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# Remarks on decay of correlations and Witten Laplacians III. Application to logarithmic Sobolev inequalities

by

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ABSTRACT. – This is the continuation of our two previous articles devoted to the use of Witten Laplacians for analyzing Laplace integrals in statistical mechanics.

The main application treated in Part I was a semi-classical one. The second application was more perturbative in spirit and gave very explicit estimates for the lower bound of the Witten Laplacian in the case of a quartic model.

We shall relate in this third part our studies of the Witten Laplacian with the existence of uniform logarithmic Sobolev inequalities through a criterion of B. Zegarlinski.

More precisely, our main contribution is to show how to control the decay of correlations uniformly with respect to various parameters, under a natural condition of strict convexity at $\infty$ of the single-spin phase and when the nearest neighbor interaction is small enough. © Elsevier, Paris

RÉSUMÉ. – Ce travail est la continuation de deux autres articles consacrés à l'utilisation des Laplaciens de Witten dans l'analyse des intégrales de Laplace dans le contexte de la mécanique statistique. La principale application traitée dans le premier était de nature semi-classique. Dans le deuxième, on donnait des estimations très explicites dans le cas du modèle quartique. Nous abordons ici l'étude des liens avec l'inégalité de Log-Sobolev par l'intermédiaire d'un critère de B. Zegarlinski.

Plus précisément, notre contribution principale est de montrer comment contrôler la décroissance des corrélations uniformément par rapport à divers paramètres sous une hypothèse très naturelle de convexité à l'infini sur la phase à un spin et lorsque l'interaction (supposée aux proches voisins) est assez petite. 

## 1. INTRODUCTION

In the recent years, a new insight has been given in the study of the decay of the correlation pairs, through the analysis of a Witten Laplacian on 1-forms ([2,12,17,21]). This gave not only a nice way to recover the Brascamp–Lieb inequality [9] but also permits the analysis of non-convex situations ([4,10–12,20] and [13]). In this spirit, it is natural to analyze if this approach gives new results concerning logarithmic Sobolev inequalities in the non convex case. The main point would be to get an extension of the Bakry–Emery criterion [3] in the case of non-convex situations by relating the logarithmic Sobolev best constant to the lowest eigenvalue of the Witten Laplacian. We discuss this question in [10] and [11] in connection with recent studies of Antoniouk–Antoniouk [1,2] but do not obtain decisive results outside the case when the gradient of the interaction is bounded and small.

Motivated by discussions with T. Bodineau and a talk by B. Zegarlinski in Brisbane (July 97), we consider here another approach initiated by B. Zegarlinski, developed by Strook and Zegarlinski [24,25] (in case of compact spins) and later on by Zegarlinski [27] who investigated unbounded spins. The method consists in deducing uniform logarithmic Sobolev inequalities from the existence of uniform decay estimates for the correlations. This is what is obtained here through the Witten Laplacian approach.

More precisely our aim is to analyze the thermodynamic properties of the measure $\exp -\Phi^{\Lambda,\omega}(X)\,dX$ in the case when $\Phi^{\Lambda,\omega}$, which is associated with cubes $\Lambda \subset \mathbb{Z}^d$ and some $\omega \in (\mathbb{R}^N)^{\mathbb{Z}^d}$ defining the boundary condition, has the form, for $X \in (\mathbb{R}^N)^\Lambda$,

$$\Phi^{\Lambda,\omega}(X) = \sum_{j\in\Lambda} \phi(x_j) + \frac{\mathcal{J}}{2} \sum_{(\{j\}\cup\{k\})\cap\Lambda\neq\emptyset,\, j\sim k} |z_j - z_k|^2, \tag{1.1}$$

where

- $X = (x_j)_{j\in\Lambda}$,
- $\phi$ is a one particle phase on $\mathbb{R}^N$ such that there exists $\tau > 0$ such that
$$\phi(x) \geqslant \frac{1}{\tau}|x|^2 - \tau, \tag{1.2}$$
-
$$\begin{aligned} z_j &= x_j \quad \text{if } j \in \Lambda \\ z_j &= \omega_j \quad \text{if } j \notin \Lambda, \end{aligned} \tag{1.3}$$
- $j \sim k$ means that $j$ and $k$ are nearest neighbors [1] for the $\ell^1$-distance in $\mathbb{Z}^d$.

We shall sometimes use the following decomposition

$$\Phi^{\Lambda,\omega} = \Phi_d^\Lambda + \mathcal{J}\, \Phi_i^{\Lambda,\omega}, \tag{1.4}$$

with

$$\Phi_d^\Lambda(X) = \sum_{j\in\Lambda} \phi(x_j), \tag{1.5}$$

and

$$\Phi_i^{\Lambda,\omega}(X) = \sum_{(\{j\}\cup\{k\})\cap\Lambda\neq\emptyset,\, j\sim k} |z_j - z_k|^2. \tag{1.6}$$

We shall also meet the free boundary condition which by definition corresponds to the phase

$$\Phi^{\Lambda,f} = \Phi_d^\Lambda + \mathcal{J}\Phi_i^{\Lambda,f}, \tag{1.7}$$

with

$$\Phi_i^{\Lambda,f}(X) = \sum_{j,k\in\Lambda,\, j\sim k} |x_j - x_k|^2. \tag{1.8}$$

[1] In our previous studies [12,13], we were mainly analyzing the case when $j$ and $k$ were nearest neighbors in $\Lambda$ considered as a discrete torus.

The sum in (1.8) is over the non-oriented pairs of $\Lambda \times \Lambda$.

If necessary, the dependence on $\mathcal{J}$ will be mentioned by the notation $\Phi^{\Lambda,\omega} = \Phi^{\Lambda,\omega,\mathcal{J}}$ or $\Phi^{\Lambda,f} = \Phi^{\Lambda,f,\mathcal{J}}$.

Let us now state the assumptions on the single-spin phase $\phi$. We assume that $\phi$ is $C^2$ on $\mathbb{R}^N$ and convex at $\infty$, so there exists $C > 0$ such that

$$\operatorname{Hess}\phi(x) \geqslant \frac{1}{C}, \quad \forall x \in \mathbb{R}^N \text{ s.t. } |x| \geqslant C. \tag{1.9}$$

We assume also (actually only when $N > 1$) the technical condition that $\phi$ is $C^\infty$ and that there exists $\rho > 0$ and, for all $\beta \in \mathbb{N}^N$, $C_\beta$ such that,

$$\left|D_x^\beta \nabla\phi(x)\right| \leqslant C_\beta < \nabla\phi(x) >^{(1-\rho|\beta|)_+}, \quad \forall x \in \mathbb{R}^N \tag{1.10}$$

where, for $u \in \mathbb{R}^N$, $< u > := (1+|u|^2)^{\frac{1}{2}}$ and, for $t \in \mathbb{R}$, $(t)_+ := \max(t, 0)$.

The typical example will be (when $N = 1$)

$$\phi(x) = \frac{1}{12}\lambda x^4 + \frac{1}{2}\nu x^2, \tag{1.11}$$

where the parameters $\lambda$ and $\nu$ satisfy

$$\lambda > 0. \tag{1.12}$$

We would like to analyze the possibility of having any sign for $\nu$. Our main problem will be to analyze the properties of the measure

$$d\mu_{\Lambda,\omega} := \exp -\Phi^{\Lambda,\omega}(X)\, dX \Big/ \left( \int_{(\mathbb{R}^N)^\Lambda} \exp -\Phi^{\Lambda,\omega}(X)\, dX \right), \tag{1.13}$$

or of the measure

$$d\mu_\Lambda := \exp -\Phi^{\Lambda,f}(X)\, dX \Big/ \left( \int_{(\mathbb{R}^N)^\Lambda} \exp -\Phi^{\Lambda,f}(X)\, dX \right). \tag{1.14}$$

We shall in particular analyze the covariance associating to $f, g \in C^\infty_{temp}((\mathbb{R}^N)^\Lambda)$

$$\operatorname{Cov}_{\Lambda,\omega}(f, g) = \langle (f - \langle f \rangle_{\Lambda,\omega})(g - \langle g \rangle_{\Lambda,\omega}) \rangle_{\Lambda,\omega}, \tag{1.15}$$

where $\langle\cdot\rangle_{\Lambda,\omega}$ denotes the mean value with respect to the measure $d\mu_{\Lambda,\omega}$ and $C^{\infty}_{temp}((\mathbb{R}^N)^{\Lambda})$ is the space of $C^{\infty}$ functions with polynomial growth.

