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Stability and other limit laws for exit times of random walks from a strip or a halfplane

by

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ABSTRACT. – We show that the passage time, $T^*(r)$, of a random walk S_n above a horizontal boundary at r ($r \geqslant 0$) is *stable* (in probability) in the sense that $T^*(r)/C(r) \xrightarrow{P} 1$ as $r \to \infty$ for a deterministic function $C(r) > 0$, if and only if the random walk is *relatively stable* in the sense that $S_n/B_n \xrightarrow{P} 1$ as $n \to \infty$ for a deterministic sequence $B_n > 0$. The stability of a passage time is an important ingredient in some proofs in sequential analysis, where it arises during applications of Anscombe's Theorem. We also prove a counterpart for the almost sure stability of $T^*(r)$, which we show is equivalent to $E|X| < \infty$, $EX > 0$. Similarly, counterparts for the exit of the random walk from the strip $\{|y| \leqslant r\}$ are proved. The conditions are further related to the relative stability of the maximal sum and the maximum modulus of the sums. Another result shows that the exit position of the random walk outside the boundaries at $\pm r$ drifts to ∞ as $r \to \infty$ if and only if the random walk drifts to ∞.

Key words: Random walks, first-passage times, exit times, relative stability, boundary crossing probabilities, Anscombe's Theorem, elementary renewal theorem

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RÉSUMÉ. – Nous montrons que le temps de passage, $T^*(r)$, d'une marche aléatoire S_n au-dessus d'une frontière r ($r \geqslant 0$) est stable (en probabilité) au sens où $T^*(r)/C(r) \xrightarrow{P} 1$ quand $r \to \infty$ pour une fonction déterministe $C(r) > 0$, si et seulement si la marche aléatoire est relativement stable au sens où $S_n/B_n \xrightarrow{P} 1$ quand $n \to \infty$ pour une suite déterministe $B_n > 0$. La stabilité d'un temps de passage est un ingrédient important dans quelques preuves d'analyse séquentielle, où elle intervient dans des applications du théorème d'Anscombe. Nous démontrons aussi un analogue pour la stabilité presque sûre de $T^*(r)$, que nous montrons être équivalente à $E|X| < \infty$, $EX > 0$. De même nous démontrons des analogues pour la sortie de la marche aléatoire hors de la bande $|y| \leqslant r$. Les conditions sont liées à la stabilité relative du maximum des sommes partielles et des sommes partielles des modules. Un autre résultat montre que la position de sortie de la marche aléatoire hors des frontières en $\pm r$ tend vers l'infini quand $r \to \infty$ si et seulement si la marche aléatoire tend vers l'infini.

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1. INTRODUCTION

The time taken for a random walk to exit a deterministic region is an important consideration in the design and analysis of a sequential trial defined in terms of that region, because the exit time is, or is closely related to, the (random) sample size at the conclusion of the trial. Thus certain aspects of its distribution assume great prominence in the theory of sequential analysis (and in other applications), and one aspect of particular importance is what we will term the "stability" of the passage time.

To formulate this concept, we first specify the two kinds of passage time to be studied here. Suppose our random walk is

$$S_n = X_1 + X_2 + \cdots + X_n, \tag{1.1}$$

composed of increments X_i which are independent with the same distribution as a random variable X whose distribution function is F.

We assume throughout that F is not degenerate at 0. Take $S_0 = 0$. Let

$$T^*(r) = \min\{n \geqslant 1\colon\ S_n > r\}, \quad r \geqslant 0 \tag{1.2}$$

(with $T^*(r) = \infty$ if $S_n \leqslant r$ for all n) be the first passage time of S_n strictly above the horizontal boundary at r. Elementary properties are that $T^*(r) < \infty$ a.s. for some (hence all) $r \geqslant 0$ if and only if $\limsup_{n\to\infty} S_n = \infty$ a.s., and $E(T^*(r)) < \infty$ for some (hence all) $r \geqslant 0$ if and only if $S_n \to \infty$ a.s. as $n \to \infty$. (See, for example, Chow and Teicher [4, pp. 145–146] and Kesten and Maller [15].)

Assuming $T^*(r) < \infty$ a.s., we say that $T^*(r)$ is *stable* as $r \to \infty$ if there is a deterministic function $C(r) > 0$ such that

$$\frac{T^*(r)}{C(r)} \xrightarrow{P} 1 \quad \text{as } r \to \infty. \tag{1.3}$$

The terminology here is borrowed from random walk theory, where it is said that S_n is *positively relatively stable* (written, $S_n \in PRS$) if there is a deterministic sequence $B_n > 0$ for which

$$\frac{S_n}{B_n} \xrightarrow{P} +1 \quad \text{as } n \to \infty. \tag{1.4}$$

We say that S_n is *negatively relatively stable*, $S_n \in NRS$, if (1.4) holds with -1 in the right hand side instead of $+1$. If $S_n \in PRS$ or $S_n \in NRS$, we say that S_n is *relatively stable*, $S_n \in RS$. The use of the terminology "stable" for (1.3) is a good choice, as it turns out, since one of our main results (Theorem 2.1 below) will show that (1.3) and (1.4) are equivalent.

We will also consider exit times from the horizontal strip with boundaries at $\pm r$, $r > 0$; thus:

$$T(r) = \min\{n \geqslant 1\colon\ |S_n| > r\} \tag{1.5}$$

(with $T(r) = \infty$ if $|S_n| \leqslant r$ for all $n \geqslant 1$). It is always the case (when the X_i are not degenerate at 0) that $T(r) < \infty$ a.s., and in fact $E(T(r)) < \infty$ for all $r > 0$ (by "Stein's Lemma", see, e.g., Woodroofe [22, p. 29]). Again we say that $T(r)$ is stable if

$$\frac{T(r)}{C(r)} \xrightarrow{P} 1 \quad \text{as } r \to \infty, \tag{1.6}$$

and we will show that this occurs if and only if $S_n \in RS$.

The stability, in the above sense, of $T^*(r)$ and of $T(r)$ (and of other passage times out of more general boundaries) is especially useful, for example, in "Anscombe's Theorem", which we discuss further below. Also applicable is the concept of *almost sure* (*a.s.*) *stability* of $T^*(r)$ or of $T(r)$. $T^*(r)$ and $T(r)$ are *a.s. stable* if $T^*(r)/C(r) \to 1$ a.s. or $T(r)/C(r) \to 1$ a.s., respectively, for some deterministic function $C(r) > 0$. We will show that these occur if and only if $0 < EX \leqslant E|X| < \infty$ or $0 < |EX| \leqslant E|X| < \infty$, respectively. Since we can then take $C(r) = r/|EX|$, as we show below, these results are closely related to the elementary renewal theorem. Other elementary properties of $T^*(r)$ and $T(r)$ are that

$$\{T^*(r) > n\} = \Big\{\max_{1 \leqslant j \leqslant n} S_j \leqslant r\Big\}$$

and

$$\{T(r) > n\} = \Big\{\max_{1 \leqslant j \leqslant n} |S_j| \leqslant r\Big\}$$

(for $r > 0$), and these suggest that the stability of $T^*(r)$ or of $T(r)$ (in probability) might be equivalent to the stability of the maximum or the maximum in modulus of the random walk, i.e., to

$$\frac{S_n^*}{B_n} := \frac{\max_{1 \leqslant j \leqslant n} S_j}{B_n} \xrightarrow{P} 1 \quad (n \to \infty) \tag{1.7}$$

or to

$$\frac{\overline{S}_n}{B_n} := \frac{\max_{1 \leqslant j \leqslant n} |S_j|}{B_n} \xrightarrow{P} 1 \quad (n \to \infty). \tag{1.8}$$

This is so, as we demonstrate, and similarly for the a.s. versions. These results are stated in Section 2 and proved in Section 4.

We go on in Section 3 to state other weak or strong convergence or divergence results for $T(r)$. These are related, as expected, to corresponding divergence or convergence properties of $\overline{S}_n$ which are relatively easy to derive. S_n^* is harder to handle, however, and we conclude with an example demonstrating that S_n^* is much less predictable in this respect than $\overline{S}_n$. These results are proved in Section 5.

The following functionals of the tails of F will be needed. For $x > 0$ define

$$H(x) = P\{|X| > x\} = 1 - F(x) + F(-x-); \tag{1.9}$$

$$\nu(x) = E\big(XI(|X| \leqslant x)\big) = \int\limits_{[-x,x]} y\,dF(y); \tag{1.10}$$

$$\begin{aligned} A(x) &= \int\limits_0^x \big(1 - F(y) - F(-y)\big)\,dy \\ &= \nu(x) + x\big[1 - F(x) - F(-x-)\big]; \end{aligned} \tag{1.11}$$

$$V(x) = E\big(X^2 I(|X| \leqslant x)\big) = \int\limits_{[-x,x]} y^2\,dF(y); \tag{1.12}$$

$$U(x) = 2\int\limits_0^x yH(y)\,dy = V(x) + x^2 H(x). \tag{1.13}$$

It will be useful to summarize here some properties concerning the relative stability of S_n; for reference, see, e.g., Rogozin [20], Maller [18], Kesten and Maller [12, especially Theorem 2.1], Kesten and Maller [14, p. 450]. First assume that

$$H(x) = P\big(|X| > x\big) > 0 \quad \text{for all } x. \tag{1.14}$$

We then have that $S_n \in PRS$, i.e., $S_n/B_n \xrightarrow{P} 1$ for a deterministic sequence $B_n > 0$, if and only if the function $A(x)$ defined in (1.11) is strictly positive for all x large enough, $x \geqslant x_0$, say, and satisfies

$$\lim_{x\to\infty} \frac{A(x)}{xH(x)} = \infty. \tag{1.15}$$

We need to remark here that this is proven in Kesten and Maller [12] only under the extra assumption that B_n is increasing (in the weak sense, that is, nondecreasing). However, this assumption is superfluous, because we may always replace B_n by $\max_{k\leqslant n} B_k$. To show that this is permissible, it suffices to show that $S_n/B_n \xrightarrow{P} 1$, or even only

$$\frac{|S_n|}{B_n} \xrightarrow{P} 1, \tag{1.16}$$

implies

$$\lim_{n\to\infty} \frac{\max_{k\leqslant n} B_k}{B_n} = 1. \tag{1.17}$$

Clearly $\max_{k \leqslant n} B_k \geqslant B_n$. Therefore, (1.17) can fail only if there exist sequences $\{m_k\}$, $\{n_k\}$ with $m_k < n_k$, and a constant $b > 1$ so that

$$\frac{B_{m_k}}{B_{n_k}} \to b \in (1, \infty] \quad (k \to \infty),$$

$$\frac{S_{n_k}}{B_{n_k}} \xrightarrow{P} 1 \quad \text{or} \quad \frac{S_{n_k}}{B_{n_k}} \xrightarrow{P} -1 \quad (k \to \infty)$$

and

$$\frac{S_{m_k}}{B_{m_k}} \xrightarrow{P} 1 \quad \text{or} \quad \frac{S_{m_k}}{B_{m_k}} \xrightarrow{P} -1 \quad (k \to \infty).$$

In Lemma 4.3 we shall prove (without using any monotonicity properties of B_n) that these relations imply

$$\frac{\overline{S}_{n_k}}{B_{n_k}} \xrightarrow{P} 1 \quad \text{and} \quad \frac{\overline{S}_{m_k}}{B_{m_k}} \xrightarrow{P} 1 \quad (k \to \infty).$$

On the other hand, $m_k < n_k$ implies $0 \leqslant \overline{S}_{m_k} \leqslant \overline{S}_{n_k}$, so that

$$\frac{\overline{S}_{m_k}}{B_{m_k}} \leqslant \frac{\overline{S}_{n_k}}{B_{m_k}} \xrightarrow{P} \frac{1}{b} < 1.$$

The last two relations clearly contradict each other, so that (1.16) does imply (1.17). We may therefore assume that B_n is increasing and (1.15) is the correct analytic condition for $S_n \in PRS$.

If $A(x) > 0$ for $x \geqslant x_0$, then there is some $x_1 \geqslant x_0$ so that the function

$$D(x) := \sup\left\{ y \geqslant x_0 \colon \frac{A(y)}{y} \geqslant \frac{1}{x} \right\} \tag{1.18}$$

is strictly positive and finite and satisfies

$$D(x) = xA\big(D(x)\big) \tag{1.19}$$

for all $x \geqslant x_1$, and in addition is easily seen to be strictly increasing on $x \geqslant x_1$ (by the continuity of $y \mapsto A(y)/y$). Thus we can define the inverse function

$$D^{-1}(y) = \sup\{x \colon D(x) \leqslant y\} = \inf\{x \colon D(x) > y\}. \tag{1.20}$$

In addition, if (1.14) holds, then $A(x)$ is slowly varying as $x \to \infty$ (Maller [18]). As a consequence $D(x)$ and $D^{-1}(x)$ are both regularly varying with index 1 as $x \to \infty$ (see Bingham, Goldie and Teugels [2, Theorem 1.5.12]). Also, when (1.15) holds, it is easy to check that $A(y)/y$ is strictly decreasing for y large enough, so $D^{-1}(y)$ is continuous and strictly increasing, and

$$D^{-1}(y) = \frac{y}{A(y)}, \tag{1.21}$$

for y large enough. Finally, we can take $B_n = D(n)$ in (1.4) (Maller [18]). For *negative* relative stability, i.e., when $S_n/B_n \xrightarrow{P} -1$ (where $B_n > 0$), the above remains true with $A(x)$ replaced by $-A(x)$. Relative stability of S_n, i.e., $S_n/B_n \xrightarrow{P} \pm 1$ as $n \to \infty$, is equivalent to

$$\lim_{x\to\infty} \frac{|A(x)|}{xH(x)} = \infty. \tag{1.22}$$

As explained in Kesten and Maller [12, p. 1806], (1.22) implies that $A(x) > 0$ for all large x or $A(x) < 0$ for all large x, corresponding to PRS or NRS. We will note in Theorem 2.3 below that $S_n \in RS$ if and only if $|S_n|/B_n \xrightarrow{P} 1$ as $n \to \infty$, a nice counterpart to (1.22).

If (1.14) fails, then $H(x_0) = 0$ for some $x_0 > 0$ (but not $H(x) = 0$ for all $x > 0$). Then the X_i are bounded, and $EX^2 < \infty$. Then (by the weak law of large numbers) $S_n \in PRS$ (NRS) if $EX > 0$ ($EX < 0$) and then in fact $S_n/(n|EX|) \to \pm 1$ a.s. Conversely, if $S_n/B_n \xrightarrow{P} \pm 1$ for some $B_n > 0$, and $EX^2 < \infty$, then $EX \neq 0$. It is possible to have $S_n \in RS$ and $EX = 0$, but we must have $EX^2 = \infty$ then, as discussed in Remark 5, p. 1812, of Kesten and Maller [12] and Maller [18, p. 143]. We will make the convention that (1.15) and (1.22) are understood to hold when $H(x_0) = 0$ for some x_0 and $EX > 0$, respectively $EX \neq 0$, so that they characterise PRS and RS (and similarly for NRS), irrespective of the validity of (1.14). Note that in the case $H(x_0) = 0$, $EX \neq 0$, we have $A(x) = EX \neq 0$ for $x \geqslant x_0$, and the regular variation of $|A(x)|/x$, $D(x)$ and $D^{-1}(x)$ as $x \to \infty$ is trivial in this case.

The *almost sure relative stability* of S_n, i.e., $S_n/B_n \to \pm 1$ a.s., can only occur if $E|X| < \infty$ and $EX \neq 0$ (see Chow and Robbins [3], Maller [18] or Kesten and Maller [12, Theorem 2.2]), and this idea transfers over to the a.s. stability of $T^*(r)$ and of $T(r)$, also.

