathbf{P}[0 < |\mathcal{C}(v,\Lambda)| < \infty]$ and

$$\sum_{u\in V_X} f(u,v)\frac{\mu_*(u)}{\mu_*(v)} \leqslant \frac{\deg(v)}{\kappa_G}\mathbf{P}[v \notin \Lambda].$$

Taking $v = v_i$ and summing over i, we obtain from Lemma 2.4 that

$$\sum_{i=1}^{L} \mathbf{P}\big[0 < \big|\mathcal{C}(v_i,\Lambda)\big| < \infty\big] \leqslant \sum_{i=1}^{L} \frac{\deg(v_i)}{\kappa_G}\mathbf{P}[v_i \notin \Lambda]. \tag{2.6}$$

Since $\mathbf{P}[|\mathcal{C}(v_i,\Lambda)| < \infty] = \mathbf{P}[0 < |\mathcal{C}(v_i,\Lambda)| < \infty] + \mathbf{P}[v_i \notin \Lambda]$, (2.2) follows. Finally, if (2.3) holds, then the right-hand side of (2.2) is less than L, so at least one of the probabilities on the left-hand side of (2.2) is less than 1. □

3. PROOF OF NONUNIQUENESS AT p_u

We will use the canonical coupling of the percolation processes for all p, obtained by equipping the edges of a graph (V,E) with i.i.d. random variables $\{U(e)\}_{e\in E}$, uniform in [0, 1]. Denote by $\mathbf{P}$ the resulting product measure on $[0,1]^E$. For each p, the edge set $\mathcal{E}(p) := \{e \in E\colon U(e) \leqslant p\}$ has the same distribution as the set of open edges under $\mathbf{P}_p$. Denote by $\mathcal{C}(w,p)$ the connected component of a vertex w in the subgraph $(V,\mathcal{E}(p))$, and for $W \subset V$, write $\mathcal{C}(W,p) := \bigcup_{w\in W}\mathcal{C}(w,p)$. We need the following easy lemma.

LEMMA 3.1. – *Consider the coupling defined above on a graph (V,E), and fix $p_1 < p_2$ in $[0,1]$. For any two sets $K, W \subset V$ and $M < \infty$, denote by $A_M(K,W;p_1)$ the event that infinitely many vertices in $\mathcal{C}(K,p_1)$ are within distance at most M from $\mathcal{C}(W,p_1)$. Then*

$$\mathbf{P}\big[K \leftrightarrow W \textit{ in } \mathcal{E}(p_2) \mid A_M(K,W;p_1)\big] = 1.$$

Proof. – On the event $A_M(K,W;p_1)$, there are infinitely many paths $\{\psi_j\}$ of length at most M from $\mathcal{C}(K,p_1)$ to $\mathcal{C}(W,p_1)$. Each of these paths intersects at most finitely many of the others, so we can extract an infinite subcollection $\{\psi_j'\}$ of edge-disjoint paths. Thus on $A_M(K, W; p_1)$,

$$\mathbf{P}\left[\psi_j' \text{ open in } \mathcal{E}(p_2) \mid \mathcal{E}(p_1)\right] \geqslant (p_2 - p_1)^M$$

for each j, and the assertion follows. □

Proof of Theorem 1.1. – We will show that in $X \times Y$, if

$$\mathbf{P}_p[\exists \text{ a unique infinite cluster}] = 1, \tag{3.1}$$

then $p > p_u$. Let $G = \mathrm{Aut}(X)$, and fix a threshold $\eta_G > 0$ as in Theorem 2.2. (By Theorem 2.3, we can take $\eta_G = \kappa_G/(\kappa_G + D_X)$ where $D_X := \max_{x \in V_X} \deg(x)$.) Denote by $\mathcal{C}_\infty(p)$ the unique infinite cluster in $\mathcal{E}(p)$, and define

$$\Gamma_1 = \Gamma_1(r) := \left\{\mathsf{v} \in V_{X\times Y}\colon B_r(\mathsf{v}) \cap \mathcal{C}_\infty(p) \neq \emptyset\right\}.$$

By (3.1) and quasi-transitivity of $X \times Y$, there exists r such that

$$\forall \mathsf{v} \in V_{X\times Y}, \quad \mathbf{P}\left[\mathsf{v} \notin \Gamma_1(r)\right] < \eta_G/6. \tag{3.2}$$

Next, define

$$\Gamma_2 = \Gamma_2(r, n) := \left\{\mathsf{v} \in V_{X\times Y}\colon \forall \mathsf{v}_0, \mathsf{v}_1 \in B_{r+1}(\mathsf{v}) \cap \mathcal{C}_\infty(p),\right. \\ \left. \operatorname{dist}\left(\mathsf{v}_0, \mathsf{v}_1; \mathcal{E}(p)\right) < n\right\}.$$

Once r is chosen, we can find n such that

$$\forall \mathsf{v} \in V_{X\times Y}, \quad \mathbf{P}\left[\mathsf{v} \notin \Gamma_2(r, n)\right] < \eta_G/6. \tag{3.3}$$

Denote by $D = D_{X\times Y}$ the maximal degree in $X \times Y$.