Similarly, we associate with $\Phi^{\Lambda,f}$ and the measure $d\mu_{\Lambda}$ the mean value $\langle\cdot\rangle_{\Lambda}$ and the covariance $\mathrm{Cov}_{\Lambda}$. Our main result is the following

THEOREM 1.1. – *Let $\Phi = \Phi^{\Lambda,f,\mathcal{J}} = \Phi^{\Lambda}_{d} + \mathcal{J}\Phi^{\Lambda,f}_{i}$ with $\phi$ satisfying* (1.9) *and* (1.10). *Then there exist constants $C$ and $\mathcal{J}_0 > 0$ such that, for $\mathcal{J} \in [-\mathcal{J}_0, \mathcal{J}_0]$ and for any cube $\Lambda \subset \mathbb{Z}^d$, we have*[2] :

$$\langle f \ln f\rangle_{\Lambda} - \langle f\rangle_{\Lambda} \ln\langle f\rangle_{\Lambda} \leqslant 2C\big\langle|\nabla f^{\frac{1}{2}}|^2\big\rangle_{\Lambda} \tag{1.16}$$

*for all non-negative function $f$ for which the right hand side is finite.*

As communicated by B. Zegarlinski, example (1.11) is also analyzed by N. Yoshida [26].

In this case, we say shortly that the uniform logarithmic Sobolev inequality (ULS inequality) is true.

The proof will be the conjunction of

- a criterion[3] by B. Zegarlinski [27] relating the existence of "uniform" decay estimates with the existence of "uniform" logarithmic Sobolev inequalities,
- the proof (mainly given in [13]), that the existence of "uniform" decay estimates may be obtained from the existence of an "uniform" lower bound for the lowest eigenvalue of a Witten Laplacian on 1-forms,
- the proof of this "uniform" lower bound in the case of a weak nearest neighbors interaction by comparison with a family of "one-particle" differential operators on $\mathbb{R}^N$. The case $N = 1$ will be easier to treat and will lead to more general theorems.

Although the main techniques were already present in the previous articles of the serie [12] and [13], we think it is worthwhile to follow in detail this problem of the control of the uniformity with respect to $\Lambda$ and $\omega$ and to show that the assumption of strict convexity at $\infty$ for the single spin phase is a natural condition under which this technique works.

[2] With $g = f^{\frac{1}{2}}$, this is equivalently written as

$$\int g^2 \ln \frac{g}{||g||_{L^2}} d\mu \leqslant C||\nabla g||^2_{L^2}.$$

[3] Note that quite recently variants or generalizations of this criterion have been obtained in [26] and [5].

More precisely, as a particular case of a more general result, we will prove the following theorem

THEOREM 1.2. – *Under the same assumptions as in Theorem* 1.1, *there exist* $\mathcal{J}_0 > 0$, $C$ *and* $\kappa$, *such that, for any* $\Lambda$, $\omega$ *and* $\mathcal{J} \in [-\mathcal{J}_0, +\mathcal{J}_0]$, *the correlation pair function satisfies*

$$\left|\operatorname{Cov}_{\Lambda,\omega}(x_i, x_j)\right| \leqslant C \exp -\kappa \|i-j\|, \quad \forall i, j \in \Lambda. \tag{1.17}$$

The present version is with many aspects different of previous ones which were diffused during the period 1997–1998. We have indeed interacted with many colleagues and particularly J.-D. Deuschel (who indicates to us an alternative proof of Theorem 1.2) and this has pushed us to extend or to modify our proofs in order to compare the possibilities of different variants of our approach in comparison with other approaches.

## 2. LOWER BOUND FOR THE SPECTRUM OF THE WITTEN LAPLACIAN

Let us recall that the Witten Laplacian on 1-forms attached to the phase $\Phi = \Phi^{\Lambda,\omega}$ is defined as,[4]

$$W_\Phi^{(1)} := \left[\sum_{j\in\Lambda, \ell\in\{1,\dots,N\}} \left(-\frac{\partial}{\partial x_{j,\ell}} + \frac{1}{2}\frac{\partial \Phi}{\partial x_{j,\ell}}\right)\left(\frac{\partial}{\partial x_{j,\ell}} + \frac{1}{2}\frac{\partial \Phi}{\partial x_{j,\ell}}\right)\right] \otimes I + \operatorname{Hess} \Phi \tag{2.1}$$

defined on the $L^2$ 1-forms with respect to the standard Lebesgue measure on $\mathbb{R}^m$, with $m = N|\Lambda|$. According to a classical theorem (see [19] and references therein), this Witten Laplacian $W_\Phi^{(1)}$ is essentially selfadjoint from $C_0^\infty$ under the assumption that $\Phi$ is $C^2$ and there is consequently a uniquely well defined selfadjoint extension which coincides in particular with the Friedrichs extension. The aim of this section is the proof of the following theorem

THEOREM 2.1. – *For any cube* $\Lambda \subset \mathbb{Z}^d$ *and* $\omega \in (\mathbb{R}^N)^{\mathbb{Z}^d}$, *let* $\Phi := \Phi^{\Lambda,\omega,\mathcal{J}}$ *be the phase on* $\mathbb{R}^\Lambda$ *with* $\phi$ *satisfying* (1.9)–(1.10). *There exist* $\mathcal{J}_0$ *and* $\sigma_1 > 0$, *such that the lowest eigenvalue* $\lambda_1^{\Lambda,\omega,\mathcal{J}}$, *of the corresponding*

[4] It was denoted by $\Delta_\Phi^{(1)}$ in [21].

*Witten Laplacian on* 1*-forms* $W_{\Phi}^{(1)}$*, satisfies, for any cube* $\Lambda$*,* $\omega \in (\mathbb{R}^N)^{\mathbb{Z}^d}$ *and* $\mathcal{J} \in [-\mathcal{J}_0, \mathcal{J}_0]$,

$$\lambda_1^{\Lambda,\omega,\mathcal{J}} \geqslant \sigma_1. \tag{2.2}$$

Similarly, we have also

THEOREM 2.2. – *For any cube* $\Lambda \subset \mathbb{Z}^d$*, let* $\Phi := \Phi^{\Lambda,f,\mathcal{J}}$ *be the phase on* $(\mathbb{R}^N)^{\Lambda}$ *with* $\phi$ *satisfying* (1.9)–(1.10)*. There exist* $\mathcal{J}_0$ *and* $\sigma_1 > 0$*, such that the lowest eigenvalue* $\lambda_1^{\Lambda,f,\mathcal{J}}$*, of the corresponding Witten Laplacian on* 1*-forms* $W_{\Phi}^{(1)}$*, satisfies, for any cube* $\Lambda$ *and* $\mathcal{J} \in [-\mathcal{J}_0, \mathcal{J}_0]$,

$$\lambda_1^{\Lambda,f,\mathcal{J}} \geqslant \sigma_1. \tag{2.3}$$

We treat first the case $N = 1$ and will explain later what has to be modified in the case $N > 1$ (see Sections 6 and 7). This will make the notations easier and some of the proofs in the case $N = 1$ are indeed simpler and lead to better results.

