

# ANNALES DE L'I. H. P., SECTION B

FUMIO HIAI

DÉNES PETZ

## **A large deviation theorem for the empirical eigenvalue distribution of random unitary matrices**

*Annales de l'I. H. P., section B*, tome 36, n° 1 (2000), p. 71-85

[http://www.numdam.org/item?id=AIHPB\\_2000\\_\\_36\\_1\\_71\\_0](http://www.numdam.org/item?id=AIHPB_2000__36_1_71_0)

© Gauthier-Villars, 2000, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section B » (<http://www.elsevier.com/locate/anihpb>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

# **A large deviation theorem for the empirical eigenvalue distribution of random unitary matrices \***

by

**Fumio HIAI**<sup>a</sup>, **Dénes PETZ**<sup>b,1</sup>

<sup>a</sup> Department of Mathematical Sciences, Ibaraki University, Mito, Ibaraki 310, Japan

<sup>b</sup> Department for Mathematical Analysis, Technical University of Budapest, H-1521  
Budapest XI. Sztoczek u. 2, Hungary

Article received on 16 November 1998

---

**ABSTRACT.** – It is shown that the empirical eigenvalue distribution of suitably distributed random unitary matrices satisfies the large deviation principle as the matrix size goes to infinity. The primary term of the rate function is the logarithmic energy (or the minus sign of Voiculescu’s free entropy). Examples of random unitaries are also discussed, one of them is related to the work of Gross and Witten in quantum physics. © 2000 Éditions scientifiques et médicales Elsevier SAS

*Key words:* Large deviation, Random unitary matrix, Eigenvalue density, Logarithmic energy

**RÉSUMÉ.** – Nous montrons que la distribution empirique des valeurs propres de matrices unitaires aléatoires (de loi convenable) satisfait un principe de grandes déviations quand la taille de la matrice tend vers l’infini. Le terme principal de la fonction de taux est l’énergie logarithmique (ou, au signe près, l’entropie libre de Voiculescu). Nous discutons aussi des exemples d’opérateurs unitaires aléatoires, dont l’un

---

\* This paper has been circulated as Preprint No. 18/1997, Mathematical Institute of the Hungarian Academy of Sciences.

<sup>1</sup> E-mail: petz@math.bme.hu.

est lié au travail de Gross et Witten en physique quantique. © 2000 Éditions scientifiques et médicales Elsevier SAS

## 1. INTRODUCTION AND STATEMENT OF RESULT

Let  $\nu$  be a measure on  $\mathbb{C}$ . The double integral

$$- \iint \log |x - y| d\nu(x) d\nu(y)$$

has been used in potential theory for a long time and it has been called logarithmic energy there [9]. It is a very remarkable fact that essentially the same quantity appeared in Voiculescu's work on free probability theory [11–14] (also [6]). When Voiculescu modeled probability laws by the eigenvalue distribution of random matrices in his random matrix heuristics, the above logarithmic energy appeared as a renormalized Boltzmann–Shannon entropy [11]. One characteristic feature of entropy-like quantities is that they can serve as a rate function in large deviation theorems. Indeed, the logarithmic energy (alias Voiculescu's free entropy) is the rate function in a very recent large deviation theorem obtained by Ben Arous and Guionnet [1], which concerns the empirical eigenvalue distribution of Gaussian symmetric random matrices as the matrix size tends to infinity. The subject of the present paper is to obtain a large deviation theorem for the empirical eigenvalue distribution of random unitary matrices. This work has been motivated very much by a preliminary version of [1], we use the method developed in that paper. The final published version of the work of Ben Arous and Guionnet treats random unitaries, however the full large deviation was not obtained in [1].

Let us recall the definition of the large deviation principle [3]. Let  $(P_n)$  be a sequence of measures on a topological space  $X$ . The large deviation principle holds with rate function  $I$  in the scale  $n^{-2}$  if

$$\liminf_n \frac{1}{n^2} \log P_n(G) \geq - \inf \{ I(x) : x \in G \}$$

for every open set  $G \subset X$  and

$$\limsup_n \frac{1}{n^2} \log P_n(F) \leq - \inf \{ I(x) : x \in F \}$$

for every closed set  $F \subset X$ . (If the latter condition holds only for compact sets  $F$ , then the weak large deviation principle is said to hold true.) Here  $I : X \rightarrow [0, \infty]$  is lower semicontinuous and is called a good rate function if  $\{x \in X : I(x) \leq c\}$  is compact for every  $c \geq 0$ .