2. STABILITY RESULTS FOR $T^*(r)$ AND $T(r)$

THEOREM 2.1. – *The following are equivalent:*
There exists a deterministic function $C(r) > 0$ such that

$$T^*(r) < \infty \text{ a.s.} \quad \text{and} \quad \frac{T^*(r)}{C(r)} \xrightarrow{P} 1 \quad \text{as } r \to \infty; \tag{2.1}$$

There exists a nonrandom sequence $B_n > 0$ such that

$$\frac{\max_{1 \leqslant j \leqslant n} S_j}{B_n} \xrightarrow{P} 1 \quad \text{as } n \to \infty; \tag{2.2}$$

There exists a nonrandom sequence $B_n > 0$ such that

$$\frac{S_n}{B_n} \xrightarrow{P} 1 \quad \text{as } n \to \infty. \tag{2.3}$$

If any of (2.1)–(2.3) *hold, we have $A(r) > 0$ for r large enough, and we may choose $C(r) = r/A(r)$. This $C(r)$ is regularly varying with index* 1 *as $r \to \infty$. Furthermore, we may choose $B_n = D(n)$ in* (2.2) *and* (2.3), *where $D(\cdot)$ is defined in* (1.18). *Then B_n is also regularly varying with index* 1 *as $n \to \infty$.*

THEOREM 2.2. – *The following are equivalent:*
There exists a deterministic function $C(r) > 0$ such that

$$T^*(r) < \infty \text{ a.s.} \quad \text{and} \quad \frac{T^*(r)}{C(r)} \to 1 \text{ a.s.} \quad \text{as } r \to \infty; \tag{2.4}$$

There exists a nonrandom sequence $B_n > 0$ such that

$$\frac{\max_{1 \leqslant j \leqslant n} S_j}{B_n} \to 1 \text{ a.s.} \quad \text{as } n \to \infty; \tag{2.5}$$

$$0 < EX \leqslant E|X| < \infty. \tag{2.6}$$

If (2.4)–(2.6) *hold we may take $C(r) = r/EX$ and $B_n = nEX$.*

Remarks. –
(i) Heyde [10, Theorem 7] showed that (2.4) (with $C(r) = r/EX$) follows from (2.6), so Theorem 2.2 provides a converse to this, as well as the extra equivalence in (2.5). When $X_i \geqslant 0$ a.s., Theorem 2.2

becomes much simpler, since $\max_{1\leqslant j\leqslant n} S_j$ equals S_n then, and the equivalence of (2.5) and (2.6) follows from the strong law of large numbers. Gut et al. [9] showed the equivalence of (2.4)–(2.6) in this special case.

(ii) There are of course analogues of Theorems 2.1 and 2.2 for passage times of S_n below $-r$, $r > 0$, and the negative relative stability of S_n, which can easily be formulated—we omit the details of these.

THEOREM 2.3. – *The following are equivalent:*
There exists a deterministic function $C(r) > 0$ *such that*

$$\frac{T(r)}{C(r)} \xrightarrow{P} 1 \quad \text{as } r \to \infty; \tag{2.7}$$

There exists a nonrandom sequence $B_n > 0$ *such that*

$$\frac{\max_{1\leqslant j\leqslant n} |S_j|}{B_n} \xrightarrow{P} 1 \quad \text{as } n \to \infty; \tag{2.8}$$

There exists a nonrandom sequence $B_n > 0$ *such that*

$$\frac{|S_n|}{B_n} \xrightarrow{P} 1, \quad \text{or, equivalently,} \quad \frac{S_n}{B_n} \xrightarrow{P} \pm 1, \quad \text{as } n \to \infty. \tag{2.9}$$

If (2.7)–(2.9) *hold, we may take* $C(r) = r/|A(r)|$, *which is strictly positive for* r *large enough, and in* (2.8), *as well as* (2.9), *we may take* $B_n = D(n)$, *where* $D(\cdot)$ *is defined in* (1.18) *with* $A(\cdot)$ *replaced by* $|A(\cdot)|$.

THEOREM 2.4. – *The following are equivalent:*
There exists a deterministic function $C(r) > 0$ *such that*

$$\frac{T(r)}{C(r)} \to 1 \text{ a.s.} \quad \text{as } r \to \infty; \tag{2.10}$$

There exists a nonrandom sequence $B_n > 0$ *such that*

$$\frac{\max_{1\leqslant j\leqslant n} |S_j|}{B_n} \to 1 \text{ a.s.} \quad \text{as } n \to \infty; \tag{2.11}$$

$$0 < |EX| \leqslant E|X| < \infty. \tag{2.12}$$

If (2.10)–(2.12) *hold we may take* $C(r) = r/|EX|$ *and* $B_n = n|EX|$.

Remark. –

(iii) Suppose $EX^2 < \infty$ and $EX = 0$. A version of Anscombe's Theorem (Theorem 1, p. 322 of Chow and Teicher [4]) tells us that $S_{t(r)}/\sqrt{t(r)} \Rightarrow N(0,1)$, if $t(r)$ are integer-valued random variables with $t(r)/C(r) \xrightarrow{P} 1$ as $r \to \infty$ for a deterministic sequence $C(r) > 0$. This cannot be true for $t(r) = T^*(r)$, of course, because $S_{T^*(r)} > r$ cannot be asymptotically normal, and Theorem 2.1 explains why Anscombe's Theorem is not applicable here: $T^*(r)/C(r) \xrightarrow{P} 1$ as $r \to \infty$ can only hold if $S_n \in PRS$, and PRS cannot apply when $EX^2 < \infty$ and $EX = 0$, as discussed in Section 1. The same argument explains why $S_{T(r)}/\sqrt{T(r)} \Rightarrow N(0,1)$ cannot be expected (and in fact is not true) when $EX^2 < \infty$, $EX = 0$.

3. OTHER WEAK AND STRONG LAWS FOR $T(r)$ AND $S_{T(r)}$

In this section we consider conditions under which $T(r)$ becomes small, or large, with respect to a deterministic norming sequence, rather than remaining comparable, as is the case with stability. Most of our results concern $T(r)$, which is easier to handle than $T^*(r)$ in this context, due in part to some inequalities of Pruitt [19] for the distribution of $T(r)$, which are given at the beginning of Section 5. It is natural to consider the behaviour of the position $S_{T(r)}$, too, for these results. The one-sided passage time is much more difficult to analyse in this way, and we restrict ourselves to some limited results and a counterexample at the end of this section.

Throughout the paper $f(n)$ will denote a strictly positive, increasing, deterministic norming sequence, with $f(n) \to \infty$ as $n \to \infty$, not necessarily connected with the $D(n)$ or B_n of the previous section. The sequence $f(n)$ will be the norming for $\max_{1 \leqslant j \leqslant n} |S_j|$, in our first two results, and

$$f^{-1}(r) = \max\{n \geqslant 1\colon f(n) \leqslant r\}, \quad r > f(1), \tag{3.1}$$

its inverse function, will be the appropriate norming for $T(r)$. Since $f(n) \to \infty$ as $n \to \infty$, $f^{-1}(r)$ is well defined (finite) for all $r > f(1)$. Note that, according to (3.1), $n = f^{-1}(r)$ is an integer for $r > 0$, with $f(n) \leqslant r$, that is, $f(f^{-1}(r)) \leqslant r$, but $f(f^{-1}(r) + 1) > r$. Also $f^{-1}(f(n)) \geqslant n$ and $f^{-1}(f(n) - \varepsilon) \leqslant n$ for $\varepsilon > 0$.

Our first theorem gives necessary and sufficient conditions for $T(r)/f^{-1}(r) \xrightarrow{P} 0$ as $r \to \infty$, which is a kind of "degenerate convergence criterion" for $T(r)$. This turns out to be equivalent to the divergence of $|S_{T(r)}|/f(T(r))$ to ∞, in probability, as $r \to \infty$. As a corollary to this, we obtain necessary and sufficient conditions for $S_{T(r)}/f(T(r)) \xrightarrow{P} \infty$. Theorem 3.2 gives a subsequential version of Theorem 3.1, Theorem 3.3 gives a version of Theorem 3.1 for divergence to ∞ of $T(r)/f^{-1}(r)$, while Theorem 3.4 gives equivalences for the almost sure divergence to ∞ of $S_{T(r)}$ and $S_{T(r)}/T(r)$. Recall that $f(n) > 0$ and $f(n) \uparrow \infty$ as $n \to \infty$, throughout.

THEOREM 3.1. – *Let $f(n)$ be such that*

$$\frac{f(2n)}{f(n)} \leqslant \Lambda, \quad n \geqslant 1, \tag{3.2}$$

for some constant Λ. Then the following statements are equivalent:

$$\frac{T(r)}{f^{-1}(r)} \xrightarrow{P} 0 \quad (r \to \infty); \tag{3.3}$$

$$\frac{f(T(r))}{r} \xrightarrow{P} 0 \quad (r \to \infty); \tag{3.4}$$

$$\frac{\max_{1 \leqslant j \leqslant n} |S_j|}{f(n)} \xrightarrow{P} \infty \quad (n \to \infty); \tag{3.5}$$

$$\frac{|S_{T(r)}|}{f(T(r))} \xrightarrow{P} \infty \quad (r \to \infty); \tag{3.6}$$

$$f^{-1}(x)\left(\frac{|A(x)|}{x} + \frac{U(x)}{x^2}\right) \to \infty \quad (x \to \infty). \tag{3.7}$$

COROLLARY TO THEOREM 3.1. – *If $f(n)$ satisfies* (3.2), *then $S_n/f(n) \xrightarrow{P} \infty$ if and only if $S_{T(r)}/f(T(r)) \xrightarrow{P} \infty$.*

Remarks. –

(iv) The assumption (3.2) is used only to show the equivalence of (3.4), (3.5) and (3.6). The corollary to Theorem 3.1 provides a full sequence version of Lemma 2 of Kesten and Maller [16]. A necessary and sufficient condition for $S_n/f(n) \xrightarrow{P} \infty$ is in Theorem 2.2 of Kesten and Maller [13].

(v) As another interesting corollary to Theorem 3.1, take $f(n)=\sqrt{n}$ to see that the following are equivalent:

$$\frac{T(r)}{r^2}\xrightarrow{P}0\quad(r\to\infty);\tag{3.8}$$

$$\frac{\max_{1\leqslant j\leqslant n}|S_j|}{\sqrt{n}}\xrightarrow{P}\infty\quad(n\to\infty);\tag{3.9}$$

$$\frac{|S_{T(r)}|}{\sqrt{T(r)}}\xrightarrow{P}\infty\quad(r\to\infty);\tag{3.10}$$

$$\lim_{x\to\infty}\big(x|A(x)|+U(x)\big)=\infty.\tag{3.11}$$

(3.11) is equivalent to:

$$EX^2=\infty\quad\text{or}\quad EX^2<\infty\quad\text{and}\quad EX\neq 0.\tag{3.12}$$

To these we can add the equivalence:

$$\frac{|S_n|}{\sqrt{n}}\xrightarrow{P}\infty\tag{3.13}$$

((3.13) obviously implies (3.9). Also, (3.13) follows from a concentration function inequality (Esseen [5, Theorem 3.1]) if $EX^2=\infty$, or from the weak law of large numbers if $EX^2<\infty$ and $EX\neq 0$.) This raises the question of how generally we may add the equivalence $|S_n|/f(n)\xrightarrow{P}\infty$ to those listed in Theorem 3.1. We do not know the answer to this even for $f(n)=n$.

THEOREM 3.2. – *Let $f(n)$ satisfy* (3.2). *Then the following statements are equivalent*:
There is a nonstochastic sequence $r_k\to\infty$ such that

$$\frac{T(r_k)}{f^{-1}(r_k)}\xrightarrow{P}0;\tag{3.14}$$

There is a nonstochastic sequence $r_k\to\infty$ such that

$$\frac{f(T(r_k))}{r_k}\xrightarrow{P}0;\tag{3.15}$$

There is a nonstochastic sequence $n_k\to\infty$ such that

$$\frac{\max_{1\leqslant j\leqslant n_k}|S_j|}{f(n_k)}\xrightarrow{P}\infty;\tag{3.16}$$

There is a nonstochastic sequence $r_k \to \infty$ *such that*

$$\frac{|S_{T(r_k)}|}{f(T(r_k))} \xrightarrow{P} \infty; \tag{3.17}$$

$$\limsup_{x\to\infty} f^{-1}(x)\left(\frac{|A(x)|}{x} + \frac{U(x)}{x^2}\right) = \infty. \tag{3.18}$$

THEOREM 3.3. – *The following are equivalent*:

$$\frac{T(r)}{f^{-1}(r)} \xrightarrow{P} \infty \quad (r \to \infty); \tag{3.19}$$

$$\frac{\max_{1\leqslant j\leqslant n}|S_j|}{f(n)} \xrightarrow{P} 0 \quad (n \to \infty); \tag{3.20}$$

$$\frac{S_n}{f(n)} \xrightarrow{P} 0 \quad (n \to \infty); \tag{3.21}$$

$$\lim_{x\to\infty} f^{-1}(x)\left(\frac{|A(x)|}{x} + \frac{U(x)}{x^2}\right) = 0. \tag{3.22}$$

As a corollary of the next theorem, we prove a "trichotomy" result for $S_{T(r)}$ which parallels Spitzer's trichotomy theorem for S_n.

THEOREM 3.4. – *We have*

$$\lim_{n\to\infty} S_n = \infty \quad \text{a.s.} \quad \text{if and only if} \quad \lim_{r\to\infty} S_{T(r)} = \infty \text{ a.s.} \tag{3.23}$$

and

$$\lim_{n\to\infty} \frac{S_n}{n} = \infty \text{ a.s.} \quad \text{if and only if} \quad \lim_{r\to\infty} \frac{S_{T(r)}}{T(r)} = \infty \text{ a.s.} \tag{3.24}$$

COROLLARY TO THEOREM 3.4. – *We have*

$$\limsup_{r\to\infty} S_{T(r)} > -\infty \text{ a.s.} \quad \text{if and only if} \quad \limsup_{n\to\infty} S_n = \infty \text{ a.s.}$$

and then

$$\limsup_{r\to\infty} S_{T(r)} = \infty \text{ a.s.}$$

Also,

$$\liminf_{r\to\infty} S_{T(r)} < \infty \text{ a.s.} \quad \text{if and only if} \quad \liminf_{n\to\infty} S_n = -\infty \text{ a.s.}$$

and then

$$\liminf_{r\to\infty} S_{T(r)} = -\infty \ a.s.$$

Consequently there are only three modes of behaviour for a random walk:

$$\lim_{r\to\infty} S_{T(r)} = \infty \ a.s. \tag{3.25}$$

or

$$-\infty = \liminf_{r\to\infty} S_{T(r)} < \limsup_{r\to\infty} S_{T(r)} = \infty \ a.s. \tag{3.26}$$

or

$$\lim_{r\to\infty} S_{T(r)} = -\infty \ a.s. \tag{3.27}$$

Remarks. –

(vi) Necessary and sufficient analytic conditions for (3.23)–(3.24) and for $\limsup_{n\to\infty} S_n = \infty$ a.s. are given in Kesten and Maller [15].

(vii) We showed in Kesten and Maller [16] that $S_{T(r)}$ often mimics S_n to some extent in its asymptotic behaviour. In particular, we found that $S_{T(r)} \xrightarrow{P} \infty$ (respectively, $S_{T(r)}/T(r) \xrightarrow{P} \infty$) as $r \to \infty$ if and only if $S_n \xrightarrow{P} \infty$ (respectively, $S_n/n \xrightarrow{P} \infty$) as $n \to \infty$. The equivalence of (3.10) and (3.13) is another instance, this time for "two-sided" divergence. On the other hand, it is obvious that this kind of correspondence does not always occur; for example, we always have $|S_{T(r)}| > r$, so $|S_{T(r)}| \to \infty$ a.s. as $r \to \infty$; but of course $|S_n| \to \infty$ a.s. if and only if the random walk S_n is transient.

We conclude with some results which show that the maximal sum $\max_{1\leqslant j\leqslant n} S_j$ can behave in a much less predictable way than the maximum modulus of the sums, $\max_{1\leqslant j\leqslant n} |S_j|$, with respect to divergence. We begin with a negative result.

