Annales de l'I. H. P., section B

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Annales de l'I. H. P., section B, tome 35, nº 5 (1999), p. 605-630

<http://www.numdam.org/item?id=AIHPB_1999__35_5_605_0>

Ann. Inst. Henri Poincaré,

Vol. 35, n° 5, 1999, p. 605-630. *Probabilités et Statistiques*

Gaussian lower bounds for random walks from elliptic regularity

by

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Article received on 16 September 1998

ABSTRACT. – We give a direct approach to off-diagonal lower bounds for reversible Markov chains on infinite graphs with regular volume growth and Poincaré inequalities, by using ideas that go back to De Giorgi and Morrey. © Elsevier, Paris

AMS classification: 60J15, 35K05

RÉSUMÉ. – On développe une théorie de la régularité elliptique pour le laplacien discret sur un graphe infini à croissance régulière du volume qui satisfait des inégalités de Poincaré, en suivant des idées de De Giorgi et Morrey. On en déduit une approche directe des bornes inférieures gaussiennes pour les marches aléatoires réversibles sur de tels graphes. © Elsevier, Paris

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1. INTRODUCTION

Consider a nearest neighbourhood random walk on an infinite graph Γ. Assume that the transition probability of the random walk is given by a Markov kernel $p(x, y)$, $x, y \in \Gamma$, which is reversible with respect to a positive measure $m(x)$ on Γ. Denote by $p_k(x, y)$ the convolution powers of $p(x, y)$, by $d(x, y)$ the combinatorial distance between the vertices x, y, and by $V(x, r)$ the volume of the ball $B(x, r) := \{y \mid d(x, y) \leqslant r\}$, that is $V(x, r) = \sum_{y \in B(x,r)} m(y)$. One says that Γ has regular volume growth, or satisfies the doubling property, if there exists C such that

$$V(x, 2r) \leqslant CV(x, r), \quad \forall x \in \Gamma, r > 0. \tag{D}$$

It is natural to expect (mainly because of the analogy with the behaviour of the heat kernel on non-compact Riemannian manifolds) that, if Γ satisfies (D) and if its geometry is sufficiently regular, then $p_k(x, y)$ is uniformly comparable from above and below to quantities of the type

$$\frac{m(y)}{V(x, \sqrt{k})} \exp\left(-c\frac{d^2(x, y)}{k}\right),$$

when $x, y \in \Gamma$, $k \in \mathbb{N}^*$ are such that $d(x, y) \leqslant k$ (otherwise $p_k(x, y)$ is zero!).

Indeed, Thierry Delmotte shows in [8] that the expected upper and lower bounds are equivalent to some geometric properties of Γ (or rather of the weighted graph associated with Γ and p), namely the doubling property and a family of Poincaré inequalities. This is the exact discrete time and discrete space analogue of the main result in [17], (see also [12], and [18] for a complete exposition).

On non-compact manifolds, the best known strategy to obtain similar heat kernel bounds goes through Moser's iteration [18]; this does not seem to work on graphs for discrete time. Delmotte's strategy is instead the following: using Moser's iteration, he proves a parabolic Harnack principle for the continuous time process p_t associated with p_k. The estimates follow for p_t, then a careful pointwise comparison between p_k and p_t gives the theorem. The method is fairly indirect, and none of its steps is straightforward. It appears therefore desirable to get a direct and truly discrete approach to the problem.

Some progress on that program is made in [4], where the upper bound is obtained for a strictly larger class of graphs than the ones with (D)

and (P). More precisely, for graphs with regular volume growth, the estimate

$$p_k(x,y) \leqslant \frac{Cm(y)}{V(x,\sqrt{k})} \exp\left(-c\frac{d^2(x,y)}{k}\right) \quad \forall x,y \in \Gamma, k \in \mathbb{N}^*$$

is shown to be equivalent to some kind of localised L^2 isoperimetric inequalities called relative Faber–Krahn inequalities. This is performed in purely discrete terms, except for a technical lemma whose only known proof uses the associated continuous-time semigroup.

In the present paper, we take up the second part of the program, namely to obtain, by purely discrete means, the lower bound of $p_k(x,y)$ for graphs with regular volume growth from the upper bound and the Poincaré inequalities. We avoid the parabolic Moser iteration by importing in this discrete setting the methods of [1].

The basic idea is to consider the parabolic heat equation $\Delta p_k = p_{k+1} - p_k$ as an inhomogeneous elliptic equation and to get Hölder estimates on p_k by applying the Morrey type elliptic regularity theory. This is done in Sections 3–5 below. This is the main part of the paper, which can be summarized by saying that it provides us with a direct derivation of the parabolic Harnack inequalities from the elliptic ones.

Indeed, the initial input needed by this method is a growth property of the Dirichlet integrals of harmonic functions on balls, called the De Giorgi property. This property turns out to be equivalent to an elliptic regularity estimate, which follows from the elliptic Harnack inequality.

The latter can be obtained by the Moser iteration which, in the elliptic case, does not meet the same obstacles as in the parabolic case (see [7]). However, in Section 6, we show that the De Giorgi method, which is classical for differential operators [6], also works in our setting and offers an alternative route to the elliptic regularity. As one can expect, the proof uses the John–Nirenberg lemma. However, assuming a slightly stronger version of the Poincaré inequalities, we also show, following the exposition by Giaquinta [11] of De Giorgi's ideas, that one can obtain the elliptic regularity without going through any kind of John–Nirenberg or Bombieri–Giusti type argument.

Our methods carry over very easily to the setting of heat kernels on non-compact manifolds, and yield an alternative approach to the results of [17] and [12]. This is, however, of less interest than in the discrete case, since when the time is continuous, the additional difficulty of the parabolic Moser iteration process with respect to the elliptic one is not so great.

Finally, these methods may also open the way to the study of operators with complex coefficients, in the spirit of [1], but in a discrete non-Euclidean context.

2. PRELIMINARIES

2.1. Notation and assumptions

Let Γ be an infinite graph. Assume that it is locally finite, i.e., every vertex of Γ has a finite number of neighbours. Write $x \sim y$ if $x, y \in \Gamma$ are neighbours. A path of length n between x and y in Γ is a sequence x_i, $0 = 1, \ldots, n$ such that $x_0 = x$, $x_n = y$ and $x_i \sim x_{i+1}$, $i = 0, \ldots, n-1$. Assume that Γ is connected, i.e., there exists a path between any two vertices. Let d be the natural metric on Γ: $d(x, y)$ is the minimal length of a path between x and y. Denote by $B(x, r)$ the closed ball of center $x \in \Gamma$ and radius $r > 0$.

Let p be a Markov kernel on Γ, reversible with respect to a measure m:

$$m(x) > 0, \quad \forall x \in \Gamma,$$

$$p(x, y) \geqslant 0, \ p(x, y)m(x) = p(y, x)m(y), \quad \forall x, y \in \Gamma,$$

$$\sum_{y \in \Gamma} p(x, y) = 1, \quad \forall x \in \Gamma.$$

Assume that $p(x, y) = 0$ if $d(x, y) \geqslant 2$ and that there exists $\kappa > 0$ such that

$$p(x, y) \geqslant \kappa \quad \text{if} \quad y \sim x \quad \text{or} \quad y = x. \tag{κ}$$

The latter hypothesis is crucial to obtain lower bounds (at least in the above form), see [8], and also [9] to see how one can deal with situations where it does not hold.