Let  $\mathcal{U}(n)$  denote the group of  $n \times n$  unitary matrices and let  $\gamma_n$  be the Haar probability measure on  $\mathcal{U}(n)$ . Moreover, let  $\mathcal{M}(\mathbb{T})$  be the space of all probability Borel measures on the unit circle  $\mathbb{T} \subset \mathbb{C}$ .

For  $U \in \mathcal{U}(n)$  we write  $\Lambda_n(U)$  for the atomic measure

$$\frac{\delta(\lambda_1(U)) + \delta(\lambda_2(U)) + \dots + \delta(\lambda_n(U))}{n},$$

where  $\lambda_1(U), \lambda_2(U), \dots, \lambda_n(U)$  are the eigenvalues of  $U$  and  $\delta(\zeta)$  denotes the Dirac measure at  $\zeta$ . In this way a mapping  $\Lambda_n : \mathcal{U}(n) \rightarrow \mathcal{M}(\mathbb{T})$  is determined. Given a measure  $\nu_n$  on  $\mathcal{U}(n)$ , there exists a unique probability measure  $P_n$  on  $\mathcal{M}(\mathbb{T})$  such that

$$P_n(H) = \nu_n(\Lambda_n^{-1}H)$$

for every Borel set  $H$  in  $\mathcal{M}(\mathbb{T})$ . Note that  $P_n$  is nothing else but the distribution of the random measure  $\Lambda_n(U)$  when  $U$  is considered to be random and distributed according to  $\nu_n$ .

Now let  $Q(\zeta)$  be a real continuous function on  $\mathbb{T}$  and for each  $n \in \mathbb{N}$  set a probability measure  $\nu_n$  on  $\mathcal{U}(n)$  as

$$\nu_n = \frac{1}{Z_n} \exp(-n \operatorname{Tr} Q(U)) d\gamma_n(U), \tag{1.1}$$

where  $Z_n$  is for normalization. Then we have the following large deviation theorem.

**THEOREM.** – *Let  $P_n$  ( $n \in \mathbb{N}$ ) be the probability measures defined above on  $\mathcal{M}(\mathbb{T})$ . Then the finite limit  $B = \lim_{n \rightarrow \infty} n^{-2} \log Z_n$  exists and  $(P_n)$  satisfies the large deviation principle in the scale  $n^{-2}$  with rate function*

$$I(\mu) = - \iint_{\mathbb{T}^2} \log |\zeta - \eta| d\mu(\zeta) d\mu(\eta) + \int_{\mathbb{T}} Q(\zeta) d\mu(\zeta) + B \tag{1.2}$$

for  $\mu \in \mathcal{M}(\mathbb{T})$ . Furthermore, there exists a unique  $\mu_0 \in \mathcal{M}(\mathbb{T})$  such that  $I(\mu_0) = 0$ .

## 2. PREPARATION

The space  $\mathcal{M}(\mathbb{T})$  of all probability measures on  $\mathbb{T}$  equipped with the weak\* topology is compact and metrizable. Set

$$F(\zeta, \eta) = -\log |\zeta - \eta| + \frac{1}{2}(Q(\zeta) + Q(\eta)),$$

$$F_\alpha(\zeta, \eta) = \min\{F(\zeta, \eta), \alpha\}$$

for  $\alpha > 0$ . Since  $F_\alpha(\zeta, \eta)$  is bounded and continuous,

$$\mu \in \mathcal{M}(\mathbb{R}) \mapsto \iint_{\mathbb{T}^2} F_\alpha(\zeta, \eta) d\mu(\zeta) d\mu(\eta)$$

is continuous in the weak\* topology. Given a real constant  $B$ , the functional  $I$  in (1.2) is written as

$$I(\mu) = \iint F(\zeta, \eta) d\mu(\zeta) d\mu(\eta) + B = \sup_{\alpha > 0} \iint F_\alpha(\zeta, \eta) d\mu(\zeta) d\mu(\eta) + B$$

and hence it is lower semicontinuous. Since the logarithmic kernel is strictly negative definite (see [2]),  $I$  is shown to be strictly convex.