The starting point for the proof is the basic identity

$$\langle W_{\Phi}^{(1)} u \mid u \rangle_{L^2} = \sum_{j,k} \|Z_k u_j\|^2 + \sum_{j,k} \int \frac{\partial^2 \Phi}{\partial x_j \, \partial x_k} u_j u_k \, dX, \tag{2.4}$$

with

$$Z_j = \partial_j + \frac{1}{2} \partial_j \Phi, \tag{2.5}$$

and

$$\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}. \tag{2.6}$$

Let us denote by $w_j^{(0)}$ and $w_j^{(1)}$ the single-spin Witten Laplacians (respectively, on 0- and 1-forms) attached to the variable $x_j$ and the phase on $\mathbb{R}$

$$\phi_j(x_j) := \Phi^{\Lambda,\omega,\mathcal{J}}(X). \tag{2.7}$$

The differential of this phase $\phi_j$ $d_j \phi_j$ depends actually only, as parameters, on the variables $z_\ell$ with $\ell \sim j$.

We recall that $z_\ell = x_\ell$ if $\ell \in \Lambda$ and $z_\ell = \omega_\ell$ if $\ell \in \mathbb{Z}^d \setminus \Lambda$. We have indeed

$$\phi_j(t) = \phi(t) + \mathcal{J} \sum_{(\{\ell\} \cup \{j\}) \cap \Lambda \neq \emptyset, \ell \sim j} |t - z_\ell|^2 + \widehat{\phi}(\widehat{z}_j), \tag{2.8}$$

where the last term is independent of $t$ and will be irrelevant in the discussion. In particular, the operators $w_j^{(0)}$ and $w_j^{(1)}$ depend only on the $z_\ell$ with $\ell \sim j$.

It will be quite important that the estimates which will be proved are independent of these parameters.

We note the relations

$$w_j^{(0)} = Z_j^* Z_j, \tag{2.9}$$

and

$$w_j^{(1)} = Z_j Z_j^* = Z_j^* Z_j + \frac{\partial^2 \Phi}{\partial x_j^2}. \tag{2.10}$$

*Remark* 2.3. – We would like to emphasize that (2.10) is specific of $N = 1$. We shall analyze in Sections 6 and 7 the situation when $N \neq 1$.

According to the context, we shall see these identities as identities between differential operators on $L^2(\mathbb{R}^\Lambda)$ or on $L^2(\mathbb{R}_{x_j})$ (the other variables being considered as parameters).

With these conventions, we have

$$\begin{aligned} \langle W_\Phi^{(1)} u \mid v \rangle_{L^2} = & \sum_{j,k \in \Lambda, j \neq k} \langle w_k^{(0)} u_j \mid v_j \rangle + \sum_{j \in \Lambda} \langle w_j^{(1)} u_j \mid v_j \rangle \\ & + \mathcal{J} \langle \mathrm{Hess}' \Phi_i u \mid v \rangle, \end{aligned} \tag{2.11}$$

where $\Phi_i = \Phi_i^{\Lambda,\omega}$ denotes the interaction phase and Hess′ means that we consider only the terms outside of the diagonal of the Hessian, that is such that, for $k, \ell \in \Lambda$,

$$(\mathrm{Hess}' \Phi_i)_{k\ell} = \begin{cases} -1 & \text{if } k \sim \ell, \\ 0 & \text{else.} \end{cases} \tag{2.12}$$

Here we observe also that Hess $\Phi^i$ is independent of $z$ and that $\mathcal{J}$ Hess $\Phi^i$ corresponds to a perturbation in $\mathcal{O}(\mathcal{J})$, where $\mathcal{O}$ is uniform with respect to $\Lambda$, using Schur's Lemma. If we observe that we deduce (as in [12]) from (2.12) and the positivity of $w_k^{(0)}$ the inequality

$$\langle W_\Phi^{(1)} u \mid u \rangle_{L^2} \geqslant \sum_{j \in \Lambda} \langle w_j^{(1)} u_j \mid u_j \rangle + \mathcal{J} \langle \mathrm{Hess}' \Phi_i u \mid u \rangle, \tag{2.13}$$

the two theorems of this section are a consequence of the general:

THEOREM 2.4. – *Let us assume that there exists $\rho_1 > 0$ such that, for any $j \in \mathbb{Z}^d$ and any $z \in \mathbb{R}^{\mathbb{Z}^d \setminus \{j\}}$, the operator*[5] $w_j^{(1)}$ *satisfies*

$$w_j^{(1)} \geqslant \rho_1, \tag{2.14}$$

*then, for any $\varepsilon > 0$, there exists $\mathcal{J}_0 > 0$ such that the Witten Laplacian $W_\Phi^{(1)}$, with $\Phi := \Phi^{\Lambda,\omega,\mathcal{J}}$ or with $\Phi := \Phi^{\Lambda,f,\mathcal{J}}$, satisfies for any $\Lambda$, $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ and $\mathcal{J} \in [-\mathcal{J}_0, +\mathcal{J}_0]$,*

$$W_\Phi^{(1)} \geqslant (\rho_1 - \varepsilon). \tag{2.15}$$

We recall that the positivity of $w_j^{(1)}$ is immediate from the definition. We also observe that it is sufficient to treat the case of a fixed $j_0$ all the families being unitary equivalent (after a simple change of the names of the parameters). We shall see that the strict positivity for a fixed value of the parameter $z$ is a consequence of general arguments (see [21] and [15]), at least for $\mathcal{J} \in [-\mathcal{J}_0, +\mathcal{J}_0]$ with $\mathcal{J}_0$ small enough. The other important point is to verify the condition of uniformity. This will be done in the next section.

*Remark* 2.5. – As shortly presented in [13], Bach–Jecko–Sjöstrand [4] give a rather general approach for estimating from below $W_\Phi^{(1)}$ uniformly with respect to $\Lambda$. It is not completely clear how this can be used for getting also the uniformity with respect to the boundary condition $\omega$.

## 3. UNIFORM ESTIMATES FOR A FAMILY OF 1-DIMENSIONAL WITTEN LAPLACIANS

Motivated by discussions with J.-D. Deuschel, we shall discuss various conditions under which these uniform estimates can be obtained. This will lead to stronger results than announced in the introduction in this particular situation when $N = 1$.

In all this section $\phi$ is at least $C^2$. If $\psi$ is a $C^2$ phase, we shall denote by $w_\psi^{(0)}$ and $w_\psi^{(1)}$ the corresponding Witten Laplacians defined in this simple case by

$$w_\psi^{(0)} = -\frac{d^2}{dt^2} + \frac{1}{4}\psi'(t)^2 - \frac{1}{2}\psi''(t), \tag{3.1}$$

[5] For a given $j \in \mathbb{Z}^d$, the effective parameters are actually the $z_\ell$ such that $\ell \sim j$.

and

$$w_\psi^{(1)} = -\frac{d^2}{dt^2} + \frac{1}{4}\psi'(t)^2 + \frac{1}{2}\psi''(t). \tag{3.2}$$

These two operators being positive are automatically selfadjoint on $L^2(\mathbb{R})$ starting from $C_0^\infty$ as proved, for example, in [19] (see also references therein).
The first condition is (for reference) the existence of $C > 0$ such that

$$(\mathrm{sc}) \quad \phi''(t) \geqslant \frac{1}{C}, \quad \forall t \in \mathbb{R}. \tag{3.3}$$

A weaker condition is

$$(\mathrm{sc}(\infty)) \quad \phi''(t) \geqslant \frac{1}{C}, \quad \forall t \in \mathbb{R}, \quad |t| \geqslant C. \tag{3.4}$$

A still weaker assumption is that there exists a bounded function $\chi$ in $C^2$ such that

$$(\mathrm{scm}) \quad \phi''(t) + \chi''(t) \geqslant \frac{1}{C}, \quad \forall t \in \mathbb{R}. \tag{3.5}$$