We stress that the above will be standing assumptions throughout this paper.

Next, set

$$\mu_{xy} := p(x, y)m(x) = \mu_{yx}.$$

Notice that all the information is contained in the function

$$\mu : \Gamma \times \Gamma \to \mathbb{R}_+,$$

since $x \sim y$ if and only if $\mu_{xy} > 0$ and $x \neq y$, then $m(x) = \sum_{y \sim x} \mu_{xy}$ and $p(x, y) = \frac{\mu_{xy}}{m(x)}$. The object under consideration in the sequel is therefore the weighted graph (Γ, μ).

Define $p_1(x, y) = p(x, y)$ and

$$p_k(x, y) = \sum_{z \in \Gamma} p_{k-1}(x, z) p(z, y), \quad k \geqslant 2.$$

The volume $|\Omega|$ of a subset Ω of Γ is defined by

$$|\Omega| = m(\Omega) = \sum_{x \in \Omega} m(x).$$

Denote as above by $V(x, r)$ the volume $|B(x, r)|$ of the ball $B(x, r)$. The ℓ^p norms on Γ are taken with respect to the measure m. For $u \in \mathbb{R}^\Gamma$ and $\Omega \subset \Gamma$, the notation

$$\sum_{\Omega} u \quad \text{stands for} \quad \sum_{x \in \Omega} u(x) m(x).$$

One says that (Γ, μ) has *regular volume growth*, or satisfies the doubling property, if there exists C such that

$$V(x, 2r) \leqslant C V(x, r), \quad \forall x \in \Gamma,\ r > 0. \tag{D}$$

Calculus on functions f defined on Γ is performed with the help of the following operators.

1. Gradient

$$\nabla_{xy} f := f(y) - f(x).$$

2. Length of the gradient

$$|\nabla f|(x) := \left(\frac{1}{2} \sum_{y \in \Gamma, x \sim y} |\nabla_{xy} f|^2 p(x, y) \right)^{1/2}.$$

Note that

$$\| \, |\nabla f| \, \|_2^2 = \frac{1}{2} \sum_{x, y \in \Gamma} |f(x) - f(y)|^2 \mu_{xy}.$$

3. Laplace operator

$$\Delta f(x) := \sum_{y\in\Gamma} p(x,y)f(y) - f(x) = \sum_{y\in\Gamma} p(x,y)\nabla_{xy} f$$
$$= \frac{1}{m(x)} \sum_{y\in\Gamma} (\nabla_{xy} f)\mu_{xy}.$$

The following integration by parts rule holds:
If one of the functions f, g on Γ has a finite support then

$$\sum_{\Gamma} (\Delta f) g = -\frac{1}{2} \sum_{x,y\in\Gamma} (\nabla_{xy} f)(\nabla_{xy} g)\mu_{xy}.$$

One says that (Γ,μ) satisfies the Poincaré inequality if there exists $C>0$ such that

$$\sum_{B(x,r)} |f - f_r(x)|^2 \leqslant Cr^2 \sum_{B(x,r)} |\nabla f|^2, \quad \forall f \in \mathbb{R}^{\Gamma},\ x \in \Gamma, r>0, \qquad (P)$$

where $f_r(x) := (1/V(x,r)) \sum_{B(x,r)} f$.

One says that $u \in \mathbb{R}^{\Gamma}$ is harmonic (respectively subharmonic) on $\Omega \subset \Gamma$ if

$$\Delta u(x) = 0 \text{ (respectively} \geqslant 0), \quad \forall x \in \Omega.$$

We shall use several times the fact that if u is harmonic on Ω and k is a real number, then $(u-k)_+$ is subharmonic on Ω.

The following Cacciopoli inequality is classical; in our setting see [7], Lemme 5.1.

PROPOSITION 2.1. – *There exists C such that, for all $x \in \Gamma$, $0<r<R$, and $u \in \mathbb{R}^{\Gamma}$ subharmonic on $B(x,R)$,*

$$\sum_{B(x,r)} |\nabla u|^2 \leqslant \frac{C}{(R-r)^2} \sum_{B(x,R)} |u|^2. \qquad (C)$$

Note that in the above inequality one can of course substract any constant to u in the right hand side, and that a good choice is $u_R(x)$.

One says that (Γ,μ) satisfies the De Giorgi property if there exists $C>0$ and $\alpha \in]0,1[$ such that for every $x \in \Gamma$, every r, R such that $1 \leqslant r \leqslant R$, and every $u \in \mathbb{R}^{\Gamma}$ which is harmonic on $B(x,R-1)$, one has

$$\sum_{B(x,r)} |\nabla u|^2 \leqslant C\left(\frac{R}{r}\right)^{2(1-\alpha)} \frac{V(x,r)}{V(x,R)} \sum_{B(x,R)} |\nabla u|^2. \qquad (DG)$$

We prove in Section 5 below that (*DG*) is a consequence of (D) and (P) through the elliptic regularity theory.

For $x \in \Gamma$ and $R \geqslant 1$, define

$$C_{R,x} = C_{R,x}(\Gamma) = \{f \in \mathbb{R}^{\Gamma};\ \operatorname{supp} f \subset B(x, R-1)\}.$$

One says that (Γ, μ) satisfies a relative Faber–Krahn inequality if there exist $C > 0$ and $\nu > 2$ such that

$$\|f\|_q \leqslant CR\big(V(x,R)\big)^{-1/\nu} \|\,|\nabla f|\,\|_2, \quad \forall x \in \Gamma,\ R \geqslant 2,\ f \in C_{R,x}, \tag{FK}$$

where $q = \frac{2\nu}{\nu-2}$. For more information and a variational formulation of (FK), see [4], §2.

Note the following easy consequence of (D): for all $R \geqslant r > 0$, for all $x \in \Gamma$ and $y \in B(x, R)$,

$$\frac{V(x,R)}{V(y,r)} \leqslant C\left(\frac{R}{r}\right)^{\theta}.$$

One can always take $\theta > 2$. The following lemma can be found in [7], p. 31, see also [4], §6.

LEMMA 2.2. – (D) *and* (P) *imply* (*FK*) *with* $\nu = \theta$.

2.2. Upper estimates

Assume that (Γ, μ) has regular volume growth. The main result of [4] is that the estimate

$$p_k(x,y) \leqslant \frac{Cm(y)}{V(x,\sqrt{k})} \exp\left(-c\frac{d^2(x,y)}{k}\right) \quad \forall x, y \in \Gamma,\ k \in \mathbb{N}^*$$

is equivalent to (*FK*). More precisely, together with (D), the on-diagonal upper estimate

$$p_k(x,x) \leqslant C\frac{m(x)}{V(x,\sqrt{k})} \tag{DUE}$$

implies (*FK*), which in turn implies (D) and the off-diagonal upper estimate

$$p_k(x,y) \leqslant \frac{Cm(y)}{V(x,\sqrt{k})} \exp\left(-c\frac{d^2(x,y)}{k}\right) \quad \forall x, y \in \Gamma,\ k \in \mathbb{N}^*. \tag{UE}$$

Recall that the latter together with (D) implies easily the on-diagonal lower bound

$$p_k(x,x) \geqslant \frac{cm(x)}{V(x,\sqrt{k})}$$

(see [4], §6). In contrast, the off-diagonal lower bound

$$p_k(x,y) \geqslant \frac{cm(y)}{V(x,\sqrt{k})} \exp\left(-c\frac{d^2(x,y)}{k}\right) \quad \forall x,y \in \Gamma,\ k \in \mathbb{N}^*$$

such that $d(x,y) \leqslant k$ may very well be false under these sole assumptions, as it is the case for the standard random walk on two copies of $\mathbb{Z}^2$ glued together by a single edge. To see this, notice first that on this graph (UE) holds (see, for instance, [3] for references) but (P) does not. Then apply Theorem 3.11 in [8].

