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Asymptotics of a dynamic random walk in a random scenery: I. Law of large numbers *

by

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ABSTRACT. – In this paper, we consider a $\mathbb{Z}$-random walk $(S_n)_{n\in\mathbb{N}}$ on nearest neighbours with dynamical quasiperiodic transition probabilities in a random scenery $\xi(\alpha)$, $\alpha \in \mathbb{Z}$, a family of i.i.d. random variables, independent of the random walk. We prove, at first, that $(S_n)_{n\in\mathbb{N}}$ verifies a local limit theorem and is recurrent on its moving average.

Then, we show, explicitly, that $Z_n = \sum_{i=0}^{n} \xi(S_i)$ satisfies a law of large numbers. © 2000 Éditions scientifiques et médicales Elsevier SAS

Key words: Random walk, Random scenery, Continued fractions, Denjoy–Koksma's inequality, Low discrepancy sequences

RÉSUMÉ. – Dans ce papier, nous considérons une marche aléatoire $(S_n)_{n\in\mathbb{N}}$ sur $\mathbb{Z}$ se déplaçant sur ses plus proches voisins avec des probabilités de transition dynamiques et quasi-périodiques ainsi qu'une scène aléatoire $\xi(\alpha), \alpha \in \mathbb{Z}$, une famille de variables aléatoires i.i.d., indépendante de la marche aléatoire. En premier lieu, nous prouvons que $(S_n)_{n\in\mathbb{N}}$ vérifie un théorème limite local et est récurrente sur sa moyenne mobile. Puis, nous montrons explicitement que $Z_n = \sum_{i=0}^{n} \xi(S_i)$ satisfait une loi des grands nombres. © 2000 Éditions scientifiques et médicales Elsevier SAS

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1. INTRODUCTION AND RESULTS

Let X_i, $i \geqslant 1$, be a sequence of independent random variables with values ± 1, let $f : \mathbb{T}^d \to [0, 1]$ be a function and τ_α the rotation on the d-dimensional torus $\mathbb{T}^d$, associated with the d-dimensional vector $\alpha = (\alpha_1, \dots, \alpha_d)$, defined by $x \mapsto x + \alpha \bmod 1$. For every i, the law of the random variable X_i is given by

$$\begin{aligned}\mathbb{P}(X_i = +1) &= f\left(\tau_\alpha^i x\right) \quad \text{where } x \in \mathbb{T}^d \\ &= 1 - \mathbb{P}(X_i = -1).\end{aligned}$$

We write

$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} X_i \quad \text{for } n \geqslant 1$$

for the random walk generated by the family $(X_i)_{i\in\mathbb{N}}$. It is worth remarking that $(S_n)_{n\in\mathbb{N}}$ is a non-homogeneous Markov chain. Furthermore, let $\xi(\alpha), \alpha \in \mathbb{Z}$, be a family of i.i.d. $\mathbb{R}$-valued random variables which are independent of the family $(X_i)_{i\in\mathbb{N}}$, with zero mean and finite positive variance σ^2. These random variables play the role of random scenery. We shall prove that $(S_n)_{n\in\mathbb{N}}$ is recurrent on its moving average and that

$$Z_n = \sum_{i=0}^{n} \xi(S_i)$$

satisfies a weak law of large numbers.

When $(S_n)_{n\in\mathbb{N}}$ is a standard $\mathbb{Z}^d$-random walk with i.i.d. increments then $(\xi(S_n))_{n\in\mathbb{N}}$ is a stationary sequence, the strong law of large numbers is evident by Birkhoff's theorem. In $d = 1$, Kesten and Spitzer [6] proved that when X and ξ belong to the domains of attraction of different stable laws of indices $1 < \alpha \leqslant 2$ and $0 < \beta \leqslant 2$, respectively, then there exists a $\delta > \frac{1}{2}$ such that $n^{-\delta} Z_{[nt]}$ converges weakly as $n \to \infty$ to a self-similar process with stationary increments, δ being related to α and β by $\delta = 1 - \alpha^{-1} + (\alpha\beta)^{-1}$. The case $0 < \alpha < 1$ and β arbitrary is easier; they showed then that $n^{-1/\beta} Z_{[nt]}$ converges weakly, as $n \to \infty$, to a stable process with index β. Bolthausen [2] gave a method to solve the more difficult case $\alpha = 1$ and $\beta = 2$ and especially, he proved that when $(S_n)_{n\in\mathbb{N}}$ is a recurrent $\mathbb{Z}^2$-random walk, $(n \log n)^{-1/2} Z_{[nt]}$ satisfies a functional central limit theorem. For arbitrary transient random walk, $n^{-1/2}Z_n$ is asymptotically normal (see [17, p. 53]).

Lin et al. [12] developed a more abstract theory about the random walks with dynamical random transitions. They considered a conservative and ergodic non-singular transformation θ of a dynamic environment (Ω, Σ, m) (the environment space), μ_ω a (random) probability on a locally compact second countable group G (the state space). They gave conditions for the convergence $\lim_{n\to\infty} \|\nu_\omega^{(n)} * (f - \delta_t * f)\|_1 = 0$ for a.e. ω and every $f \in L_1(G)$ and $t \in G$ given, in the case where G is abelian and $\nu_\omega^{(n)}$ is given by

$$\nu_\omega^{(n)} = \mu_{\theta^{n-1}\omega} * \mu_{\theta^{n-2}\omega} * \cdots * \mu_{\theta\omega} * \mu_\omega.$$

This means that, having chosen ω, the inhomogeneous Markov chain does not distinguish the initial absolutely continuous distribution in the long run.

Besides their mathematical interest, the walks we consider here are of some relevance in the statistical mechanics of quasiperiodic systems in the presence of external spatial disorder (see [5] and [8]).

Our main results are summarised in the following theorems. We use the notation $a_n \sim b_n$ for $a_n, b_n > 0$ if $\lim_{n\to\infty} a_n/b_n = 1$. Let α be an irrational. We denote a_n the nth partial quotient of α, i.e.

$$\alpha = [\alpha] + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}.$$

This continued fraction expansion will be denoted $\alpha = [\alpha] + [a_1, \ldots, a_n, \ldots]$.

DEFINITION 1.1. – *A Markov chain $(S_n)_{n\in\mathbb{N}}$ is said to be recurrent on its moving average if for all $\varepsilon > 0$,*

$$\mathbb{P}\Big(\limsup_n \big\{\big|S_n - \mathbb{E}(S_n)\big| < \varepsilon\big\}\Big) = 1.$$

Remark. – This definition is not trivial when the process $(S_n)_{n\in\mathbb{N}}$ is not a stationary sequence.

THEOREM 1.2. – *Let $f : \mathbb{T}^1 \to [0, 1]$ be a function of bounded variation such that $a = \int_0^1 4f(t)(1 - f(t))\,dt > 0$.*

(1) *For every $x \in [0, 1[$ and for every irrational α, the inhomogeneous Markov chain $(S_n)_{n\in\mathbb{N}}$ is recurrent on its moving average.*

(2) *Assume that* $\int_0^1 f(t)\,\mathrm{d}t = \frac{1}{2}$. *Then, for every* $x \in [0,1[$ *and for every irrational* α *such that the inequality*

$$a_m < m^{1+\varepsilon}$$

is satisfied for any m *large enough, with* $\varepsilon > 0$,

$$\mathbb{P}(S_{2n} = 0) \sim (a\pi n)^{-1/2}.$$

Remark. – The previous theorem is formulated in a weak form. Actually, it can be shown that $(S_n)_{n\in\mathbb{N}}$ is recurrent on 0 for all irrationals α such that $a_m < m^{1+\varepsilon}$, eventually for all m, with $\varepsilon > 0$, but the proof of this statement needs some technical details that will be developed only on part II of this paper.