These three conditions are ordered:

$$(\mathrm{sc}) \Rightarrow (\mathrm{sc}(\infty)) \Rightarrow (\mathrm{scm}).$$

Another family of conditions corresponds to assumptions on the operator

$$w^{red} := -\frac{d^2}{dx^2} + \frac{1}{2}\phi'', \tag{3.6}$$

or more precisely to the quadratic form associated to $w^{red}$ on $C_0^\infty$:

$$u \mapsto q^{red}(u) = \langle w^{red} u \mid u \rangle. \tag{3.7}$$

We consider a new family of conditions starting with:

$$(qsc) \quad q^{red}(u) \geqslant \frac{1}{C}\,||u||^2, \quad \forall u \in C_0^\infty(\mathbb{R}). \tag{3.8}$$

A weaker condition is the existence of $C > 0$, such that:

$$(\mathrm{qsc}(\infty)) \quad q^{red}(u) \geqslant \frac{1}{2C}||u||^2, \quad \forall u \in C_0^\infty(\mathbb{R} \setminus ]-C, +C[). \tag{3.9}$$

Finally a still weaker assumption is the existence of a bounded function $\chi$ in $C^2$ such that

$$\text{(qscm)}\quad q^{red}(u)+\frac{1}{2}\int_{\mathbb{R}}\chi''(t)\,|u(t)|^2\,dt\geqslant\frac{1}{2C}\,||u||^2,\quad \forall u\in C_0^\infty(\mathbb{R}). \tag{3.10}$$

These three new conditions are again ordered:

$$\text{(qsc)}\Rightarrow\text{(qsc}(\infty))\Rightarrow\text{(qscm)}$$

and it is also clear that (qsc) is weaker than (sc). The condition (qsc) permits to treat roughly speaking functions which are strictly convex in mean value. An explicit criterion for verifying (qsc) can be obtained from a criterion given by N. Lerner and J. Nourrigat ([16], see also the references therein to the fundamental work by C. Fefferman and his collaborators). For the reader's convenience, let us give here a variant of their result obtained by a small modification of their proof.

PROPOSITION 3.1. – *Let $V\in L^1_{loc}(\mathbb{R})$. Let us assume that there exist $\alpha\in\,]0,1]$ and $C>0$ such that, for any interval $I=\,]a,b[$, the following inequality is satisfied*

$$\left(\frac{1}{b-a}\int_a^b V_+(t)\,dt\right)-(1+\alpha)\left(\frac{1}{b-a}\int_a^b V_-(t)\,dt\right)+\frac{\alpha^2}{(\alpha^2+16)(b-a)^2}\geqslant\frac{1}{C}, \tag{3.11}$$

*then the operator $-d^2/dt^2+V(t)$ is strictly positive.*

*Remark* 3.2. – It is important to observe that if $\phi$ satisfies one of the three conditions (sc), ..., (scm), for some constant $C$, then the same condition [6] will still be true for the phase

$$\phi_{\mathcal{J},\alpha}(t):=\phi(t)+\mathcal{J}\,dt^2-\alpha\,t \tag{3.12}$$

for any $\mathcal{J}$ such that

$$2\mathcal{J}d>-\frac{1}{C}. \tag{3.13}$$

[6] This is also essentially true for the other family of conditions, if we keep the condition (1.2), uniformly satisfied for the phase $\phi_{\mathcal{J},0}$.

In the preceding section, we have shown that the proof of a uniform lower bound for the Witten Laplacian $W_\Phi^{(1)}$ can be deduced from the study of one-dimensional Witten Laplacians. In this sense, we are not far of Dobrushin's approach as described for example in [2]. We want to analyze

$$w_{\phi_{\mathcal{J},\alpha}}^{(1)} := -\frac{d^2}{dt^2} + \frac{1}{4}\left(\phi'_{\mathcal{J},\alpha}(t)\right)^2 + \frac{1}{2}\left(\phi''_{\mathcal{J},\alpha}(t)\right), \tag{3.14}$$

with

$$\alpha = -2\mathcal{J}\sum_{0\sim k} z_k. \tag{3.15}$$

The first theorem is the following.

THEOREM 3.3. – *Let $\phi$ be a phase satisfying* (3.8). *Then, there exist $\mathcal{J}_0$ and $\rho_1 > 0$ such that, for any $(\alpha, \mathcal{J}) \in \mathbb{R} \times [-\mathcal{J}_0, +\mathcal{J}_0]$, we have*

$$w_{\phi_{\mathcal{J},\alpha}}^{(1)} \geqslant \rho_1. \tag{3.16}$$

The proof is trivial if we observe that

$$w_{\phi_{\mathcal{J},\alpha}}^{(1)} \geqslant w^{red} + \mathcal{J}d, \tag{3.17}$$

in the sense that we compare the corresponding quadratic forms on $C_0^\infty$. This result was already presented in [12]. Note that in the strictly convex case (sc), we simply write

$$w_{\phi_{\mathcal{J},\alpha}}^{(1)} \geqslant \phi'' + 2\mathcal{J}d. \tag{3.18}$$

The second theorem is

THEOREM 3.4. – *Let $\phi$ be a phase satisfying* (3.10). *Then, there exist $\mathcal{J}_0$ and $\rho_1 > 0$ such that, for any $(\alpha, \mathcal{J}) \in \mathbb{R} \times [-\mathcal{J}_0, +\mathcal{J}_0]$, we have*

$$w_{\phi_{\mathcal{J},\alpha}}^{(1)} \geqslant \rho_1. \tag{3.19}$$

We were inspired for the proof of this theorem by discussions with J.-D. Deuschel and V. Bach.

We start by the following lemma

LEMMA 3.5. – *If, for a phase $\psi$ in $C^2$, $\lambda_1(\psi)$ is the bottom of the spectrum of the Witten Laplacian $w_\psi^{(1)}$ attached to $\psi$, then, for any $C^2$ functions $\chi$ and $\varphi$ such that $\chi$ is bounded, we have*

$$\lambda_1(\varphi) \geqslant (\exp -2\|\chi\|_{L^\infty}) \cdot \lambda_1(\varphi + \chi). \tag{3.20}$$

*Proof of Lemma 3.5.* – One proof (suggested by Deuschel) consists in observing that in the case of dimension 1, the lowest eigenvalue $\lambda_1$ of $w^{(1)}$ is exactly the second eigenvalue of the Witten Laplacian $w^{(0)}$. One can then use standard comparison theorems related to the variance.

A second proof (suggested by V. Bach) consists just in starting from the identity

$$\begin{aligned}\langle w_\varphi^{(1)}u \mid u\rangle = \langle &\exp -\tfrac{\chi}{2}\left(-\tfrac{d}{dt}+\tfrac{1}{2}(\varphi'+\chi')\right)\exp\tfrac{\chi}{2}u \mid \\ &\exp -\tfrac{\chi}{2}\left(-\tfrac{d}{dt}+\tfrac{1}{2}(\varphi'+\chi')\right)\exp\tfrac{\chi}{2}u\rangle. \end{aligned}\tag{3.21}$$

We then deduce, that, for any $u\in C_0^\infty(\mathbb{R})$,

$$\begin{aligned}\langle w_\varphi^{(1)}u \mid u\rangle &\geqslant \exp -\|\chi\|_{L^\infty}\langle w_{\varphi+\chi}^{(1)}\exp\tfrac{\chi}{2}u \mid \exp\tfrac{\chi}{2}u\rangle \\ &\geqslant \exp -\|\chi\|_{L^\infty}\lambda_1(\varphi+\chi)\left\|\exp\tfrac{\chi}{2}u\right\|^2, \end{aligned}\tag{3.22}$$

and consequently

$$\langle w_\varphi^{(1)}u \mid u\rangle \geqslant \exp -2\|\chi\|_{L^\infty}\lambda_1(\varphi+\chi)\|u\|^2. \tag{3.23}$$

The minimax principle and (3.23) give (3.20). This ends the proof of Lemma 3.5. □

*Proof of Theorem 3.4.* – We simply apply the previous lemma and Theorem 3.3 with $\phi_{\alpha,\mathcal{J}}$ replaced by $\phi_{\alpha,\mathcal{J}}+\chi$. □

*Remark* 3.6. – As observed by Deuschel, these estimates are still valid for a large class of interactions with bounded range and bounded second derivatives.