Example 3.5. – There is a random walk S_n with

$$\frac{S_n}{n} \xrightarrow{P} -\infty, \quad \text{yet} \quad \frac{\max_{1\leqslant j\leqslant n} S_j}{n} \xrightarrow{P} \infty \quad \text{as } n \to \infty. \tag{3.28}$$

The next theorem shows that (3.28) is, in some sense, not atypical.

THEOREM 3.6. – *If* $P(S_n \leqslant 0) \to 1$ *and* $\max_{1\leqslant j\leqslant n} S_j/n \xrightarrow{P} \infty$, *then* $S_n/n \xrightarrow{P} -\infty$, *as* $n \to \infty$.

As a necessary condition for $\max_{1\leqslant j\leqslant n} S_j/n \xrightarrow{P} \infty$, we find:

THEOREM 3.7. – *If* $\max_{1\leqslant j\leqslant n} S_j/n \xrightarrow{P} \infty$ *then*

$$\max\left\{\frac{A(x)}{1+\sqrt{U(x)(1-F(x))}}, \sqrt{U(x)(1-F(x))}\right\} \to \infty$$
$$(x\to\infty). \tag{3.29}$$

4. PROOFS FOR SECTION 2

We will prove Theorems 2.1–2.4 via a series of lemmas. It will be convenient to use the abbreviations S_n^* and $\overline{S}_n$ introduced in (1.7)–(1.8) for the maximal sum and the maximum in modulus of the sums. Throughout the proofs we will understand that convergences related to S_n are as $n\to\infty$, and those related to $T^*(r)$, $T(r)$, $S_{T^*(r)}$ and $S_{T(r)}$ are as $r\to\infty$.

LEMMA 4.1. – *If n is such that $P\{S_n\geqslant 0\}\geqslant\delta$, then for any $m\geqslant 2$ and $0<\varepsilon<1$,*

$$P\left\{\min_{2n\leqslant j\leqslant mn} S_j\geqslant 0\right\}\geqslant \delta^{m+2}(1-\varepsilon)^m\varepsilon. \tag{4.1}$$

Proof of Lemma 4.1. – This is proved in a similar way as Lemma 1 in Bertoin and Doney [1]; we omit the details. □

LEMMA 4.2. – *Suppose $T^*(r)<\infty$ a.s. and $T^*(r)/C(r)\xrightarrow{P} 1$ as $r\to\infty$ for a nonstochastic function $C(r)>0$. Then $C(\cdot)$ may be chosen to be increasing and to satisfy, for $\lambda>1$,*

$$1<\liminf_{r\to\infty}\frac{C(\lambda r)}{C(r)}\leqslant\limsup_{r\to\infty}\frac{C(\lambda r)}{C(r)}<\infty. \tag{4.2}$$

Proof of Lemma 4.2. – Let $T^*(r)<\infty$ a.s. and $T^*(r)/C(r)\xrightarrow{P} 1$. We first prove that

$$\limsup_{r\to\infty}\frac{\sup_{s\leqslant r} C(s)}{C(r)}=1 \tag{4.3}$$

and hence

$$\frac{\sup_{s\leqslant r} C(s)}{C(r)}\to 1. \tag{4.4}$$

If (4.3) fails then $C(s_k)/C(r_k) \to a$ for some $r_k \geqslant s_k \to \infty$ and $a > 1$. But $r_k \geqslant s_k$ implies $T^*(r_k) \geqslant T^*(s_k)$, so

$$\frac{C(s_k)}{C(r_k)} \leqslant \frac{T^*(r_k)}{C(r_k)} \frac{C(s_k)}{T^*(s_k)} \xrightarrow{P} 1,$$

a contradiction. Thus (4.3) holds and we can replace $C(r)$ by $\sup_{s \leqslant r} C(s)$, if necessary, to obtain an increasing $C(r)$. We assume this has been done. Next, by the strong Markov property,

$$T^*(r+s) \leqslant T^*(r) + (T^*)'(s)$$

for $r, s > 0$, where $(T^*)'(s)$ is an independent copy of $T^*(s)$. Thus for $\varepsilon > 0$

$$P\{T^*(r+s) \leqslant (1+\varepsilon)(C(r)+C(s))\}$$
$$\geqslant P\{T^*(r) \leqslant (1+\varepsilon)C(r)\} P\{T^*(s) \leqslant (1+\varepsilon)C(s)\} \to 1$$

as $r \wedge s \to \infty$. Consequently, for $r \wedge s$ large enough,

$$C(r+s) \leqslant (1+2\varepsilon)(C(r)+C(s)),$$

and thus for $\lambda > 1$

$$\limsup_{r\to\infty} \frac{C(\lambda r)}{C(r)} \leqslant \lambda + 1. \tag{4.5}$$

This proves the right hand inequality in (4.2). In fact, (4.5) holds for all $\lambda > 0$ since $C(\cdot)$ is increasing.

Next we prove the left hand inequality in (4.2). Fix $\lambda > 1$. Note that

$$\liminf_{r\to\infty} \frac{C(\lambda r)}{C(r)} \geqslant 1, \tag{4.6}$$

because we took $C(\cdot)$ increasing. Suppose equality holds in (4.6), so for some sequence $r_k \to \infty$,

$$\frac{C(\lambda r_k)}{C(r_k)} \to 1. \tag{4.7}$$

Then also $T^*(\lambda r_k)/T^*(r_k) \xrightarrow{P} 1$. We claim that this implies

$$P\{S_{T^*(r_k)} > (\lambda+1)r_k/2\} \to 1. \tag{4.8}$$

To see this, use (4.5) to write

$$C(r_k) \leqslant \big(2/(\lambda-1)+1\big)^{1+\delta} C\big((\lambda-1)r_k/2\big)$$

for some $\delta > 0$, r_k large enough. Then

$$\begin{aligned}
&P\Big\{T^*(\lambda r_k) \geqslant \frac{1}{2}\big(2/(\lambda-1)+1\big)^{-1-\delta} C(r_k) + (1-\varepsilon)C(r_k)\Big\} \\
&\quad + P\big\{T^*(r_k) \leqslant (1-\varepsilon)C(r_k)\big\} \\
&\geqslant P\Big\{T^*(\lambda r_k) \geqslant \frac{1}{2}\big(2/(\lambda-1)+1\big)^{-1-\delta} C(r_k) + T^*(r_k)\Big\} \\
&\geqslant P\Big\{T^*(\lambda r_k) - T^*(r_k) \geqslant \frac{1}{2} C\big((\lambda-1)r_k/2\big)\Big\} \\
&\geqslant P\Big\{S_{T^*(r_k)} \leqslant (\lambda+1)r_k/2, \\
&\qquad \max_{T^*(r_k) < n \leqslant T^*(r_k) + \frac{1}{2}C((\lambda-1)r_k/2)} \big(S_n - S_{T^*(r_k)}\big) \leqslant (\lambda-1)r_k/2\Big\} \\
&\geqslant P\big\{S_{T^*(r_k)} \leqslant (\lambda+1)r_k/2\big\} \\
&\quad \times P\Big\{T^*\big((\lambda-1)r_k/2\big) > \frac{1}{2} C\big((\lambda-1)r_k/2\big)\Big\} \\
&\sim P\big\{S_{T^*(r_k)} \leqslant (\lambda+1)r_k/2\big\}. \qquad (4.9)
\end{aligned}$$

Since $T^*(\lambda r_k)/C(\lambda r_k) \xrightarrow{P} 1$ and $C(\lambda r_k) \sim C(r_k)$ under (4.7), the left hand side of (4.9) tends to 0 as $r_k \to \infty$, provided

$$0 < \varepsilon < \frac{1}{2}\big(2/(\lambda-1)+1\big)^{-1-\delta}.$$

This proves (4.8).

Now (4.8), together with

$$S_{T^*(r_k)} = S_{T^*(r_k)-1} + X_{T^*(r_k)} \leqslant r_k + X_{T^*(r_k)},$$

implies

$$P\big\{X_{T^*(r_k)} > (\lambda-1)r_k/2\big\} \to 1. \qquad (4.10)$$

Moreover, the left hand side of (4.10) is at most

$$\begin{aligned}
&P\big\{X_n > (\lambda-1)r_k/2 \quad \text{for some } n \leqslant 2C(r_k)\big\} + P\big\{T^*(r_k) > 2C(r_k)\big\} \\
&\quad = 1 - \big(F\big((\lambda-1)r_k/2\big)\big)^{2C(r_k)} + o(1),
\end{aligned}$$

so we see that

$$C(r_k)\big(1 - F\big((\lambda - 1)r_k/2\big)\big) \to \infty. \tag{4.11}$$

Define $s_k \to \infty$ by

$$s_k = \sup\{r\colon\ C(r)\big(1 - F\big((\lambda - 1)r_k/2\big)\big) \leqslant 1\}.$$

Then $C(s_k)/C(r_k) \to 0$, so by the monotonicity of $C(\cdot)$, $s_k \leqslant r_k$. In fact we must have

$$\frac{s_k}{r_k} \to 0. \tag{4.12}$$

If (4.18) fails, choose $\ell \geqslant 1$ so that $s_k/r_k \geqslant 2^{-\ell}$. Then by virtue of (4.5), $C(s_k) \geqslant C(r_k/2^\ell) \geqslant 2^{-\ell-1}C(r_k)$, so

$$C(s_k)\big(1 - F\big((\lambda - 1)r_k/2\big)\big) \geqslant 2^{-\ell-1}C(r_k)\big(1 - F\big((\lambda - 1)r_k/2\big)\big) \to \infty,$$

giving a contradiction. Thus (4.12) holds. We note also that

$$C(s_k + 1) \geqslant \big[1 - F\big((\lambda - 1)r_k/2\big)\big]^{-1} \to \infty,$$

so that also

$$C(s_k) \to \infty, \quad k \to \infty \tag{4.13}$$

(by (4.5)). Further, since $T^*(r)/C(r) \xrightarrow{P} 1$ and $S_{T^*(r)} \geqslant r > 0$,

$$\begin{aligned} &P\big\{S_n > 0 \text{ for some } n \in \big[C(s_k)/2, 2C(s_k)\big]\big\} \\ &\quad \geqslant P\big\{C(s_k)/2 \leqslant T^*(s_k) \leqslant 2C(s_k)\big\} \to 1. \end{aligned} \tag{4.14}$$

Next we claim that, for each $m \geqslant 4$,

$$\min_{C(s_k)/m \leqslant n \leqslant 2C(s_k)/m} P\{S_n \geqslant 0\} \to 1. \tag{4.15}$$

If this fails, there is a $\delta > 0$ and infinitely many integers $n_k \in [C(s_k)/m, 2C(s_k)/m]$ such that

$$P\{S_{n_k} \leqslant 0\} \geqslant \delta > 0 \quad (k \to \infty). \tag{4.16}$$

Applying (4.1) with X_i replaced by $-X_i$ and m replaced by $2m$, we deduce from this that for infinitely many k and $0 < \varepsilon < 1$,

$$P\Big\{\max_{\frac{1}{2}C(s_k) \leqslant j \leqslant 2C(s_k)} S_j \leqslant 0\Big\} \geqslant \delta^{2m+2}(1 - \varepsilon)^{2m}\varepsilon.$$

This is not possible by (4.14), so we have (4.15). But then for $m \geqslant 4$ and k so large that $s_k \leqslant (\lambda - 1)r_k/2$,

$$P\left\{T^*(s_k) \in \left[\frac{C(s_k)}{m} + 1, \frac{2C(s_k)}{m} + 1\right]\right\}$$
$$\geqslant \sum_{C(s_k)/m \leqslant n \leqslant 2C(s_k)/m} P\{n < T^*(s_k), S_n \geqslant 0, X_{n+1} > (\lambda - 1)r_k/2\}$$

(because $T^*(s_k) \leqslant T^*((\lambda - 1)r_k/2) \leqslant n + 1$ when $S_n \geqslant 0$ and $X_{n+1} > (\lambda - 1)r_k/2$). Thus

$$P\left\{T^*(s_k) \in \left[\frac{C(s_k)}{m} + 1, \frac{2C(s_k)}{m} + 1\right]\right\}$$
$$\geqslant \sum_{C(s_k)/m \leqslant n \leqslant 2C(s_k)/m} \left(P\{S_n \geqslant 0\} - P\{T^*(s_k) \leqslant n\}\right)$$
$$\times \left(1 - F\left((\lambda - 1)r_k/2\right)\right)$$
$$\geqslant \left(1 + o(1)\right)\frac{C(s_k)}{m}\left(1 - F\left((\lambda - 1)r_k/2\right)\right)$$

(because $P\{T^*(s_k) \leqslant 2C(s_k)/m\} \to 0$ for $m \geqslant 4$, and by (4.15) and (4.13))

$$\geqslant \left(1 + o(1)\right)\frac{C(2s_k)}{3m}\left(1 - F\left((\lambda - 1)r_k/2\right)\right) \quad \text{(by (4.5))}$$
$$\geqslant \left(1 + o(1)\right)\frac{1}{4m} \quad \text{(by definition of } s_k\text{)}.$$

But since $2C(s_k)/m + 1 < 3C(s_k)/4$ for large k and $m \geqslant 4$, this contradicts $T^*(s_k)/C(s_k) \xrightarrow{P} 1$. □

LEMMA 4.3. – *Let $S_{n_k}/f_k \xrightarrow{P} a$ for some deterministic sequences $n_k \to \infty$ and $f_k > 0$, where $-\infty < a < \infty$ is a constant. Then*

$$\frac{\overline{S}_{n_k}}{f_k} \xrightarrow{P} |a| \quad \textit{and} \quad \frac{S^*_{n_k}}{f_k} \xrightarrow{P} \max(a, 0). \tag{4.17}$$

Proof of Lemma 4.3. – First let $S_{n_k}/f_k \xrightarrow{P} a \in [0, \infty)$ as $k \to \infty$. Then necessarily $f_k \to \infty$, because $\{S_{n_k}\}$ is not tight. By the degenerate convergence criterion (cf. Gnedenko and Kolmogorov [7, p. 134])

$$n_k H(x f_k) \to 0, \quad \frac{n_k \nu(x f_k)}{f_k} \to a \quad \text{and} \quad \frac{n_k V(x f_k)}{f_k^2} \to 0 \tag{4.18}$$

for all $x > 0$. In particular $n_k P\{|X| > f_k\} \to 0$ and

$$j|\nu(f_k)| \leqslant n_k|\nu(f_k)| \leqslant (a+\varepsilon)f_k$$

for k large enough and $1 \leqslant j \leqslant n_k$. We can therefore write, for $\varepsilon > 0$,

$$\begin{aligned} &P\{\overline{S}_{n_k} > (a+2\varepsilon)f_k\} \\ &\quad\leqslant P\left\{\max_{1\leqslant j\leqslant n_k}\left|\sum_{i=1}^{j} X_i I(|X_i| \leqslant f_k)\right| > (a+2\varepsilon)f_k\right\} + n_k P\{|X| > f_k\} \\ &\quad\leqslant P\left\{\max_{1\leqslant j\leqslant n_k}\left|\sum_{i=1}^{j} X_i I(|X_i| \leqslant f_k) - j\nu(f_k)\right| > \varepsilon f_k\right\} + \mathrm{o}(1). \end{aligned} \tag{4.19}$$

Applying Kolmogorov's inequality to (4.19) gives, by (4.18),

$$P\{\overline{S}_{n_k} > (a+2\varepsilon)f_k\} \leqslant \frac{n_k V(f_k)}{\varepsilon^2 f_k^2} + \mathrm{o}(1) \to 0. \tag{4.20}$$

(Recall that V is defined in (1.12).) If $a = 0$ this proves the first half of (4.17) in this case. If $a > 0$, simply combine (4.20) with

$$P\{\overline{S}_{n_k} \leqslant (a-\varepsilon)f_k\} \leqslant P\{S_{n_k} \leqslant (a-\varepsilon)f_k\} \to 0 \tag{4.21}$$

(where $0 < \varepsilon < a$) to prove the first half of (4.17) in this case.