THEOREM 1.3. – *Under the same hypothesis as for item* (2) *of the previous theorem,*

$$\frac{Z_n}{n} \to 0 \quad \textit{in } \mathbb{P}\textit{-probability}.$$

Example 1.4. – The function defined by $f(x) = \cos^2(2\pi x)$, $x \in \mathbb{T}^1$, verifies the conditions of Theorems 1.2 and 1.3. For this particular function, the results are valid for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (see Example 5.13).

Remarks. –

(1) The case $a = 0$ which corresponds to the situation where $f = 0$ or 1 almost everywhere is not considered and must be treated by another way.
(2) The set of irrationals whose partial quotients satisfy condition $a_m < m^{1+\varepsilon}$ for any m large enough, with $\varepsilon > 0$, is of full measure (see [7, Theorem 30]).
(3) It is worth remarking that $\mathbb{E}(X_i) = 2f(\tau_\alpha^i x) - 1$ and $\operatorname{var}(X_i) = 4f(\tau_\alpha^i x)(1 - f(\tau_\alpha^i x))$. The vector α being irrational, the translation τ_α is uniquely ergodic, so

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(X_i) = 2\int_0^1 f(t)\,\mathrm{d}t - 1$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \operatorname{var}(X_i) = \int_0^1 4f(t)\big(1 - f(t)\big)\,\mathrm{d}t.$$

The above one-dimensional results can be generalised in $d > 1$ under some additional hypotheses on the mutual irrationality of the components of $\alpha = (\alpha_1, \dots, \alpha_d)$. To formulate precisely these results, some definitions are needed. In order to facilitate the exposition, these definitions are postponed until section 5. Here, we give the main result, valid in $d \geqslant 1$,

THEOREM 1.5. – *Let $f : \mathbb{T}^d \to [0, 1]$ be a function of bounded variation in the sense of Hardy and Krause such that $a = \int_{\mathbb{T}^d} 4f(t)(1 - f(t))\,\mathrm{d}t > 0$.*

(1) *For every $x \in \mathbb{T}^d$ and for every irrational vector $\alpha = (\alpha_1, \dots, \alpha_d)$, the inhomogeneous Markov chain $(S_n)_{n \in \mathbb{N}}$ is recurrent on its moving average.*
(2) *Assume that $\int_{\mathbb{T}^d} f(t)\,\mathrm{d}t = \frac{1}{2}$. Then, for every $x \in \mathbb{T}^d$ and for every irrational vector $\alpha = (\alpha_1, \dots, \alpha_d)$ of type η such that $1 \leqslant \eta < 1 + \frac{1}{d}$, we have*
   (a) $\mathbb{P}(S_{2n} = 0) \sim (a\pi n)^{-1/2}$.
   (b) *$Z_n/n \to 0$ in $\mathbb{P}$-probability.*

In the case where α is a rational vector, which is easier and will be treated in the sixth section, we obtain the following

THEOREM 1.6. – *Let $f : \mathbb{T}^d \to [0, 1]$ and $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i = p_i/q_i$, $\tilde{q} = s.c.m.(q_1, \dots, q_d)$. Let x be a point of $\mathbb{T}^d$. We denote $p = \frac{1}{\tilde{q}} \sum_{k=1}^{\tilde{q}} f(\tau_\alpha^k x)$. Then, if $\sum_{k=1}^{\tilde{q}} 4f(\tau_\alpha^k x)(1 - f(\tau_\alpha^k x)) > 0$, we have the following results:*

(1) *when $p = \frac{1}{2}$, $(S_n)_{n \in \mathbb{N}}$ is recurrent, otherwise $(S_n)_{n \in \mathbb{N}}$ is transient.*
(2) $Z_n/n \overset{a.s}{\to} 0$.

All the previous results are obtained pointwise for some fixed initial value x for the dynamics. It is also possible to consider a smeared initialisation of the dynamics and examine the evolution of the process on the product space $\mathbb{R}^{\mathbb{Z}} \times \mathbb{Z}^{\mathbb{N}} \times \mathbb{T}^d$. For every $x \in \mathbb{T}^d$, we denote $\mathbb{P}_x$ the product measure on the cartesian product of the set of sceneries and the set of dynamics paths $(\mathbb{R}^{\mathbb{Z}} \times \mathbb{Z}^{\mathbb{N}}, \mathcal{F})$, where $\mathcal{F}$ is the σ-field generated by the cylinder sets. Now, for $A \in \mathcal{B}([0, 1]^d)$, for $F \in \mathcal{F}$, let

$$\mathbb{P}(F \times A) = \int_A \mathbb{P}_x(F)\,\mathrm{d}x.$$

$\mathbb{P}$ is a probability measure on the space $(\mathbb{R}^{\mathbb{Z}} \times \mathbb{Z}^{\mathbb{N}} \times \mathbb{T}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{T}^d))$. The smearing of the initialisation greatly simplifies the problem since by a straightforward calculation we can prove that $(\xi(S_k))_{k\in\mathbb{N}}$ is a stationary sequence. Hence by the ergodic theorem, we have

PROPOSITION 1.7. – *With probability one,*

$$\frac{Z_n}{n} \to 0.$$

2. PRELIMINARY RESULTS

Let α be an irrational. We call a rational $\frac{p}{q}$ with p, q relatively prime such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2},$$

a rational approximation of α. When α has the continued fraction expansion $\alpha = [\alpha] + [a_1, \ldots, a_n, \ldots]$, the nth principal convergent of α is $\frac{p_n}{q_n}$ where, $\forall n \geqslant 2$,

$$p_n = a_n p_{n-1} + p_{n-2},$$
$$q_n = a_n q_{n-1} + q_{n-2};$$

the recurrence is given by defining the values of p_0, p_1 and q_0, q_1.

DEFINITION 2.1. – *A partition P of* $[0, 1]$ *is defined by a sequence of points $x_0, \ldots, x_n$ with $0 = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = 1$. For a function f on* $[0, 1]$, *we set*

$$V(f) = \sup_P \sum_{i=0}^{n-1} \left|f(x_{i+1}) - f(x_i)\right|,$$

where the supremum is extended over all partitions P of $[0, 1]$. *If $V(f)$ is finite, then f is said to be with* bounded variation.

THEOREM 2.2 (Denjoy–Koksma's inequality). – *Let $f : \mathbb{R} \to [0, 1]$ be a function with bounded variation $V(f)$ and $\frac{p}{q}$ a rational approximation of α. Then, for every $x \in \mathbb{T}^1$,*

$$\left|\sum_{l=1}^{q} f(\tau_\alpha^l x) - q \int_0^1 f(t)\,\mathrm{d}t\right| \leqslant 2V(f).$$

(See [11] where the theorem is given in a more general case.)

PROPOSITION 2.3. – *Let f be a 1-periodic function with bounded variation $V(f)$. For every irrational α such that the inequality $a_m < m^{1+\varepsilon}$, where $\varepsilon > 0$, is satisfied eventually for all m and for every $x \in \mathbb{R}$,*

$$\sum_{l=1}^{n} f\left(\tau_\alpha^l x\right) - n\int_0^1 f(t)\,\mathrm{d}t = \mathcal{O}\left(\log^{2+\varepsilon} n\right).$$

Proof. – The sequence of integers $(q_i)_{i\geqslant 1}$ being strictly increasing, for a given $n \geqslant 1$, there exists $m = m(n) \geqslant 0$ such that

$$q_m \leqslant n < q_{m+1}.$$

By Euclidean division, we have $n = b_m q_m + n'$ with $0 \leqslant n' < q_m$. We can use the usual relations

$$\begin{aligned} &q_0 = 1, \quad q_1 = a_1, \\ &q_n = a_n q_{n-1} + q_{n-2}, \quad n \geqslant 2. \end{aligned} \tag{1}$$