## 4. UNIFORM DECAY ESTIMATES

Although everything is written in this proof for simplicity in the case $N=1$, there are no specific difficulties to treat the case $N$ general. See, however, Section 7 for a complementary discussion.

We recall the argument developed in [12] (see also [17,21] (Theorem B) or more recently [4,22]) and control its uniformity with respect to $\Lambda$ and $\omega$. The analysis in [12] was only given for the periodic case and for $d=1$ and we consider here other boundary conditions.

Our starting point is the following formula for the covariance of two functions $f$ and $g$ introduced in (1.15):

$$\mathrm{Cov}_{\Lambda,\omega}(f,g)=\int\left[(A_1^{-1})\nabla f\right]\cdot\nabla g\,d\mu_{\Lambda,\omega}. \tag{4.1}$$

Here $A_1$ is unitary equivalent to $W_{\Phi}^{(1)}$ by conjugation by the map

$$\omega \mapsto U\omega = \exp -\frac{\Phi}{2}\omega,$$

and in particular isospectral. We have in mind to take $f = x_i$, $g = x_j$ with $i$ and $j$ in $\Lambda$. The idea is reminiscent of [6] and consists in the introduction of weighted spaces on $\Lambda$, associated with strictly positive weights satisfying

$$\exp -\kappa \leqslant \rho(\ell)/\rho(k) \leqslant \exp \kappa, \tag{4.2}$$

where $\ell \sim k$ (this means that $\ell$ and $k$ are nearest neighbors in $\mathbb{Z}^d$) and $\kappa$ will be determined later.

For a given $i \in \Lambda$, the functions $\rho(\ell) = \exp -\kappa d(i,\ell)$ where $d$ is a usual distance on $\mathbb{R}^d$ satisfy this condition.

For a given $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| = m$, let us now associate with a given weight $\rho$ on $\Lambda$ the $m \times m$ diagonal matrix $M = M^{\Lambda}$ defined by

$$M_{k\ell} = \delta_{k\ell}\rho(\ell), \quad \text{for } \ell, k \in \Lambda. \tag{4.3}$$

For any slowly increasing functions $f$, $g$, we can rewrite (4.1) in the form

$$\mathrm{Cov}_{\Lambda,\omega}(f,g) = \int \left((M^{-1}A_1^{-1}M)M^{-1}\nabla f\right)\cdot (M\nabla g)\, d\mu_{\Lambda,\omega} \tag{4.4}$$

and we deduce the estimate

$$\left|\mathrm{Cov}_{\Lambda,\omega}(f,g)\right| \leqslant \left\|M^{-1}A_1^{-1}M\right\| \cdot \left\|M^{-1}\nabla f\right\| \cdot \|M\nabla g\|. \tag{4.5}$$

We now take $f(X) = x_i$, $g(X) = x_j$ and choose $\rho = \rho_i = \exp -\kappa d(i,\cdot)$, so that (4.2) is satisfied. We immediately observe that for this choice

$$\left\|M^{-1}\nabla f\right\| = 1, \qquad \|M\nabla g\| = \exp -\kappa d(i,j). \tag{4.6}$$

Everything is then reduced to the control, uniformly with respect to $\Lambda$ and $\omega$, of $M^{-1}A_1^{-1}M$ in suitable $L^2$-norms. We have only here to analyze the effect of the "distorsion" by $M$. This will be done by a simple perturbation argument, once we have characterized the domain of the selfadjoint operator $A_1$ and verified that the domain is conserved in the distorsion. This is easily done under the assumptions (1.1)–(1.10) as proved in [15]. We observe that, for all $X \in \mathbb{R}^{\Lambda}$,

$$\begin{aligned}&\|\operatorname{Hess}\Phi(X)-M^{-1}\operatorname{Hess}\Phi(X)M\|_{\mathcal{L}(\ell^2)}\\&\quad=|\mathcal{J}|\|\operatorname{Hess}\Phi_i(X)-M^{-1}\operatorname{Hess}\Phi_i(X)M\|_{\mathcal{L}(\ell^2)}.\end{aligned}\tag{4.7}$$

In this example, observing that the coefficients of

$$\delta_M(\operatorname{Hess}\Phi_i)=:\operatorname{Hess}\Phi_i-M^{-1}\operatorname{Hess}\Phi_i M$$

vanish if $k\not\sim\ell$, it is immediate, using Schur's Lemma, to get that

$$\begin{aligned}\|\delta_M(\operatorname{Hess}\Phi_i)\|_{\mathcal{L}(\ell^2)}&\leqslant 2d\sup_{\ell\sim k}\left|\left(1-\frac{\rho(\ell)}{\rho(k)}\right)\right|\\&\leqslant 2d\max\big((1-\exp-\kappa),(\exp\kappa-1)\big)\\&=2d\theta,\end{aligned}\tag{4.8}$$

with

$$\theta:=(\exp\kappa-1),\tag{4.9}$$

and this is clearly uniform with respect to the lattice.

We now estimate the operator $M^{-1}A_1^{-1}M$. An immediate computation gives

$$M^{-1}A_1^{-1}M=A_1^{-1}\big[I+\mathcal{J}\delta_M(\operatorname{Hess}\Phi_i)A_1^{-1}\big]^{-1},\tag{4.10}$$

where $\delta_M(\operatorname{Hess}\Phi_i)$ is now considered as an operator (of order 0) on the $L^2$ 1-forms. According to (4.6) and (4.8), we finally obtain that, if

$$0<2\mathcal{J}d\,\theta\leqslant\frac{1}{2}\sigma_1,$$

then

$$\|M^{-1}A_1^{-1}M\|\leqslant\frac{2}{\sigma_1}.\tag{4.11}$$

We have obtained the following:

THEOREM 4.1. – *Under the same assumptions as in Theorem* 3.4, *there exist $\mathcal{J}_0>0$, $C$ and $\kappa$, such that, for any $\Lambda$, $\omega$ and $\mathcal{J}\in[-\mathcal{J}_0,+\mathcal{J}_0]$, the correlation pair function satisfies*

$$\big|\operatorname{Cov}_{\Lambda,\omega}(x_i,x_j)\big|\leqslant C\exp-\kappa d(i,j),\quad\forall i,j\in\Lambda.\tag{4.12}$$

*Remark* 4.2. – Theorem 1.2 is an immediate corollary of Theorem 4.1 if we observe that the strict convexity at $\infty$ is a sufficient condition for having (3.10).

*Remark* 4.3. – The proof gives actually an exponential decay with a rate $\kappa$ such that $\exp \kappa \cdot \mathcal{J}$ is small enough. This is coherent with Dobrushin's approach (see [11], for a discussion about the links between this approach and the Witten Laplacian approach). The transfer matrix approach (in the case $d = 1$) or the formal perturbative approach suggest also that $\kappa \asymp \ln \frac{1}{\mathcal{J}}$.

## 5. PROOF OF THE LOGARITHMIC SOBOLEV ESTIMATES

We can just finish the proof by recalling the following theorem by B. Zegarlinski [27]. B. Zegarlinski considers a Gaussian measure on $\mathbb{R}^{\mathbb{Z}^d}$ with zero mean and a covariance $G$ whose inverse $A = G^{-1}$ satisfies that there exists $R > 0$ such that

$$A_{ij} \equiv 0 \quad \text{if } d(i, j) > R, \tag{5.1}$$

and that

$$\|A\| := \sup_j \sum_i |A_{ij}| < +\infty. \tag{5.2}$$

In the case when (1.9) is assumed, this contains our case, with $R = 1$, $A_{ii} = \mathcal{J} + \frac{1}{C}$ and $A_{ij} = -\mathcal{J}$ when $i \sim j$.