When $a = 0$, the second half of (4.17) follows from

$$P\{S^*_{n_k} \leqslant -\varepsilon f_k\} \leqslant P\{X_1 \leqslant -\varepsilon f_k\} \to 0 \tag{4.22}$$

together with

$$P\{S^*_{n_k} > (a+2\varepsilon)f_k\} \leqslant P\{\overline{S}_{n_k} \geqslant (a+2\varepsilon)f_k\} \to 0.$$

When $a > 0$, combine the last relation with

$$P\{S^*_{n_k} \leqslant (a-\varepsilon)f_k\} \leqslant P\{S_{n_k} \leqslant (a-\varepsilon)f_k\} \to 0$$

(where $0 < \varepsilon < a$) to prove the second half of (4.17) in this case.

When $S_{n_k}/f_k \xrightarrow{P} a$, where $-\infty < a < 0$, we can replace a by $|a|$ and X_i by $-X_i$ in (4.19)–(4.21) to see that $\overline{S}_{n_k}/f_k \xrightarrow{P} |a|$, proving the first half of (4.17) in this case. It only remains to show that $S^*_{n_k}/f_k \xrightarrow{P} 0$ in this case. Note that (4.18) gives $\nu(f_k) < 0$ for large k. Thus if $\varepsilon > 0$ and k is large enough, then

$$P\{S^*_{n_k} > \varepsilon f_k\} \leqslant P\left\{\max_{1\leqslant j\leqslant n_k}\left(\sum_{i=1}^{j} X_i I(|X_i| \leqslant f_k) - j\nu(f_k)\right) > \varepsilon f_k\right\}$$
$$+ n_k P\{|X| > f_k\},$$

and the right hand side here converges to 0, as shown in (4.19), (4.20). Together with (4.22), this proves the second convergence in (4.17) when $a < 0$. □

Remark. –
(viii) If $S_{n_k}/f_k \xrightarrow{P} a = \pm\infty$, then the first half of (4.17) with $|a| = \infty$ is obviously true, as is the second half of (4.17) if $a = +\infty$. But if $a = -\infty$, the second half of (4.17) is not true in general, as is shown by Example 3.5.

LEMMA 4.4. – *Suppose $S^*_n/B_n \xrightarrow{P} 1$ as $n \to \infty$, for some nonstochastic sequence $B_n > 0$. Then $S_n/B_n \xrightarrow{P} 1$ as $n \to \infty$.*

Proof of Lemma 4.4. – First note that we must have

$$\lim_{n\to\infty} \frac{\max_{k\leqslant n} B_k}{B_n} = 1$$

for the same reasons as in (4.3). We may therefore assume that B_n is increasing in n, in fact, $B_n \uparrow \infty$, since $S^*_n \xrightarrow{P} \infty$.

Next, we show that

$$\liminf_{n\to\infty} \frac{B_{2n}}{B_n} > 1. \tag{4.23}$$

Suppose (4.23) fails, so (in view of the monotonicity of B_n) there is a sequence $n_k \to \infty$ with $B_{2n_k}/B_{n_k} \to 1$. Let $r_k = 3B_{n_k}/4$. Then

$$P\{T^*(r_k) > n_k\} = P\{S^*_{n_k} \leqslant r_k\} = P\{S^*_{n_k} \leqslant 3B_{n_k}/4\} \to 0, \tag{4.24}$$

because $S^*_{n_k}/B_{n_k} \xrightarrow{P} 1$. Also

$$P\{T^*(2r_k) \leqslant 2n_k\} = P\{S^*_{2n_k} > 2r_k = 3B_{n_k}/2\} \to 0 \tag{4.25}$$

because we have $S^*_{2n_k}/B_{n_k} \xrightarrow{P} 1$ by virtue of the facts that $S^*_{2n_k}/B_{2n_k} \xrightarrow{P} 1$ and $B_{2n_k}/B_{n_k} \to 1$. However

$$T^*(2r_k) \leqslant T^*(r_k) + (T^*)'(r_k),$$

where $(T^*)'(r)$ is an independent copy of $T^*(r)$, so

$$\begin{aligned} P\{T^*(2r_k) > 2n_k\} &\leqslant P\{T^*(r_k) + (T^*)'(r_k) > 2n_k\} \\ &\leqslant P\{T^*(r_k) + n_k > 2n_k\} + P\{(T^*)'(r_k) > n_k\} \\ &\to 0 \quad (\text{by } (4.24)). \end{aligned} \tag{4.26}$$

(4.25) and (4.26) are contradictory, so we have established (4.23).

Now note that, for $\varepsilon > 0$

$$P\{S_n > (1+\varepsilon)B_n\} \leqslant P\{S^*_n > (1+\varepsilon)B_n\} \to 0 \quad (n \to \infty). \tag{4.27}$$

Next consider

$$\begin{aligned} S^*_{2n} &= S^*_n \vee \Big(\max_{n+1 \leqslant j \leqslant 2n} S_j\Big) \\ &= S^*_n \vee \left(S_n + \max_{1 \leqslant j \leqslant n}\left(\sum_{i=1}^{j} X_{n+i}\right)\right) \\ &= S^*_n \vee \big(S_n + (S^*_n)'\big), \end{aligned}$$

where $(S^*_n)'$ is an independent copy of S^*_n. Consequently, for $0 < \varepsilon < 1$,

$$\begin{aligned} &P\{S^*_{2n} \leqslant (1-\varepsilon)B_{2n}\} = P\{S^*_n \vee \big(S_n + (S^*_n)'\big) \leqslant (1-\varepsilon)B_{2n}\} \\ &\quad = \int_{(-\infty,\infty)} P\{S^*_n \vee (S_n + yB_n) \leqslant (1-\varepsilon)B_{2n}\}\, dP\{S^*_n \leqslant yB_n\} \\ &\quad = \int_{(-\infty,\infty)} P\{S^*_n \leqslant (1-\varepsilon)B_{2n}, S_n + yB_n \leqslant (1-\varepsilon)B_{2n}\}\, dP\{S^*_n \leqslant yB_n\} \\ &\quad \geqslant \int_{|y-1| \leqslant \varepsilon} P\{S^*_n \leqslant (1-\varepsilon)B_{2n}, S_n + (1+\varepsilon)B_n \leqslant (1-\varepsilon)B_{2n}\} \\ &\qquad \times dP\{S^*_n \leqslant yB_n\} \\ &\quad = P\left\{\frac{S^*_n}{B_n} \leqslant (1-\varepsilon)\frac{B_{2n}}{B_n}, \frac{S_n}{B_n} \leqslant (1-\varepsilon)\frac{B_{2n}}{B_n} - (1+\varepsilon)\right\} \\ &\qquad \times P\left\{\left|\frac{S^*_n}{B_n} - 1\right| \leqslant \varepsilon\right\}. \end{aligned} \tag{4.28}$$

Now by (4.23), $B_{2n}/B_n \geqslant a > 1$ for some $a > 1$ once $n \geqslant$ some $n_0(a)$. Choose $\varepsilon \in (0, 1)$ so small that $(1-\varepsilon)a > 1$ and $(1-\varepsilon)(a-1) > 3\varepsilon$. Then from (4.28)

$$\begin{aligned} &P\{S_{2n}^* \leqslant (1-\varepsilon)B_{2n}\} \\ &\geqslant P\left\{\frac{S_n^*}{B_n} \leqslant (1-\varepsilon)a,\ \frac{S_n}{B_n} \leqslant (1-\varepsilon)a - (1+\varepsilon)\right\}(1+\mathrm{o}(1)) \\ &\geqslant \left[P\left\{\frac{S_n}{B_n} \leqslant (1-\varepsilon)(a-1) - 2\varepsilon\right\} - P\left\{\frac{S_n^*}{B_n} > (1-\varepsilon)a\right\}\right](1+\mathrm{o}(1)) \\ &\geqslant \left[P\left\{\frac{S_n}{B_n} \leqslant \varepsilon\right\} - \mathrm{o}(1)\right](1+\mathrm{o}(1)). \end{aligned} \tag{4.29}$$

Here $P\{S_n^* > (1-\varepsilon)aB_n\} \to 0$ because we chose $(1-\varepsilon)a > 1$.

Since the left hand side of (4.29) converges to 0 as $n \to \infty$, we have

$$P\left\{\frac{S_n}{B_n} \leqslant \varepsilon\right\} \to 0.$$

Together with (4.27) this gives

$$1 = \lim_{n\to\infty} P\left\{\varepsilon < \frac{S_n}{B_n} \leqslant 1+\varepsilon\right\} \leqslant \liminf_{n\to\infty} P\left\{\varepsilon \leqslant \frac{|S_n|}{B_n} \leqslant 1+\varepsilon\right\},$$

and this implies that S_n is relatively stable by Lemma 5.3 of Kesten and Maller [14]. Thus $S_n/B_n^* \xrightarrow{P} 1$ for some B_n^*. But then $S_n^*/B_n^* \xrightarrow{P} 1$ by Lemma 4.3, so $B_n^* \sim B_n$ and $S_n/B_n \xrightarrow{P} 1$. □

Proof of Theorem 2.1. – Let $T^*(r) < \infty$ a.s. and $T^*(r)/C(r) \xrightarrow{P} 1$ as $r \to \infty$. By Lemma 4.2 we can choose $C(r)$ to be increasing and to satisfy (4.2). Define $B_n^* \uparrow \infty$ by

$$B_n^* = \inf\{r > 0\colon\ C(r) \geqslant n\},$$

so that $C(B_n^*+) \geqslant n$. Here $C(x+) = \lim_{y\downarrow x} C(y)$, which exists by the monotonocity of $C(\cdot)$. Fix $0 < \varepsilon < 1$ and let $r = r(n,\varepsilon) = (1+\varepsilon)B_n^*$. Because of (4.2), we have

$$\frac{C(r)}{C(B_n^*+)} = \frac{C(r)}{C((r/(1+\varepsilon))+)} \geqslant a_\varepsilon$$

for some $a_\varepsilon > 1$ and r large enough. Then

$$\begin{aligned} P\{S_n^* > (1+\varepsilon)B_n^*\} &= P\{T^*(r) \leqslant n\} \leqslant P\{T^*(r) \leqslant C(B_n^*+)\} \\ &\leqslant P\{T^*(r) \leqslant C(r)/a_\varepsilon\} \to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{4.30}$$

By the definition of B_n^*, we have $n \geqslant C((1-\varepsilon/2)B_n^*)$. Let $s = (1-\varepsilon)B_n^*$. Because of (4.2), we then have

$$n \geqslant C((1-\varepsilon/2)B_n^*) = C\left(s\frac{(1-\varepsilon/2)}{(1-\varepsilon)}\right) \geqslant b_\varepsilon C(s)$$

for some $b_\varepsilon > 1$ and large s. Thus as $n \to \infty$

$$\begin{aligned} P\{S_n^* \leqslant (1-\varepsilon)B_n^*\} &= P\{T^*(s) > n\} \\ &\leqslant P\{T^*(s) \geqslant b_\varepsilon C(s)\} \to 0. \end{aligned} \tag{4.31}$$

(4.30) and (4.31) prove that $S_n^*/B_n^* \xrightarrow{P} 1$, which is (2.2).

Next, (2.2) implies (2.3) by Lemma 4.4, while (2.3) implies (2.2) by Lemma 4.3.

Now, as we already stated at the end of Section 1, when (2.3) holds for some sequence $\{B_n\}$, then $A(r) > 0$ for large r, and $A(r)$ is slowly varying as $r \to \infty$. Moreover, (2.3) will hold with B_n replaced by $D(n)$ (as defined in (1.18)), and $D(x)$ is regularly varying with index 1 as $x \to \infty$. In particular, we have $B_n \to \infty$ as $n \to \infty$.

Finally let (2.2), or, equivalently, (2.3), hold, and let $C(r) = r/A(r)$. Now (2.3) implies $B_n \to \infty$ and so $S_n \xrightarrow{P} \infty$. Thus $T^*(r) < \infty$ a.s. for each $r \geqslant 0$. Take $\varepsilon \in (0,1)$ and let $n = n(r,\varepsilon) = \lfloor (1+\varepsilon)C(r) \rfloor$. ($\lfloor x \rfloor$ denotes the largest integer less than or equal to x; $\lceil x \rceil$ will denote the smallest integer greater than or equal to x.) Then

$$P\{T^*(r) > (1+\varepsilon)C(r)\} = P\{S_n^* \leqslant r\} \to 0 \quad (r \to \infty),$$

because $r/A(r) = C(r) \sim n/(1+\varepsilon)$ and $r \sim D(n/(1+\varepsilon)) \sim D(n)/(1+\varepsilon)$, by the regular variation of $D(\cdot)$, and because (2.2) holds with B_n replaced by $D(n)$. Similarly, with $m = \lfloor (1-\varepsilon)C(r) \rfloor$,

$$P\{T^*(r) \leqslant (1-\varepsilon)C(r)\} = P\{S_m^* > r\} \to 0 \quad (r \to \infty),$$

because $r \sim C^{-1}(m/(1-\varepsilon)) \sim B_m/(1-\varepsilon)$. This proves (2.1), with $C(r)$ and B_n chosen as claimed. □

Proof of Theorem 2.2. – Let $T^*(r) < \infty$ a.s. and $T^*(r)/C(r) \to 1$ a.s. as $r \to \infty$. Then $T^*(r)/C(r) \xrightarrow{P} 1$ as $r \to \infty$, so we know from

Theorem 2.1 that we may take $C(r) = r/A(r)$, and if $B_n = D(n)$, then $S_n/B_n \xrightarrow{P} 1$, and $C(r)$ and $D(x)$ enjoy the properties listed in Section 1. Now if $\varepsilon > 0$ and $r = (1+\varepsilon)D(n) = (1+\varepsilon)B_n$, we have

$$\begin{aligned} P\{S_n^* > (1+\varepsilon)B_n \text{ i.o.}\} &= P\{T^*(r) \leqslant n \sim D^{-1}(r/(1+\varepsilon)) \text{ i.o.}\} \\ &\leqslant P\{T^*(r) \leqslant C(r/(1+\varepsilon/2)) \text{ i.o. (in } r)\} = 0, \end{aligned} \tag{4.32}$$

because

$$D^{-1}(r/(1+\varepsilon)) \sim C(r/(1+\varepsilon)) \sim C(r)/(1+\varepsilon)$$

as $r \to \infty$ by (1.21) and the regular variation of $C(\cdot)$. Similarly, we see that $P\{S_n^* \leqslant (1-\varepsilon)B_n \text{ i.o.}\} = 0$, so (2.5) holds. Conversely, (2.5) implies (2.4) by similar arguments.