For the rest of this section we shall assume that (Γ,μ) satisfies (UE), which is in particular the case if (P) holds together with (D).

Fix $y \in \Gamma$ and set

$$u_k^y(x) = u_k(x) = \frac{p_k(x,y)}{m(y)} = \frac{p_k(y,x)}{m(x)}, \quad x \in \Gamma.$$

Note that u_k is a solution of the discrete heat equation

$$\Delta u_k = u_{k+1} - u_k, \quad k \in \mathbb{N}^*.$$

The following estimate, whose consequences are to be used in Section 4, is derived from (UE) in [16], Lemma 5. The proof relies in a crucial way on the hypothesis (κ) on p. A much stronger and more difficult estimate can be found in [2], but we shall not need it here.

LEMMA 2.3. – *There exists C independent of y such that for every $k \in \mathbb{N}^*$, $x \in B(y,\sqrt{k})$,*

$$\left|u_k(x) - u_{k+1}(x)\right| \leqslant \frac{C}{kV(y,\sqrt{k})}.$$

COROLLARY 2.4. – *There exists C independent of $y \in \Gamma$ such that, for every $k \in \mathbb{N}^*$, r such that $0 \leqslant r \leqslant \sqrt{k}$, and $x \in B(y,\sqrt{k})$,*

$$\sum_{B(x,r)} |u_{k+1} - u_k|^2 \leqslant C\frac{V(x,r)}{k^2V^2(x,\sqrt{k})}, \tag{i}$$

and

$$\sum_{\Gamma} |\nabla u_k|^2 \leqslant \frac{C}{kV(x,\sqrt{k})}. \tag{ii}$$

Proof. – Notice first that

$$V(y,\sqrt{k}) \sim V(x,\sqrt{k})$$

since $x \in B(y,\sqrt{k})$. Then (i) is straightforward from Lemma 2.3. The estimate (ii) is in [16], Lemma 7, where one takes $\gamma = 0$, but in this particular case the proof follows from Lemma 2.3 and the discrete heat equation by a simple integration by parts. □

2.3. Statement of the main result

The following statement is contained in [8]. Our main goal is to prove it in a more direct way.

THEOREM 2.5. – *Suppose that the weighted graph (Γ,μ) satisfies (D) and (P). Then*

$$p_k(x,y) \geqslant \frac{cm(y)}{V(x,\sqrt{k})} \exp\left(-C\frac{d^2(x,y)}{k}\right), \quad \forall x,y \in \Gamma,\ k \in \mathbb{N}^* \tag{LE}$$

for all $x, y \in \Gamma$ such that $d(x,y) \leqslant k$.

Remember that, with our assumptions, $p_k(x,y) = 0$ if $d(x,y) > k$.

2.4. The Morrey–Campanato embedding

In this section, we only assume that (Γ,μ) satisfies (D). Let $R \geqslant 1$, $x \in \Gamma$ and $\gamma > 0$. For $f \in \mathbb{R}^\Gamma$, define the Morrey–Campanato norms

$$[f]_{x,R,\gamma} = \sup_{0 \leqslant r \leqslant R} \left(\frac{r^{-\gamma}}{V(x,r)} \sum_{B(x,r)} |f - f_r(x)|^2 \right)^{1/2}.$$

The following kind of characterisation of Hölder continuous functions through such norms is classical (see [10], III, 1).

PROPOSITION 2.6. – *Let $\gamma > 0$. There exists C depending only on the doubling constant and on γ such that, for every $R \geqslant 1$, $x, y \in \Gamma$ such that $d(x,y) \leqslant R$, and $f \in \mathbb{R}^\Gamma$, one has*

$$|f(x) - f(y)| \leqslant C([f]_{x,R,\gamma} + [f]_{y,R,\gamma})\, d(x,y)^{\gamma/2}.$$

Proof. – Fix first $x \in \Gamma$. Let $0 \leqslant r \leqslant R$ and $i \in \mathbb{N}$. Write

$$\begin{aligned}
&\left|f_{2^{-i}r}(x) - f_{2^{-i-1}r}(x)\right| \\
&\quad = \frac{1}{V(x,2^{-i-1}r)} \sum_{B(x,2^{-i-1}r)} \left|f_{2^{-i}r}(x) - f + f - f_{2^{-i-1}r}(x)\right| \\
&\quad \leqslant \left(\frac{1}{V(x,2^{-i-1}r)} \sum_{B(x,2^{-i}r)} \left|f - f_{2^{-i}r}(x)\right|^2\right)^{1/2} \\
&\qquad + \left(\frac{1}{V(x,2^{-i-1}r)} \sum_{B(x,2^{-i-1}r)} \left|f - f_{2^{-i-1}r}(x)\right|^2\right)^{1/2} \\
&\quad \leqslant [f]_{x,R,\gamma}\left(\sqrt{\frac{V(x,2^{-i}r)}{V(x,2^{-i-1}r)}}(2^{-i}r)^{\gamma/2} + (2^{-i-1}r)^{\gamma/2}\right) \\
&\quad \leqslant C[f]_{x,R,\gamma}(2^{-i}r)^{\gamma/2}.
\end{aligned}$$

It follows that

$$\left|f_r(x) - f(x)\right| \leqslant \sum_{i \in \mathbb{N}} \left|f_{2^{-i}r}(x) - f_{2^{-i-1}r}(x)\right| \leqslant C[f]_{x,R,\gamma} r^{\gamma/2}.$$

Now, if $x, y \in \Gamma$ and $r = d(x,y) \leqslant R$, one has

$$\left|f(x) - f(y)\right| \leqslant C\left([f]_{x,R,\gamma} + [f]_{y,R,\gamma}\right)(2r)^{\gamma/2} + \left|f_r(x) - f_r(y)\right|.$$

Finally

$$\begin{aligned}
&\left|B(x,r) \cap B(y,r)\right| \left|f_r(x) - f_r(y)\right|^2 \\
&\quad \leqslant 2\left(\sum_{B(x,r)} \left|f_r(x) - f\right|^2 + \sum_{B(y,r)} \left|f_r(y) - f\right|^2\right) \\
&\quad \leqslant 2\left(V(x,r)[f]^2_{x,R,\gamma} + V(y,r)[f]^2_{y,R,\gamma}\right) r^{\gamma},
\end{aligned}$$

and thanks to doubling, $|B(x,r) \cap B(y,r)|$, $V(x,r)$ and $V(y,r)$ are uniformly comparable.

