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Necessary conditions for the bootstrap of the mean of a triangular array*

by

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ABSTRACT. – Although necessary conditions for the bootstrap of the mean to work have already been given, only the case of an i.i.d. sequence has been exhaustively considered. We study such necessary conditions for (not necessarily infinitesimal) triangular arrays showing that the existence of a limit law in probability leads to infinitesimality and to the Central Limit Theorem to hold for a rescaled subarray.

Our setup is based on a triangular array of row-wise independent identically distributed random variables and any resampling size. While our results are similar to those obtained in Arcones and Giné [*Ann. Inst. Henri Poincaré* 25 (1989) 457–481] for an i.i.d. sequence, our proof is based on symmetrizations and the consideration of U-statistics and allows a unified treatment without moment assumptions. © Elsevier, Paris

Key words: necessary condition, bootstrap mean, arbitrary resampling sizes, triangular arrays, U-statistics, Central Limit Theorem.

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RÉSUMÉ. – Bien que les conditions nécessaires au fonctionnement du bootstrap de la moyenne aient déjà été données, seul le cas d'une séquence de variables aléatoires indépendantes et de même loi (i.i.d.) a été considéré exhaustivement. Nous étudions de telles conditions nécessaires pour un tableau triangulaire (pas nécessairement infinitésimal) en montrant que l'existence d'une loi limite en probabilité conduit à l'infinitésimalité et à la vérification du Théorème Central du Limite pour un sous-tableau normalisé.

Notre cadre se base sur un tableau triangulaire de variables aléatoires indépendantes et de même loi sur chaque ligne et de n'importe quel taille d'enchantillonnage. Bien que nos resultats sont similaires à ceux obtenus dans Arcones et Giné [*Ann. Inst. Henri Poincaré* 25 (1989) 457–481] pour une séquence i.i.d., notre démonstration se base sur la symétrisation et la prise en considération des U-statistiques et permet un traitement unifié sans hypothèses sur les moments.

1. INTRODUCTION

The differences between the asymptotic distribution of the bootstrap sample mean for infinitesimal arrays and for sequences of independent identically distributed (i.i.d.) random variables (r.v.'s) have been shown in Cuesta-Albertos and Matrán (1998). After this work it becomes mathematically natural to follow it by providing necessary conditions to assure a bootstrap limit law.

We would like to emphasize, however, the interest of necessary conditions from the point of view of statistical applications of the bootstrap. Recall that one of the achievements of the bootstrap techniques since their introduction by Efron (1979) is that it allows, via Monte Carlo method, to approximate the distribution of the statistic of interest. In the simplest case of the mean, given n observations $X_1 = x_1, ..., X_n = x_n$ of a random variable X, this leads to obtain a large set of resamples of size m_n from $\{x_1, ..., x_n\}$:

$$x^*_{1,1}, ..., x^*_{1,m_n} \text{ with sample mean } t^*_1$$

$$x^*_{2,1}, ..., x^*_{2,m_n} \text{ with sample mean } t^*_2$$

$$\cdots\cdots\cdots\cdot$$

$$x^*_{r,1}, ..., x^*_{r,m_n} \text{ with sample mean } t^*_r$$

with the guess that the sample distribution of $t_1^*, t_2^*, ... t_r^*$ will be a good approximation to the asymptotic distribution, if any!, of $\bar{X}$.

Thus, as a matter of fact our interest is to ensure that if the bootstrap distribution is, say approximately normal, then $\bar{X}$ will be approximately normal. The mathematical justification of this procedure must be given by a necessary condition.