The phase $U := \phi - \frac{x^2}{2C}$ is indeed a semibounded function which can be represented as the sum

$$U = v + w, \tag{5.3}$$

where $w$ has a bounded first and second derivative and $v$ is with non negative second derivative.

This corresponds actually to the condition (scm).

THEOREM 5.1. – *Let us assume that* (1.9) *and* (1.10) *are satisfied. Suppose, that for some $\mathcal{J}$, there are constants $C$, $\kappa \in ]0, +\infty[$ such that for any sufficiently large cube $\Lambda_0 \subset \mathbb{Z}^d$ and any $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ we have*

$$\left|\mathrm{Cov}_{\Lambda_0,\omega}(x_i, x_j)\right| \leqslant C \exp -\kappa d(i, j), \quad \forall i, j \in \Lambda_0. \tag{5.4}$$

*Then there exists a constant $c \in ]0, +\infty[$ such that for any cube $\Lambda \subset \mathbb{Z}^d$ we have*

$$\langle f \ln f \rangle_\Lambda \leqslant 2c \langle |\nabla f^{\frac{1}{2}}|^2 \rangle_\Lambda + \langle f \rangle_\Lambda \ln \langle f \rangle_\Lambda, \tag{5.5}$$

*for all nonnegative functions $f$ for which the right-hand side is finite.*

Combining all the statements Theorems 2.1, 2.4, 3.3, 4.1 and 5.1, we get Theorem 1.1 in the case $N = 1$.

*Remark* 5.2. – The preprint [26] considers generalizations going in other directions. Let us mention some of them where our proof is still relevant. We can consider more generally, for any $h \in \mathbb{Z}^d$, the phases

$$\Phi^{\Lambda,D,\mathcal{J},h}(X) = \Phi^{\Lambda,D,\mathcal{J}}(X) + \sum_{i\in\Lambda} h_i \cdot x_i$$

and get the previous result uniformly with respect to $h$.

We can also consider more generals $A_{ij}$ satisfying the assumptions of B. Zegarlinski [27]. The condition on $\mathcal{J}$ is then replaced by a condition on $\|A - \frac{1}{C}I\|$.

N. Yoshida mentioned also the case when, for fixed $\mathcal{J}$, one considers weak perturbations of convex situations. Here the techniques and results of our previous papers [12] and [13] can be used for verifying the assumptions of Theorem 5.1 of [27].

On the other hand, our assumptions on the single spin phase are weaker than in Yoshida [26]. His proof permits to analyze only the quartic model and other polynomials like

$$\phi(x) = a_{2n}x^{2n} + \cdots + a_2x^2$$

with non negative $a_4, \ldots, a_{2(n-1)} \geqslant 0$ and $a_{2n} > 0$. On the other hand, this is not limited to perturbative situations.

## 6. THE CASE WHEN THE ONE PARTICLE PHASE IS DEFINED ON $\mathbb{R}^N$

Some of the methods for proving decay estimates use standard estimates in statistical mechanics (for example, GHS) where the dimension $N$ of the space on which the one particle phase is defined plays a role. For example, the condition $N \leqslant 4$ is mentioned by A. Sokal in [23] (see the discussion after his Theorem 3). We shall show here that, in the Witten Laplacian approach, the extension from $N = 1$ to general $N$ leads to some specific difficulty. If it seems easy to get a uniform control for the Witten Laplacian on 1-forms restricted to exact 1-forms (which is sufficient for obtaining the Poincaré estimates), the argument presented in Section 4 for analyzing the decay needs explicitly the existence of a lower bound for the Witten Laplacians on **all** 1-forms and this leads us to find another proof when $N > 1$. We shall show in Section 7 another way for circumventing this difficulty by modifying the proof given in Section 4.

Let us assume now that our phase $\phi$ is now defined on $\mathbb{R}^N$ and satisfies the two following natural conditions.

- There exists [7] $C$ such that, $\forall x \in \mathbb{R}^N$ with $|x| \geqslant C$, we have

$$\operatorname{Hess}\phi \geqslant \frac{1}{C}. \tag{6.1}$$

- There exists $\rho > 0$ and, for all $\beta \in \mathbb{N}^N$, a constant $C_\beta$ such that

$$\left|\partial_x^\beta \nabla\phi(x)\right| \leqslant C_\beta < \nabla\phi(x) >^{(1-\rho|\beta|)_+}. \tag{6.2}$$

The second condition is probably technical.

THEOREM 6.1. – *Under the two conditions* (6.1) *and* (6.2), *there exists* $\mathcal{J}_0 > 0$ *such that, for* $|\mathcal{J}| \leqslant \mathcal{J}_0$, *the Witten Laplacian on the* 1*-forms attached to the phase* $\Phi$ *on* $\mathbb{R}^{N|\Lambda|}$ *is uniformly strictly positive with respect to the cubes* $\Lambda$, *the boundary conditions and* $\mathcal{J} \in [-\mathcal{J}_0, +\mathcal{J}_0]$.

The basic example satisfying these assumptions appears in the so called lattice vector field theory (see [8]). We take $N = 4$, $d = 4$ and

$$\phi(x) = \frac{1}{12}\lambda|x|^4 + \frac{\nu}{2}|x|^2, \tag{6.3}$$

with $\lambda > 0$ and $\nu < 0$.

The proof is a consequence of the following

LEMMA 6.2. – *Let* $\phi$ *satisfy* (6.1) *and* (6.2) *and* $w^{(1)}_{\phi_{\mathcal{J},\alpha}}$ *the Witten Laplacian on* 1*-forms on* $\mathbb{R}^N$ *attached to the phase*

$$\mathbb{R}^N \ni x \mapsto \phi_{\mathcal{J},\alpha}(x) = \phi(x) + 2\mathcal{J}d|x|^2 - \alpha\cdot x,$$

*where* $\alpha \in \mathbb{R}^N$ *and* $\mathcal{J} \in ]-\frac{1}{C}, +\frac{1}{C}[$.

*Then there exist* $\mathcal{J}_0 > 0$ *and* $\varrho_1 > 0$ *such that, for all* $(\mathcal{J}, \alpha)$ *in* $]-\mathcal{J}_0, +\mathcal{J}_0[\, \times \mathbb{R}^N$, *the lowest eigenvalue of* $w^{(1)}_{\phi_{\mathcal{J},\alpha}}$ *is larger than* $\varrho_1$.

[7] We note that this condition implies the superstability in the sense of D. Ruelle for the corresponding phase $\Phi$ defined in (1.1), that is the inequality (1.2).

*Proof of Lemma 6.2.* – The Witten Laplacian on 1-forms on $\mathbb{R}^N$ attached to the phase $\phi_{\mathcal{J},\alpha}$ takes now the form

$$w^{(1)}_{\phi_{\mathcal{J},\alpha}} := \left(-\Delta + \frac{1}{4}|\nabla\phi_{\mathcal{J},\alpha}|^2 - \frac{1}{2}\Delta\phi_{\mathcal{J},\alpha}\right) \otimes I + \operatorname{Hess}\phi_{\mathcal{J},\alpha},$$

and will simply denoted by $w$ when no confusion is possible. We emphasize that this operator has no more the simple structure used, in the case $N = 1$, for comparing Witten Laplacians with various phases.