CLAIM. – *Fix r, n as above. If*

$$p_* > p - \frac{\eta_G}{6D^{r+n}}, \tag{3.4}$$

then

$$\lim_{R\to\infty} \inf_{\mathsf{v}^1, \mathsf{v}^2 \in V_{X\times Y}} \mathbf{P}_{p_*}\left[B_R(\mathsf{v}^1) \leftrightarrow B_R(\mathsf{v}^2)\right] = 1. \tag{3.5}$$

By Theorem 2.1, the last equation yields that $p_u \leqslant p_*$, so the claim implies that

$$p_u \leqslant p - \frac{\eta_G}{6D^{r+n}}. \tag{3.6}$$

To prove the claim, choose p_1, p_2 such that

$$p_1 < p_2 < p_* \quad \text{and} \quad p - p_1 < \frac{\eta_G}{6D^{r+n}}. \tag{3.7}$$

Use the canonical coupling variables $\{U(e)\}$ to define

$$\Gamma_3 = \Gamma_3(r, n, p_1) := \big\{ \mathsf{v} \in V_{X\times Y}\colon\ U(e) \notin [p_1, p] \text{ for all edges } e \text{ in } B_{r+n}(\mathsf{v}) \big\}.$$

Since D^{r+n} bounds the number of edges in a ball of radius $r+n$ in $X \times Y$, (3.7) gives

$$\forall \mathsf{v} \in V_{X\times Y}, \quad \mathbf{P}\big[\mathsf{v} \notin \Gamma_3(r, n, p_1)\big] < \eta_G/6.$$

Let $\Gamma_\diamond := \Gamma_1(r) \cap \Gamma_2(r, n) \cap \Gamma_3(r, n, p_1)$, and note that $\mathbf{P}[(x, y) \notin \Gamma_\diamond] < \eta_G/2$ for any $(x, y) \in V_{X\times Y}$. The "shadowing method" which is the key to our argument, is based on defining a site percolation on X that requires "good behavior" simultaneously in two levels, $X \times \{y_0\}$ and $X \times \{y_1\}$. Fix $y_0, y_1 \in V_Y$, and consider

$$\Lambda := \big\{ x \in V_X\colon\ (x, y_0) \in \Gamma_\diamond \text{ and } (x, y_1) \in \Gamma_\diamond \big\}.$$

Λ is a G-invariant site percolation on X, with $\mathbf{P}[x \notin \Lambda] < \eta_G$ for every vertex x. Thus

$$\mathbf{P}[\Lambda \text{ has an infinite component}] > 0, \tag{3.8}$$

by Theorem 2.2. Since the event in (3.8) is G-invariant and determined by the i.i.d. variables in the canonical coupling, it must have probability 1. (The action of G on X has infinite orbits, whence the induced action on the random field $\{U_e\}_{e\in E_X}$ is ergodic.)

Our next task is to verify that for any infinite path with vertices $\{x_j\}_{j\geqslant 1}$ in Λ, its lift $\xi_0 := \{(x_j, y_0)\}_{j\geqslant 1}$ to $X \times \{y_0\}$, is "shadowed" by an infinite path with edges in $\mathcal{E}(p_1)$, that remains a bounded distance from ξ_0. Indeed, the ball $B_r(x_j, y_0)$ contains a point v_j^0 in $\mathcal{C}_\infty(p)$ by the definition of Γ_1, and there is a path in $\mathcal{E}(p_1)$ from v_j^0 to v_{j+1}^0 by the definitions of Γ_2 and Γ_3. Concatenating these finite paths gives an infinite path with edges in $\mathcal{E}(p_1)$, that intersects $B_r(x_j, y_0)$ for each $j \geqslant 1$. Similarly, there is an infinite path with edges in $\mathcal{E}(p_1)$, that intersects $B_r(x_j, y_1)$ for each $j \geqslant 1$.

Therefore, Lemma 3.1 with $M = 2r + \text{dist}(y_0, y_1; E_Y)$ implies that for any $x_1 \in V_X$,

$$\mathbf{P}\big[B_r(x_1, y_0) \leftrightarrow B_r(x_1, y_1) \text{ in } \mathcal{E}(p_2) \mid \mathcal{C}(x_1, \Lambda) \text{ is infinite}\big] = 1. \tag{3.9}$$

Let $\varepsilon > 0$. Since the event in (3.8) has probability 1, there exists R_0 such that for all $x \in V_X$,

$$\mathbf{P}\big[B_{R_0}(x) \text{ intersects an infinite component of } \Lambda\big] > 1 - \varepsilon. \tag{3.10}$$

Let $R = R_0 + r$. By (3.9), (3.10) and the triangle inequality,

$$\mathbf{P}\big[B_R(x, y_0) \leftrightarrow B_R(x, y_1) \text{ in } \mathcal{E}(p_2)\big] > 1 - \varepsilon. \tag{3.11}$$