Next, let (2.5) hold. Then (2.2) holds so $D(x)$ is regularly varying with index 1 by Theorem 2.1. Let $T_0^* = 0$ and let T_j^*, $j \geqslant 1, \ldots$, be the strict ascending ladder times of S, i.e.,

$$T_j^* = \min\{n > T_{j-1}^*\colon S_n > S_{T_{j-1}^*}\}, \quad j = 1, 2, \ldots.$$

(2.5) implies that $\limsup_{n\to\infty} S_n = \infty$ a.s., so $T_j^* < \infty$ a.s. and $T_j^* \to \infty$ a.s. as $j \to \infty$. Define further

$$\Delta_j = T_j^* - T_{j-1}^*, \quad j \geqslant 1.$$

We claim that (2.5) forces

$$ET_1^* = E\Delta_j < \infty. \tag{4.33}$$

Indeed, the Δ_j, $j \geqslant 1$, are i.i.d. Therefore, if (4.33) fails, then by Kesten [11] or Chow and Teicher [4, Section 7.1, Example 1],

$$\limsup_{j\to\infty} \frac{\Delta_j}{\sum_{i=1}^{j-1} \Delta_i} = \infty \text{ a.s.} \tag{4.34}$$

But, by definition of the T_j^*,

$$S_n^* = \max_{1\leqslant \ell \leqslant n} S_\ell = S_{T_{j-1}}^* \quad \text{for } T_{j-1}^* \leqslant n < T_j^*. \tag{4.35}$$

In particular,

$$S_{T_j^*-1}^* = S_{T_{j-1}^*}^*.$$

Thus, if also (2.5) holds, then (recall $B_n = D(n)$)

$$D(T_j^* - 1)/D(T_{j-1}^*) \to 1 \text{ a.s.}, \quad \text{as } j \to \infty.$$

By the regular variation of $D(\cdot)$ this implies that

$$\frac{T_j^* - 1}{T_{j-1}^*} = \frac{\sum_{i=1}^{j} \Delta_i - 1}{\sum_{i=1}^{j-1} \Delta_i} \to 1 \text{ a.s.}, \quad \text{as } j \to \infty.$$

This contradicts (4.34), so that (4.33) must hold. Note that (4.33) and the strong law of large numbers implies

$$\frac{1}{j} T_j^* \to E\Delta_1 < \infty \text{ a.s.}, \quad \text{as } j \to \infty. \tag{4.36}$$

(2.5) also implies

$$E\left[S_{T_j^*} - S_{T_{j-1}^*}\right] = E(S_{T_1^*}) < \infty, \quad j \geqslant 1. \tag{4.37}$$

To see this, note that the $S_{T_j^*} - S_{T_{j-1}^*}$ are also i.i.d., so that if (4.37) fails, then, as in (4.34),

$$\limsup_{j\to\infty} \frac{S_{T_j^*} - S_{T_{j-1}^*}}{S_{T_{j-1}^*}} = \limsup_{j\to\infty} \frac{S_{T_j^*} - S_{T_{j-1}^*}}{\sum_{i=1}^{j-1} [S_{T_i^*} - S_{T_{i-1}^*}]} = \infty \text{ a.s.} \tag{4.38}$$

But then, (4.35), (2.5), (4.36) and the regular variation of $D(\cdot)$ imply

$$S_{T_j^*} = S_{T_j^*}^* \sim D(T_j^*) \sim D(jE\Delta_1) \sim D(T_{j-1}^*) \sim S_{T_{j-1}^*}^* = S_{T_{j-1}^*}.$$

This contradicts (4.38), so that (4.37) must hold. Since $ES_{T_1^*} \geqslant E\{X_1; X_1 > 0\}$, it follows that $EX^+ < \infty$. Finally, Theorem 2.1 and Eq. (1.4) of Kesten and Maller [15] show that $S_n \to \infty$ a.s., so that we must have

$$0 < EX \leqslant E|X| < \infty$$

(compare Lemma 1.1 in Kesten and Maller [15]). Hence (2.6) is proved.

The final implication, that $0 < EX \leqslant E|X| < \infty$ implies $T^*(r)/(rEX) \to 1$ a.s. and $S_n^*/(nEX) \to 1$ a.s., follows from Theorem 7 of Heyde [10] and the strong law of large numbers, respectively. □

LEMMA 4.5. – *Let* $\overline{S}_{n_k}/f_k \xrightarrow{P} a$, *where* $0 \leqslant a < \infty$, *for some nonstochastic sequences* $n_k \to \infty$ *and* $f_k > 0$. *Then* $|S_{n_k}|/f_k \xrightarrow{P} a$.

Proof of Lemma 4.5. – For the same reasons as in Lemma 4.3, $f_k \to \infty$. Now for each $\varepsilon > 0$

$$P\{|S_{n_k}| > (a+\varepsilon)f_k\} \leqslant P\{\overline{S}_{n_k} > (a+\varepsilon)f_k\} \to 0. \tag{4.39}$$

Thus, if $a = 0$ the lemma is obvious, so take $a > 0$. Let $\{m_k\}$ be any subsequence of integers. $\{m_k\}$ has a further subsequence, denoted $\{m'_k\}$, so that $S_{n'_k}/f_{m'_k}$ converges weakly to some Z', where $n'_k = n_{m'_k}$. By (4.39), $|Z'| \leqslant a$ a.s., so Z' is a proper, infinitely divisible random variable. Furthermore, as a bounded infinitely divisible random variable it must degenerate to a point, $Z' = a'$, say (see Feller [6, p. 177]). Thus $S_{n'_k}/f_{m'_k} \xrightarrow{P} a'$. By Lemma 4.3 this means

$$\overline{S}_{n'_k}/f_{m'_k} \xrightarrow{P} |a'|.$$

Thus $|a'| = a$, so $|S_{n'_k}|/f_{m'_k} \xrightarrow{P} a$. This is true for all subsequences $\{m_k\}$, so in fact $|S_{n_k}|/f_k \xrightarrow{P} a$. □

Proof of Theorem 2.3. – Let $|S_n|/B_n \xrightarrow{P} 1$. By the argument following (1.16) we may assume that B_n is increasing. It then follows from Lemma 5.3 of Kesten and Maller [14] that $S_n/\overline{B}_n \xrightarrow{P} \pm 1$ for some $\overline{B}_n$, and clearly we can choose $\overline{B}_n = B_n$. So $\overline{S}_n/B_n \xrightarrow{P} 1$ by Lemma 4.3, and (2.8) holds. Conversely, (2.8) implies (2.9) by Lemma 4.5.

Next assume that (2.9) holds, and for the sake of definiteness suppose that $S_n/B_n \xrightarrow{P} 1$. Then $\overline{S}_n/B_n \xrightarrow{P} 1$ by (2.8). Choose $C(r) = r/A(r)$, which is strictly positive for large r. If $\varepsilon > 0$ let

$$n = n(r,\varepsilon) = \lfloor (1+\varepsilon)C(r) \rfloor,$$

so that

$$r \leqslant C^{-1}\big((n+1)/(1+\varepsilon)\big) = D\big((n+1)/(1+\varepsilon)\big).$$

Then

$$\begin{aligned} P\{T(r) > (1+\varepsilon)C(r)\} &= P\{\overline{S}_{\lfloor (1+\varepsilon)C(r)\rfloor} \leqslant r\} \\ &\leqslant P\{\overline{S}_n \leqslant D\big((n+1)/(1+\varepsilon)\big)\} \to 0, \end{aligned}$$

since $D((n+1)/(1+\varepsilon)) \sim D(n)/(1+\varepsilon)$ by the regular variation of $D(\cdot)$, and also $D(n) \sim B_n$, by Theorem 2.1. In a similar way we get

$$P\{T(r) \leqslant (1-\varepsilon)C(r)\} \to 0,$$

so (2.7) holds.

Conversely let $T(r)/C(r) \xrightarrow{P} 1$. Let us first dispose of the case in which X is bounded, that is $H(x_0)=0$ for some x_0. In this case we must have $EX \neq 0$. Indeed, if $EX=0$, $EX^2<\infty$, then the weak convergence of $\{S_{\lfloor tn \rfloor}/\sqrt{n}\}_{t \geqslant 0}$ to Brownian motion shows that $T(r)/C(r) \xrightarrow{P} 1$ is impossible. However, if $EX \neq 0$, then (2.9) is clearly true with $B_n = n|EX|$.

For the remainder of this proof we therefore assume that $H(x)>0$ for all x. We shall prove that then (1.22) must hold, and as stated in Section 1, this implies (2.9). Assume, to derive a contradiction, that (1.22) fails and let $r_k \to \infty$ be such that

$$\frac{|A(r_k)|}{r_k H(r_k)} \leqslant C \tag{4.40}$$

for some constant C. Now

$$P\{T(r) \leqslant \tfrac{1}{2}C(r)\} = P\{\overline{S}_{\lfloor \frac{1}{2}C(r) \rfloor} > r\} \geqslant P\{|S_{\lfloor \frac{1}{2}C(r) \rfloor}| > r\}$$

shows that

$$P\{|S_{\lfloor \frac{1}{2}C(r) \rfloor}| \leqslant r\} \to 1. \tag{4.41}$$

Therefore, by going over to a subsequence, if necessary, we may assume that $S_{\lfloor \frac{1}{2}C(r_k) \rfloor}/r_k \Rightarrow Z$, where Z is an infinitely divisible random variable with $P\{|Z| \leqslant 1\}=1$. If follows that Z is degenerate at some point, c say, with $|c| \leqslant 1$ (see Feller [6, p. 177]). By the conditions for degenerate convergence (Gnedenko and Kolmogorov [7, p. 134]) we then have

$$\begin{gathered}\lfloor \tfrac{1}{2}C(r_k) \rfloor H(r_k) \to 0, \quad \frac{\lfloor \frac{1}{2}C(r_k) \rfloor V(r_k)}{(r_k)^2} \to 0 \quad \text{and} \\ \frac{\lfloor \frac{1}{2}C(r_k) \rfloor \nu(r_k)}{r_k} \to c.\end{gathered} \tag{4.42}$$

First suppose $c=0$. Then we can replace $\frac{1}{2}$ by λ, for any $\lambda>1$, in (4.42), to deduce that $S_{\lfloor \lambda C(r_k) \rfloor}/r_k \xrightarrow{P} 0$. Then by Lemma 4.3, $\overline{S}_{\lfloor \lambda C(r_k) \rfloor}/r_k \xrightarrow{P} 0$ for all $\lambda>1$. But then

$$P\{T(r_k) \leqslant \lambda C(r_k)\} = P\{\overline{S}_{\lfloor \lambda C(r_k) \rfloor} > r_k\} \to 0$$

so $T(r_k)/C(r_k) \xrightarrow{P} \infty$, which is impossible. Thus $c \neq 0$ in (4.42).

But then

$$\frac{|A(r_k)|}{r_k H(r_k)} = \frac{\lfloor \frac{1}{2} C(r_k) \rfloor [|\nu(r_k)| + O(r_k H(r_k))]}{r_k \lfloor \frac{1}{2} C(r_k) \rfloor H(r_k)} \to \infty,$$

in contradiction to (4.40). Thus (1.22) and (2.9) hold.

Finally, the beginning of this proof shows that we can take $C(r) = r/A(r)$ whenever (2.7) holds. If (2.9) holds we can take $B_n = D(n)$ by Theorem 2.1. □

Proof of Theorem 2.4. – If $T(r)/C(r) \to 1$ a.s., then $T(r)/C(r) \xrightarrow{P} 1$, so we can take $C(r) = r/A(r)$ as in Theorem 2.3, and this is increasing and regularly varying with index 1. Just as in the proof of Theorem 2.2, we then see that (2.10) and (2.11) are equivalent. Furthermore, when (2.10) holds, then for $\varepsilon > 0$, a.s.

$$\begin{aligned} r < |S_{T(r)}| = \overline{S}_{T(r)} &\leqslant \overline{S}_{\lfloor (1+\varepsilon)C(r) \rfloor} \leqslant \overline{S}_{\lfloor C((1+2\varepsilon)r) \rfloor} \\ &\leqslant (1+3\varepsilon)r \quad \text{eventually} \end{aligned}$$

(since $T(r) < (1+\varepsilon)C(r) \leqslant C((1+2\varepsilon)r) < T((1+3\varepsilon)r)$ eventually). Thus $|S_{T(r)}|/r \to 1$ a.s. when (2.10) or (2.11) holds. Thus by Theorem 3.1 of Griffin and Maller [8], (2.12) holds or $EX^2 < \infty$ and $EX = 0$. But the latter cannot hold when (2.11), and hence $|S_n|/B_n \xrightarrow{P} 1$, holds. Conversely, (2.12) implies (2.11), with $B_n = n|EX|$, by the strong law of large numbers and Theorem 2.3. Hence (2.12) implies (2.10) with $C(r) = r/|EX|$. □

5. PROOFS FOR SECTION 3

The following inequalities, essentially due to Pruitt [19], will be helpful. Define

$$k(x) = \frac{|A(x)|}{x} + \frac{U(x)}{x^2} = \frac{x|A(x)| + U(x)}{x^2}. \tag{5.1}$$

Then, for some constants $c_i > 0$ and for all $n \geqslant 1$ and $x > 0$,

$$P\{\overline{S}_n \geqslant x\} \leqslant c_1 n k(x) \quad \text{and} \quad P\{\overline{S}_n \leqslant x\} \leqslant \frac{c_2}{nk(x)}. \tag{5.2}$$

Also, for $0 < \varepsilon < 1$ and r large

$$P\left\{\frac{\varepsilon}{k(r)} \leqslant T(r) \leqslant \frac{1}{\varepsilon k(r)}\right\} \geqslant 1 - c_3\varepsilon. \tag{5.3}$$

Finally, for each fixed $L > 0$,

$$\sum_{n\geqslant 1} P\{\overline{S}_n \leqslant Lx\} \asymp \frac{1}{k(x)} \quad (\text{as } x \to \infty). \tag{5.4}$$

The notation "$\asymp$" here means that the ratio of the two sides of (5.4) is bounded away from 0 and ∞ (by constants which in general depend on L) as $x \to \infty$. See Pruitt [19, Eq. (1.2) and Theorem 1], for (5.2)–(5.4). Actually, Pruitt used the function $x|\nu(x)| + U(x)$ where we have $x|A(x)| + U(x)$, but these are equivalent in the context of (5.2)–(5.4), as remarked by Griffin and Maller [8, Eq. (4.1) and Lemma 4.1].

Proof of Theorem 3.1. – Let (3.5) hold for an f with $f(n) \uparrow \infty$, and suppose (3.7) fails, so there is a constant $c > 0$ and a sequence $x_k \uparrow \infty$ such that

$$f^{-1}(x_k)k(x_k) = f^{-1}(x_k)\frac{|A(x_k)|}{x_k} + f^{-1}(x_k)\frac{U(x_k)}{x_k^2} \leqslant c < \infty. \tag{5.5}$$

Define $n_k = f^{-1}(x_k) + 1$, so $f(n_k) > x_k$. As remarked in Section 3, $n_k < \infty$, $k = 1, 2, \ldots$, and $n_k \to \infty$ as $k \to \infty$. Now (5.5) implies, for large k,

$$\begin{aligned} \frac{n_k|\nu(x_k)|}{x_k} + \frac{n_k V(x_k)}{x_k^2} &\leqslant \frac{n_k|A(x_k)|}{x_k} + n_k H(x_k) + \frac{n_k U(x_k)}{x_k^2} \\ &\leqslant \frac{4f^{-1}(x_k)|A(x_k)|}{x_k} + \frac{4f^{-1}(x_k)U(x_k)}{x_k^2} \leqslant 4c. \end{aligned} \tag{5.6}$$

Fix $\lambda \geqslant 8c$ and note that $n_k|\nu(x_k)| \leqslant 4cx_k \leqslant x_k\lambda/2$ by (5.6). Thus $j|\nu(x_k)| \leqslant x_k\lambda/2$ for $j \leqslant n_k$. Then use Kolmogorov's inequality and (5.6) to obtain

$$\begin{aligned} &P\left\{\max_{1\leqslant j\leqslant n_k}\left|\sum_{i=1}^{j} X_i I(|X_i| \leqslant x_k)\right| > \lambda x_k\right\} \\ &\leqslant P\left\{\max_{1\leqslant j\leqslant n_k}\left|\sum_{i=1}^{j} X_i I(|X_i| \leqslant x_k) - j\nu(x_k)\right| > \frac{1}{2}\lambda x_k\right\} \\ &\leqslant \frac{4n_k V(x_k)}{\lambda^2 x_k^2} \leqslant \frac{16c}{\lambda^2}. \end{aligned}$$

Thus

$$\begin{aligned}
&P\left\{\max_{1\leqslant j\leqslant n_k}\left|\sum_{i=1}^{j} X_i I\big(|X_i| > x_k\big)\right| \leqslant \lambda x_k\right\}\\
&\quad\leqslant P\left\{\max_{1\leqslant j\leqslant n_k}\left|\sum_{i=1}^{j} X_i I\big(|X_i| > x_k\big)\right| \leqslant \lambda x_k,\right.\\
&\qquad\left.\max_{1\leqslant j\leqslant n_k}\left|\sum_{i=1}^{j} X_i I\big(|X_i| \leqslant x_k\big)\right| \leqslant \lambda x_k\right\}\\
&\qquad + P\left\{\max_{1\leqslant j\leqslant n_k}\left|\sum_{i=1}^{j} X_i I\big(|X_i| \leqslant x_k\big)\right| > \lambda x_k\right\}\\
&\quad\leqslant P\left\{\max_{1\leqslant j\leqslant n_k}|S_j| \leqslant 2\lambda x_k\right\} + \frac{16c}{\lambda^2}.
\end{aligned}\tag{5.7}$$

We have $f(n_k) \geqslant x_k$, so (3.5) and $n_k \to \infty$ imply that

$$\max_{1\leqslant j\leqslant n_k}\frac{|S_j|}{x_k} \xrightarrow{P} \infty.$$

Thus from (5.7) we obtain

$$P\left\{\max_{1\leqslant j\leqslant n_k}\left|\sum_{i=1}^{j} X_i I\big(|X_i| > x_k\big)\right| \leqslant \lambda x_k\right\} \leqslant \mathrm{o}(1) + \frac{16c}{\lambda^2}. \tag{5.8}$$

This means that

$$\begin{aligned}
P\left\{\max_{1\leqslant j\leqslant n_k}|X_j| \leqslant x_k\right\} &\leqslant P\left\{\max_{1\leqslant j\leqslant n_k}\left|\sum_{i=1}^{j} X_i I\big(|X_i| > x_k\big)\right| = 0\right\}\\
&\leqslant P\left\{\max_{1\leqslant j\leqslant n_k}\left|\sum_{i=1}^{j} X_i I\big(|X_i| > x_k\big)\right| \leqslant \lambda x_k\right\}\\
&= \mathrm{o}(1) + \frac{16c}{\lambda^2}.
\end{aligned}$$

Since λ is arbitrary, this shows that

$$P\left\{\max_{1\leqslant j\leqslant n_k}|X_j| \leqslant x_k\right\} = \mathrm{e}^{-n_k H(x_k)(1+\mathrm{o}(1))} \to 0,$$

hence $n_k H(x_k) \to \infty$. Thus $n_k U(x_k)/x_k^2 \to \infty$, which contradicts (5.6). Thus (3.5) implies (3.7).