We introduce a partition of unity starting of a $C^\infty$ function $\chi(x)$ satisfying $0 \leqslant \chi \leqslant 1$, equal to 1 on $B_{\mathbb{R}^N}(0,1)$ and with support in $B_{\mathbb{R}^N}(0,2)$. We introduce also a (large) parameter $R$ and define

$$\chi_{1;R}(x) = \chi\left(\frac{x}{R}\right), \qquad \chi_{2;R}(x) = \sqrt{1 - \chi_{1;R}(x)^2}.$$

We use the standard identity associated to this partition of unity

$$\langle wu \mid u\rangle_{L^2(\mathbb{R}^N)} = \big\langle \widehat{w}(\chi_1 u) \mid (\chi_1 u)\big\rangle_{L^2(\mathbb{R}^N)} + \big\langle \widehat{w}(\chi_2 u) \mid (\chi_2 u)\big\rangle_{L^2(\mathbb{R}^N)},$$

where $\widehat{w} = w - (|\nabla\chi_1|^2 + |\nabla\chi_2|^2) \otimes I$.

We observe also that $\widehat{w} - w = \mathcal{O}(\frac{1}{R^2})$. We first observe that if $R \geqslant C$ large enough such that

$$\langle w(\chi_2 u) \mid \chi_2 u\rangle \geqslant \frac{1}{C}\|\chi_2 u\|^2.$$

We can then find $R_0 \geqslant C$ and $0 < \mathcal{J}_0 < \frac{1}{C}$ such that, for any $\mathcal{J} \in [-\mathcal{J}_0, +\mathcal{J}_0]$, any $\alpha \in \mathbb{R}^N$ and $R \geqslant R_0$

$$\big\langle \widehat{w}(\chi_2 u) \mid (\chi_2 u)\big\rangle \geqslant \frac{1}{2C}\|\chi_2 u\|^2. \tag{6.4}$$

We fix now $R$. It is easy to get, for any $A > 0$, the existence of $D$ such that for all $(\alpha, \mathcal{J}) \in \mathbb{R}^N \times [-\mathcal{J}_0, +\mathcal{J}_0]$ such that $|\alpha| \geqslant D$, we have

$$\big\langle \widehat{w}(\chi_1 u) \mid (\chi_1 u)\big\rangle \geqslant A\|\chi_1 u\|^2. \tag{6.5}$$

Putting (6.4) and (6.5) together, we have proved the existence of $\varrho_0 > 0$, $\mathcal{J}_0 > 0$ and $D > 0$ such that, for all $(\alpha, \mathcal{J}) \in \mathbb{R}^N \times [-\mathcal{J}_0, +\mathcal{J}_0]$ with $|\alpha| \geqslant D$, we have

$$\langle wu \mid u\rangle \geqslant \varrho_0\|u\|^2. \tag{6.6}$$

We now only need the local control of the lower bound with respect to $(\alpha, \mathcal{J})$. We observe that, for any $(\alpha, \mathcal{J})$, the lowest eigenvalue of $w$ is strictly positive (see [8] [21] and [15]) and continuous (under the assumptions (6.1) and (6.2)) with respect to $(\alpha, \mathcal{J}) \in \mathbb{R}^N \times [-\mathcal{J}_0, +\mathcal{J}_0]$. This proves the lemma. □

*Proof of Theorem 6.1.* – Once we have this lemma, one deduces Theorem 6.1 from the lemma along the same lines as when $N = 1$. □

*Proof of Theorem 1.1.* – Under the assumptions of this theorem, one first deduces the uniform control of the decay as in the case when $N = 1$ (see Section 4). In order to obtain the uniform logarithmic Sobolev inequality for $\mathcal{J}$ small enough and any $N$, there is a need of an extension of Zegarlinski's Theorem 5.1, for $N > 1$. As confirmed directly by him, [9] this extension is true. □

*Remark* 6.3. – As in the case when $N = 1$, we can find variants of the condition of strict convexity by introducing (in the spirit of ([12])) the condition that there exists $\varepsilon \in ]0, 1]$ such that the operator

$$W_\varepsilon^{red} := -\varepsilon\Delta - \frac{\varepsilon}{2}\Delta\phi_{\mathcal{J},0} + \operatorname{Hess}\Phi_{\mathcal{J},0}, \tag{6.8}$$

is strictly positive.

## 7. A MORE POWERFUL APPROACH FOR THE DECAY OF CORRELATIONS

In answer to a question by J.-D. Deuschel, we shall now present a variant of the proof of the decay which just uses the uniform Poincaré inequality for the Witten Laplacian associated to the single-spin phase $w^{(0)}_{\phi_{\mathcal{J},\alpha}}$ (in other words the existence of a uniform lower bound for the second eigenvalue $\lambda_2^{(0)}(\alpha, \mathcal{J})$ of this Laplacian) instead of the uniform control of the lowest eigenvalue $\lambda_1^{(1)}(\alpha, \mathcal{J})$ of the single-spin Laplacian

[8] We observe here that the assumption (1.9) implies, for some $D > 0$, the condition of Sjöstrand–Johnsen:

$$x \cdot \nabla\phi(x) \geqslant \frac{1}{D}\langle x\rangle^2 - D. \tag{6.7}$$

[9] Personal communication, January 8, 1998.

$w^{(1)}_{\Phi_{\mathcal{J},\alpha}}$ on all 1-forms. This make no difference when $N=1$ because in this case we have the equality

$$\lambda_2^{(0)}(\alpha,\mathcal{J})=\lambda_1^{(1)}(\alpha,\mathcal{J}),\quad \text{if } N=1, \tag{7.1}$$

but when $N>1$ we may have the strict inequality $\lambda_2^{(0)}(\alpha,\mathcal{J})>\lambda_1^{(1)}(\alpha,\mathcal{J})$, and consequently the infimum over $\alpha$ of $\lambda_1^{(1)}(\alpha,\mathcal{J})$ or equivalently of the lowest eigenvalue of the single-spin Laplacian $w^{(1)}_{\Phi_{\mathcal{J},\alpha}}$ restricted to the exact 1-forms may be strictly higher than the infimum over $\alpha$ of $\lambda_1^{(1)}(\alpha,\mathcal{J})$ .

*Remark* 7.1. – We recall that another proof using Dobrushin's approach has been obtained by J.-D. Deuschel [10] (see also [11]) under essentially the same assumptions.

Our starting point is again formula (4.1) or formula (4.4). It is, however, better to come back to the origin of (4.1). Let us denote by $L^2_\Phi$ the Hilbert space

$$L^2_\Phi := L^2\big((\mathbb{R}^N)^\Lambda;\exp-\Phi\,dX\big)$$

and by $\Omega^{k,2}_\Phi$ the Hilbert space of the $k$-forms with coefficients in $L^2_\Phi$.

We were starting from the operator $A_0=d^{*,\Phi}d$ where $d$ is the differential on the 0-forms and $d^{*,\Phi}$ is the adjoint defined on the one-forms:

$$\langle d_\Phi u\mid\sigma\rangle_{\Omega^{1,2}_\Phi}=\langle u\mid d^{*,\Phi}\sigma\rangle_{L^2_\Phi},$$

for all $u\in C_0^\infty((\mathbb{R}^N)^\Lambda)$ and $\sigma\in C_0^\infty(((\mathbb{R}^N)^\Lambda);((\mathbb{R}^N)^\Lambda))$.

The initial remark was that we can solve

$$f-\langle f\rangle=A_0u, \tag{7.2}$$

from which we get the identity

$$df=dA_0u=A_1\,du. \tag{7.3}$$

We consequently can write

$$\begin{aligned}\mathrm{Cov}_{\Lambda,\omega}(f,g)&=\langle du\cdot dg\rangle\\&=\big\langle(M^{-1}du)\cdot(M\,dg)\big\rangle.\end{aligned} \tag{7.4}$$

[10] Personal communication, January 30, 1998.