If  $X$  is compact and  $\mathcal{A}$  is a base for the topology, then the large deviation principle is equivalent to the following two conditions

$$\begin{aligned} -I(x) &= \inf \left\{ \limsup_{n \rightarrow \infty} \frac{1}{n^2} \log P_n(G) : G \in \mathcal{A}, x \in G \right\} \\ &= \inf \left\{ \liminf_{n \rightarrow \infty} \frac{1}{n^2} \log P_n(G) : G \in \mathcal{A}, x \in G \right\}. \end{aligned} \quad (2.1)$$

(See [3, Theorem 4.1.11].) We apply this result in the case  $X = \mathcal{M}(\mathbb{T})$  and we may choose

$$G = G(\mu; m, \varepsilon) = \{\mu' \in \mathcal{M}(\mathbb{T}) : |m_k(\mu') - m_k(\mu)| < \varepsilon \text{ for } |k| \leq m\},$$

where  $m_k$  denotes the  $k$ th moment ( $k \in \mathbb{Z}$ ), i.e.,  $m_k(\mu) = \int_{\mathbb{T}} \zeta^k d\mu(\zeta)$ . For  $\mu \in \mathcal{M}(\mathbb{T})$  the sets  $G(\mu; m, \varepsilon)$  form a neighborhood base of  $\mu$  for the weak\* topology of  $\mathcal{M}(\mathbb{T})$  where  $m \in \mathbb{N}$  and  $\varepsilon > 0$ .

The mapping  $\mathcal{U}(n) \rightarrow \mathbb{T}^n$  sending a unitary to the  $n$ -tuples of its eigenvalues induces a measure  $\bar{\nu}_n$  on  $\mathbb{T}^n$  when  $\mathcal{U}(n)$  is endowed by the measure  $\nu_n$  given as (1.1). We have

$$\begin{aligned}
 &P_n(G(\mu; m, \varepsilon)) \\
 &= \nu_n \left( \left\{ U \in \mathcal{U}(n) : \left| \frac{1}{n} \sum_{j=1}^n \lambda_j(U)^k - m_k(\mu) \right| < \varepsilon \text{ for } |k| \leq m \right\} \right) \quad (2.2) \\
 &= \bar{\nu}_n \left( \left\{ \zeta \in \mathbb{T}^n : |m_k(\mu_\zeta) - m_k(\mu)| < \varepsilon \text{ for } |k| \leq m \right\} \right),
 \end{aligned}$$

where for  $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{T}^n$  the atomic measure

$$\frac{\delta(\zeta_1) + \delta(\zeta_2) + \dots + \delta(\zeta_n)}{n}$$

is denoted by  $\mu_\zeta$ .

It is known [10, p. 195] that the measure  $\bar{\gamma}_n$  on  $\mathbb{T}^n$  induced by  $\gamma_n$  has the density  $\frac{1}{n!} \prod_{i < j} |\zeta_i - \zeta_j|^2$  with respect to  $d\zeta_1 \cdots d\zeta_n$  where  $\zeta_j = e^{i\theta_j}$  and  $d\zeta_j = \frac{1}{2\pi} d\theta_j$ . Hence the density of  $\bar{\nu}_n$  with respect to  $d\zeta_1 \cdots d\zeta_n$  is

$$\frac{1}{Z_n} \exp \left( -n \sum_{i=1}^n Q(\zeta_i) \right) \prod_{i < j} |\zeta_i - \zeta_j|^2.$$

To obtain the theorem, we have to prove that

$$\begin{aligned}
 -I(\mu) &\geq \inf \left\{ \limsup_n \frac{1}{n^2} \log P_n(G) : G \right\}, \\
 -I(\mu) &\leq \inf \left\{ \liminf_n \frac{1}{n^2} \log P_n(G) : G \right\},
 \end{aligned} \quad (2.3)$$

where  $G$  runs over neighborhoods of  $\mu$ .

### 3. PROOF OF THEOREM

Our aim is to show that conditions (2.3) hold true.

LEMMA 1. –

$$\limsup_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n \leq - \inf_{\mu \in \mathcal{M}(\mathbb{T})} \iint_{\mathbb{T}^2} F(x, y) d\mu(x) d\mu(y). \quad (3.1)$$

*Proof.* – We get

$$\begin{aligned}
 Z_n &= \int \cdots \int_{\mathbb{T}^n} \exp\left(-\sum_{i=1}^n Q(\zeta_i)\right) \\
 &\quad \times \exp\left\{-\sum_{i<j} (Q(\zeta_i) + Q(\zeta_j))\right\} \prod_{i<j} |\zeta_i - \zeta_j|^2 d\zeta_1 \cdots d\zeta_n \\
 &= \int \cdots \int_{\mathbb{T}^n} \exp\left(-\sum_{i=1}^n Q(\zeta_i)\right) \exp\left\{-2\sum_{i<j} F(\zeta_i, \zeta_j)\right\} d\zeta_1 \cdots d\zeta_n \\
 &\leq \int \cdots \int_{\mathbb{T}^n} \exp\left(-\sum_{i=1}^n Q(\zeta_i)\right) \\
 &\quad \times \exp\left\{-n^2 \iint_{\{x \neq y\}} F(x, y) d\mu_\zeta(x) d\mu_\zeta(y)\right\} d\zeta_1 \cdots d\zeta_n \\
 &\leq \exp\left\{-n^2 \inf_{\mu} \iint_{\{x \neq y\}} F(x, y) d\mu(x) d\mu(y)\right\} \\
 &\quad \times \int \cdots \int_{\mathbb{T}^n} \exp\left(-\sum_{i=1}^n Q(\zeta_i)\right) d\zeta_1 \cdots d\zeta_n \\
 &= \left(\int e^{-Q(x)} dx\right)^n \exp\left\{-n^2 \inf_{\mu} \iint F(x, y) d\mu(x) d\mu(y)\right\},
 \end{aligned}$$