Conversely, (3.7) implies (3.5). Indeed, an application of (5.2) tells us that for all $\lambda > 0$

$$P\Big\{\max_{1\leqslant j\leqslant n}|S_j|\leqslant\lambda f(n)\Big\}\leqslant\frac{c_2\lambda^2(f(n))^2}{n[\lambda f(n)|A(\lambda f(n))|+U(\lambda f(n))]}. \quad (5.9)$$

Letting $x=f(n)-1$ and using $f^{-1}(f(n)-1)\leqslant n$ in (3.7) we obtain that

$$\frac{n|A(f(n)-1)|}{f(n)}+\frac{nU(f(n)-1)}{(f(n))^2}\to\infty. \quad (5.10)$$

It is easy to check that

$$x\big|A(x)\big|+U(x)\asymp\lambda x\big|A(\lambda x)\big|+U(\lambda x) \quad (5.11)$$

for each $\lambda>0$ as $x\to\infty$, so the right hand side of (5.9) converges to 0 as $n\to\infty$ for each $\lambda>0$ when (3.7) holds. This proves (3.5).

Next we observe that (3.3) is equivalent to (3.7). This is almost immediate from (5.3) and the fact that (3.7) says $f^{-1}(x)k(x)\to\infty$ as $x\to\infty$. In one direction, assume that (3.3) holds. Then by (5.3), for all $0<\varepsilon<1/(2c_3+2)$, and large r,

$$P\Big\{T(r)\geqslant\frac{\varepsilon}{k(r)}\text{ and }T(r)\leqslant\varepsilon^2 f^{-1}(r)\Big\}\geqslant 1-(c_3+1)\varepsilon\geqslant\frac{1}{2}.$$

Therefore,

$$\varepsilon^2 f^{-1}(r)\geqslant\frac{\varepsilon}{k(r)}\quad\text{or}\quad k(r)f^{-1}(r)\geqslant\frac{1}{\varepsilon}$$

for all large r. Thus (3.7) holds. In the other direction, if (3.7) holds, then for any fixed $\varepsilon>0$, $\eta>0$ we have for large r,

$$P\{T(r)>\varepsilon f^{-1}(r)\}\leqslant P\Big\{T(r)>\frac{1}{\eta k(r)}\Big\}\leqslant c_3\eta\quad\text{(by (5.3)).}$$

Thus (3.3) follows.

For the proof that (3.5) implies (3.4) the following observation is useful: for $x\geqslant 0$, $0<\eta<1$ and $\eta n\geqslant 1$

$$\begin{aligned}&P\Big\{\max_{1\leqslant j\leqslant n}|S_j|\leqslant x\Big\}\\&\geqslant P\Big\{\max_{\ell\lfloor\eta n\rfloor<j\leqslant(\ell+1)\lfloor\eta n\rfloor}|S_j-S_{\ell\lfloor\eta n\rfloor}|\leqslant\frac{\eta x}{2+\eta}\text{ for }0\leqslant\ell\leqslant 2/\eta\Big\}\\&\geqslant\Big[P\Big\{\max_{1\leqslant j\leqslant\lfloor\eta n\rfloor}|S_j|\leqslant\frac{\eta x}{2+\eta}\Big\}\Big]^{(2+\eta)/\eta}. \end{aligned}\quad (5.12)$$

Now assume (3.5) again, and take $r > 0$, $0 < \varepsilon < 1$. Also let $n = n(\varepsilon r) = f^{-1}(\varepsilon r)$, so $r \leqslant f(n(\varepsilon r) + 1)/\varepsilon$ and $f(n) \leqslant \varepsilon r$. Then $f(T) > \varepsilon r$ implies $T \geqslant n$. So, for r large enough,

$$\begin{aligned} P\{f(T(r)) > \varepsilon r\} &\leqslant P\{T(r) \geqslant n\} \leqslant P\{T(r) > n-1\} \\ &\leqslant P\Big\{\max_{1\leqslant j\leqslant n-1} |S_j| \leqslant f(n+1)/\varepsilon\Big\} \\ &\leqslant P\Big\{\max_{1\leqslant j\leqslant \lfloor (n+1)/2\rfloor} |S_j| \leqslant f(n+1)/\varepsilon\Big\} \\ &\leqslant P\Big\{\max_{1\leqslant j\leqslant n+1} |S_j| \leqslant 5f(n+1)/\varepsilon\Big\}^{1/5} \to 0 \end{aligned}$$

by (5.12) and (3.5). Thus $f(T(r))/r \xrightarrow{P} 0$ and (3.4) holds.

Trivially, (3.4) implies (3.6), because $|S_{T(r}| > r$. Finally, we complete the proof by showing that (under (3.2)) (3.6) implies (3.5). Indeed, if $\varepsilon > 0$, $\alpha > 0$, then for large r, (3.6) and (5.3) give

$$\begin{aligned} &P\{|\overline{S}_{\lfloor 1/(\varepsilon k(r))\rfloor}| \geqslant \alpha f(\varepsilon/k(r))\} \\ &\geqslant P\Big\{|\overline{S}_{\lfloor 1/(\varepsilon k(r))\rfloor}| \geqslant \alpha f(\varepsilon/k(r));\ \frac{\varepsilon}{k(r)} \leqslant T(r) \leqslant \frac{1}{\varepsilon k(r)}\Big\} \\ &\geqslant P\Big\{|S_{T(r)}| \geqslant \alpha f(T(r));\ \frac{\varepsilon}{k(r)} \leqslant T(r) \leqslant \frac{1}{\varepsilon k(r)}\Big\} \\ &\geqslant 1-(c_3+1)\varepsilon. \end{aligned} \tag{5.13}$$

This time we first choose n and then find $r = r(n, \varepsilon)$ so that $\lfloor 1/(\varepsilon k(r))\rfloor = n$ (since $k(\cdot)$ is continuous and $k(r) \to 0$ as $r \to \infty$ such an r will exist for all large n). Further, let p be such that $\varepsilon 2^p > 1/\varepsilon$. Then by (3.2)

$$f\left(\frac{\varepsilon}{k(r)}\right) \geqslant \frac{1}{\Lambda^p} f\left(\frac{\varepsilon 2^p}{k(r)}\right) \geqslant \frac{1}{\Lambda^p} f\left(\left\lfloor \frac{1}{\varepsilon k(r)}\right\rfloor\right) = \frac{1}{\Lambda^p} f(n). \tag{5.14}$$

Consequently,

$$P\Big\{\overline{S}_n \geqslant \frac{\alpha}{\Lambda^p} f(n)\Big\} \geqslant P\{\overline{S}_{\lfloor 1/(\varepsilon k(r))\rfloor} \geqslant \alpha f(\varepsilon/k(r))\} \geqslant 1-(c_3+1)\varepsilon$$

for large n. Since $\varepsilon, \alpha > 0$ are arbitrary, this implies (3.5) and completes the proof. □

Proof of corollary to Theorem 3.1. – Let $S_n/f(n) \xrightarrow{P} \infty$. Then (3.5) holds so (3.6) holds. Also $P\{S_{T(r)} > 0\} \to 1$ by Theorem 3 of Kesten and Maller [16], so $S_{T(r)}/f(T(r)) \xrightarrow{P} \infty$.

Conversely $S_{T(r)}/f(T(r)) \xrightarrow{P} \infty$ implies (3.6), hence (3.7), under (3.2). Also $P\{S_n > 0\} \to 1$ by Theorem 3 of Kesten and Maller [16]. Now assume first that $U(\infty) := \lim_{x\to\infty} U(x) = \infty$. Then by Theorem 2.1 and Lemma 4.3 of Kesten and Maller [13], $A(x) > 0$ and $U(x) \leqslant 3xA(x)$ for x large enough. We then obtain from (3.7) that $f^{-1}(x)A(x)/x \to \infty$ as $x \to \infty$. Taking $x = f(n) - \varepsilon$ and letting first $\varepsilon \downarrow 0$ and then $n \to \infty$, we see that $nA(f(n))/f(n) \to \infty$ as $n \to \infty$, and $S_n/f(n) \xrightarrow{P} \infty$ then follows from Theorems 2.1 and 2.2 of Kesten and Maller [13]. If $U(\infty) < \infty$, so that $E|X| < \infty$, then by Theorem 2.1 of Kesten and Maller [13], $P\{S_n > 0\} \to 1$ can occur only if $EX > 0$. In this case $T(r) \sim r/EX$ a.s. by Theorem 2.4, and then (3.4) shows that $f(r)/r \to 0$. Consequently,

$$S_n/f(n) = [S_n/n] \cdot [n/f(n)] \xrightarrow{P} \infty. \qquad \square$$

Proof of Theorem 3.2. – (This is a modification of the proof of Theorem 3.1 but we will give some details for completeness.) Let (3.16) hold for a sequence n_k and suppose (3.18) fails, so there is a constant $c > 0$ such that

$$f^{-1}(x)\frac{|A(x)|}{x} + f^{-1}(x)\frac{U(x)}{x^2} \leqslant c < \infty \tag{5.15}$$

for all large x. Let $x_k = f(n_k)$, so $n_k \leqslant f^{-1}(x_k)$. (5.15) then implies, for large k,

$$\begin{aligned}\frac{n_k|\nu(x_k)|}{x_k} + \frac{n_k V(x_k)}{x_k^2} &\leqslant \frac{n_k|A(x_k)|}{x_k} + n_k H(x_k) + \frac{n_k U(x_k)}{x_k^2}\\ &\leqslant \frac{2n_k|A(x_k)|}{x_k} + \frac{2n_k U(x_k)}{x_k^2} \leqslant 2c.\end{aligned} \tag{5.16}$$

This implies (5.6), and now exactly as in the proof of Theorem 3.1 (but with (3.16) replacing (3.5)) we obtain a contradiction and thus establish (3.18).

For the converse, assume that (3.18) holds. Then, for some $x_k \to \infty$,

$$\frac{x_k^2}{f^{-1}(x_k)(x_k|A(x_k)| + U(x_k))} \to 0. \tag{5.17}$$

Let $n_k = f^{-1}(x_k)+1$, so that $f(n_k) > x_k$. Fix $\lambda > 0$. Then by (5.17), (5.2) and (5.11)

$$P\Big\{\max_{1\leqslant j\leqslant n_k}|S_j| \leqslant \lambda f(n_k)\Big\} \leqslant P\Big\{\max_{1\leqslant j\leqslant n_k}|S_j| \leqslant \lambda x_k\Big\} \leqslant \frac{c_\lambda x_k^2}{n_k(x_k|A(x_k)| + U(x_k))} = \frac{c_\lambda x_k^2}{(f^{-1}(x_k)+1)[x_k|A(x_k)| + U(x_k)]} \to 0.$$

This proves (3.16).

The equivalence between (3.14) and (3.18) is proved in exactly the same way as that of (3.3) and (3.7). One merely has to restrict r to an appropriate subsequence.

To go from (3.16) to (3.15) assume that (3.16) holds for some sequence $\{n_k\}$. Then take $r_k = f(n_k)/\varepsilon$. Of course $f(T) > \varepsilon r_k = f(n_k)$ implies $T > n_k$, so

$$P\{f(T(r_k)) > \varepsilon r_k\} \leqslant P\{T(r_k) > n_k\} = P\Big\{\max_{1\leqslant j\leqslant n_k}|S_j| \leqslant f(n_k)/\varepsilon\Big\} \to 0$$

by (3.16). Hence (3.15) holds.

Now $f(T(r_k))/r_k \xrightarrow{P} 0$ trivially implies (3.17) since $|S_{T(r_k)}| > r_k$.

The last step is to prove that (3.17) implies (3.16). We still have (5.13) and (5.14) when r is sufficiently far out in the sequence $\{r_k\}$ for which (3.17) holds. We apply these with $\alpha = \Lambda^p/\varepsilon$. We find that for each $\varepsilon > 0$ and $p = p(\varepsilon)$ such that $2^p > \varepsilon^{-2}$,

$$P\Big\{\overline{S}_{\lfloor 1/(\varepsilon k(r_k))\rfloor} \geqslant \frac{1}{\varepsilon} f\Big(\Big\lfloor \frac{1}{\varepsilon k(r_k)}\Big\rfloor\Big)\Big\} \geqslant P\Big\{\overline{S}_{\lfloor 1/(\varepsilon_k k(r_k))\rfloor} \geqslant \frac{\Lambda^p}{\varepsilon} f(\varepsilon/k(r_k))\Big\} \geqslant 1-(c_3+1)\varepsilon \quad \text{for } k \geqslant \text{some } k_0(\varepsilon).$$

We can therefore pick a sequence ε_k which tends to 0 sufficiently slowly that

$$P\Big\{\overline{S}_{\lfloor 1/(\varepsilon_k k(r_k))\rfloor} \geqslant \frac{1}{\varepsilon_k} f\Big(\Big\lfloor \frac{1}{\varepsilon_k k(r_k)}\Big\rfloor\Big)\Big\} \to 1 \quad \text{as } k \to \infty.$$

This implies (3.16) for $n_k = \lfloor 1/(\varepsilon_k k(r_k))\rfloor$. □

Proof of Theorem 3.3. – To prove that (3.19) implies (3.20) we use the following simple observation, which is a kind of converse to (5.12): Let $x \geqslant 0$ and define δ by

$$P\Big\{\max_{1\leqslant j\leqslant n}|S_j|>x\Big\}=2\delta.$$

Then at least one of the relations

$$P\Big\{\max_{1\leqslant j\leqslant n}S_j>x\Big\}\geqslant\delta \tag{5.18}$$

or

$$P\Big\{\max_{1\leqslant j\leqslant n}(-S_j)>x\Big\}\geqslant\delta \tag{5.19}$$

must hold. For the sake of definiteness, assume that (5.18) holds. Then define the stopping times

$$\widehat{T}_0=0,\quad \widehat{T}_j=\min\{n>\widehat{T}_{j-1}\colon S_n>S_{\widehat{T}_{j-1}}+x\},\quad j\geqslant 1$$

($\widehat{T}_j=\infty$ if $\widehat{T}_{j-1}=\infty$ or $S_n\leqslant S_{\widehat{T}_{j-1}}+x$ for all $n>\widehat{T}_{j-1}$). Then

$$\begin{aligned}P\{\widehat{T}_k\leqslant kn\}&\geqslant P\{\widehat{T}_j-\widehat{T}_{j-1}\leqslant n,\ 1\leqslant j\leqslant k\}\\&=\big[P\{\widehat{T}_1\leqslant n\}\big]^k\\&=\Big[P\Big\{\max_{1\leqslant j\leqslant n}S_j>x\Big\}\Big]^k\geqslant\delta^k.\end{aligned}$$

Thus, if (5.18) holds, then

$$P\Big\{\max_{1\leqslant j\leqslant kn}|S_j|>kx\Big\}\geqslant\Big[\frac{1}{2}P\Big\{\max_{1\leqslant j\leqslant n}|S_j|>x\Big\}\Big]^k. \tag{5.20}$$

In fact, this inequality is always valid, since one can replace X by $-X$ when (5.19) holds.