Let us suppose that $B_1, ..., B_n$ is a sample of Bernoulli trials. Two approximations to the probability law of $S_n = \sum_{i=1}^n B_i$ are usually considered. First, the celebrated De Moivre's Central Limit Theorem asserts that, when properly normalized, the law of S_n is nearly normal. On the other hand, the (not less celebrated) Poisson's Theorem of Rare Events allows to use the Poisson distribution to approximate the law of S_n. As it is well known, from the mathematical point of view, with the law of convergence of types in mind avoiding both convergences for the same sequence (even rescaled), the problem is circunvented through the consideration of triangular arrays of random variables. This allows to give mathematical meaning to the expression 'rare events' through a compromise between the sample size and the probability of success in the trials.

These observations should make clear that necessary conditions obtained in the setup of a sequence of i.i.d. random variables can only give answers to the following question:

If we assume that the distribution type of a summand does not depend on n and if for a wide range of sample sizes n the distribution of the corresponding bootstrap sum can be approximated by a law, what can we say about the distribution of the sum?

However, from the point of view of applications the interesting question should be

If, given a sample, the distribution of the bootstrap sum can be approximated by a law, what can we say about the distribution of the sum of the original sample?

and the appropriate setup to discuss this problem is given by triangular arrays of row-wise i.i.d. random variables, which will be the setup considered in this paper.

The first necessary condition for the bootstrap of the mean for i.i.d. sequences and resampling size equal to the sample size, i.e. $m_n = n$, was given in Giné and Zinn (1986) showing that the bootstrap works a.s. if and only if the common distribution of the sequence has finite second moment, while it works in probability if and only if that distribution belongs to the domain of attraction of the normal law. Hall (1990) completes the analysis in this setup showing that when there exists a bootstrap limit law (in probability) then either the parent distribution belongs to the domain of attraction of the normal law or it has slowly varying tails and one of the two tails completely dominates the other.

The interest of considering resampling sizes different to the sample size was noted among others by Swanepoel (1986) and Athreya (1987). For general arbitrary resampling sizes, but still in the setup of an i.i.d. sequence, Arcones and Giné (1989) obtained:

Let $X^*_{n,1}, X^*_{n,2}, ...X^*_{n,m_n}$ be a bootstrap sample obtained from i.i.d.r.v.'s $X_1, X_2, ...X_n$, and set $S^*_n = \sum_{i=1}^{m_n} X^*_{n,i}$. Denote by $\mathcal{L}^*$ the conditional law given the sample. Then, provided that $m_n \nearrow \infty$, if there exist a non-degenerated random probability measure μ, constants $\{a_n\}_{n=1}^{\infty}$, $a_n \nearrow \infty$, and r.v's c_n, measurable on the σ-fields $\sigma(X_1, X_2, ..., X_n), n \in \mathbb{N}$, satisfying

$$\mathcal{L}^*\left(\frac{1}{a_n}(S^*_n - c_n)\right) \xrightarrow{w} \mu \text{ in probability, then}$$

(a) there exist a Lévy measure ν, and $\sigma^2 \geq 0$ such that for every $\tau > 0$ with $\nu\{-\tau, \tau\} = 0$,

$$\mathcal{L}^*\left(\frac{1}{a_n}(S^*_n - \frac{m_n}{n}\sum_{i=1}^{n} X_i I_{\{|X_i| \leq \tau a_n\}})\right) \xrightarrow{w} N(0, \sigma^2) * c_\tau \text{Pois } \nu$$

in probability.

(b) If $c > 0$ and

$$b_{n,c} = \begin{cases} a_n, & \text{if } m_n \leq cn \\ a_n(n/m)^{1/2}, & \text{if } m_n > cn \end{cases}, r_{n,c} = \begin{cases} m_n & \text{if } m_n \leq cn \\ n & \text{if } m_n > cn \end{cases},$$

then,

$$\frac{1}{b_{n,c}}\sum_{i=1}^{r_{n,c}}(X_i - EX_i I_{\{|X_i| \leq \tau b_{n,c}\}}) \xrightarrow{w} N(0, \sigma^2) * c_\tau \text{Pois } \nu.$$

(c) If $\limsup m_n/n > 0$ then $\nu = 0$ and $\sigma^2 > 0$.