implying (3.1).  $\square$

LEMMA 2. – For every  $\mu \in \mathcal{M}(\mathbb{T})$ ,

$$\begin{aligned}
 &\inf\left\{\limsup_{n \rightarrow \infty} \frac{1}{n^2} \log P_n(G): G\right\} \\
 &\leq -\iint_{\mathbb{T}^2} F(x, y) d\mu(x) d\mu(y) - \liminf_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n,
 \end{aligned} \tag{3.2}$$

where  $G$  runs over a neighborhood base of  $\mu$ .

*Proof.* – For any neighborhood  $G$  of  $\mu \in \mathcal{M}(\mathbb{T})$  put

$$G_0 = \{\zeta \in \mathbb{T}^n: \mu_\zeta \in G\}.$$

As in the proof of Lemma 3.1 we get

$$\begin{aligned}
 P_n(G) &= \bar{v}_n(G_0) = \frac{1}{Z_n} \int \cdots \int_{G_0} \exp\left(-\sum_{i=1}^n Q(\zeta_i)\right) \\
 &\quad \times \exp\left\{-2 \sum_{i<j} F(\zeta_i, \zeta_j)\right\} d\zeta_1 \cdots d\zeta_n \\
 &\leq \frac{1}{Z_n} \int \cdots \int_{G_0} \exp\left(-\sum_{i=1}^n Q(\zeta_i)\right) \\
 &\quad \times \exp\left\{-n^2 \iint F_\alpha(x, y) d\mu_\zeta(x) d\mu_\zeta(y) + n\alpha\right\} d\zeta_1 \cdots d\zeta_n \\
 &= \frac{1}{Z_n} \left(\int e^{-Q(x)} dx\right)^n \\
 &\quad \times \exp\left\{-n^2 \inf_{\mu' \in G} \iint F_\alpha(x, y) d\mu'(x) d\mu'(y) + n\alpha\right\}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 &\limsup_{n \rightarrow \infty} \frac{1}{n^2} \log P_n(G) \\
 &\leq - \inf_{\mu' \in G} \iint F_\alpha(x, y) d\mu'(x) d\mu'(y) - \liminf_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n.
 \end{aligned}$$

Thanks to the weak\* continuity of  $\mu' \mapsto \iint F_\alpha(x, y) d\mu'(x) d\mu'(y)$  we get

$$\begin{aligned}
 &\inf \left\{ \limsup_{n \rightarrow \infty} \frac{1}{n^2} \log P_n(G) : G \right\} \\
 &\leq - \iint F_\alpha(x, y) d\mu(x) d\mu(y) - \liminf_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n.
 \end{aligned}$$

Letting  $\alpha \rightarrow +\infty$  yields inequality (3.2).  $\square$

A measure  $\mu$  on  $\mathbb{T}$  may be identified with the distribution

$$\frac{1}{2\pi} \mu(\theta) = \mu(\{e^{it} : 0 \leq t \leq \theta\}),$$

so that we write  $\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\mu(\theta)$  for  $\int_{\mathbb{T}} f d\mu$ .

LEMMA 3. – For any  $\mu \in \mathcal{M}(\mathbb{T})$  and  $0 < r < 1$  define

$$f_r(e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) d\mu(t), \quad \mu_r = \frac{1}{2\pi} f_r(e^{i\theta}) d\theta,$$

where  $P_r(\theta) = (1 - r^2)/(1 - 2r \cos \theta + r^2)$ , the Poisson kernel. Then  $\mu_r \rightarrow \mu$  in the weak\* topology as  $r \rightarrow 1$  and

$$\iint_{\mathbb{T}^2} F(\zeta, \eta) d\mu(\zeta) d\mu(\eta) = \lim_{r \rightarrow 1} \iint_{\mathbb{T}^2} F(\zeta, \eta) d\mu_r(\zeta) d\mu_r(\eta).$$