This yields

$$\left|f(x) - f(y)\right| \leqslant C\left([f]_{x,R,\gamma} + [f]_{y,R,\gamma}\right) r^{\gamma/2},$$

which is the claim. □

3. FROM THE PARABOLIC OSCILLATION ESTIMATE TO THE OFF-DIAGONAL LOWER BOUND

Assume that (UE) holds, and that there exist $\beta, \delta > 0$, and $C > 0$ independent of y such that, in the notation of Section 2.3,

$$\left|u_k(x) - u_k(y)\right| \leqslant \left(\frac{d(x,y)}{\sqrt{k}}\right)^{\beta} \frac{C}{V(y,\sqrt{k})},$$
$$\forall x \in \Gamma \text{ such that } d(x,y) \leqslant \delta\sqrt{k}. \qquad (PO)$$

In other terms, $\forall x, y \in \Gamma$ such that $d(x,y) \leqslant \delta\sqrt{k}$,

$$\left|p_k(x,y) - p_k(y,y)\right| \leqslant C\left(\frac{d(x,y)}{\sqrt{k}}\right)^{\beta} \frac{m(y)}{V(x,\sqrt{k})}.$$

Now recall that the on-diagonal lower bound

$$p_k(y,y) \geqslant c\frac{m(y)}{V(y,\sqrt{k})}$$

is a consequence of the upper bound and the doubling property ([4], §6).

It follows that

$$\left|p_k(x,y) - p_k(y,y)\right| \leqslant C'\left(\frac{d(x,y)}{\sqrt{k}}\right)^{\beta} p_k(y,y),$$
$$\forall x, y \in \Gamma \text{ such that } d(x,y) \leqslant \delta\sqrt{k},$$

and if one chooses $a \leqslant \delta$ such that $C'a^{\beta} \leq \frac{1}{2}$, one has

$$p_k(x,y) \geqslant \frac{1}{2}p_k(y,y) \geqslant c'\frac{m(y)}{V(y,\sqrt{k})} \geqslant c''\frac{m(y)}{V(x,\sqrt{k})},$$

for all $x, y \in \Gamma$, $k \in \mathbb{N}^*$, as soon as $d(x,y) \leqslant a\sqrt{k}$.

From there a classical iteration argument ([5,14]) yields the full off-diagonal lower bound

$$p_k(x,y) \geqslant c\frac{m(y)}{V(x,\sqrt{k})}\exp\left(-c\frac{d^2(x,y)}{k}\right)$$

for all $x, y \in \Gamma$ such that $d(x,y) \leqslant k$.

Our main task is therefore to obtain the parabolic oscillation estimate (PO). To this end, thanks to Proposition 2.7, it is enough to control the norm of the functions u_k in some suitable Morrey–Campanato spaces.

4. FROM DE GIORGI'S PROPERTY TO THE PARABOLIC OSCILLATION ESTIMATE

From now on we assume that (Γ, μ) satisfies (FK) (therefore (D), see [4], §2).

Let us begin with two lemmas. The first one follows directly from (FK) by Hölder. Recall that $C_{R,x} = \{f \in \mathbb{R}^\Gamma;\ \operatorname{supp} f \subset B(x, R-1)\}$.

LEMMA 4.1. – *There exists $C > 0$ such that, for every $x \in \Gamma$, $R \geqslant 1$, $u \in C_{R,x}(\Gamma)$,*

$$\sum_{B(x,R)} |u|^2 \leqslant C R^2 \sum_{B(x,R)} |\nabla u|^2.$$

Our second lemma is an application of Lax–Milgram.

LEMMA 4.2. – *Let $u \in \mathbb{R}^\Gamma$, $x \in \Gamma$, $R \geqslant 1$. There exists a unique $v \in C_{R,x}$ such that $\Delta v = \Delta u$ on $B(x, R-1)$. Moreover,*

$$\sum_{B(x,R)} |\nabla v|^2 \leqslant \sum_{B(x,R)} |\nabla u|^2.$$

Proof. – Set

$$\langle f, g\rangle = \frac{1}{2} \sum_{y,z\in\Gamma} (\nabla_{yz} f)(\nabla_{yz} g)\mu_{yz}, \quad f, g \in \mathbb{R}^\Gamma.$$

This bilinear form restricted to $C_{R,x}$ is a scalar product:

$$\langle f, f\rangle^{1/2} = \Big(\sum_{B(x,R)} |\nabla f|^2\Big)^{1/2}$$

yields a norm by Lemma 4.1.

The representation theorem of Riesz shows the existence of a unique $v \in C_{R,x}$ such that

$$\langle v, g\rangle = \langle u, g\rangle, \quad \forall g \in C_{R,x}.$$

Now

$$\sum_{B(x,R)} |\nabla v|^2 = \langle v, v\rangle = \langle u, v\rangle$$
$$\leqslant \Big(\frac{1}{2} \sum_{y,z\in B(x,R)} |\nabla_{yz} u|^2 \mu_{yz}\Big)^{1/2} \Big(\sum_{B(x,R)} |\nabla v|^2\Big)^{1/2}.$$

Since one clearly has

$$\sum_{y,z\in B(x,R)} |\nabla_{yz}u|^2\mu_{yz} \leqslant \sum_{B(x,R)} |\nabla u|^2,$$

this yields the final assertion of the lemma.

Finally, if $f\in\mathbb{R}^\Gamma$ and $g\in C_{R,x}$, integration by parts gives

$$\langle f, g\rangle = -\sum_{B(x,R-1)} (\Delta f)g.$$

which implies

$$\Delta u = \Delta v \quad \text{on } B(x, R-1).$$

This ends the proof. □

Note that $u-v$ can also be seen as the solution to the Dirichlet problem in $B(x, R)$ with boundary values equal to u.

We now state a first estimate for the inhomogeneous equation $\Delta u = f$.

LEMMA 4.3. – *Let $f\in\mathbb{R}^\Gamma$ and $u\in C_{R,x}$ be such that $\Delta u = f$ on $B(x, R-1)$. Then*

$$\sum_{B(x,R)} |\nabla u|^2 \leqslant CR^2 \sum_{B(x,R)} |f|^2.$$

Proof. – By integration by parts,

$$\begin{aligned}\sum_{B(x,R)} |\nabla u|^2 &= \frac{1}{2}\sum_{y,z\in\Gamma} (\nabla_{yz}u)(\nabla_{yz}u)\mu_{yz}\\ &= -\sum(\Delta u)u\\ &= -\sum_{B(x,R)} fu.\end{aligned}$$

Cauchy–Schwarz gives then

$$\Big(\sum_{B(x,R)} |\nabla u|^2\Big)^2 \leqslant \Big(\sum_{B(x,R)} |f|^2\Big)\Big(\sum_{B(x,R)} |u|^2\Big).$$

The conclusion follows by Lemma 4.1. □

The following proposition is the key to our analysis. It is similar to Morrey's fundamental estimate in his treatment of inhomogeneous elliptic equations (see [15]). The idea to use it in order to obtain parabolic estimates comes from [1].

PROPOSITION 4.4. – *Assume that* (Γ, μ) *satisfies* (*FK*) *and* (*DG*). *If* $u, f \in \mathbb{R}^\Gamma$ *and* $\Delta u = f$ *on* $\mathbb{R}^\Gamma$, *then,* $\forall x \in \Gamma$, $1 \leqslant r \leqslant R$, *one has*

$$\sum_{B(x,r)} |\nabla u|^2 \leqslant 8C\left(\frac{R}{r}\right)^{2(1-\alpha)} \frac{V(x,r)}{V(x,R)} \sum_{B(x,R)} |\nabla u|^2 + 2C'R^2 \sum_{B(x,R)} |f|^2,$$

where $C > 0$ *and* $0 < \alpha < 1$ *are the constants in De Giorgi's property, and* C' *is the constant in Lemma* 4.3.