(d) If $\liminf m_n/n > 0$ then X belongs to the domain of attraction of the normal law with norming constants $b_n = a_n(n/m_n)^{1/2}$.

As observed earlier, the fact that these results concern a sequence of i.i.d.r.v.'s prevents them from properly discussing the question of interest above. To our best knowledge, only Mammen (1992) addresses the problem (in the setup of linear statistics) considering triangular arrays, but with resampling sizes, m_n, equal to the original sample size $k_n(= n)$. Our general theorem on necessary conditions for the bootstrap of the mean will cover any resampling rate.

Before stating our main result, we introduce some notation. We will consider a triangular array, $\{X_{n,j} : j = 1, ..., k_n;\ n \in \mathbb{N}\}$, $k_n \to \infty$, of row-wise i.i.d.r.v.'s (from now on a triangular array). $X^*_{n,1}, X^*_{n,2}, ..., X^*_{n,m_n} (m_n \to \infty)$ will be a bootstrap sample, i.e., an i.i.d. sample given $X_{n,1}, ..., X_{n,k_n}$ with law $\frac{1}{k_n}\sum_{j=1}^{k_n} \delta_{X_{n,j}}$ and (bootstrap) sum $S^*_n = \sum_{j=1}^{m_n} X^*_{n,j}$, where δ_x denotes Dirac's measure on x. As usual, P^* and E^* will respectively denote the conditional probability and expectation given the sample, while $\mathcal{L}(Z)$ is the distribution of the r.v. Z and $\mathcal{L}^*(Z)$ is the corresponding conditional distribution of Z.

As in Araujo and Giné (1980), $N(a, \sigma^2) * c_\tau\text{Pois } \nu$ is a general infinitely divisible law written as the convolution of a normal law and the generalized Poisson law associated to the Lévy measure ν. Moreover, for any given r.v. X, and $\delta > 0$, we denote $X_\delta := XI_{\{|X| \le \delta\}}$ and $X^\delta := XI_{\{|X| > \delta\}}$.

Convergence in distribution, in law or weak convergence will be terms indistinctly used through the paper, and will be denoted by $\xrightarrow{w}$, while $\xrightarrow{P}$ will mean convergence in probability.

We state now our main result.

THEOREM 1.1. – *Let $m_n \nearrow \infty$, and $m_n/k_n \to c \in [0, \infty]$. If there exist a probability measure ρ, constants $\{r_n\}_{n=1}^\infty$, $r_n \nearrow \infty$, and random variables A_n, measurable on the σ-fields $\sigma(X_{n,j} : j = 1, ..., k_n), n \in \mathbb{N}$ satisfying*

$$\mathcal{L}^*\left(\frac{1}{r_n}(S^*_n - A_n)\right) \xrightarrow{w} \rho$$

in probability, then:

(i) *ρ is infinitely divisible. Thus, $\rho = N(a, \alpha^2) * c_\tau\text{Pois } \nu$.*

(ii) *If $c = 0$ then there exist constants $\{b_n\}_{n=1}^\infty$ such that*

$$\frac{1}{r_n}\sum_{j=1}^{m_n} X_{n,j} - b_n \xrightarrow{w} N(a, \alpha^2) * c_\tau\text{Pois } \nu$$

The constants b_n must satisfy

$$\lim_{n\to\infty}\left\{\frac{m_n \pi_n}{r_n} + m_n E\left(\frac{X_{n,1} - \pi_n}{r_n}\right)_\tau - b_n\right\} = a,$$

where π_n is a median for $X_{n,j}$.