Finally, consider two arbitrary vertices $\mathsf{v}^1 = (x^1, y^1)$ and $\mathsf{v}^2 = (x^2, y^2)$ in $V_{X\times Y}$. For $y \in V_Y$, let

$$\begin{aligned} H_y := \big\{ & B_R(x^1, y^1) \leftrightarrow B_R(x^1, y) \text{ and} \\ & B_R(x^2, y^2) \leftrightarrow B_R(x^2, y) \text{ in } \mathcal{E}(p_2)\big\}. \end{aligned}$$

By (3.11), $\mathbf{P}[H_y] > 1 - 2\varepsilon$ for any $y \in V_Y$. Consequently,

$$\mathbf{P}[H_y \text{ for infinitely many } y] > 1 - 2\varepsilon. \tag{3.12}$$

On this event, the sets $\mathcal{C}(B_R(\mathsf{v}^1), p_2)$ and $\mathcal{C}(B_R(\mathsf{v}^2), p_2)$ come infinitely often within distance $\text{dist}(x^1, x^2; E_X) + 2R$ from each other. As $p_* > p_2$, we obtain from Lemma 3.1 and (3.12) that

$$\mathbf{P}\big[B_R(\mathsf{v}^1) \leftrightarrow B_R(\mathsf{v}^2) \text{ in } \mathcal{E}(p_*)\big] > 1 - 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have established (3.5) and the claim. This implies (3.6) and the theorem. □

4. CONCLUDING REMARKS

- *Nonamenability and isoperimetric inequalities.* Say that an infinite graph X is nonamenable if

$$\inf\left\{\frac{|\partial K|}{|K|} \colon K \subset V_X \text{ is finite nonempty}\right\} > 0. \tag{4.1}$$

In Theorem 1.1 we assumed that the group $\text{Aut}(X)$ is nonamenable. Could this assumption be replaced by the weaker assumption that the graph X is nonamenable? (These assumptions are equivalent if $\mathrm{Aut}(X)$ is quasi-transitive and unimodular, see Salvatori [17].)

- *Intrinsic distance within the infinite cluster.* In the setup of Theorem 1.1, denote by D the maximal degree in $X \times Y$. For $p > p_u = p_u(X \times Y)$, choose $r = r(p)$ and $n = n(p)$ to satisfy (3.2) and (3.3). Then (3.4) implies that

$$D^{r+n} > \frac{\eta_G}{6(p - p_u)}. \tag{4.2}$$

  If $p_u > p_c$ then $\sup_{p>p_u} r(p) < \infty$, so (4.2) yields a bound on the distribution of the intrinsic distance between vertices in the unique infinite cluster.
- *Kazhdan groups.* Lyons and Schramm [13] proved that $p_u < 1$ for Cayley graphs of Kazhdan groups. The present author observed that their argument can be modified to prove nonuniqueness at p_u on these graphs; see [13].
- *Planar graphs.* Benjamini and Schramm (unpublished) showed that for i.i.d. percolation on a planar nonamenable transitive graph, there is a unique infinite cluster for $p = p_u$. (As noted by the referee, for Cayley graphs of cocompact Fuchsian groups of genus at least 2, this can be infered from [11].) It is an open problem to find a geometric characterization of nonamenable transitive graphs that satisfy uniqueness at p_u.
- *Minimal spanning forests and p_u.* The impetus for this note was a suggestion by I. Benjamini and O. Schramm, that uniqueness for i.i.d. percolation at $p = p_u$ on a transitive graph X, should be closely related to connectedness of the "free minimal spanning forest" (FMSF) on X; this is a random subgraph (V_X, F) of X, obtained by labeling the edges in E_X by i.i.d. uniform variables, and removing any edge that has the highest label in a cycle. Indeed, Schramm (personal communication) has recently observed that connectedness of the FMSF implies uniqueness at p_u; the converse fails for certain free products, but it is open whether it holds for transitive graphs that satisfy $p_c < p_u < 1$.
- *The contact process.* Let T_d be a regular tree of degree $d \geqslant 3$. Pemantle [15] considered the contact process on T_d with infection rate λ. He showed that if $d \geqslant 4$, then the critical parameter for global survival, $\lambda_1(T_d)$, is strictly smaller than the critical parameter for local survival, $\lambda_2(T_d)$; the result was extended to T_3 by Liggett [12]. Zhang [21] showed that the contact process on T_d does not survive

locally at the parameter $\lambda_2(T_d)$, and that for larger values of λ, the so called "complete convergence theorem" holds. The proof by Schonmann [20] of nonuniqueness for percolation at level p_u on $T \times \mathbb{Z}$, was motivated by these results of Zhang and alternative proofs of them in Salzano and Schonmann [18]. Can the proof of Theorem 1.1 be adapted to show that for any graph X with $\mathrm{Aut}(X)$ nonamenable, the contact process does not survive locally at the parameter $\lambda_2(X)$?

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