Proof. – By Lemma 4.2, there exists $v \in C_{x,R}(\Gamma)$ such that $\Delta v = f$ on $B(x, R-1)$. Set $w = u - v$. Write

$$\sum_{B(x,r)} |\nabla u|^2 \leqslant 2\left(\sum_{B(x,r)} |\nabla w|^2 + \sum_{B(x,r)} |\nabla v|^2\right).$$

Since $\Delta w = 0$ on $B(x, R-1)$, (*DG*) yields

$$\sum_{B(x,r)} |\nabla w|^2 \leqslant \left(\frac{R}{r}\right)^{2(1-\alpha)} \frac{V(x,r)}{V(x,R)} \sum_{B(x,R)} |\nabla w|^2.$$

Lemma 4.2 says that

$$\sum_{B(x,R)} |\nabla v|^2 \leqslant \sum_{B(x,R)} |\nabla u|^2,$$

hence

$$\sum_{B(x,R)} |\nabla w|^2 \leqslant 4 \sum_{B(x,R)} |\nabla u|^2.$$

This gives the first term in our estimate. Now Lemma 4.3 yields

$$\sum_{B(x,r)} |\nabla v|^2 \leqslant \sum_{B(x,R)} |\nabla v|^2 \leqslant CR^2 \sum_{B(x,R)} |f|^2.$$

The conclusion follows. □

We are now able to prove our key result. In the next section we shall see that in fact (*DG*) follows from (*D*) and (*P*).

PROPOSITION 4.5. – *Suppose that* (Γ, μ) *satisfies* (D), (P), *and* (DG). *Then the parabolic oscillation estimate* (PO), *therefore the lower bound* (LE), *holds.*

Proof. – By Lemma 2.2, (D) and (P) imply (FK), therefore we can use Proposition 4.4. Let $y \in \Gamma$. Define $u_k = u_k^y$ as in Section 2.2. For $x \in \Gamma$, $k \in \mathbb{N}^*$ and $r > 0$, set

$$\psi_k^x(r) = \psi_k(r) = \sum_{B(x,r)} |\nabla u_k|^2.$$

Since

$$\Delta u_k = u_{k+1} - u_k,$$

Proposition 4.4 yields, for $1 \leqslant r \leqslant R$,

$$\psi_k(r) \leqslant 8C\left(\frac{R}{r}\right)^{2(1-\alpha)} \frac{V(x,r)}{V(x,R)}\psi_k(R) + 2C'R^2 \sum_{B(x,R)} |u_{k+1} - u_k|^2,$$

with $0 < \alpha < 1$. According to Corollary 2.4, (i), ψ_k satisfies, if $1 \leqslant r \leqslant R \leqslant \sqrt{k}$,

$$\psi_k(r) \leqslant 8C\left(\frac{R}{r}\right)^{2(1-\alpha)} \frac{V(x,r)}{V(x,R)}\psi_k(R) + C''\frac{R^2V(x,R)}{k^2V^2(x,\sqrt{k})}. \qquad (*)$$

Fix β, $0 < \beta < \alpha$ and choose $\tau \in]0,1[$ such that $8C\tau^{2(\alpha-1)} \leqslant \tau^{2(\beta-1)}$. Then, if $\tau R \geqslant 1$ and $R \leqslant \sqrt{k}$,

$$\psi_k(\tau R) \leqslant \tau^{2(\beta-1)} \frac{V(x,\tau R)}{V(x,R)}\psi_k(R) + C''\frac{R^2V(x,R)}{k^2V^2(x,\sqrt{k})}.$$

By iterating, one obtains

$$\psi_k(\tau^{j+1}R) \leqslant \tau^{2(\beta-1)(j+1)} \frac{V(x,\tau^{j+1}R)}{V(x,R)}\psi_k(R) + \frac{C''}{k^2V^2(x,\sqrt{k})} \times \sum_{\ell=0}^{j} \tau^{2(\beta-1)(j-\ell)} \frac{V(x,\tau^{j+1}R)}{V(x,\tau^{\ell+1}R)}(\tau^\ell R)^2 V(x,\tau^\ell R).$$

By doubling, $\frac{V(x,\tau^\ell R)}{V(x,\tau^{\ell+1}R)} \leqslant C(\tau)$, so that the second term is smaller than

$$C'(\tau)\frac{R^2V(x,\tau^{j+1}R)}{k^2V^2(x,\sqrt{k})}\tau^{2(\beta-1)j}.$$

Set now $R=\sqrt{k}$, and, for $r\in[1,\sqrt{k}]$, let $j\in\mathbb{N}$ be such that

$$\tau^{j+2}\sqrt{k}\leqslant r\leqslant\tau^{j+1}\sqrt{k}.$$

One has

$$\begin{aligned}\psi_k(r)\leqslant C_1(\tau)\left(\frac{\sqrt{k}}{r}\right)^{2(1-\beta)}\frac{V(x,r)}{V(x,\sqrt{k})}\psi_k(\sqrt{k})\\+C_2(\tau)\frac{V(x,r)}{kV^2(x,\sqrt{k})}\left(\frac{\sqrt{k}}{r}\right)^{2(1-\beta)}.\end{aligned}\tag{$**$}$$

Since Corollary 2.4, (ii) tells us that $\psi_k(\sqrt{k})\leqslant\frac{C}{kV(x,\sqrt{k})}$, we obtain, for all $x\in\Gamma$, $k\in\mathbb{N}^*$, and $1\leqslant r\leqslant\tau\sqrt{k}$,

$$\sum_{B(x,r)}|\nabla u_k|^2\leqslant C'\left(\frac{\sqrt{k}}{r}\right)^{2(1-\beta)}\frac{V(x,r)}{kV^2(x,\sqrt{k})}.$$

By Poincaré,

$$\sum_{B(x,r)}\left|u_k-(u_k)_r(x)\right|^2\leqslant C'\left(\frac{r}{\sqrt{k}}\right)^{2\beta}\frac{V(x,r)}{V^2(x,\sqrt{k})},$$

i.e., in the notation of Section 2.4,

$$[u_k]_{x,\tau\sqrt{k},2\beta}\leqslant\frac{C}{k^{\beta/2}V(x,\sqrt{k})}.$$

Recall now that $u_k=u_k^y$, with $d(x,y)\leqslant\sqrt{k}$, hence $V(x,\sqrt{k})\sim V(y,\sqrt{k})$. We have therefore

$$[u_k]_{x,\tau\sqrt{k},2\beta}\leqslant\frac{C}{k^{\beta/2}V(y,\sqrt{k})},\quad\forall x\in B(y,\sqrt{k}),$$

and Proposition 2.7 yields

$$\left|u_k(x)-u_k(y)\right|\leqslant\frac{C}{k^{\beta/2}V(y,\sqrt{k})}d(x,y)^\beta,$$
$$\forall x\text{ such that }d(x,y)\leqslant\tau\sqrt{k},$$

i.e., the estimate *(PO)*. The lower bound *(LE)* follows from Section 3. □

The step from $(*)$ to $(**)$ in the above proof was very much inspired by the Lemma in [11], p. 44.

5. FROM THE ELLIPTIC REGULARITY TO DE GIORGI'S PROPERTY

In [7], Delmotte proves the following result (Propositions 5.3 and 6.2), as a consequence of the elliptic Moser iteration process.