*Proof.* – The first assertion is well known [8, p. 13]. A basic fact on harmonic extension and Poisson integral (see [8]) is used in the following computations. For any  $\eta \in \mathbb{T}$ , since  $\zeta \mapsto \log |\zeta - \eta|$  is integrable on  $\mathbb{T}$ , we get

$$\begin{aligned} \int_{\mathbb{T}} \log |\zeta - \eta| d\mu_r(\zeta) &= \frac{1}{2\pi} \int_0^{2\pi} f_r(e^{i\theta}) \log |e^{i\theta} - \eta| d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - s) \log |e^{i\theta} - \eta| d\theta \right) d\mu(s) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log |r e^{is} - \eta| d\mu(s), \end{aligned}$$

and hence

$$\begin{aligned} &\iint_{\mathbb{T}^2} \log |\zeta - \eta| d\mu_r(\zeta) d\mu_r(\eta) \\ &= \frac{1}{2\pi} \int_0^{2\pi} f_r(e^{i\theta}) \left( \frac{1}{2\pi} \int_0^{2\pi} \log |r e^{is} - e^{i\theta}| d\mu(s) \right) d\theta \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) \log |r e^{is} - e^{i\theta}| d\theta \right) d\mu(s) d\mu(t) \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log |r^2 e^{is} - e^{it}| d\mu(s) d\mu(t), \end{aligned}$$

which tends to  $\iint_{\mathbb{T}^2} \log |\zeta - \eta| d\mu(\zeta) d\mu(\eta)$  as  $r \rightarrow 1$  and the lemma is shown.  $\square$

LEMMA 4. – For every  $\mu \in \mathcal{M}(\mathbb{T})$ ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n \geq - \iint_{\mathbb{T}^2} F(x, y) d\mu(x) d\mu(y) \tag{3.3}$$

and

$$\begin{aligned} & \inf \left\{ \liminf_{n \rightarrow \infty} \frac{1}{n^2} \log P_n(G) : G \right\} \\ & \geq - \iint_{\mathbb{T}^2} F(x, y) d\mu(x) d\mu(y) - \limsup_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n, \end{aligned} \tag{3.4}$$

where  $G$  runs over a neighborhood base of  $\mu$ .

*Proof.* – Thanks to Lemma 3.3 we may assume that  $\mu$  has a continuous density  $f > 0$  on  $\mathbb{T}$  so that  $\mu = \frac{1}{2\pi} f(e^{i\theta}) d\theta$ . Let  $\delta > 0$  be taken so that  $\delta \leq f(x) \leq \delta^{-1}$  for  $x \in \mathbb{T}$ . The following proof is a modification of that of [12, Proposition 4.5].

For each  $n \in \mathbb{N}$  choose a partition

$$0 = b_0^{(n)} < a_1^{(n)} < b_1^{(n)} < a_2^{(n)} < \dots < a_n^{(n)} < b_n^{(n)} = 2\pi$$

such that

$$\frac{1}{2\pi} \int_0^{a_j^{(n)}} f(e^{i\theta}) d\theta = \frac{j - \frac{1}{2}}{n}, \quad \frac{1}{2\pi} \int_0^{b_j^{(n)}} f(e^{i\theta}) d\theta = \frac{j}{n} \quad (1 \leq j \leq n).$$

Then it is immediate that

$$\frac{\pi \delta}{n} \leq b_j^{(n)} - a_j^{(n)} \leq \frac{\pi}{n\delta} \quad (1 \leq j \leq n). \tag{3.5}$$

Define

$$\Delta_n = \{ (e^{i\theta_1}, \dots, e^{i\theta_n}) \in \mathbb{T}^n : a_j^{(n)} \leq \theta_j \leq b_j^{(n)} \text{ for } 1 \leq j \leq n \},$$

$$\Delta'_n = \{ (\theta_1, \dots, \theta_n) : a_j^{(n)} \leq \theta_j \leq b_j^{(n)} \text{ for } 1 \leq j \leq n \},$$

$$\xi_i^{(n)} = \max \{ Q(e^{i\theta}) : a_i^{(n)} \leq \theta \leq b_i^{(n)} \},$$

$$d_{ij}^{(n)} = \min \{ |e^{is} - e^{it}| : a_i^{(n)} \leq s \leq b_i^{(n)}, a_j^{(n)} \leq t \leq b_j^{(n)} \}.$$