(iii) *If* $c > 0$ *then* $\nu \equiv 0$ *and there exists* $\{c_n\}_{n=1}^{\infty}$ *such that*

$$\sqrt{\frac{m_n}{k_n}}\frac{1}{r_n}\sum_{j=1}^{k_n} X_{n,j} - c_n \xrightarrow{w} N(a, \alpha^2).$$

The constants c_n *must satisfy*

$$\lim_{n\to\infty}\left\{\frac{\sqrt{m_n k_n}}{r_n}\left(\pi_n + E(X_{n,j} - \pi_n)_{\tau r_n \sqrt{\frac{k_n}{m_n}}}\right) - c_n\right\} = a$$

where π_n *is a median of* $X_{n,j}$.

Remark 1. – Note that the sum in (ii) in the previous theorem does not include all the values in the sample. □

Even though our results are similar to those in Arcones and Giné (1989), the difficulties that arise in the triangular array setting are very different. In addition to the usual techniques in the study of the Central Limit Theorem, to carry out the proof we handle a uniform approximation between some U-statistics and their corresponding projections made possible through Lemma 2.1. The use of U-statistics is a consequence of the impossibility of avoiding the use of symmetrizations to get independent summands in some steps (see Remark 2).

2. PROOFS

Our proof of Theorem 1.1 is based on symmetrization. Let $X_{n,1}^{**}, ..., X_{n,m_n}^{**}$ be a new bootstrap sample independent of $X_{n,1}^{*}, ..., X_{n,m_n}^{*}$ (i.e., $X_{n,1}^{**}, ..., X_{n,m_n}^{**}$ are i.i.d. r.v's with law $k_n^{-1}\sum_{j=1}^{k_n}\delta_{X_{n,j}}$ and independent of $X_{n,1}^{*}, ..., X_{n,m_n}^{*}$). Let $S_n^{**} = \sum_{j=1}^{m_n} X_{n,j}^{**}$. Then

$$\mathcal{L}^*\left(\frac{1}{r_n}(S_n^* - S_n^{**})\right) \xrightarrow{w} \rho * \bar{\rho} \tag{2.1}$$

in probability, where $\bar{\rho}$ is the probability measure such that $\bar{\rho}(A) = \rho(-A)$ for every Borel set A. Let $\{Y_{n,j} : j = 1, ..., m_n, \quad n \in \mathbb{N}\}$ be a triangular array of row-wise i.i.d. r.v.'s such that $\mathcal{L}(Y_{n,j}) = \mathcal{L}\left(\frac{X_{n,j} - X'_{n,j}}{r_n}\right)$, where $X'_{n,j}$ is an independent copy of $X_{n,j}$. When the limit resampling rate, c, is finite, we will use (2.1) to show that, $\{Y_{n,j} : j = 1, ..., m_n, n \in \mathbb{N}\}$ verifies the hypotheses of the Central Limit Theorem. Then we will obtain the convergence of the original array. In order to prove the convergence for $\{Y_{n,j} : j = 1, ..., m_n,\ n \in \mathbb{N}\}$ we need a lemma concerning the approximation of a U -statistic by its Hoeffding projection. First we introduce some notation.

Let $h : \mathbb{R}^m \to \mathbb{R}$ be a symmetric integrable function. With the notation $P_1 \times \cdots \times P_m\, h = \int h\, d(P_1 \times \cdots \times P_m)$, the Hoeffding projections of h are defined as

$$\pi_k\, h(x_1, ..., x_k) = (\delta_{x_1} - P) \times \cdots \times (\delta_{x_k} - P) \times P^{m-k}\, h$$

for $x_i \in \mathbb{R}$ and $1 \leq k \leq m$. For convenience, we denote $\pi_0\, h = P^m\, f$. These projections induce a decomposition of the U -statistic

$$U_n(h) = U_n^{(m)}(h) = \frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < \cdots < i_m \leq n} h(X_{i_1}, ..., X_{i_m})$$

into the sum of U-statistics of orders $k \leq m$, namely, the Hoeffding decomposition:

$$U_n^{(m)}(h) = \sum_{k=0}^{m} \binom{m}{k} U_n^{(k)}(\pi_k\, h).$$

We will use this decomposition to prove the following lemma, which provides a bound for the L_2-distance between a U-statistic of order 2 and a sum of independent r.v's. The main interest of this result relies on the fact that this bound is valid for all symmetric kernels.