PROPOSITION 5.1. – *Assume that (Γ,μ) satisfies (D) and (P). Then there exists $\alpha, C>0$ such that for every $x_0\in\Gamma$, $R\geqslant 1$, $u\in\mathbb{R}^\Gamma$ harmonic in $B(x_0,R)$ and $x,y\in B(x_0,R/4)$, one has*

$$\begin{aligned}&\left|u(x)-u(y)\right|\\&\quad\leqslant C\left(\frac{d(x,y)}{R}\right)^{\alpha}\left(\frac{1}{V(x_0,R)}\sum_{B(x_0,R)}|u-u_R(x_0)|^2\right)^{1/2}.\qquad (ER)\end{aligned}$$

The estimate (*DG*) follows from this result. Indeed, if u is harmonic in $B(x_0,R)$ (we change $R-1$ to R for convenience) and $1\leqslant r\leqslant R/8$, write, for $x\in B(x_0,2r)$,

$$\begin{aligned}&\left|u(x)-u_{2r}(x_0)\right|\\&\quad\leqslant\frac{1}{V(x_0,2r)}\sum_{y\in B(x_0,2r)}\left|u(x)-u(y)\right|m(y)\leqslant\frac{C}{V(x_0,2r)}\\&\qquad\times\sum_{y\in B(x_0,2r)}\left(\frac{d(x,y)}{R}\right)^{\alpha}\left(\frac{1}{V(x_0,R)}\sum_{B(x_0,R)}\left|u-u_R(x_0)\right|^2\right)^{1/2}m(y)\\&\quad\leqslant C\left(\frac{r}{R}\right)^{\alpha}\left(\frac{1}{V(x_0,R)}\sum_{B(x_0,R)}\left|u-u_R(x_0)\right|^2\right)^{1/2},\end{aligned}$$

thus

$$\begin{aligned}&\left(\sum_{x\in B(x_0,2r)}\left|u(x)-u_{2r}(x_0)\right|^2m(x)\right)^{1/2}\\&\quad\leqslant C\left(\frac{r}{R}\right)^{\alpha}\left(\frac{V(x_0,2r)}{V(x_0,R)}\sum_{B(x_0,R)}\left|u-u_R(x_0)\right|^2\right)^{1/2}.\end{aligned}$$

Applying doubling, Poincaré and Cacciopoli (Proposition 2.1), one obtains

$$\left(\sum_{B(x_0,r)} |\nabla u|^2\right)^{1/2} \leqslant C\left(\frac{r}{R}\right)^{\alpha-1}\left(\frac{V(x_0,r)}{V(x_0,R)}\sum_{B(x_0,R)} |\nabla u|^2\right)^{1/2},$$

if $1 \leqslant r \leqslant R/8$. The case $R/8 < r \leqslant R$ being trivial, this yields (DG).

Theorem 2.5 is proved.

Note that under (D) and (P), one can also easily deduce (ER) from (DG), with the help of Section 2.4.

6. ALTERNATIVE PROOFS OF THE ELLIPTIC REGULARITY

In this section, we present an alternative proof of Proposition 5.1, following the scheme of the De Giorgi method (or rather its exposition by Giaquinta) for regularity of solutions of elliptic PDEs (see [6,11]). An additional ingredient needed is a John-Nirenberg type lemma taken from [7]. We also show that, if one makes a slightly stronger assumption than (P), one can give a self-contained and elementary proof of (ER), therefore of the lower bound in Theorem 2.5, that does not rely on the John–Nirenberg lemma (or alternatively the Bombieri–Giusti lemma).

We first use the fact that (FK) implies an L^2 mean value property for harmonic functions. This is also a step in the proof of (ER) in [7] (Proposition 5.3). However, we give a slightly different proof, inspired by [11], Theorem 5.1.

PROPOSITION 6.1. – *Assume that* (Γ,μ) *satisfies* (FK). *Then there exists* $C>0$ *such that, for all* $R>0$, $x_0 \in \Gamma$, *and* $u \in \mathbb{R}^\Gamma$ *harmonic in* $B(x_0,R)$,

$$\max_{x\in B(x_0,R/2)} u(x) \leqslant C\left(\frac{1}{V(x_0,R)}\sum_{B(x_0,R)} u_+^2\right)^{1/2}.$$

Proof. – By Hölder, for $q=\frac{2\theta}{\theta-2}$, $\rho>0$ and $h\in\mathbb{R}$,

$$\sum_{B(x_0,\rho)} (u-h)_+^2 \leqslant a(h,\rho)^{2/\theta}\left(\sum_{B(x_0,\rho)} (u-h)_+^q\right)^{2/q}.$$

Apply now (FK), to obtain

$$\sum_{B(x_0,\rho)} (u-h)_+^2 \leqslant C\rho^2 V(x_0,\rho)^{-2/\theta} a(h,\rho)^{2/\theta}\Big(\sum_{B(x_0,\rho)} |\nabla (u-h)_+|^2\Big).$$

Now $(u-h)_+$ is subharmonic and Proposition 2.1 yields, for $r \in]\rho, R]$,

$$\sum_{B(x_0,\rho)} (u-h)_+^2 \leqslant C\frac{\rho^2}{(r-\rho)^2} V(x_0,\rho)^{-2/\theta} a(h,\rho)^{2/\theta}\Big(\sum_{B(x_0,r)} (u-h)_+^2\Big).$$

Denote by $A^u(h,\rho) = A(h,\rho)$ the set $\{x \in B(x_0,\rho); u(x) \geqslant h\}$ and by $a(h,\rho)$ its measure. Set

$$u(h,\rho) = \sum_{x\in B(x_0,\rho)} (u-h)_+^2 = \sum_{A(h,\rho)} (u-h)^2.$$

One has

$$u(h,\rho) \leqslant C\frac{\rho^2}{(r-\rho)^2} V(x_0,\rho)^{-2/\theta} a(h,\rho)^{2/\theta} u(h,r). \tag{1}$$

Moreover, for $h > k$,

$$(h-k)^2 a(h,\rho) \leq \sum_{A(h,\rho)} (u-k)^2 \leqslant \sum_{A(k,\rho)} (u-k)^2,$$

that is

$$a(h,\rho) \leqslant \frac{1}{(h-k)^2} u(k,\rho). \tag{2}$$

Using the fact that $u(h,\rho)$ is non-increasing in h and non-decreasing in ρ, (1) and (2) yield

$$u(h,\rho) \leqslant C\frac{\rho^2}{(r-\rho)^2} V(x_0,\rho)^{-2/\theta} \frac{1}{(h-k)^{4/\theta}} u(k,r)^{1+\frac{2}{\theta}}. \tag{3}$$

Set $k_n = (1-\frac{1}{2^n})d$ and $\rho_n = \frac{R}{2}(1+\frac{1}{2^n})$, $n \in \mathbb{N}$, where

$$d = C^{\theta/4} 2^{\frac{\theta^2}{4}+\theta+1} V(x_0,R)^{-1/2} u(0,R)^{1/2}.$$

Inequality (3) yields

$$u(k_{n+1}, \rho_{n+1}) \leqslant C \frac{2^{2(n+2)+\frac{4}{\theta}(n+1)} \rho_{n+1}^2}{R^2 d^{4/\theta}} V(x_0, \rho_{n+1})^{-2/\theta} u(k_n, \rho_n)^{1+\frac{2}{\theta}}$$

therefore, since $\frac{V(x,R)}{V(x,\rho_{n+1})} \leqslant C\left(\frac{R}{\rho_{n+1}}\right)^{\theta}$,

$$u(k_{n+1}, \rho_{n+1}) \leqslant C \frac{2^{2(n+2)+\frac{4}{\theta}(n+1)}}{d^{4/\theta}} V(x_0, R)^{-2/\theta} u(k_n, \rho_n)^{1+\frac{2}{\theta}}. \tag{4}$$

From (4) one proves easily by induction that

$$u(k_n, \rho_n) \leqslant \frac{u(0, R)}{2^{\mu n}}, \quad \forall n \in \mathbb{N}, \tag{5}$$

with $\mu = \theta + 2$.