For any neighborhood  $G$  of  $\mu$ , it is clear that

$$\Delta_n \subset G_0 = \{ \zeta \in \mathbb{T}^n : \mu_\zeta \in G \}$$

for all  $n$  large enough. Therefore for large  $n$  we have

$$\begin{aligned} P_n(G) &= \bar{v}_n(G_0) \geq \bar{v}_n(\Delta_n) \\ &= \frac{1}{Z_n(2\pi)^n} \int \cdots \int_{\Delta'_n} \exp\left(-n \sum_{i=1}^n Q(e^{i\theta_i})\right) \prod_{i < j} |e^{i\theta_i} - e^{i\theta_j}|^2 d\theta_1 \cdots d\theta_n \\ &\geq \frac{1}{Z_n(2\pi)^n} \exp\left(-n \sum_{i=1}^n \xi_i^{(n)}\right) \prod_{i < j} (d_{ij}^{(n)})^2 \int \cdots \int_{\Delta'_n} d\theta_1 \cdots d\theta_n \\ &\geq \frac{1}{Z_n} \left(\frac{\delta}{2n}\right)^n \exp\left(-n \sum_{i=1}^n \xi_i^{(n)}\right) \prod_{i < j} (d_{ij}^{(n)})^2 \end{aligned}$$

thanks to (3.5).

Now let  $h : [0, 1] \rightarrow [0, 2\pi]$  be the inverse function of  $\theta \in [0, 2\pi] \mapsto \frac{1}{2\pi} \int_0^\theta f(e^{it}) dt$ . Since  $a_j^{(n)} = h((j - \frac{1}{2})/n)$  and  $b_j^{(n)} = h(j/n)$ , we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2}{n^2} \sum_{i < j} \log(d_{ij}^{(n)})^2 &= 2 \iint_{0 \leq u < v \leq 1} \log |e^{ih(u)} - e^{ih(v)}| du dv \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(e^{is}) f(e^{it}) \log |e^{is} - e^{it}| ds dt \\ &= \iint_{\mathbb{T}^2} f(x) f(y) \log |x - y| dx dy \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \xi_i^{(n)} = \frac{1}{2\pi} \int_0^{2\pi} Q(e^{is}) f(e^{is}) ds = \int_{\mathbb{T}} Q(x) d\mu(x).$$

Therefore

$$0 \geq \limsup_{n \rightarrow \infty} \frac{1}{n^2} \log P_n(G) \geq - \iint F(x, y) \, d\mu(x) \, d\mu(y) - \liminf_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n$$

and

$$\liminf_{n \rightarrow \infty} \frac{1}{n^2} \log P_n(G) \geq - \iint F(x, y) \, d\mu(x) \, d\mu(y) - \limsup_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n,$$

as desired.  $\square$

*End of proof of Theorem.* – By (3.1) and (3.3) we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n \leq - \inf_{\mu} \iint F(x, y) \, d\mu(x) \, d\mu(y) \leq \liminf_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n.$$

Since the functional  $\mu \mapsto \iint F(x, y) \, d\mu(x) \, d\mu(y)$  is lower semicontinuous and takes the finite infimum on the compact space  $\mathcal{M}(\mathbb{T})$ , the finite limit  $B = \lim_{n \rightarrow \infty} n^{-2} \log Z_n$  exists and (1.2) gives a proper rate function  $I$ . Recall that proof was reduced above to verification of (2.3) and these two conditions were proven as (3.2) and (3.4). Also the uniqueness of  $\mu_0 \in \mathcal{M}(\mathbb{T})$  satisfying  $I(\mu_0) = 0$  follows from the strict convexity of  $I$ .  $\square$

#### 4. DISCUSSIONS

First we give two examples of our large deviation result for unitary random matrices.

(1) For  $\alpha \in \mathbb{C}$ ,  $|\alpha| < 1$ , let  $Q(\zeta) = \log |\zeta - \alpha|^2$  ( $\zeta \in \mathbb{T}$ ). Then the probability measure  $\nu_n$  on  $\mathcal{U}(n)$  is given by

$$\nu_n = \frac{1}{Z_n} \frac{d\gamma_n(U)}{\det |U - \alpha I|^{2n}}.$$

Hence

$$\bar{\nu}_n = \frac{1}{Z_n} \frac{\prod_{i < j} |\zeta_i - \zeta_j|^2}{\prod_{i=1}^n |\zeta_i - \alpha|^{2n}} \, d\zeta_1 \cdots d\zeta_n.$$