LEMMA 2.1. – *Let $X_1, ..., X_n$ be independent identicaly distributed random variables. Assume that $h : \mathbb{R}^2 \to \mathbb{R}$ is a symmetric function such that $E\, h^2(X_1, X_2) < \infty$. With the above notation, let $\hat{U}_n(h) = U_n^{(0)}(\pi_0\, h) + 2U_n^{(1)}(\pi_1\, h)$. Then*

$$E\Big(U_n(h) - \hat{U}_n(h)\Big)^2 \leq \frac{2}{n(n-1)} Var\, h(X_1, X_2).$$

Proof. – According to Hoeffding's expansion, $U_n(h) - \hat{U}_n(h) = U_n^{(2)}(\pi_2\, h)$. Note that $E\, U_n^{(2)}(\pi_2\, h) = 0$, so that it suffices to prove that

$$Var\left(U_n^{(2)}(\pi_2\, h)\right) \leq \frac{2}{n(n-1)} Var\, h(X_1, X_2).$$

Now observe that $U_n^{(2)}(\pi_2\, h) = \binom{n}{2}^{-1} \sum_{1 \leq i_1 < i_2 \leq n} \pi_2\, h(X_{i_1}, X_{i_2})$. We can easily check that $Var\ \ \pi_2\, h(X_1, X_2) = Var\ \ h(X_1, X_2) - 2Var\ \pi_1\, h(X_1)$ and $Cov(\pi_2 h(X_1, X_2),\ \pi_2 h(X_1, X_3)) = 0$. Therefore

$$Var\Big(U_n^{(2)}(\pi_2\, h)\Big) = \frac{2}{n(n-1)} Var\, \pi_2 h(X_1, X_2) \leq \frac{2}{n(n-1)} Var\, h(X_1, X_2),$$

which completes the proof. □

Now we prove convergence for $\{Y_{n,j} : j = 1, ..., m_n, n \in \mathbb{N}\}$ when $c < \infty$.

LEMMA 2.2. – *Under the hypotheses of Theorem 1.1 and using the above notation, if $c < \infty$ and $T_n = \sum_{j=1}^{m_n} Y_{n,j}$ then*

$$T_n \xrightarrow{w} \rho * \bar{\rho}.$$

Proof. – Conditionally given $X_{n,1}, ..., X_{n,k_n}$, $r_n^{-1}(S_n^* - S_n^{**})$ is a sum of i.i.d r.v's. From every subsequence $\{n'\}$ we can extract a further subsequence $\{n''\}$ such that $r_{n''}^{-1}(S_{n''}^* - S_{n''}^{**})$ converges weakly in a probability one set. In that set, $\{r_{n''}^{-1}(X_{n'',j}^* - X_{n'',j}^{**}) : j = 1, 2, ..., m_{n''}\}$ is infinitesimal (see e.g. Breiman (1968), p. 191). Hence, we can conclude that

$$P^*\left(\left|X_{n,1}^* - X_{n,1}^{**}\right| > \delta r_n\right) \xrightarrow{P} 0 \qquad \forall \delta > 0 \tag{2.2}$$

and that $\rho * \bar{\rho}$ is infinitely divisible, i.e. there exist $\sigma \geq 0$ and a symmetric Lévy measure, μ, such that $\rho * \bar{\rho} = N(0, \sigma^2) * c_\tau \text{Pois}\ \mu$. By infinitesimality, necessary conditions in the CLT (see e.g. Araujo and Giné (1980), p. 61) imply that there exists a Lévy measure μ such that

$$m_n P^*\left(X_{n,1}^* - X_{n,1}^{**} > \delta r_n\right) \xrightarrow{P} \mu(\delta, \infty) \tag{2.3}$$

for every $\delta > 0$ such that $\mu\{\delta\} = 0$ and

$$m_n E^*\left[\left(\frac{X_{n,1}^* - X_{n,1}^{**}}{r_n}\right)_{\tau_n}\right]^2 \xrightarrow{P} \sigma^2 \tag{2.4}$$

for every sequence $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \searrow 0$.