Now assume that (3.19) holds. Then for fixed $0<\varepsilon<1$, and n sufficiently large,

$$\begin{aligned}&\Big[\frac{1}{2}P\Big\{\max_{1\leqslant j\leqslant n}|S_j|>\varepsilon f(n)\Big\}\Big]^{\lceil 1/\varepsilon\rceil}\\&\quad\leqslant P\Big\{\max_{1\leqslant j\leqslant(\lceil 1/\varepsilon\rceil)n}|S_j|>\big(\lceil 1/\varepsilon\rceil\big)\varepsilon f(n)\Big\}\\&\quad\leqslant P\Big\{\max_{1\leqslant j\leqslant 2n/\varepsilon}|S_j|>f(n)\Big\}.\end{aligned}$$

Now write $r = r(n) = f(n)$. Then $f^{-1}(r) \geqslant n$ and

$$\begin{aligned} P\Big\{\max_{1\leqslant j\leqslant n} |S_j| > \varepsilon f(n)\Big\} &\leqslant 2\big[P\{T(f(n)) \leqslant 2n/\varepsilon\}\big]^{1/\lceil 1/\varepsilon\rceil} \\ &\leqslant 2\big[P\{T(r) \leqslant (2/\varepsilon) f^{-1}(r)\}\big]^{1/\lceil 1/\varepsilon\rceil}. \end{aligned}$$

The right hand side here tends to 0 as n, and hence r, tend to ∞. Thus (3.20) holds.

(3.20) and (3.21) are equivalent by Lemmas 4.3 and 4.5.

(3.21) implies

$$\frac{n|A(f(n))|}{f(n)} + \frac{nU(f(n))}{(f(n))^2} \to 0 \tag{5.21}$$

by the degenerate convergence criterion (cf. Gnedenko and Kolmogorov [7, p. 134]). Conversely (5.21) implies (3.21) by replacing X_i by $(X_i \wedge f(n)) \vee (-f(n))$ and an application of Chebychev's inequality. Also (5.21) and (3.22) are easily seen to be equivalent by the definition of f^{-1} and the fact that for $f(n) \leqslant x < f(n+1)$ we have $f^{-1}(x) = n$,

$$\frac{U(x)}{x^2} \leqslant \frac{U(f(n))}{(f(n))^2}$$

(since $U(x)/x^2$ decreases) and

$$\begin{aligned} \frac{|A(x)|}{x} &\leqslant \frac{|A(f(n))| + [x - f(n)]H(f(n))}{x} \\ &\leqslant \frac{|A(f(n))|}{f(n)} + H(f(n)) \leqslant \frac{|A(f(n))|}{f(n)} + \frac{U(f(n))}{(f(n))^2}. \end{aligned}$$

Finally, assume that (3.22) holds. Let $\lambda > 0$, $r > 0$ and $n = f^{-1}(r)$ (so $f(n) \leqslant r$), and use (5.2) and (5.11) to write

$$\begin{aligned} P\{T(r) \leqslant \lambda f^{-1}(r)\} = P\{\overline{S}_{\lfloor \lambda n\rfloor} > r\} &\leqslant P\{\overline{S}_{\lfloor \lambda n\rfloor} > f(n)\} \\ &\leqslant c_\lambda n\left(\frac{|A(f(n))|}{f(n)} + \frac{U(f(n))}{(f(n))^2}\right) \end{aligned}$$

for some $c_\lambda > 0$. The last expression converges to 0 as $n \to \infty$ by (5.21), so (3.19) holds. □

Remark. –

(ix) Note that the above proof yields the equivalence of (3.22) and (5.21). In a similar way, noting (5.10) and (5.11), we see that (3.7) implies

$$\frac{n|A(f(n))|}{f(n)} + \frac{nU(f(n))}{(f(n))^2} \to \infty. \tag{5.22}$$

In fact, (3.7) is equivalent to (5.22). To see this assume that (5.22) holds but (3.7) fails. Then there exists a sequence $x_k \uparrow \infty$ and a constant c such that

$$f^{-1}(x_k)\left(\frac{|A(f(x_k))|}{f(x_k)} + \frac{U(f(x_k))}{(f(x_k))^2}\right) \leqslant c. \tag{5.23}$$

Take $n_k = f^{-1}(x_k)$, so that $n_k \leqslant x_k < f(n_k + 1)$. Then we have

$$n_k H(f(x_k)) \leqslant n_k U(f(x_k))/(f(x_k))^2 \leqslant c,$$

and

$$\begin{aligned} &n_k\left(\frac{|A(f(x_k))|}{f(x_k)} + \frac{U(f(x_k))}{(f(x_k))^2}\right) \\ &\quad \geqslant n_k\left(\frac{|A(f(n_k+1))| - f(n_k+1)H(f(x_k))}{f(n_k+1)} + \frac{U(f(n_k+1))}{(f(n_k+1))^2}\right) \\ &\quad \geqslant n_k\left(\frac{|A(f(n_k+1))|}{f(n_k+1)} + \frac{U(f(n_k+1))}{(f(n_k+1))^2}\right) - c. \end{aligned}$$

Together with (5.23) this contradicts (5.22). This proves the claimed equivalence of (3.7) and (5.22).

Proof of Theorem 3.4. – If $S_n \to \infty$ a.s., then obviously $S_{T(r)} \to \infty$ a.s., so assume $S_{T(r)} \to \infty$ a.s. Then $S_{T(r)} \xrightarrow{P} \infty$ so $S_n \xrightarrow{P} \infty$ by Theorem 3 of Kesten and Maller [16]. If $U(\infty) < \infty$, then Theorem 2.1 of Kesten and Maller [13] shows that we must have $0 < EX \leqslant E|X| < \infty$ and consequently $S_n \to \infty$ a.s. (compare proof of corollary to Theorem 3.1). We therefore may assume that $U(\infty) = \infty$. Then we see from Theorem 2.1 and Lemma 4.3 in Kesten and Maller [13] that $A(x) > 0$ and $U(x) \leqslant 3xA(x)$ for large x, $x \geqslant x_0$, say. The proof now is like that of Theorem 3.1 of Griffin and Maller [8], but we give the details for completeness. The key point is to consider $T(r)$ along the 'right' subsequence of r's. Let $D(x)$ be as defined in (1.18), so that

$$\frac{nA(D(n))}{D(n)} = 1. \tag{5.24}$$

Define $k(x)$ as in (5.1). Then since $S_n \xrightarrow{P} \infty$, we have

$$\frac{A(x)}{x} \leqslant k(x) = \frac{A(x)}{x} + \frac{U(x)}{x^2} \leqslant \frac{4A(x)}{x} \quad (\text{for } x \geqslant x_0). \tag{5.25}$$

Thus from (5.24) we have, for large n,

$$1 \leqslant nk\big(D(n)\big) \leqslant 4. \tag{5.26}$$

Now define events $E(\ell)$ for $\ell = 1, 2, 3, \ldots$ by

$$E(\ell) = \{X_{T(D(2^\ell))} \leqslant -2LD(2^\ell)\}, \tag{5.27}$$

where $L > 2$ is some fixed integer. If $E(\ell)$ occurs then $S_{T(D(2^\ell))} < 0$, so by hypothesis, $P\{E(\ell)\text{ i.o.}\} = 0$. We shall prove later that

$$\sum_{\ell \geqslant 1} P\{E(\ell)\} < \infty. \tag{5.28}$$

Notice that, for $x > 2r$

$$\begin{aligned}
&P\{X_{T(r)} \leqslant -x\} \\
&\quad = \sum_{j \geqslant 1} P\{X_j \leqslant -x, T(r) = j\} \\
&\quad = \sum_{j \geqslant 1} P\Big\{X_j \leqslant -x, \max_{1 \leqslant \ell \leqslant j-1} |S_\ell| \leqslant r < |S_j|\Big\} \\
&\quad = \sum_{j \geqslant 1} P\Big\{X_j \leqslant -x, \max_{1 \leqslant \ell \leqslant j-1} |S_\ell| \leqslant r\Big\} \\
&\quad = F(-x) \sum_{j \geqslant 1} P\Big\{\max_{1 \leqslant \ell \leqslant j-1} |S_\ell| \leqslant r\Big\}.
\end{aligned}$$

Thus, by virtue of (5.4) there are constants $c_4 > 0$, $c_5 > 0$, such that

$$\frac{c_4 F(-x)}{k(r)} \leqslant P\{X_{T(r)} \leqslant -x\} \leqslant \frac{c_5 F(-x)}{k(r)}. \tag{5.29}$$

This estimate, combined with (5.28), gives

$$\begin{aligned}\infty &> \sum_{\ell\geqslant 1} P\{X_{T(D(2^\ell))} \leqslant -2LD(2^\ell)\}\\ &\geqslant c_4 \sum_{\ell>1} \frac{F(-2LD(2^\ell))}{k(D(2^\ell))} \quad \text{(by (5.29))}\\ &\geqslant \frac{c_4}{4}\sum_{\ell\geqslant 1} 2^\ell F\big(-2LD(2^\ell)\big) \quad \text{(by (5.26))}.\end{aligned}$$

But now we see from

$$\begin{aligned}\sum_{n\geqslant 2L} F\big(-2LD(n)\big) &= \sum_{\ell\geqslant 0}\ \sum_{2L2^\ell\leqslant n<2L2^{\ell+1}} F\big(-2LD(n)\big)\\ &\leqslant 2L\sum_{\ell\geqslant 0} 2^\ell F\big(-2LD(2^\ell)\big) \end{aligned}\tag{5.30}$$

that $\sum_{n\geqslant 1} F(-2LD(n))$ converges. It is necessary for our purposes to remove the factor $2L$ in this sum, that is, to prove

$$\sum_{n\geqslant 1} F\big(-D(n)\big) < \infty. \tag{5.31}$$

To this end we use the fact that for any constant $L > 0$

$$k(2Lx) \asymp k(x) \tag{5.32}$$

(see Griffin and Maller [8, Eq. 4.1 and Eq. 4.3]. Combined with (5.25) this shows that there exists some constant $c_6 = c_6(L)$ such that

$$\frac{A(2Lx)}{2Lx} \geqslant c_6\frac{A(x)}{x}, \quad x \geqslant x_0\left(1+\frac{1}{2L}\right).$$

But then, for large x

$$\begin{aligned}D(x) = \sup\left\{y \geqslant x_0\colon \frac{A(y)}{y} \geqslant \frac{1}{x}\right\} &\leqslant \sup\left\{y \geqslant x_0\colon \frac{A(2Ly)}{2Ly} \geqslant \frac{c_6}{x}\right\}\\ = \frac{1}{2L}\sup\left\{z \geqslant 2Lx_0\colon \frac{A(z)}{z} \geqslant \frac{c_6}{x}\right\} &= \frac{1}{2L}D\left(\frac{x}{c_6}\right).\end{aligned}$$

Since $D(\cdot)$ is increasing, it follows that

$$F\big(-2LD(n)\big) \geqslant F\big(-D(\lceil 1/c_6\rceil n)\big)$$

for large n. But then (5.31) follows immediately from the convergence of $\sum F(-2LD(n))$. We can now use Lemma 3.3 and Theorem 2.1 of Kesten and Maller [15], to conclude from (5.31) that $S_n \to \infty$ a.s.

To complete the proof of (3.23) we must show that $\sum_{\ell \geqslant 1} P\{E(\ell)\}$ converges, given that $P\{E(\ell)\,\text{i.o.}\} = 0$. To do this we use a generalised Borel–Cantelli lemma of Kochen and Stone [17] (see also Spitzer [21, p. 317]). Suppose by way of contradiction that $\sum P\{E(\ell)\} = \infty$. Take $j > k$ and consider

$$
\begin{aligned}
&P\{E(k) \cap E(j)\} \\
&\quad = \sum_{m=1}^{\infty} \sum_{\ell=m+1}^{\infty} P\{X_m \leqslant -2LD(2^k), T\big(D(2^k)\big) = m, X_\ell \leqslant -2LD(2^j), \\
&\qquad\qquad T\big(D(2^j)\big) = \ell\} \\
&\qquad + \sum_{m=1}^{\infty} P\{X_m \leqslant -2LD(2^j), T\big(D(2^k)\big) = T\big(D(2^j)\big) = m\} \\
&\quad = \Sigma_1(k, j) + \Sigma_2(k, j), \quad \text{say.}
\end{aligned} \tag{5.33}
$$

Now

$$
\begin{aligned}
&\Sigma_1(k, j) \\
&\quad = \sum_{m=1}^{\infty} \sum_{\ell=m+1}^{\infty} P\Big\{X_m \leqslant -2LD(2^k), \max_{1 \leqslant i \leqslant m-1} |S_i| \leqslant D(2^k) < |S_m|, \\
&\qquad\qquad X_\ell \leqslant -2LD(2^j), \max_{1 \leqslant i \leqslant \ell-1} |S_i| \leqslant D(2^j) < |S_\ell|\Big\} \\
&\quad = \sum_{m=1}^{\infty} \sum_{\ell=m+1}^{\infty} P\Big\{X_m \leqslant -2LD(2^k), \max_{1 \leqslant i \leqslant m-1} |S_i| < D(2^k), \\
&\qquad\qquad X_\ell \leqslant -2LD(2^j), \max_{1 \leqslant i \leqslant \ell-1} |S_i| \leqslant D(2^j)\Big\}
\end{aligned} \tag{5.34}
$$

$$
\begin{aligned}
&\quad \leqslant \sum_{m=1}^{\infty} \sum_{\ell=m+1}^{\infty} P\left\{X_m \leqslant -2LD(2^k), \max_{1 \leqslant i \leqslant m-1} |S_i| \leqslant D(2^k), \right. \\
&\qquad\qquad X_\ell \leqslant -2LD(2^j), \\
&\qquad\qquad \left. \max_{m+1 \leqslant i \leqslant \ell-1} \left| \sum_{s=m+1}^{i} X_s \right| \leqslant 2D(2^j) \right\}.
\end{aligned} \tag{5.35}
$$

The last step follows because, on the event in (5.34), $|S_m| \leqslant D(2^j)$ and $|S_i| \leqslant D(2^j)$ for $m+1 \leqslant i \leqslant \ell - 1$, so

$$\left|\sum_{s=m+1}^{i} X_i\right| = |S_i - S_m| \leqslant 2D(2^j).$$

From (5.27) we have

$$\begin{aligned}
&\Sigma_1(k,j) \\
&\leqslant \sum_{m=1}^{\infty}\sum_{\ell=m+1}^{\infty} P\Big\{X_m \leqslant -2LD(2^k),\ \max_{1\leqslant i\leqslant m-1}|S_i| \leqslant D(2^k)\Big\} \\
&\quad \times P\left\{X_\ell \leqslant -2LD(2^j),\ \max_{m+1\leqslant i\leqslant \ell-1}\left|\sum_{s=m+1}^{i} X_s\right| \leqslant 2D(2^j)\right\} \\
&= \sum_{m=1}^{\infty}\sum_{\ell=m+1}^{\infty} P\Big\{X_m \leqslant -2LD(2^k),\ \max_{1\leqslant i\leqslant m-1}|S_i| \leqslant D(2^k) < |S_m|\Big\} \\
&\quad \times P\left\{X_\ell \leqslant -2LD(2^j),\right. \\
&\qquad\qquad \left.\max_{m+1\leqslant i\leqslant \ell-1}\left|\sum_{s=m+1}^{i} X_s\right| \leqslant 2D(2^j) < \left|\sum_{s=m+1}^{\ell} X_s\right|\right\} \\
&= \sum_{m=1}^{\infty} P\{X_m \leqslant -2LD(2^k), T\big(D(2^k)\big) = m\} \\
&\quad \times \sum_{\ell=1}^{\infty} P\{X_\ell \leqslant -2LD(2^j), T\big(2D(2^j)\big) = l\} \\
&= P\big(E(k)\big) P\{X_{T(2D(2^j))} \leqslant -2LD(2^j)\}.
\end{aligned}$$