We obtain that $(a_{m+1} + 1)q_m > q_{m+1} > n$ and so $b_m \leqslant a_{m+1}$. If $m > 0$, we may write $n' = b_{m(n')}q_{m(n')} + n''$ with $0 \leqslant n'' < q_{m(n')}$. Again, we find $b_{m(n')} \leqslant a_{m(n')+1}$. Continuing in this manner, we arrive at a representation for n of the form

$$n = \sum_{i=0}^{m} b_i q_i$$

with $0 \leqslant b_i \leqslant a_{i+1}$ for $0 \leqslant i \leqslant m$ and $b_m \geqslant 1$. The non-null terms in this representation of n are those which come from the sub-sequence m. Using the Denjoy–Koksma's inequality, we get

$$\left|\sum_{l=1}^{n} f\left(\tau_\alpha^l x\right) - n\int_0^1 f(t)\,\mathrm{d}t\right| \leqslant 2V(f)\sum_{i=0}^{m} b_i \leqslant 2V(f)\sum_{i=0}^{m} a_{i+1}.$$

By hypothesis, there exists $m_0 \geqslant 1$ such that,

$$a_m < m^{1+\varepsilon}, \quad \forall m \geqslant m_0.$$

Let n be such that $m > m_0$. Thus,

$$\left|\sum_{l=1}^{n} f\left(\tau_\alpha^l x\right) - n\int_0^1 f(t)\,\mathrm{d}t\right| \leqslant 2V(f)\left(\sum_{i=0}^{m_0-1} a_{i+1} + (m+1)^{2+\varepsilon}\right).$$

We need to know the asymptotic behaviour of $m = m(n)$. When α is the golden ratio, $a_n = 1$, $\forall n \geqslant 1$ and the relation (1) implies that $q_n \sim \frac{1}{\sqrt{5}}\alpha^{n+1}$. Let α' be another irrational; its partial quotients a'_n satisfy necessarily $a'_n \geqslant 1$. Using the relation (1), we see that $q'_n \geqslant q_n$, $\forall n \geqslant 1$. Therefore, $m(n) = \mathcal{O}(\log n)$ and the proposition is proved. □

Remark. – Consider ϕ a homeomorphism of the torus $\mathbb{T}^1$ and π the projection from $\mathbb{R}$ onto $\mathbb{T}^1$. We choose $\tilde{\phi}(0)$ such that $\pi\tilde{\phi}(0) = \phi(0)$ and we can gradually define a continuous map $\tilde{\phi}:\mathbb{R} \to \mathbb{R}$ such that $\pi\tilde{\phi} = \phi\pi$. The function $\tilde{\phi}$ is called a *lifting* of ϕ; and ϕ *preserves the orientation* of $\mathbb{T}^1$ if its liftings are nondecreasing functions. Then, $(\frac{1}{n}(\tilde{\phi}^n(t) - t))_{n\geqslant 1}$ converges uniformly to a number $\alpha(\tilde{\phi})$ when n goes to infinity. The fractional part $\pi\alpha(\tilde{\phi})$ does not depend on the lifting and is called the *rotation number* of the homeomorphism ϕ. Let μ be a ϕ-invariant probability measure. Using the Denjoy–Koksma's theorem (see [9]), we may generalise the work made in this section to the case where the law of the random variable X_i, $i \geqslant 1$, is given by

$$\begin{aligned}\mathbb{P}(X_i = +1) &= f\left(\phi^i(x)\right) \quad \text{where } x \in \mathbb{T}^1\\ &= 1 - \mathbb{P}(X_i = -1),\end{aligned}$$

where ϕ is an orientation preserving homeomorphism with irrational rotation number α, f and α being defined as in Theorem 1.2. It will be easy to verify that Theorems 1.2 and 1.3 remain valid with these new hypotheses.

3. PROOF OF THEOREM 1.2

Introduce the characteristic function $\phi_n(\theta)$, $\theta \in [-\pi, \pi)$ for the r.v. S_{2n}

$$\phi_n(\theta) = \mathbb{E}\big(\exp(\mathrm{i}\theta S_{2n})\big),$$

reading in the present case

$$\phi_n(\theta) = \prod_{j=1}^{2n}\big(\cos\theta + \mathrm{i}\big(2f\big(\tau_\alpha^j x\big) - 1\big)\sin\theta\big). \tag{2}$$

The probability of return to the origin is expressed by the standard inversion formula

$$\mathbb{P}(S_{2n} = 0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_n(\theta)\,\mathrm{d}\theta. \tag{3}$$

The integrand in (3) is π-periodic and we can perform the change of variable $\theta\sqrt{n}=u$, so that

$$(a\pi n)^{1/2}\mathbb{P}(S_{2n}=0)=\left(\frac{a}{\pi}\right)^{1/2}\int\limits_{|u|\leqslant\frac{\pi}{2}\sqrt{n}}\phi_n\left(\frac{u}{\sqrt{n}}\right)\mathrm{d}u.$$

We decompose this integral into a dominant part and three correction terms I_1, I_2 and I_3,

$$(a\pi n)^{1/2}\mathbb{P}(S_{2n}=0)=\left(\frac{a}{\pi}\right)^{1/2}\int\limits_{\mathbb{R}}\mathrm{e}^{-au^2}\,\mathrm{d}u+I_1(n)+I_2(n)+I_3(n).$$

The correction terms are given by

$$I_1(n)=\left(\frac{a}{\pi}\right)^{1/2}\int\limits_{|u|\leqslant\log n}\left(\phi_n\left(\frac{u}{\sqrt{n}}\right)-\mathrm{e}^{-au^2}\right)\mathrm{d}u,$$

$$I_2(n)=-\left(\frac{a}{\pi}\right)^{1/2}\int\limits_{|u|>\log n}\mathrm{e}^{-au^2}\,\mathrm{d}u,$$

$$I_3(n)=\left(\frac{a}{\pi}\right)^{1/2}\int\limits_{\log n<|u|\leqslant\frac{\pi}{2}\sqrt{n}}\phi_n\left(\frac{u}{\sqrt{n}}\right)\mathrm{d}u.$$

LEMMA 3.1. – *Under the conditions of Theorem* 1.2, *we have*

$$I_1(n)\underset{n\to\infty}{\to}0.$$

Proof. – Using (2), we get

$$I_1(n)=\left(\frac{a}{\pi}\right)^{1/2}\int\limits_{|u|\leqslant\log n}\left[\exp\left(\frac{1}{2}\sum_{j=1}^{2n}\log\left(1-4f(\tau_\alpha^j x)(1-f(\tau_\alpha^j x))\right.\right.\right.$$
$$\left.\left.\times\sin^2\left(\frac{u}{\sqrt{n}}\right)\right)\right)\exp\left(\mathrm{i}\sum_{j=1}^{2n}\arctan\left((2f(\tau_\alpha^j x)-1)\tan\left(\frac{u}{\sqrt{n}}\right)\right)\right)$$
$$\left.-\exp(-au^2)\right]\mathrm{d}u.$$

The absolute value of the integrand is bounded by using the following trick. Let $F(t) = \mathrm{e}^{tz}$, $t \in \mathbb{R}$, $z \in \mathbb{C}$. Then

$$F(0) = 1, \quad F(1) = \mathrm{e}^{z}$$

and

$$\left|F(1) - F(0)\right| = \left|\int_0^1 F'(t)\,\mathrm{d}t\right| \leqslant |z|\,\mathrm{e}^{|z|}, \quad \forall z \in \mathbb{C}.$$

Using this inequality, we can write, for $n \geqslant 1$,

$$\left|I_1(n)\right| \leqslant \left(\frac{a}{\pi}\right)^{1/2} \int_{|u| \leqslant \log n} \exp\left(-au^2\right)\left|h_n(u)\right| \exp\left|h_n(u)\right| \mathrm{d}u$$

with

$$h_n(u) = au^2 + \frac{1}{2}\sum_{j=1}^{2n} \log\left(1 - 4f\left(\tau_\alpha^j x\right)\left(1 - f\left(\tau_\alpha^j x\right)\right)\sin^2\left(\frac{u}{\sqrt{n}}\right)\right)$$
$$+\,\mathrm{i}\sum_{j=1}^{2n} \arctan\left(\left(2f\left(\tau_\alpha^j x\right) - 1\right)\tan\left(\frac{u}{\sqrt{n}}\right)\right).$$