The conditional distribution of $X_{n,1}^* - X_{n,1}^{**}$ given $X_{n,1}, ..., X_{n,k_n}$ is

$$\left(\frac{1}{k_n}\sum_{i=1}^{k_n}\delta_{X_{n,i}}\right) * \left(\frac{1}{k_n}\sum_{i=1}^{k_n}\delta_{-X_{n,i}}\right) = \frac{1}{k_n^2}\sum_{1 \leq i,j \leq k_n}\delta_{X_{n,i}-X_{n,j}},$$

hence

$$P^*\left(\left|X_{n,1}^* - X_{n,1}^{**}\right| > \delta r_n\right) = \frac{1}{k_n^2}\sum_{i \neq j} I_{\{|X_{n,i}-X_{n,j}| > \delta\, r_n\}}$$

and we can use Lebesgue's Theorem and (2.2) to obtain

$$EP^*\left(\left|X_{n,1}^* - X_{n,1}^{**}\right| > \delta\, r_n\right) = \frac{k_n - 1}{k_n} P(|X_{n,1} - X_{n,2}| > \delta\, r_n) \underset{n\to\infty}{\longrightarrow} 0,$$

which implies that $\{Y_{n,j} : j = 1, ..., m_n \, n \in \mathbb{N}\}$ is infinitesimal.

Now observe that, using the same notation as in Lemma 2.1 and the fact that the law of $X^*_{n,1} - X^{**}_{n,1}$ is symmetric,

$$m_n P^*\left(X^*_{n,1} - X^{**}_{n,1} > \delta \ \ r_n\right) = \frac{m_n}{2} P^*\left(|X^*_{n,1} - X^{**}_{n,1}| > \delta \ \ r_n\right)$$
$$= \frac{k_n - 1}{k_n} \frac{m_n}{2} U_{k_n}(h_n),$$

where $h_n(x,y) = I_{\{|x-y|>\delta \, r_n\}}$, while

$$m_n E^* \left[\left(\frac{X^*_{n,1} - X^{**}_{n,1}}{r_n} \right)_{\tau_n} \right]^2 = m_n \frac{k_n - 1}{k_n} U_{k_n}(g_n),$$

where $g_n(x,y) = \left(\frac{x-y}{r_n}\right)^2 I_{\{|x-y| \le \tau_n \, r_n\}}$. Thus, we can rewrite (2.3) and (2.4) as

$$\frac{m_n}{2} U_{k_n}(h_n) \xrightarrow{P} \mu(\delta, \infty) \tag{2.5}$$

for every $\delta > 0$ such that $\mu\{\delta\} = 0$ and

$$m_n U_{k_n}(g_n) \xrightarrow{P} \sigma^2 \tag{2.6}$$

for every $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \searrow 0$.

Since

$$Var \ \ h_n(X_{n,1}, X_{n,2}) = P(|X_{n,1} - X_{n,2}| > \delta \, r_n) P(|X_{n,1} - X_{n,2}| \le \delta \, r_n)$$

and

$$Var \, g_n(X_{n,1}, X_{n,2}) \le \tau_n^2,$$

we can use Lemma 2.1 to conclude that

$$E\left[m_n \left(U_{k_n}(h_n) - \hat{U}_{k_n}(h_n)\right)\right]^2$$
$$\le 2 \frac{m_n^2}{k_n(k_n - 1)} P(|X_{n,1} - X_{n,2}| > \delta \, r_n) \underset{n\to\infty}{\rightarrow} 0 \tag{2.7}$$