By letting n go to infinity, one concludes that

$$u(d, R/2) = 0.$$

This means that, for all $x \in B(x_0, R/2)$,

$$u(x) \leqslant d = C\left(\frac{1}{V(x_0, R)} \sum_{A(0,R)} u^2\right)^{1/2}.$$

The lemma is proved. □

Set $M(r) = \max_{x \in B(x_0,r)} u(x)$ and $m(r) = \min_{x \in B(x_0,r)} u(x)$. Applying Proposition 6.1 to $u - h$, where $h \in \mathbb{R}$ is such that $h \leqslant M(R)$, one obtains

$$M(R/2) - h \leqslant C(M(R) - h)\left(\frac{a(h, R)}{V(x_0, R)}\right)^{1/2}. \tag{6}$$

Here is the crucial lemma.

LEMMA 6.2. – *Assume that (Γ, μ) satisfies (D) and (P). Let $u \in \mathbb{R}^\Gamma$ be harmonic in $B(x_0, R)$, $x_0 \in \Gamma$, $R > 0$. Set $M = M(R/2)$, $m = m(R/2)$, and $h_0 = h_0(R/2) = \frac{M+m}{2}$. There exists C independent of x_0, R and u such that, if $a(h_0, R/2) \leqslant \frac{V(x_0,R/2)}{2}$, then for all h with $m < h < M$,*

$$\frac{a(h, R/2)}{V(x_0, R/2)} \leqslant C\left[\log\left(\frac{M-m}{M-h}\right)\right]^{-1}.$$

Proof. – Normalize u so that $M = 1$ and $m = 0$. Then

$$\log(1-h)\frac{a(h,R/2)}{V(x_0,R/2)} \geqslant \frac{1}{V(x_0,R/2)} \sum_{A(h,R/2)} \log(1-u)$$
$$\geqslant \frac{1}{V(x_0,R/2)} \sum_{B(x_0,R/2)} \log(1-u).$$

Hence, for every $\alpha > 0$,

$$\exp\left(-\alpha \log(1-h)\frac{a(h,R/2)}{V(x_0,R/2)}\right)$$
$$\leqslant \frac{1}{V(x_0,R/2)} \sum_{B(x_0,R/2)} \exp\big(-\alpha \log(1-u)\big).$$

Now, since $1-u$ is a positive harmonic function,

$$\|\log(1-u)\|_{BMO(B(x_0,3R/4))} \leqslant C$$

([7], p. 28; this is where (P) is used), therefore, by the John–Nirenberg lemma ([7], p. 25), one can choose α for which there exists C such that

$$\left(\frac{1}{V(x_0,R/2)} \sum_{B(x_0,R/2)} \exp\big(-\alpha \log(1-u)\big)\right)$$
$$\times \left(\frac{1}{V(x_0,R/2)} \sum_{B(x_0,R/2)} \exp\big(\alpha \log(1-u)\big)\right) \leqslant C.$$

Next, by the hypothesis,

$$\frac{1}{V(x_0,R/2)} \sum_{B(x_0,R/2)} \exp\big(\alpha \log(1-u)\big) \geqslant \frac{1}{V(x_0,R/2)} \sum_{u \leqslant 1/2} 2^{-\alpha} \geqslant 2^{-\alpha-1}.$$

One obtains

$$\exp\left(\alpha \log(1-h)\frac{a(h,R/2)}{V(x_0,R/2)}\right) \leqslant C2^{\alpha+1}.$$

and the desired inequality follows. □

Proof of Proposition 5.1. – Fix $x_0 \in \Gamma$, $R \geqslant 1$, and let $u \in \mathbb{R}^\Gamma$ be harmonic in $B(x_0,R)$. Define $M(r)$, $m(r)$, $h_0 = h_0(r)$ and $a(h,r)$ as above. Let $r \in [1, R/2]$. Apply (6) with R replaced by r and h replaced by

$$h_i = h_i(r) = M(r) - \frac{1}{2^i}\big(M(r) - h_0(r)\big) = M(r) - \frac{1}{2^{i+1}}\big(M(r) - m(r)\big),$$

to obtain

$$M(r/2) \leqslant h_i + C\big(M(r) - h_i\big)\left(\frac{a(h_i, r)}{V(x_0, r)}\right)^{1/2}. \tag{7}$$

Assume that $a(h_0, r) \leqslant \frac{V(x_0,r)}{2}$, otherwise work with $-u$. Lemma 6.2 says that

$$\frac{a(h_i, r)}{V(x_0, r)} \leqslant \frac{C'}{i},$$

therefore one can choose i large enough so that

$$C\left(\frac{a(h_i, r)}{V(x_0, r)}\right)^{1/2} \leqslant 1/2.$$

One obtains

$$M(r/2) \leqslant M(r) - \frac{1}{2^{i+2}}\big(M(r) - m(r)\big),$$

hence

$$M(r/2) - m(r/2) \leqslant \big(M(r) - m(r)\big)\left(1 - \frac{1}{2^{i+2}}\right).$$

Set $\omega(r) = M(r) - m(r)$. One has

$$\omega(r/2) \leqslant \eta\omega(r), \quad \forall r \in]0, R/2],$$

where $\eta = 1 - \frac{1}{2^{i+2}} \in]0, 1[$. It follows that there exists $C, \alpha > 0$ such that

$$\omega(\rho) \leqslant C\left(\frac{\rho}{R}\right)^{\alpha} \omega(R/2), \quad \forall \rho,\ 0 < \rho \leqslant R/2.$$

In particular,

$$\big|u(x) - u(y)\big| \leqslant C'\left(\frac{d(x, y)}{R}\right)^{\alpha} \max_{B(x_0, R/2)} |u|, \quad \forall x, y \in B(x_0, R/2).$$

Now, it follows easily from Proposition 6.1 that

$$\max_{x\in B(x_0,R/2)} |u(x)| \leqslant C\left(\frac{1}{V(x_0,R)}\sum_{B(x_0,R)} |u-u_R(x_0)|^2\right)^{1/2}.$$

Proposition 5.1 is proved. □

Now, instead of $(P)=(P_2)$, we assume, for $\varepsilon\in]0,1]$,

$$\left(\sum_{B(x,r)} |f-f_r(x)|^{2-\varepsilon}\right)^{\frac{1}{2-\varepsilon}} \leqslant Cr\left(\sum_{B(x,r)} |\nabla f|^{2-\varepsilon}\right)^{\frac{1}{2-\varepsilon}},$$
$$\forall f\in\mathbb{R}^\Gamma,\ x\in\Gamma,\ r>0. \qquad (P_{2-\varepsilon})$$