Then, we use expansions into entire series of the functions $\log(1-x)$ and $\arctan(x)$ for x such that $|x| < 1$ using the fact that $|u| \leqslant \log n$ and we get the majorization

$$\left|I_1(n)\right| \leqslant A(n)\exp\left(A(n)\right),$$

where

$$A(n) = a\frac{\log^4 n}{n}\sum_{m=0}^{\infty}\left(\frac{\log^2 n}{n}\right)^m \frac{2^{2m+3}}{(2(m+2))!}$$
$$+\,\frac{\log^2 n}{2n}\left|\sum_{j=1}^{2n} 4f\left(\tau_\alpha^j x\right)\left(1 - f\left(\tau_\alpha^j x\right)\right) - 2na\right|$$
$$+\,\frac{\log^4 n}{n}\sum_{m=0}^{\infty}\left(\frac{\log^2 n}{n}\right)^m \frac{1}{m+2}$$
$$+\,\tan\left(\frac{\log n}{\sqrt{n}}\right)\left|\sum_{j=1}^{2n}\left(2f\left(\tau_\alpha^j x\right) - 1\right)\right|$$
$$+\,2n\tan^3\left(\frac{\log n}{\sqrt{n}}\right)\sum_{m=0}^{\infty}\left(\tan^2\left(\frac{\log n}{\sqrt{n}}\right)\right)^m \frac{1}{2m+3};$$

The three series in the expression $A(n)$ are all convergent with factors getting to zero as n goes to infinity. Moreover, using Proposition 2.3, both other terms tend to 0 as n goes to infinity, so $I_1(n)$ goes to 0 as $n \to \infty$. □

LEMMA 3.2. – *Under the same conditions as the previous lemma, the correction terms $I_2(n)$ and $I_3(n)$ tend to zero as $n \to \infty$.*

Proof. – The assertion is evident for $I_2(n)$. To prove the lemma for $I_3(n)$, we write

$$\left|I_3(n)\right| \leqslant \left(\frac{a}{\pi}\right)^{1/2} \int\limits_{\log n<|u|\leqslant\frac{\pi}{2}\sqrt{n}} \left|\phi_n\left(\frac{u}{\sqrt{n}}\right)\right| \mathrm{d}u.$$

We have seen that

$$\left|\phi_n\left(\frac{u}{\sqrt{n}}\right)\right| = \exp\left(\frac{1}{2}\sum_{j=1}^{2n}\log\left(1-4f(\tau_\alpha^j x)(1-f(\tau_\alpha^j x))\sin^2\left(\frac{u}{\sqrt{n}}\right)\right)\right).$$

Using the fact that $\log(1-x) \leqslant -x$, $1 > x \geqslant 0$ and $\sin x \geqslant \frac{2}{\pi}x$, $0 \leqslant x \leqslant \frac{\pi}{2}$, we get

$$\begin{aligned}
\left|I_3(n)\right| &\leqslant 2\left(\frac{a}{\pi}\right)^{1/2} \int_{\log n}^{\frac{\pi}{2}\sqrt{n}} \exp\left(-\frac{4u^2}{\pi^2 2n}\sum_{j=1}^{2n}\left(4f(\tau_\alpha^j x)(1-f(\tau_\alpha^j x))\right)\right)\mathrm{d}u \\
&\leqslant \sqrt{a\pi n}\exp\left(-\frac{4\log^2 n}{\pi^2}\left(a-\left|R(n)\right|\right)\right),
\end{aligned}$$

where $R(n) = \frac{1}{2n}\sum_{j=1}^{2n}(4f(\tau_\alpha^j x)(1-f(\tau_\alpha^j x))) - a$. The parameter a being strictly positive, the lemma follows. □

Let us define the random variable

$$Y_n = \frac{S_n - \mathbb{E}(S_n)}{\sqrt{\operatorname{var}(S_n)}}.$$

In order to prove the recurrence of the Markov chain $(S_n)_{n\in\mathbb{N}}$ on its moving average, it is enough to prove that the event

$$A = \left\{\limsup_n Y_n = \infty;\ \liminf_n Y_n = -\infty\right\}$$

has probability one. Let

$$A_c = \left\{\limsup_n Y_n \geqslant c\right\} \cap \left\{\liminf_n Y_n \leqslant -c\right\} = A'_c \cap A''_c, \quad c > 0.$$

Then, $A_c \searrow A$, $c \to \infty$ and A, A_c, A'_c, A''_c are in the tail algebra $\mathcal{G} = \bigcap \mathcal{G}_n$, where $\mathcal{G}_n = \sigma(X_n, X_{n+1}, \ldots)$. Let us show that $\mathbb{P}(A'_c) = \mathbb{P}(A''_c) = 1$ for each $c > 0$. Because of the Kolmogorov's zero-one law, since $A'_c \in \mathcal{G}$ and $A''_c \in \mathcal{G}$, it is sufficient to show only that $\mathbb{P}(A'_c) > 0$ and $\mathbb{P}(A''_c) > 0$. The events

$$\limsup\{Y_n \geqslant c\} \quad \text{and} \quad \limsup\{Y_n \leqslant -c\}$$

being respectively included into A'_c and A''_c, we obtain that

$$\mathbb{P}(A'_c) \geqslant \limsup_n \mathbb{P}(Y_n \geqslant c)$$

and

$$\mathbb{P}(A''_c) \geqslant \limsup_n \mathbb{P}(Y_n \leqslant -c).$$

Now, the Lyapunov condition being satisfied, we obtain that for every $x \in [0, 1]$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$,

$$Y_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1).$$

Therefore,

$$\limsup_n \mathbb{P}(Y_n \geqslant c) > 0$$

and

$$\limsup_n \mathbb{P}(Y_n \leqslant -c) > 0.$$

Thus, $\mathbb{P}(A_c) = 1$, $\forall c > 0$ and

$$\mathbb{P}(A) = \lim_{c \to \infty} \mathbb{P}(A_c) = 1.$$

From the definition of the event A, we obtain that for every $x \in [0, 1]$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\forall \varepsilon > 0$,

$$\mathbb{P}\left(\limsup_n \left\{\left|S_n - \sum_{i=1}^{n} (2f(\tau_\alpha^i x) - 1)\right| < \varepsilon\right\}\right) = 1,$$

so, the recurrence of $(S_n)_{n \in \mathbb{N}}$ on its moving average is proved.

4. PROOF OF THEOREM 1.3

We define

$$V_n = \sum_{i,j=0}^{n} \mathbf{1}_{\{S_i=S_j\}}.$$

LEMMA 4.1. –

$$\operatorname{var} Z_n = \sigma^2 \mathbb{E}(V_n).$$

Proof. – Obviously, $\mathbb{E}(Z_n)=0$. Then

$$\operatorname{var} Z_n = \sum_{k,l=0}^{n} \mathbb{E}\big(\xi(S_k)\xi(S_l)\big) = \sum_{k\neq l} \mathbb{E}\big(\xi(S_k)\xi(S_l)\big) + \sigma^2(n+1).$$

Let $\mathcal{A}$ be the σ-field generated by the r.v. $X_1, X_2, \ldots$. We have

$$\begin{aligned}\mathbb{E}\big(\xi(S_k)\xi(S_l)|\mathcal{A}\big) &= \sum_{i,j\in\mathbb{Z}\ i\neq j} \mathbf{1}_{\{S_k=i\}}\mathbf{1}_{\{S_l=j\}}\mathbb{E}\big(\xi(i)\xi(j)\big) + \sigma^2\mathbf{1}_{\{S_k=S_l\}}\\ &= \sigma^2\mathbf{1}_{\{S_k=S_l\}}.\end{aligned}$$