If  $P_n$  is the empirical eigenvalue distribution of the associated unitary random matrix, then our theorem says that  $(P_n)$  satisfies the large deviation principle with rate function

$$I(\mu) = - \iint_{\mathbb{T}^2} \log |\zeta - \eta| \, d\mu(\zeta) \, d\mu(\eta) + \int \log |\zeta - \alpha|^2 \, d\mu(\zeta) - \log(1 - |\alpha|^2)$$

for  $\mu \in \mathcal{M}(\mathbb{T})$ , and the Poisson kernel measure  $p_\alpha = (1 - |\alpha|^2) |\zeta - \alpha|^{-2} \, d\zeta$  is a unique minimizer of  $I$ . We have

$$\iint_{\mathbb{T}^2} \log |\zeta - \eta| \, dp_\alpha(\zeta) \, dp_\alpha(\eta) = \log(1 - |\alpha|^2),$$

and also

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \log \int_{\mathcal{U}(n)} \frac{d\gamma_n(U)}{\det |U - \alpha I|^{2n}} = -\log(1 - |\alpha|^2).$$

It does not seem easy to directly compute the above asymptotic limit of integrals. In particular when  $\alpha = 0$  (hence  $Q = 0$ ), the eigenvalue distribution of a unitary random matrix distributed according to the Haar measure on  $\mathcal{U}(n)$  (called a standard unitary random matrix) converges to the Haar measure on  $\mathbb{T}$ .

(2) Let  $\lambda > 0$  and set another function  $Q(\zeta) = -\frac{2}{\lambda} \operatorname{Re} \zeta$  ( $\zeta \in \mathbb{T}$ ). Then  $\nu_n$  on  $\mathcal{U}(n)$  is

$$\nu_n = \frac{1}{Z_n} \exp\left(\frac{n}{\lambda} \operatorname{Tr}(U + U^*)\right) d\gamma_n(U),$$

and  $\bar{\nu}_n$  on  $\mathbb{T}^n$  is

$$\bar{\nu}_n = \frac{1}{Z_n} \exp\left(\frac{2n}{\lambda} \sum_{i=1}^n \cos \theta_i\right) \prod_{i < j} |e^{i\theta_i} - e^{i\theta_j}|^2 \, d\theta_1 \cdots d\theta_n.$$

By our theorem the associated sequence of empirical eigenvalue distributions satisfies the large deviation principle with rate function

$$I(\mu) = - \iint_{\mathbb{T}^2} \log |\zeta - \eta| \, d\mu(\zeta) \, d\mu(\eta) - \frac{2}{\lambda} \int \operatorname{Re} \zeta \, d\mu(\zeta) + B$$

for  $\mu \in \mathcal{M}(\mathbb{T})$ , where

$$B = \begin{cases} \frac{1}{\lambda^2} & \text{if } \lambda \geq 2, \\ \frac{1}{2} \log \frac{\lambda}{2} + \frac{2}{\lambda} - \frac{3}{4} & \text{if } 0 < \lambda < 2. \end{cases}$$

In [3], Gross and Witten calculated the above value of  $B$  and showed that a unique minimizer  $\rho_\lambda$  of  $I$  is given by

$$\rho_\lambda = \begin{cases} \frac{1}{2\pi} (1 + \frac{2}{\lambda} \cos \theta) \, d\theta & \text{if } \lambda \geq 2, \\ \frac{2}{\pi\lambda} \cos \frac{\theta}{2} \sqrt{\frac{\lambda}{2}} - \sin^2 \frac{\theta}{2} \chi_{[-2 \arcsin \sqrt{\lambda/2}, 2 \arcsin \sqrt{\lambda/2}]}(\theta) \, d\theta & \text{if } 0 < \lambda < 2. \end{cases}$$

Incidentally, we have

$$\iint_{\mathbb{T}^2} \log |\zeta - \eta| \, d\rho_\lambda(\zeta) \, d\rho_\lambda(\eta) = \begin{cases} -\frac{1}{\lambda^2} & \text{if } \lambda \geq 2, \\ \frac{1}{2} \log \frac{\lambda}{2} - \frac{1}{4} & \text{if } 0 < \lambda < 2, \end{cases}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \log \int_{\mathcal{U}(n)} \exp\left(\frac{n}{\lambda} \operatorname{Tr}(U + U^*)\right) \, d\gamma_n(U) = B.$$

More details about the minimization of the rate function in the two examples are found also in [6].