Now by (5.29) and (5.32)

$$\begin{aligned}
&P\{X_{T(2D(2^j))} \leqslant -2LD(2^j)\} \\
&\quad \leqslant \frac{c_5 F(-2LD(2^j))}{k(2D(2^j))} \leqslant c_7 \frac{F(-2LD(2^j))}{k(D(2^j))} \\
&\quad \leqslant \left(\frac{c_7}{c_4}\right) P\{X_{T(D(2^j))} \leqslant -2LD(2^j)\} = \left(\frac{c_7}{c_4}\right) P\big(E(j)\big).
\end{aligned}$$

Consequently we have shown that

$$\Sigma_1(k, j) \leqslant \left(\frac{c_7}{c_4}\right) P(E(k)) P(E(j)). \tag{5.36}$$

For the second term,

$$\begin{aligned}
\Sigma_2(k, j) &\leqslant P\{X_{T(D(2^k))} \leqslant -2LD(2^j)\} \\
&\leqslant \frac{c_5 F(-2LD(2^j))}{k(D(2^k))} \quad \text{(by (5.29))} \\
&\leqslant c_5 2^k F(-2LD(2^j)) \quad \text{(by (5.26))} \\
&= \frac{c_5 2^j F(-2LD(2^j))}{2^{j-k}} \\
&\leqslant \frac{4c_5 F(-2LD(2^j))}{k(D(2^j))2^{j-k}} \quad \text{(by (5.20))} \\
&\leqslant \left(\frac{4c_5}{c_4}\right) \frac{P\{X_{T(D(2^j))} \leqslant -2LD(2^j)\}}{2^{j-k}} \quad \text{(by (5.29))} \\
&= \left(\frac{4c_5}{c_4}\right) \frac{P(E(j))}{2^{j-k}}.
\end{aligned}$$

Combining this with (5.36) gives, by (5.33),

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sum_{j=k+1}^{N} P(E(k) \cap E(j)) \\
&\quad \leqslant \left(\frac{c_7}{c_4}\right) \sum_{k=1}^{N-1} \sum_{j=k+1}^{N} P(E(k)) P(E(j)) + \left(\frac{4c_5}{c_4}\right) \sum_{j=2}^{N} \sum_{k=1}^{j-1} \frac{P(E(j))}{2^{j-k}} \\
&\quad \leqslant \frac{c_7}{c_4} \left(\sum_{k=1}^{N} P(E(k))\right)^2 + \left(\frac{4c_5}{c_4}\right) \sum_{j=1}^{N} P(E(j)) \\
&\quad \leqslant \left(\frac{c_7}{c_4} + o(1)\right) \left(\sum_{k=1}^{N} P(E(k))\right)^2, \quad \text{as } N \to \infty.
\end{aligned}$$

By Kochen and Stone [17] or Spitzer [21, p. 317] we obtain $P(E(\ell)\text{ i.o.}) \geqslant c_4/c_7 > 0$, a contradiction. Thus indeed $S_n \to \infty$ a.s., as argued above. This completes the proof of (3.23).

Next we prove (3.24). Note that $S_n/n \to \infty$ a.s. implies $S_{T(r)}/T(r) \to \infty$ a.s., of course. So assume $S_{T(r)}/T(r) \to \infty$ a.s. Then $S_{T(r)} \to \infty$

a.s. So by what we've already proved, $S_n \to \infty$ a.s. This is equivalent to $0 < EX \le E|X| < \infty$ or $J_- < \infty = EX^+$, where

$$J_- = \int_{[0,\infty)} \frac{y}{\int_0^y [1 - F(u)]du} |dF(-y)|,$$

by (1.18) of Kesten and Maller [15]. Also $S_{T(r)}/T(r) \xrightarrow{P} \infty$ so $S_n/n \xrightarrow{P} \infty$ and $A(x) \to \infty$ by Theorem 4 of Kesten and Maller [16]. So we must have $J_- < \infty = EX^+$, equivalently, $S_n/n \to \infty$ a.s. by (1.15) of Kesten and Maller [15]. □

Proof of corollary to Theorem 3.4. – If $\limsup_{r\to\infty} S_{T(r)} > -\infty$ a.s. then trivially $\limsup_{n\to\infty} S_n > -\infty$ a.s., so $\limsup_{n\to\infty} S_n = +\infty$ a.s. Conversely let $\limsup_{n\to\infty} S_n = \infty$ a.s. and $\limsup_{r\to\infty} S_{T(r)} < \infty$ a.s. Then $P(S_{T(r)} > r \text{ i.o.}) = 0$ so $P(S_{T(r)} \geqslant -r \text{ i.o.}) = 0$ thus $S_{T(r)} \to -\infty$ a.s. By Theorem 3.4 (with + and – interchanged) we have $S_n \to -\infty$ a.s., a contradiction. Thus $\limsup_{r\to\infty} S_{T(r)} = \infty$ a.s. □

Demonstration of Example 3.5. We will find a negatively relatively stable S_n, i.e., such that

$$\frac{S_n}{D(n)} \xrightarrow{P} -1 \quad (n \to \infty) \tag{5.37}$$

(where $D(\cdot)$ is defined by (1.18) with $A(x)$ replaced by $-A(x)$, and enjoys the properties listed in Section 1), and such that

$$\frac{D(n)}{n} \to \infty, \tag{5.38}$$

but with

$$\sum_{j=D^{-1}(n)}^{n} \big(1 - F(xD(j))\big) \to \infty \quad (n \to \infty) \tag{5.39}$$

for all $x > 0$. (5.37) then implies $S_n/n \xrightarrow{P} -\infty$. We claim that in addition (5.37)–(5.39) imply

$$\frac{S_n^*}{n} \xrightarrow{P} \infty. \tag{5.40}$$

To prove (5.40), fix $x > 1$ and write, for any integer $\ell \leqslant n$,

$$P\left\{S_n^* > x \min_{\ell \leqslant j \leqslant n} D(j)\right\}$$
$$\geqslant \sum_{j=\ell}^{n} P\left\{S_{j-1} > -xD(j),\ \max_{j+1 \leqslant k \leqslant n} \frac{X_k}{D(k)} \leqslant 2x < \frac{X_j}{D(j)}\right\}. \quad (5.41)$$

We will choose $\ell = \ell(n) = D^{-1}(n)$. Since

$$P\{S_{j-1} \geqslant -xD(j)\} \to 1$$

when $j \to \infty$, as a result of (5.37), the regular variation of $D(\cdot)$ and $x > 1$, we see from (5.41) that

$$P\{S_n^* > (x-1)n\} \geqslant P\{S_n^* > xD(\ell(n))\}$$
$$\geqslant \sum_{j=\ell(n)}^{n} (1-\mathrm{o}(1)) P\left\{\max_{j+1 \leqslant k \leqslant n} \frac{X_k}{D(k)} \leqslant 2x < \frac{X_j}{D(j)}\right\}$$
$$= (1-\mathrm{o}(1)) P\{X_j > 2xD(j) \text{ for some } j \in [\ell(n), n]\}$$
$$= (1-\mathrm{o}(1)) \left(1 - \prod_{j=\ell(n)}^{n} F(2xD(j))\right).$$

Thus (5.40) will follow from (5.37)–(5.39), as claimed.

It remains to give an example where (5.37)–(5.39) hold. Define

$$L(x) = \mathrm{e}^{(\log x)^\beta}, \quad x > \mathrm{e},$$

and keep $\frac{1}{2} < \beta < 1$. Choose a distribution function F which satisfies

$$1 - F(x) = L'(x) = \frac{\beta(\log x)^{\beta-1} L(x)}{x} \text{ and } F(-x) = 2(1 - F(x))$$

for x large enough, $x \geqslant x_0$, say. (Note that $1 - F(x)$ and $F(-x)$ decrease to 0 as $x \to \infty$, as they should.) Then for $x > 0$

$$A(x) = \int_0^x [1 - F(y) - F(-y)]\, dy$$
$$= \int_0^{x_0} [1 - F(y)]\, dy + \int_{x_0}^{x} L'(y)\, dy - \int_0^{x_0} F(-y)\, dy - 2\int_{x_0}^{x} L'(y)\, dy$$
$$= -L(x) + c_0,$$

where c_0 is a constant. Thus $A(x) \to -\infty$ as $x \to \infty$ and

$$\frac{-A(x)}{x[1-F(x)+F(-x)]} \sim \frac{(\log x)^{1-\beta}}{3\beta} \to \infty \quad \text{as } x \to \infty. \tag{5.42}$$

Thus by (1.22), S_n is negatively relatively stable and (5.37) holds. The norming sequence $D(n)$ can be chosen to satisfy

$$D(n) = n\big[-A\big(D(n)\big)\big] \quad \text{and} \quad D^{-1}(n) = n/\big[-A(n)\big].$$

Since $-A(x) \to \infty$ as $x \to \infty$, (5.38) also holds. Now $L(x)$ is slowly varying, so $1 - F(x)$ is regularly varying as $x \to \infty$. It therefore suffices to check (5.39) with $x = 1$. Note also that $\log(D(n)) \sim \log n$ as $n \to \infty$, because $D(n)/[-A(D(n))] = n$ and $A(\cdot)$ is slowly varying. Hence

$$\begin{aligned}
\sum_{j=D^{-1}(n)}^{n} \big(1-F\big(D(j)\big)\big) &= \sum_{j=D^{-1}(n)}^{n} \frac{D(j)(1-F(D(j))}{-jA(D(j))} \\
&\sim \sum_{j=D^{-1}(n)}^{n} \frac{\beta(\log D(j))^{\beta-1}}{j} \\
&\geqslant \beta\big(\log D(n)\big)^{\beta-1} \sum_{j=D^{-1}(n)}^{n} \frac{1}{j} \\
&\geqslant c(\log n)^{\beta-1}\big(\log n - \log\big(D^{-1}(n)\big)\big)
\end{aligned}$$

for some constant $c > 0$. Since $D^{-1}(n) = n/[-A(n)]$ we have

$$\begin{aligned}
(\log n)^{\beta-1}\big(\log n - \log D^{-1}(n)\big) &= (\log n)^{\beta-1} \log\big(-A(n)\big) \\
&\sim \beta(\log n)^{\beta-1}(\log n)^{\beta} \\
&= \beta(\log n)^{2\beta-1} \to \infty \quad (n \to \infty)
\end{aligned}$$

(because $\beta > \frac{1}{2}$). Thus (5.39) holds too.

Proof of Theorem 3.6. – Suppose $P\{S_n \leqslant 0\} \to 1$ and $S_n^*/n \xrightarrow{P} \infty$. Note that the latter forces $1 - F(x) > 0$ for all x. By Kesten and Maller [16, Theorem 3 and Remark 3(iii)], the former then implies

$$\frac{A(x)}{\sqrt{U(x)(1-F(x))}} \to -\infty \quad (x \to \infty), \tag{5.43}$$

and consequently that $A(x) < 0$ for large x. We show, further, that $A(x) \to -\infty$ as $x \to \infty$. If not, there is a sequence $x_k \to \infty$ with

$x_k \geqslant 0$ and $-c \leqslant A(x_k) < 0$ for some constant c. Then by (5.43), $U(x_k)(1-F(x_k)) \to 0$ as $x_k \to \infty$. Take

$$n_k = \lceil U(x_k) \rceil. \tag{5.44}$$

Then, for $x > 0$,

$$P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} X_i I(X_i \leqslant x_k) > xn_k\right\} \leqslant P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} (X_i \wedge x_k) \vee (-x_k) > xn_k\right\},$$

because

$$X_i I(X_i \leqslant x_k) \leqslant X_i \wedge x_k \leqslant (X_i \wedge x_k) \vee (-x_k).$$

Set

$$X_i^k = (X_i \wedge x_k) \vee (-x_k),$$

and note that $E(X_i^k) = A(x_k)$ and $E(X_i^k)^2 = U(x_k)$. Since $A(x_k) < 0$ we have, by Kolmogorov's inequality,

$$P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} X_i I(X_i \leqslant x_k) > xn_k\right\} \leqslant P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} \left(X_i^k - A(x_k)\right) > xn_k\right\} \leqslant \frac{n_k U(x_k)}{x^2 n_k^2} \leqslant \frac{1}{x^2}.$$

Hence

$$\begin{aligned} P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} X_i I(X_i > x_k) \leqslant xn_k\right\} &\leqslant P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} X_i I(X_i > x_k) \leqslant xn_k, \right. \\ &\qquad \left. \max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} X_i I(X_i \leqslant x_k) \leqslant xn_k\right\} \\ &\quad + P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} X_i I(X_i \leqslant x_k) > xn_k\right\} \\ &\leqslant P\left\{\max_{1\leqslant j\leqslant n_k} S_j \leqslant 2xn_k\right\} + x^{-2} = x^{-2} + \mathrm{o}(1). \end{aligned}$$

It follows that

$$P\Big\{\max_{1\leqslant j\leqslant n_k} X_j \leqslant x_k\Big\} \leqslant P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} X_i I(X_i > x_k) = 0\right\}$$

$$\leqslant P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} X_i I(X_i > x_k) \leqslant x n_k\right\}$$

$$\leqslant x^{-2} + o(1).$$

First let $n_k \to \infty$, then $x \to \infty$. This shows that

$$P\Big\{\max_{1\leqslant j\leqslant n_k} X_j \leqslant x_k\Big\} = F^{n_k}(x_k) \to 0 \quad (n_k \to \infty).$$

Hence $n_k(1 - F(x_k)) \to \infty$. But then

$$U(x_k)\big(1 - F(x_k)\big) = \frac{U(x_k)}{n_k} n_k\big(1 - F(x_k)\big) \to \infty,$$

contradicting $U(x_k)(1 - F(x_k)) \to 0$. Thus indeed we have $A(x) \to -\infty$ and then $S_n/n \xrightarrow{P} -\infty$ follows from this, (5.43), and Theorem 4 and Remark 3(iii) of Kesten and Maller [16]. □

Proof of Theorem 3.7. – Let $S_n^*/n \xrightarrow{P} \infty$ and suppose

$$\max\left\{\frac{A(x_k)}{1 + \sqrt{U(x_k)(1 - F(x_k))}}, \sqrt{U(x_k)(1 - F(x_k))}\right\} \leqslant c \qquad (5.45)$$

for some $c < \infty$ and $x_k \to \infty$. Again define $n_k \to \infty$ by (5.44) and use the same notation as in the proof of Theorem 3.6 to write, for $x > 0$,

$$P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} X_i I(X_i \leqslant x_k) > x n_k\right\} \leqslant P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} X_i^k > x n_k\right\}.$$

We choose $x > 2c(1 + c)$ here. Now $A(x_k) \leqslant c(1 + c) < x/2$ by (5.45), so

$$P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} X_i^k > x n_k\right\} \leqslant P\left\{\max_{1\leqslant j\leqslant n_k} \sum_{i=1}^{j} \big(X_i^k - A(x_k)\big) > x n_k/2\right\}.$$

By Kolmogorov's inequality we get

$$P\left\{\max_{1\leqslant j\leqslant n_k}\sum_{i=1}^{j}X_i^k > xn_k\right\} \leqslant \frac{4n_kU(x_k)}{x^2n_k^2} \leqslant \frac{4}{x^2}.$$

Now follow exactly the same argument as in the proof of Theorem 3.6 to see that $F^{n_k}(x_k) \to 0$ and $n_k(1 - F(x_k)) \to \infty$. But, then, again, $U(x_k)(1 - F(x_k)) \to \infty$, contradicting (5.45). □

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