Consequently,

$$\operatorname{var} Z_n = \sigma^2\left[\sum_{k\neq l}\mathbb{P}(S_k=S_l)+(n+1)\right] = \sigma^2\mathbb{E}(V_n). \qquad \square$$

LEMMA 4.2. –

$$\mathbb{E}(V_n) \sim \frac{4(2^{3/2}-1)}{3\sqrt{a\pi}} n^{3/2}.$$

Proof. –

$$\mathbb{E}(V_n) = \sum_{k\neq l}\mathbb{P}(S_k=S_l)+(n+1) = 2\sum_{l=1}^{n}\sum_{k=0}^{l-1}\mathbb{P}(S_k=S_l)+(n+1)$$

$(S_n)_{n\in\mathbb{N}}$ being homogeneous in space, for every $j\geqslant 0$,

$$\mathbb{P}(S_{k+2j}=S_k) = \mathbb{P}(S_{k+2j}=0|S_k=0) \quad \text{if } k \text{ even},$$

$$\mathbb{P}(S_{k+2j}=S_k) = \mathbb{P}(S_{k+2j}=1|S_k=1) \quad \text{if } k \text{ odd}.$$

Let us denote $[x]$ the integer part of x. Therefore,

$$\mathbb{E}(V_n) = (n+1) + 2\sum_{k=0}^{[\frac{(n-1)}{2}]}\sum_{j=1}^{[\frac{(n-k)}{2}]} \mathbb{P}(S_{2(k+j)} = 0 | S_{2k} = 0)$$
$$+ 2\sum_{k=0}^{[\frac{(n-2)}{2}]}\sum_{j=1}^{[\frac{(n-k-1)}{2}]} \mathbb{P}(S_{2k+1+2j} = 1 | S_{2k+1} = 1).$$

We are only interested in the asymptotic behaviour of the first double sum in the above expression, the second one can be treated in the same way. Let $\phi_j^{(k)}$ the characteristic function of $\widetilde{S}_{2j}^{(k)}$, where $(\widetilde{S}_j^{(k)})_{j\in\mathbb{N}}$ is defined in the same way as the Markov chain $(S_n)_{n\in\mathbb{N}}$ but with transition probabilities given by

$$\mathbb{P}\big(\widetilde{S}_j^{(k)} = l+1 | \widetilde{S}_{j-1}^{(k)} = l\big)$$
$$= f\big(\tau_\alpha^{j+2k} x\big) = 1 - \mathbb{P}\big(\widetilde{S}_j^{(k)} = l-1 | \widetilde{S}_{j-1}^{(k)} = l\big), \quad \forall l \in \mathbb{Z},\ \forall j \geqslant 1.$$

Using the results of Section 3, we can write

$$\mathbb{P}(S_{2(k+j)} = 0 | S_{2k} = 0)$$
$$= (a\pi)^{-1/2}\frac{1}{\sqrt{j}} + R_1(k,j) + R_2(k,j) + R_3(k,j),$$

where $R_i(k,j)$, $i = 1,2,3$, are given by

$$R_1(k,j) = \frac{1}{\pi}\frac{1}{\sqrt{j}}\int_{|u|\leqslant \log j}\left(\phi_j^{(k)}\left(\frac{u}{\sqrt{j}}\right) - \mathrm{e}^{-au^2}\right)\mathrm{d}u,$$
$$R_2(k,j) = -\frac{1}{\pi}\frac{1}{\sqrt{j}}\int_{|u|>\log j}\mathrm{e}^{-au^2}\,\mathrm{d}u,$$
$$R_3(k,j) = \frac{1}{\pi}\frac{1}{\sqrt{j}}\int_{\log j<|u|\leqslant \frac{\pi}{2}\sqrt{j}}\phi_j^{(k)}\left(\frac{u}{\sqrt{j}}\right)\mathrm{d}u.$$

First we have the following equivalence

$$\sum_{k=0}^{[\frac{(n-1)}{2}]}\sum_{j=1}^{[\frac{(n-k)}{2}]}\frac{1}{\sqrt{j}} \sim \frac{(2^{3/2}-1)}{3}n^{3/2}.$$

So Lemma 4.2 will be proved if we show that for $i = 1,2,3$,

$$\sum_{k=0}^{[\frac{(n-1)}{2}]}\sum_{j=1}^{[\frac{(n-k)}{2}]} R_i(k,j) = \mathrm{o}(n^{3/2}).$$

Applying the method developed in the proof of Lemma 3.1, we have

$$\begin{aligned}
\left|R_1(k,j)\right| &\leqslant \frac{1}{\sqrt{\pi a j}} A(j,k)\exp\bigl(A(j,k)\bigr)\\
&= \frac{1}{\sqrt{\pi a j}} \sum_{p=0}^{\infty} \frac{A(j,k)^{p+1}}{p!}, \quad \forall j \geqslant 1,\ \forall k \geqslant 0,
\end{aligned}$$

where

$$\begin{aligned}
A(j,k) &= a\,\frac{\log^4 j}{j} \sum_{m=0}^{\infty} \left(\frac{\log^2 j}{j}\right)^m \frac{2^{2m+3}}{(2(m+2))!}\\
&\quad + \frac{\log^2 j}{2j} \left| \sum_{l=2k+1}^{2k+2j} 4 f(\tau_\alpha^l x)\bigl(1 - f(\tau_\alpha^l x)\bigr) - 2ja \right|\\
&\quad + \frac{\log^4 j}{j} \sum_{m=0}^{\infty} \left(\frac{\log^2 j}{j}\right)^m \frac{1}{m+2}\\
&\quad + \tan\left(\frac{\log j}{\sqrt{j}}\right) \left| \sum_{l=2k+1}^{2k+2j} \bigl(2f(\tau_\alpha^l x) - 1\bigr) \right|\\
&\quad + 2j\tan^3\left(\frac{\log j}{\sqrt{j}}\right) \sum_{m=0}^{\infty} \left(\tan^2\left(\frac{\log j}{\sqrt{j}}\right)\right)^m \frac{1}{2m+3}.
\end{aligned}$$

We can use again Proposition 2.3. There exists $n_0 \geqslant 1$ and a positive constant M such that

$$\begin{aligned}
&\left| \frac{1}{n^{3/2}} \sum_{k=0}^{[\frac{(n-1)}{2}]} \sum_{j=1}^{[\frac{(n-k)}{2}]} R_1(k,j) \right|\\
&\quad \leqslant \frac{1}{\sqrt{\pi a n}} \sum_{j=n_0+1}^{n} \frac{1}{\sqrt{j}} \sum_{p=0}^{\infty} \frac{1}{p!} \left(M\,\frac{\log^{3+\varepsilon} 2j}{\sqrt{j}} \right)^{p+1} + \mathrm{o}(1) = \mathrm{o}(1),\\
&\left| \frac{1}{n^{3/2}} \sum_{k=0}^{[\frac{(n-1)}{2}]} \sum_{j=1}^{[\frac{(n-k)}{2}]} R_2(k,j) \right|\\
&\quad \leqslant \frac{2}{\pi} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} \int_{\log j}^{\infty} \mathrm{e}^{-au^2}\,\mathrm{d}u = \frac{1}{\pi} \sqrt{\frac{2}{an}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} \int_{\sqrt{2a}\log j}^{\infty} \mathrm{e}^{-v^2/2}\,\mathrm{d}v\\
&\quad \leqslant \frac{1}{\pi a \sqrt{n}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} \frac{\mathrm{e}^{-a\log^2 j}}{\log j} = \mathrm{o}(1).
\end{aligned}$$

Then,

$$\left|\frac{1}{n^{3/2}}\sum_{k=0}^{[\frac{(n-1)}{2}]}\sum_{j=1}^{[\frac{(n-k)}{2}]}R_3(k,j)\right| \leqslant \frac{2}{\pi}\frac{1}{n^{3/2}}\sum_{k=0}^{n-1}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}$$

$$\times \int_{\log j}^{\frac{\pi}{2}\sqrt{j}} \exp\left(-\frac{4u^2}{\pi^2 2j}\sum_{l=2k+1}^{2k+2j}\left(4f(\tau_\alpha^l x)\left(1-f(\tau_\alpha^l x)\right)\right)\right)\mathrm{d}u.$$

Proposition 2.3 says that there exists $n_0 \geqslant 1$ and a positive constant M such that

$$\left|\frac{1}{n^{3/2}}\sum_{k=0}^{[\frac{(n-1)}{2}]}\sum_{j=1}^{[\frac{(n-k)}{2}]}R_3(k,j)\right|$$

$$\leqslant \frac{1}{\sqrt{n}}\sum_{j=n_0+1}^{n}\exp\left(\frac{4M\log^{4+\varepsilon}2j}{\pi^2 2j}\right)\exp\left(-\frac{4a\log^2 j}{\pi^2}\right)+\mathrm{o}(1)=\mathrm{o}(1).$$

□

The Tchebychev's inequality and Lemmata 4.1 and 4.2 imply Theorem 1.3.