We shall use in fact the following formulation of $(P_{2-\varepsilon})$:

$$\left(\sum_{B(x,r)} |f|^{2-\varepsilon}\right)^{\frac{1}{2-\varepsilon}}$$
$$\leqslant Cr\frac{V(x,r)}{m(\{y\in B(x,r);\ f(y)=0\})}\left(\sum_{B(x,r)} |\nabla f|^{2-\varepsilon}\right)^{\frac{1}{2-\varepsilon}}, \quad (\widetilde{P}_{2-\varepsilon})$$

$\forall f\in\mathbb{R}^\Gamma$, $x\in\Gamma$, $r>0$. It is easily seen by checking the elementary inequality

$$\sum_{B(x,r)} |f|^{2-\varepsilon} \leqslant C_\varepsilon\frac{V(x,r)}{m(\{y\in B(x,r);\ f(y)=0\})}\sum_{B(x,r)} |f-f_r(x)|^{2-\varepsilon}.$$

For the fact that $(P_{2-\varepsilon})$ implies (P_2), but that, given $\varepsilon>0$ there exists a graph Γ such that (P_2) holds but $(P_{2-\varepsilon})$ fails, see [13].

We are going to prove the following weak form of Proposition 5.1.

PROPOSITION 6.3. – *For every $\varepsilon\in]0,1]$, (D) and $(P_{2-\varepsilon})$ imply (ER).*

In order to by-pass the John–Nirenberg type argument, one first replaces Lemma 6.3 by the following one.

LEMMA 6.4. – *Assume that (Γ,μ) satisfies (D) and $(P_{2-\varepsilon})$. Let $x_0\in\Gamma$, $R>0$ and $u\in\mathbb{R}^\Gamma$ harmonic in $B(x_0,R)$. Set*

$$k_i = M(R)-\left(\frac{M(R)-m(R)}{2^{i+1}}\right),\quad i\in\mathbb{N}.$$

Assume that $a(k_0, R/2) \leqslant \frac{V(x_0,R/2)}{2}$. *Then*

$$\frac{a(k_i, R/2)}{V(x_0, R/2)} \leqslant \frac{C}{i^{\varepsilon/2}},$$

where C *does not depend on* x_0, R, *or* u.

Proof. – For $h > k > k_0$, set $v = (u-k)_+ \wedge (h-k)$. By assumption,

$$\begin{aligned} m\{x \in B(x_0, R/2); v(x) = 0\} &= m\{B(x_0, R/2) \setminus A(k, R/2)\} \\ &\geqslant m\{B(x_0, R/2) \setminus A(k_0, R/2)\} \\ &\geqslant \frac{V(x_0, R/2)}{2}. \end{aligned}$$

The Poincaré inequality $(\widetilde{P}_{2-\varepsilon})$ yields therefore

$$\sum_{B(x_0,R/2)} |v|^{2-\varepsilon} \leqslant C R^{2-\varepsilon} \sum_{B(x_0,R/2)} |\nabla v|^{2-\varepsilon}.$$

Hence

$$(h-k)^{2-\varepsilon} a(h, R/2) \leqslant C R^{2-\varepsilon} \sum_{B(x_0,R/2)} |\nabla v|^{2-\varepsilon}.$$

Now one easily checks that $|\nabla v|(x) \leqslant |\nabla u|(x)$, $\forall x \in \Gamma$, that $|\nabla v|(x) = 0$ if x is not in $A(k, R/2) \setminus A(h, R/2)$ and has no neighbour there, and finally that if $x \notin A(k, R/2) \setminus A(h, R/2)$, but there exists $y \sim x$ belonging to $A(k, R/2) \setminus A(h, R/2)$, then $|\nabla v|(x) \leqslant C|\nabla u|(y)$, where C only depends on the constant in (D). Therefore

$$(h-k)^{2-\varepsilon} a(h, R/2) \leqslant C' R^{2-\varepsilon} \sum_{A(k,R/2)\setminus A(h,R/2)} |\nabla u|^{2-\varepsilon}.$$

By Hölder,

$$\begin{aligned} &(h-k)^{2-\varepsilon} a(h, R/2) \\ &\quad \leqslant C R^{2-\varepsilon} \big(a(k, R/2) - a(h, R/2)\big)^{\varepsilon/2} \Bigg(\sum_{A(k,R/2)\setminus A(h,R/2)} |\nabla u|^2 \Bigg)^{\frac{2-\varepsilon}{2}}. \end{aligned}$$

Now

$$\sum_{A(k,R/2)\setminus A(h,R/2)} |\nabla u|^2 \leqslant \sum_{A(k,R/2)} |\nabla u|^2 = \sum_{B(x_0,R/2)} |\nabla (u-k)_+|^2.$$

Since $(u-k)_+$ is harmonic, one infers from Proposition 2.1 that

$$\sum_{B(x_0,R/2)} \left|\nabla(u-k)_+\right|^2 \leqslant \frac{C}{R^2} \sum_{B(x_0,R)} (u-k)_+^2 \leqslant C' \frac{V(x_0,R)}{R^2} \big(M(R)-k\big)^2.$$

Thus

$$\begin{aligned}&(h-k)^{2-\varepsilon} a(h,R/2)\\ &\quad \leqslant C V(x_0,R)^{\frac{2-\varepsilon}{2}} \big(M(R)-k\big)^{2-\varepsilon} \big(a(k,R/2)-a(h,R/2)\big)^{\frac{\varepsilon}{2}}.\end{aligned}$$

Since $k_i - k_{i-1} = \frac{M(R)-k_0}{2^i}$ and $M(R) - k_{i-1} = \frac{M(R)-k_0}{2^{i-1}}$, the above inequality yields

$$a(k_i,R/2)^{\frac{2}{\varepsilon}} \leqslant C' V(x_0,R)^{\frac{2-\varepsilon}{\varepsilon}} \big(a(k_{i-1},R/2)-a(k_i,R/2)\big).$$

Using the fact that $a(k_i,R/2)$ is non-increasing in i, one obtains

$$\begin{aligned} i a(k_i,R/2)^{\frac{2}{\varepsilon}} &\leqslant \sum_{j=0}^{i} a(k_j,R/2)^{\frac{2}{\varepsilon}}\\ &\leqslant C' V(x_0,R)^{\frac{2-\varepsilon}{\varepsilon}} \big(a(k_0,R/2)-a(k_i,R/2)\big)\\ &\leqslant C' V(x_0,R)^{\frac{2-\varepsilon}{\varepsilon}} a(k_0,R/2).\end{aligned}$$

Hence

$$\frac{a(k_i,R/2)}{V(x_0,R)} \leqslant \left(C' i^{-1} \frac{a(k_0,R/2)}{V(x_0,R)}\right)^{\varepsilon/2} \leqslant \left(\frac{C' i^{-1}}{2}\right)^{\varepsilon/2},$$

and the claim follows. □

Then the proof of Proposition 6.3 is similar that of Proposition 5.1 above except that, instead of (7), one begins with the inequality

$$M(r/2) \leqslant k_i + C\big(M(2r)-k_i(2r)\big)\left(\frac{a(k_i,r)}{V(x_0,r)}\right)^{1/2},$$

with $k_i(r) = M(r) - (\frac{M(r)-m(r)}{2^{i+1}})$.

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