The next comment is about the relation of this work to Voiculescu’s entropy of noncommutative variables. Our result may be considered from the viewpoint of free entropy for unitary random variables. Let  $(\mathcal{M}, \tau)$  be a  $W^*$ -probability space consisting of a von Neumann algebra  $\mathcal{M}$  and a faithful normal tracial state  $\tau$ , [14]. Let  $U$  be a unitary element in  $\mathcal{M}$  and  $\mu \in \mathcal{M}(\mathbb{T})$  be the distribution of  $U$  with respect to  $\tau$ . For  $n, m \in \mathbb{N}$  and  $\varepsilon > 0$  we define

$$\Gamma_n(U; m, \varepsilon) = \{V \in \mathcal{U}(n): |\tau_n(V^k) - \tau(U^k)| < \varepsilon \text{ for } |k| \leq m\},$$

where  $\tau_n$  denotes the normalized trace on  $n \times n$  matrices. In case of  $Q = 0$ , since  $\gamma_n(\Gamma_n(U; m, \varepsilon)) = P_n(G(\mu; m, \varepsilon))$  by (2.2), equality (2.1) means that

$$\iint_{\mathbb{T}^2} \log |\zeta - \eta| \, d\mu(\zeta) \, d\mu(\eta) = \lim_{\substack{m \rightarrow \infty \\ \varepsilon \rightarrow 0}} \left\{ \limsup_{n \rightarrow \infty} \frac{1}{n^2} \log \gamma_n(\Gamma_n(U; m, \varepsilon)) \right\}. \tag{4.1}$$

The above right-hand side can serve as the definition of the free entropy of  $U$ .

Any operator  $X$  can be written in the form  $A + iB$  with selfadjoint  $A$  and  $B$ . Voiculescu's entropy of the non-selfadjoint  $X$  would be the entropy of the pair  $(A, B)$ , i.e.,  $\chi(A, B)$  in his notation. For a normal operator (particularly a unitary)  $X$ ,  $A$  commutes with  $B$  and this entropy is always  $-\infty$  [13]. Roughly speaking, our entropy (4.1) for unitary random variables takes finite values because we condition Voiculescu's entropy with respect to the conditions  $AB = BA$  and  $A^2 + B^2 = I$ .

Furthermore, we can naturally extend the free entropy of unitaries to the case of multi-unitaries  $(U_1, \dots, U_N)$  as Voiculescu's free entropy  $\chi(X_1, \dots, X_N)$  in the selfadjoint case was introduced in [12]. The free entropy of multi-unitaries is not directly related to probability theory, hence it is not discussed here, see [7]. We confine ourselves to pointing out that the unitary (or conditioned) version of multiple free entropy has properties similar to those in [12].

### ACKNOWLEDGEMENT

The authors would like to thank Professors I. Csiszár and M. Izumi for their useful suggestions and fruitful conversations on the subject of the paper. D. Petz acknowledges financial supports from the Canon Foundation, OTKA T016924 and AKP 96/2-6782.

### REFERENCES

- [1] G. BEN AROUS and A. GUIONNET, Large deviation for Wigner's law and Voiculescu's non commutative entropy, *Probab. Theory Related Fields* 108 (1997) 517–542.
- [2] C. BERG, J.P.R. CHRISTENSEN and P. RESSEL, *Harmonic Analysis on Semigroups. Theory of Positive Definite and Related Functions*, Springer, New York, 1984.
- [3] A. DEMBO and O. ZEITOUNI, *Large Deviations Techniques and Applications*, Jones and Bartlett, Boston, 1993.
- [4] D.J. GROSS and E. WITTEN, Possible third-order phase transition in the large- $N$  lattice gauge theory, *Phys. Rev. D* 21 (1980) 446–453.
- [5] F. HIAI and D. PETZ, Maximizing free entropy, *Acta Math. Hungar.* 80 (1998) 325–346.
- [6] F. HIAI and D. PETZ, *The Semicircle Law, Free Random Variables and Entropy*, to be published.
- [7] F. HIAI and D. PETZ, Properties of free entropy related to polar decomposition, *Comm. Math. Phys.* 202 (1999) 421–444.

- [8] P. KOOSIS, *Introduction to  $H_p$  Spaces*, Cambridge Univ. Press, Cambridge, 1980.
- [9] N.S. LANDKOF, *Foundations of Modern Potential Theory*, Springer, Berlin, 1972.  
(Translated from the Russian).
- [10] M.L. MEHTA, *Random Matrices*, Academic Press, Boston, 1991.
- [11] D. VOICULESCU, The analogues of entropy and of Fisher's information measure in free probability theory, I, *Comm. Math. Phys.* 155 (1993) 71–92.
- [12] D. VOICULESCU, The analogues of entropy and of Fisher's information measure in free probability theory, II, *Invent. Math.* 118 (1994) 411–440.
- [13] D. VOICULESCU, The analogues of entropy and of Fisher's information measure in free probability theory III: The absence of Cartan subalgebras, *Geom. Funct. Anal.* 6 (1996) 172–199.
- [14] D.V. VOICULESCU, K.J. DYKEMA and A. NICA, *Free Random Variables*, CRM Monograph Ser., Vol. 1, Amer. Math. Soc., 1992.