5. GENERALISATION IN DIMENSION d

We recall some definitions and well known results from the method of low discrepancy sequences in dimension $d \geqslant 1$ (see, for example, [9] or [13]).

Suppose we are given a function $f(x)=f(x^{(1)},\ldots,x^{(d)})$ with $d \geqslant 1$. By a partition P of $[0,1]^d$, we mean a set of d finite sequences $\eta_0^{(j)},\eta_1^{(j)},\ldots,\eta_{m_j}^{(j)}$ $(j=1,\ldots,d)$, with $0=\eta_0^{(j)}\leqslant\eta_1^{(j)}\leqslant\cdots\leqslant\eta_{m_j}^{(j)}=1$ for $j=1,\ldots,d$. In connection with such a partition, we define, for $j=1,\ldots,d$ an operator Δ_j by

$$\begin{aligned}&\Delta_j f\left(x^{(1)},\ldots,x^{(j-1)},\eta_i^{(j)},x^{(j+1)},\ldots,x^{(d)}\right)\\&\quad=f\left(x^{(1)},\ldots,x^{(j-1)},\eta_{i+1}^{(j)},x^{(j+1)},\ldots,x^{(d)}\right)\\&\qquad-f\left(x^{(1)},\ldots,x^{(j-1)},\eta_i^{(j)},x^{(j+1)},\ldots,x^{(d)}\right),\end{aligned}$$

for $0\leqslant i<m_j$. We denote $\Delta_{1,\ldots,d}$ the operator $\Delta_1\ldots\Delta_d$.

DEFINITION 5.1. –
(1) *For a function* f *on* $[0,1]^d$, *we set*

$$V^{(d)}(f)=\sup_P \sum_{i_1=0}^{m_1-1}\cdots\sum_{i_d=0}^{m_d-1}\left|\Delta_{1,\dots,d} f\left(\eta_{i_1}^{(1)},\dots,\eta_{i_d}^{(d)}\right)\right|,$$

where the supremum is extended over all partitions P *of* $[0,1]^d$. *If* $V^{(d)}(f)$ *is finite, then* f *is said to be of bounded variation on* $[0,1]^d$ *in the sense of Vitali.*

(2) *For* $1 \leqslant p \leqslant d$ *and* $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant d$, *we denote by* $V^{(p)}(f; i_1,\dots,i_p)$ *the* p*-dimensional variation in the sense of Vitali of the restriction of* f *to*

$$E^d_{i_1\dots i_p}=\big\{(t_1,\dots,t_d)\in[0,1]^d;\ t_j=1 \text{ whenever } j \text{ is none of the } i_r,\ 1\leqslant r\leqslant p\big\}.$$

If all the variations $V^{(p)}(f; i_1,\dots,i_p)$ *are finite, the function* f *is said to be of bounded variation on* $[0,1]^d$ *in the sense of Hardy and Krause.*

Let $x_1,\dots,x_n$ be a finite sequence of points in $[0,1]^d$ with $x_l=(x_{l,1},\dots,x_{l,d})$ for $1\leqslant l\leqslant n$. We introduce the function

$$R_n(t_1,\dots,t_d)=\frac{A(t_1,\dots,t_d;x_1,\dots,x_n)}{n}-t_1\dots t_d$$

for $(t_1,\dots,t_d)\in[0,1]^d$, where $A(t_1,\dots,t_d;x_1,\dots,x_n)$ denotes the number of elements x_l, $1\leqslant l\leqslant n$, for which $x_{l,i}<t_i$ for $1\leqslant i\leqslant d$.

DEFINITION 5.2. – *The discrepancy* D_n^* *of the sequence* $x_1,\dots,x_n$ *in* $[0,1]^d$ *is defined to be*

$$D_n^*=\sup_{(t_1,\dots,t_d)\in[0,1]^d}\left|R_n(t_1,\dots,t_d)\right|.$$

For a real number t, let $\|t\|$ denote its distance to the nearest integer, namely,

$$\|t\|=\inf_{n\in\mathbb{Z}}|t-n|=\inf\big(\{t\},1-\{t\}\big),$$

where $\{t\}$ is the fractional part of t.

DEFINITION 5.3. – *A vector* $\alpha=(\alpha_1,\dots,\alpha_d)$, $\alpha_i\in\mathbb{R}$, *is said irrational if* $1,\alpha_1,\alpha_2,\dots,\alpha_d$ *are linearly independent over* $\mathbb{Z}$.

In the sense of the Lebesgue's measure, almost every vector in $\mathbb{R}^d$ is irrational.

DEFINITION 5.4. – *For a real number η, a d-tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$ of irrationals is said to be of type η if η is the infimum of all numbers σ for which there exists a positive constant $c = c(\sigma; \alpha_1, \ldots, \alpha_d)$ such that*

$$r^{\sigma}(h)\big\|\langle h, \alpha\rangle\big\| \geqslant c$$

holds for all $h \neq 0$ in $\mathbb{Z}^d$, where $r(h) = \prod_{i=1}^{d} \max(1, |h_i|)$ and $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^d$.

Remark. – The type η of α is also equal to

$$\sup\Big\{\gamma\colon \inf_{h\in(\mathbb{Z}^d)^*} r^{\gamma}(h)\big\|\langle h, \alpha\rangle\big\| = 0\Big\}.$$

We will always have $\eta \geqslant 1$ (see [13]). We give now a result (see [9]) which gives us the asymptotic behaviour of the discrepancy of the sequence $w = (x_1 + l\alpha_1, \ldots, x_d + l\alpha_d)$, $l = 1, 2, \ldots,$ in function of the mutual irrationality of the components of α.

PROPOSITION 5.5. – *Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be an irrational vector. Suppose there exists $\eta \geqslant 1$ and $c > 0$ such that*

$$r^{\eta}(h)\big\|\langle h, \alpha\rangle\big\| \geqslant c$$

for all $h \neq 0$ in $\mathbb{Z}^d$. Then, for every $x \in [0,1]^d$, the discrepancy of the sequence $w = ((x_1 + l\alpha_1) \bmod 1, \ldots, (x_d + l\alpha_d) \bmod 1)$, $l = 1, 2, \ldots,$ satisfies $D_n^*(w) = \mathcal{O}(n^{-1}\log^{d+1} n)$ for $\eta = 1$ and $D_n^*(w) = \mathcal{O}(n^{-1/((\eta-1)d+1)}\log n)$ for $\eta > 1$.

The proof of this result is identical to this of Exercice 3.17 in [9] when $x = (0, \ldots, 0)$. The following result can be found in [13].