We have consequently to control

$$\sigma := M^{-1}du. \tag{7.5}$$

In order to get this control, we rewrite (7.3) in the form

$$M^{-1}df = M^{-1}A_1MM^{-1}du = A_1\sigma + (M^{-1}A_1M - A_1)\sigma. \tag{7.6}$$

We now take the scalar product in $\Omega_\Phi^{1,2}$ with $\sigma$ in the identity (7.6) and get

$$\langle (M^{-1}df)\cdot\sigma\rangle \geqslant \langle (A_1\sigma)\cdot\sigma\rangle - C\mathcal{J}\|\sigma\|^2_{\Omega_\Phi^{1,2}}. \tag{7.7}$$

Here we have used the point-wise estimate of $\|M^{-1}\operatorname{Hess}\Phi_i M\|$ in $\mathcal{L}(\ell^2(\Lambda;\mathbb{R}^N))$ (cf. (4.7) and (4.8)). Observing that $\sigma$ was no more exact we thought in the previous versions (see Section 4) that we really need a lower bound for $A_1$ on all forms and we develop consequently a technique for getting this lower bound in Section 6. We show here that it is actually not necessary for obtaining the decay. Coming back to what was done before we can rewrite (7.7) in the form

$$\langle (M^{-1}df)\cdot\sigma\rangle \geqslant \sum_j \langle (a_{\phi_j}^{(1)}\sigma_j)\cdot\sigma_j\rangle - C\mathcal{J}\|\sigma\|^2_{\Omega_\Phi^{1,2}}, \tag{7.8}$$

where $a_{\phi_j}^{(1)}$ is unitary equivalent to $w_j^{(1)}$ (through the map $u \mapsto \exp -\frac{\phi_j}{2}u$) and $C$ is uniform with respect to all the parameters.

*The crucial observation is that*

$$\sigma_j := \rho(j)^{-1}d_ju = d_j(\rho(j)^{-1}u)$$

*is consequently exact as a function of $x_j$ in $\mathbb{R}^N$.*

Using this property, we can write

$$\langle (M^{-1}df)\cdot\sigma\rangle \geqslant \sum_j \rho(j)^{-2}\|a_{\phi_j}^{(0)}u\|^2_{\Omega_\Phi^{1,2}} - C\mathcal{J}\|\sigma\|^2_{\Omega_\Phi^{1,2}}, \tag{7.9}$$

where $C$ is uniform with respect to all parameters and

$$a_{\phi_j}^{(0)} = d_j^{*,\phi_j}d_j = \sum_{\ell=1}^N(-\partial_{x_{j\ell}} + \partial_{x_{j\ell}}\phi_j)\partial_{x_{j\ell}} \tag{7.10}$$

which is unitary equivalent (through the conjugation $u \mapsto \exp -\frac{\phi_j}{2}u$) to $w_j^{(0)}$ and will be more shortly denoted as $a_j^{(0)}$. We would like to consider (in addition to the parameter $\omega$) the $x_k$ $(k \neq j)$ as parameters (we write $\widehat{x}_j$) and consider the single-spin Hilbert spaces of $k$-forms on $\mathbb{R}^N$ constructed on $L^2_{\phi_j}(\mathbb{R}^N) : \Omega^{k,2}_{\phi_j}$.

We recall the observation that

$$(\partial_{x_{jl}}\phi_j)(x_j) = (\partial_{x_{jl}}\Phi)(x_j, \widehat{x}_j).$$

In particular, the operator $a_j^{(0)}$ initially defined as a selfadjoint operator on $L^2_\Phi((\mathbb{R}^N)^\Lambda)$ can also be considered as a family (depending on the parameter $\widehat{x}_j$) of selfadjoint operators on $L^2_{\phi_j}(\mathbb{R}^N)$.

We denote shortly by $u_j$ the map $x_j \mapsto u(x_j, \widehat{x}_j)$ and define $\widetilde{u}_j = u_j - \langle u_j \rangle_j$ where the mean value $\langle \cdot \rangle_j$ is respect with the measure on $\mathbb{R}^N$ $\exp -\phi_j dx_j$. We have of course

$$d_j u_j = d_j \widetilde{u}_j,$$

and

$$a_j^{(0)} u_j = a_j^{(0)} \widetilde{u}_j.$$

Using the standard inequalities, we get, denoting by $\lambda_2^{(0);j,z_k}$ the second (that is the first positive) eigenvalue of $a_j^{(0)}$ (or equivalently of $w_j^{(0)}$) considered as an operator on $L^2(\mathbb{R}^N)$

$$\begin{aligned}\lambda_2^{(0);j,z_k} \|\sigma_j\|^2_{\Omega^{1,2}_{\phi_j}} &= \lambda_2^{(0);j,z_k} \langle a_j^{(0)} \widetilde{u}_j \mid \widetilde{u}_j \rangle_{L^2_{\phi_j}} \\ &\leqslant \lambda_2^{(0);j,z_k} \|a_j^{(0)} \widetilde{u}_j\|_{L^2_{\phi_j}} \cdot \|\widetilde{u}_j\|_{L^2_{\phi_j}} \\ &\leqslant \|a_j^{(0)} u_j\|^2_{L^2_{\phi_j}}. \end{aligned} \tag{7.11}$$

We then multiply by $\exp -(\Phi - \phi_j)$ this inequality, integrate over $(\mathbb{R}^N)^{\Lambda \setminus \{j\}}$ with respect to the other variables $\widehat{x}_j$, and obtain

$$\langle (M^{-1} df) \cdot \sigma \rangle \geqslant \sum_j \Big( \inf_{z_k; k \neq j} \lambda_2^{(0);j,z_k} \Big) \|\sigma_j\|^2_{\Omega^{1,2}_\Phi} - C\mathcal{J} \|\sigma\|^2_{\Omega^{1,2}_\Phi}. \tag{7.12}$$

This shows that it is sufficient to get a uniform lower bound of the splitting between the two first eigenvalues for the family of Witten Laplacians on 0-forms on $\mathbb{R}^N$. But this is now easy in this case by one of the two proofs done in the case when $N = 1$. Under the assumption (6.1), we first prove the existence of a bounded function $S(t)$ in $C^2(\mathbb{R}^N)$, such that

$$\operatorname{Hess}\phi + \operatorname{Hess} S \geqslant \frac{1}{C}. \tag{7.13}$$

We can then reduce the problem to the case when $\phi$ is strictly convex everywhere for estimating $\lambda_2^{(0);j,z_k}$ and finding a lower bound.

For a suitable constant $C$ and for $\mathcal{J}$ small enough, we get, after use of Cauchy–Schwarz in (7.12), the inequality

$$\left\|(M^{-1}\,df)\right\|_{\Omega_\Phi^{1,2}} \cdot \|\sigma\|_{\Omega_\Phi^{1,2}} \geqslant \frac{1}{C}\|\sigma\|^2_{\Omega_\Phi^{1,2}}, \tag{7.14}$$

and finally

$$\left\|M^{-1}\,du\right\|_{\Omega_\Phi^{1,2}} \leqslant C\|M^{-1}\,df\|_{\Omega_\Phi^{1,2}}. \tag{7.15}$$

The end of the proof is the same as in Section 4 if we replace (4.5) by

$$\left|\operatorname{Cov}_{\Lambda,\omega}(f,g)\right| \leqslant \|M^{-1}\,du\|_{\Omega_\Phi^{1,2}} \cdot \|M\,dg\|_{\Omega_\Phi^{1,2}}, \tag{7.16}$$

and use (7.15).

*Remark* 7.2. – The advantage of this proof of the decay is also that it can be extended to general interaction with finite range and bounded Hessian. The proof in Section 6 was indeed limited to quadratic interactions or suitable perturbations of this quadratic interaction. On the other hand, the constants obtained by the different approachs may be different, particularly in the semi-classical context.

But an open question remains: Is this extension true for the logarithmic Sobolev estimates? This depends on the proof of an extension of Zegarlinski's theorem in this more general case. A partial answer to this question is given in [5].

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