THEOREM 5.6 (Hlawka, Zaremba). – *Let f be of bounded variation on $[0,1]^d$ in the sense of Hardy and Krause, and let ω be a finite sequence of points $x_1, \ldots, x_n$ in $[0,1]^d$. Then, we have*

$$\left|\frac{1}{n}\sum_{l=1}^{n} f(x_l) - \int_{\mathbb{T}^d} f(t)\,\mathrm{d}t\right| \leqslant \sum_{p=1}^{d}\ \sum_{1\leqslant i_1<i_2<\cdots<i_p\leqslant d} V^{(p)}(f; i_1, \ldots, i_p)D_n^*(\omega_{i_1\ldots i_p}),$$

where $D_n^*(\omega_{i_1\ldots i_p})$ *is the discrepancy in* $E^d_{i_1\ldots i_p}$ *of the sequence* $\omega_{i_1\ldots i_p}$ *obtained by projecting* ω *onto* $E^d_{i_1\ldots i_p}$.

Proof of Theorem 1.5. – The first part of the theorem is proved by the same method as this used in the unidimensional case (see end of Section 3). Let us prove the item (2). Let η' be such that $\eta \leqslant \eta' < 1 + \frac{1}{d}$. There exists $c > 0$ such that

$$r^{\eta'}(h)\|\langle h, \alpha\rangle\| \geqslant c$$

holds for all $h \neq 0$ in $\mathbb{Z}^d$. Suppose we are given a p-tuple $\alpha_p = (\alpha_{i_1}, \ldots, \alpha_{i_p})$, $1 \leqslant p \leqslant d$, of α, then

$$r^{\eta'}(h)\|\langle h, \alpha_p\rangle\| \geqslant c$$

holds for all $h \neq 0$ in $\mathbb{Z}^p$, $1 \leqslant p \leqslant d$. Thus, every p-tuple, $1 \leqslant p \leqslant d$, is of type δ such that $1 \leqslant \delta \leqslant \eta$ and $(\alpha_{i_1}, \ldots, \alpha_{i_p})$ is an irrational vector. For every p, $1 \leqslant p \leqslant d$, we define $w_{i_1\ldots i_p}$ by the projection of w on $E^d_{i_1\ldots i_p}$. By Proposition 5.5, we have for every p, $1 \leqslant p \leqslant d$,

$$\begin{cases} nD_n^*(w_{i_1\ldots i_p}) = \mathcal{O}(\log^{p+1} n) & \text{if } \delta = 1, \\ nD_n^*(w_{i_1\ldots i_p}) = \mathcal{O}\big(n^{1-\frac{1}{((\delta-1)p+1)}} \log n\big) & \text{if } 1 < \delta \leqslant \eta. \end{cases}$$

Now, $\forall p = 1, \ldots, d$,

$$0 \leqslant 1 - \frac{1}{(\delta-1)p+1} \leqslant 1 - \frac{1}{(\eta-1)d+1} \leqslant 1.$$

Therefore, using the Hlawka–Zaremba's theorem, we obtain, for every function f with bounded variation in the sense of Hardy and Krause,

$$\sum_{l=1}^{n} f(\tau_\alpha^l x) - n \int_{\mathbb{T}^d} f(t)\,\mathrm{d}t = \begin{cases} \mathcal{O}(\log^{d+1} n) & \text{if } \eta = 1, \\ \mathcal{O}\big(n^{1-\frac{1}{((\eta-1)d+1)}} \log n\big) & \text{if } \eta > 1. \end{cases}$$

The proof of item (2) of Theorem 1.5 is the same as this developed in the unidimensional case but instead of using Proposition 2.3 we apply the above result and Theorem 1.5 follows as soon as α is of type η such that $1 \leqslant \eta < 1 + \frac{1}{d}$. □

The class of functions with bounded variations in the sense of Hardy and Krause (H&K) turns out to be rather technical to define and uneasy to handle, especially for large values of d. A more pleasant, and almost identical class, is constructed by using bounded variations in the measure sense (see [14]).

We would like to know whether a lot of irrational vectors are of type 1. The following proposition permits us to answer this question.

PROPOSITION 5.7. – *Almost every irrational vector is of type* 1.

In order to prove the proposition, we use the following lemma.

LEMMA 5.8. – *Let $h \in (\mathbb{Z}^d)^*$, $\varepsilon \in]0, \frac{1}{2}[$. The set of points $x \in [0, 1[^d$ such that $\|\langle h, \alpha \rangle\| \leqslant \varepsilon$ is of d-dimensional Lebesgue's measure equal to 2ε.*

Proof. – We set λ the Lebesgue's measure on the interval $[0, 1[$. Suppose for example that $h_1 > 0$. If $y \in \mathbb{R}$, we have for the one-dimensional Lebesgue's measure,

$$\lambda\bigl(\{x_1 \in [0, 1[;\ \|h_1 x_1 + y\| \leqslant \varepsilon\}\bigr) = \frac{1}{h_1}\lambda\bigl(\{x_1' \in [0, h_1[;\ \|x_1' + y\| \leqslant \varepsilon\}\bigr)$$
$$= 2\varepsilon.$$

So, for every $(x_2, x_3, \ldots, x_d) \in [0, 1[^{d-1}$, we obtain, denoting $x = (x_1, x_2, \ldots, x_d)$,

$$\lambda\bigl(\{x_1 \in [0, 1[;\ \bigl\|\langle h, x \rangle\bigr\| \leqslant \varepsilon\}\bigr) = 2\varepsilon.$$

Now, we conclude with the Fubini's theorem. □

Proof of the proposition. – We know that the type of an irrational vector is always greater than 1. Let σ be a real strictly greater than 1. So, we must prove that almost every irrational vector α is of type $< \sigma$. To do this, it suffices to prove that for almost every α, for every $h \in (\mathbb{Z}^d)^*$ except perhaps for a finite family,

$$r^\sigma(h)\bigl\|\langle h, \alpha \rangle\bigr\| > 1. \tag{4}$$

We denote λ_d the Lebesgue's measure on $[0, 1[^d$ and for every $h \in \mathbb{Z}^d$,

$$A_h = \bigl\{x \in [0, 1[^d;\ r^\sigma(h)\bigl\|\langle h, \alpha \rangle\bigr\| \leqslant 1\bigr\}.$$

From the previous lemma, we obtain

$$\lambda_d(A_h) \leqslant \frac{2}{r^\sigma(h)}.$$

Then,

$$\sum_{h\in\mathbf{Z}^d}\lambda_d(A_h)\leqslant 2^d\sum_{h\in\mathbf{N}^d}\lambda_d(A_h)\leqslant 2^{d+1}\sum_{h_1,h_2,\dots,h_d\geqslant 0}\prod_{i=1}^{d}\frac{1}{\max(1,h_i)^{\sigma}}$$
$$=2^{d+1}\left(\sum_{h\geqslant 0}\frac{1}{\max(1,h)^{\sigma}}\right)^{d}<+\infty,$$

σ being strictly greater than 1.

Using the Borel–Cantelli's lemma, we conclude that almost every vector $x\in[0,1[^d$ is only in a finite number of sets A_h. This proves inequality (4). □

Example 5.9. – W. Schmidt [15] has shown that $\alpha=(\alpha_1,\dots,\alpha_d)$ with real algebraic numbers α_i for which $1,\alpha_1,\dots,\alpha_d$ are linearly independent over the rationals, is of type $\eta=1$. Moreover, Baker [1] has proved that $\alpha=(\mathrm{e}^{r_1},\dots,\mathrm{e}^{r_d})$, with distinct nonzero rationals $r_1,\dots,r_d$, is of type $\eta=1$.

Example 5.10. – We can give an example of function for which the theorem is verified. Let

$$f(x)=\frac{1}{d}\sum_{i=1}^{d}\cos^2(2\pi x_i),\qquad x=(x_1,\dots,x_d)\in\mathbb{T}^d.$$

After calculations, we obtain that for every $x\in\mathbb{T}^d$, for every $\alpha=(\alpha_1,\dots,\alpha_d)$ a d-dimensional vector of irrational numbers such that 1, α_i and α_m be independent on $\mathbb{Q}$, there exists two positive constants C_1 and C_2 such that $\forall k\geqslant 0$, $\forall j\geqslant 1$,

$$\left|\sum_{l=2k+1}^{2k+2j}\Bigl(4f(\tau_\alpha^l x)\bigl(1-f(\tau_\alpha^l x)\bigr)\Bigr)-2ja\right|\leqslant C_1$$

and

$$\left|\sum_{l=2k+1}^{2k+2j}\bigl(2f(\tau_\alpha^l x)-1\bigr)\right|\leqslant C_2.$$

Consequently, our result is valid, for this particular function, for every α defined as above.

Remarks. –

(1) We could have used the Fourier Series Method to estimate the asymptotic behaviour of $\frac{1}{n}\sum_{l=1}^{n} f(\tau_\alpha^l x) - \int_{\mathbb{T}^d} f(t)\,\mathrm{d}t$. Let f be a function on $\mathbb{R}^d$, 1-periodic in each variable and which can be expanded into a multiple Fourier series $f(t) = \sum_{h\in\mathbb{Z}^d} c_h\, \mathrm{e}^{2\pi \mathrm{i}\langle h,t\rangle}$. The function f belongs to the class $\mathcal{E}^s$, $s > 1$ if there exists a constant M such that $|c_h| \leqslant M r^{-s}(h)$ for every $h \neq 0$. Then, Theorem 8.1 of [13] says that if α is a d-tuple of irrationals of type η with $1, \alpha_1, \ldots, \alpha_d$ linearly independent over the rationals, we have, for every $x \in \mathbb{T}^d$,

$$\frac{1}{n}\sum_{l=1}^{n} f(\tau_\alpha^l x) - \int_{\mathbb{T}^d} f(t)\,\mathrm{d}t = \mathcal{O}(n^{-1}),$$

for all $f \in \mathcal{E}^{\eta+\lambda}$, where $\lambda > 0$ is arbitrary. Here, the majorizations are better than in the first method, but the hypothesis on f are stronger. But, in the case where α is of type η with $\eta \geqslant 1 + 1/d$, this method can be applied.

(2) The restrictive conditions on α are needed to obtain a sequence of points $(x_1 + l\alpha_1, \ldots, x_d + l\alpha_d)$, $l = 1, 2, \ldots, n$, with an adequate asymptotic behaviour of his discrepancy. But, $(x_1 + l\alpha_1, \ldots, x_d + l\alpha_d)$, $l = 1, 2, \ldots, n$, with α and x correctly chosen, are not the unique sequences for which the theory developed here is valid. The sequences which generalise the Halton's sequences (or Van der Corput's sequences in dimension one) are sequences with low discrepancy (see [14] and [10] for the definitions). A way to define them is to set a pseudo-addition on the p-adic rationals from [0, 1]. For every $p \geqslant 2$, if $x, y \in [0,1] \cap \mathbb{Q}_p$ (p-adic rationals), we define $\Phi_{p,y}(x) = x \oplus_p y$ like in a classical addition, but from the left to the right (see [14] and [10] for some improvements). If $p_1, \ldots, p_d$ are d integers $\geqslant 2$ which are relatively prime to each other, $y_1, \ldots, y_d \in [0, 1[\cap \mathbb{Q}_p$ and $x_1, \ldots, x_d \in [0, 1]$, all sequences defined by $\xi_l = (\Phi_{p_1,y_1}^{l-1}(x_1), \ldots, \Phi_{p_d,y_d}^{l-1}(x_d))$, $l = 1, \ldots, n$ have a discrepancy $D_n(\xi) = \mathcal{O}(\log^d(n)/n)$ (uniformly in $x = (x_1, \ldots, x_d) \in [0,1]^d$) ([14]). Then, Theorem 1.5 is still valid for these sequences with the same conditions on the function f.

6. CASE OF α RATIONAL VECTOR

We prove Theorem 1.6. The function f being 1-periodic, the vectors $(X_{n\tilde{q}+1},\dots,X_{(n+1)\tilde{q}})$, $n\geqslant 0$, have the same distribution. So, if we denote $Y_j=\sum_{l=1}^{\tilde{q}}X_{j\tilde{q}+l}$, $j=0,\dots,n-1$, we can write that

$$S_0=0,\quad S_{n\tilde{q}}=\sum_{j=0}^{n-1}Y_j,\quad n\geqslant 1,$$

where $(Y_j)_{j\geqslant 0}$ is a family of random variables independent, identically distributed with $\mathbb{E}(Y_0)=\sum_{l=1}^{\tilde{q}}(2f(\tau_\alpha^l x)-1)=(2p-1)\tilde{q}$ and $\operatorname{var}(Y_0)=\sum_{l=1}^{\tilde{q}}4f(\tau_\alpha^l x)(1-f(\tau_\alpha^l x))<\infty$. Therefore, when $p=\frac{1}{2}$ (respectively $p\neq\frac{1}{2}$), $(S_{n\tilde{q}})_{n\geqslant 0}$ is a recurrent (respectively transient) $\mathbb{Z}$-random walk embedded in the random walk $(S_n)_{n\geqslant 0}$, which implies the recurrence (respectively the transience) of the Markov chain $(S_n)_{n\geqslant 0}$.

For every $i=0,\dots,\tilde{q}-1$, $S_{n\tilde{q}+i}-S_i$ is a sum of independent, identically distributed random variables with mean $(2p-1)\tilde{q}$ and finite variance. Therefore, for every $i=0,\dots,\tilde{q}-1$, $(\xi(S_{n\tilde{q}+i}))_{n\geqslant 0}$ is a stationary sequence and by the Birkhoff's theorem,

$$\frac{1}{n}\sum_{k=0}^{n}\xi(S_{k\tilde{q}+i})\overset{a.s}{\to}0. \tag{5}$$

Now, there exists two sequences of integers $(m_n)_{n\geqslant 0}$, $(r_n)_{n\geqslant 0}$ such that $n=m_n\tilde{q}+r_n$, $0\leqslant r_n\leqslant\tilde{q}-1$. And,

$$Z_n=\sum_{i=1}^{r_n}\sum_{k=0}^{m_n}\xi(S_{k\tilde{q}+i})+\sum_{i=r_n+1}^{\tilde{q}-1}\sum_{k=0}^{m_n-1}\xi(S_{k\tilde{q}+i})+\sum_{i=0}^{m_n}\xi(S_{i\tilde{q}}).$$

Then,

$$\left|\frac{Z_n}{n}\right|\leqslant\sum_{i=1}^{\tilde{q}-1}\frac{1}{n}\left|\sum_{k=0}^{m_n}\xi(S_{k\tilde{q}+i})\right|+\sum_{i=1}^{\tilde{q}-1}\frac{1}{n}\left|\sum_{k=0}^{m_n-1}\xi(S_{k\tilde{q}+i})\right|+\frac{1}{n}\left|\sum_{i=0}^{m_n}\xi(S_{i\tilde{q}})\right|.$$

Using (5), the second part of the theorem is proved. □

Remark. – For example, we can take the function $f(x)=\cos^2(2\pi\sum_{i=1}^{d}x_i)$, $x\in\mathbb{T}^d$, with α such that $\sum_{i=1}^{d}\alpha_i\notin\mathbb{Z}$.

7. OPEN PROBLEMS AND FURTHER DEVELOPMENTS

(1) The constants of normalisation in the theorems proved in [6] and [2] are not standard and depend on the asymptotic behaviour of the probability of return to the origin of the considered random walks. In our case, using Kesten's and Spitzer's method, it is possible to prove that $n^{-3/4}Z_{[nt]}$ converges weakly as $n \to \infty$ to a self-similar process with stationary increments (see [4]). The central limit theorem proved by Bolthausen in dimension two can be also verified for these random walks.

(2) We can also consider random walk in random environment in the sense developed by Solomon (see [16]). In this situation, the transition probabilities do not depend on the time, but on the site visited by the random walk. These random walks are recurrent under some conditions and it would be interesting to study them in random sceneries.

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