and also that

$$E\left[m_n \left(U_{k_n}(g_n) - \hat{U}_{k_n}(g_n)\right)\right]^2 \le 2 \frac{m_n^2}{k_n(k_n - 1)} \tau_n^2 \underset{n\to\infty}{\rightarrow} 0, \tag{2.8}$$

which, together with (2.5) and (2.6), provide

$$\frac{m_n}{2}\,\hat{U}_{k_n}(h_n) \xrightarrow{P} \mu(\delta,\infty) \tag{2.9}$$

for every $\delta > 0$ such that $\mu\{\delta\} = 0$ and

$$m_n\,\hat{U}_{k_n}(g_n) \xrightarrow{P} \sigma^2 \tag{2.10}$$

for every $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \searrow 0$. Note that $\hat{U}_{k_n}(h_n)$ is a sum of independent r.v's, namely

$$\hat{U}_{k_n}(h_n) = \theta_n + \frac{2}{k_n}\sum_{i=1}^{k_n} \pi_1\, h_n(X_{n,i}).$$

Since

$$\begin{aligned} P(|\pi_1\, h_n(X_{n,i})| > \varepsilon) &\le \frac{1}{\varepsilon} E|\pi_1\, h_n(X_{n,i})| \\ &\le \frac{2}{\varepsilon} P(|X_{n,1} - X_{n,2}| > \delta\, r_n) \underset{n\to\infty}{\to} 0, \end{aligned}$$

$\left\{\frac{m_n}{k_n}\pi_1 h_n(X_{n,i}) : i = 1, \ldots, k_n\; n \in \mathbb{N}\right\}$ is an infinitesimal array. By the CLT (see e.g. Araujo and Giné (1980), pg. 61, Theorem 4.7(iii)) and (2.9), $\frac{m_n}{2}\ E\ \hat{U}_{k_n}(h_n) \to \mu(\delta,\infty)$. A similar reasoning shows that $m_n\ E\ \hat{U}_{k_n}(g_n) \to \sigma^2$. Now (2.7) and (2.8) imply

$$\frac{m_n}{2}\, E\, U_{k_n}(h_n) \to \mu(\delta,\infty) \tag{2.11}$$

for every $\delta > 0$ such that $\mu\{\delta\} = 0$ and

$$m_n E\, U_{k_n}(g_n) \to \sigma^2 \tag{2.12}$$

for every $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \searrow 0$.

Note that $E\ U_{k_n}(h_n) = P(|Y_{n,1}| > \delta)$ and that $E\ U_{k_n}(g_n) = Var\ Y_{n,j,\tau_n}$. Observe also that (2.12) implies

$$\lim_{\tau\to 0}\left\{\begin{matrix}\limsup_n \\ \liminf_n\end{matrix}\right\} m_n Var Y_{n,j,\tau} = \sigma^2,$$

which completes the proof. □

Now we are ready to prove our main result.

Proof of Theorem 1.1. – Assume first that $c < \infty$. Let $Y_{n,1}$ be as in Lemma 2.2. If π_n is a median of $X_{n,1}$ then Lévy inequalities (see e.g. Feller (1966), pag. 147) imply that

$$P(|X_{n,1} - \pi_n| > r_n\varepsilon) \le 2\,P(|Y_{n,1}| > \varepsilon) \underset{n\to\infty}{\rightarrow} 0,$$

that is, $\left\{\frac{X_{n,j}-\pi_n}{r_n} : j = 1, 2, ..., m_n\; n \in \mathbb{N}\right\}$ is an infinitesimal array. On the other hand, since

$$E\,P^*\left(\left|X^*_{n,1} - \pi_n\right| > r_n\varepsilon\right) = P(|X_{n,1} - \pi_n| > r_n\varepsilon) \to 0,$$

then

$$P^*\left(|X^*_{n,1} - \pi_n| > r_n\varepsilon\right) \xrightarrow{P} 0$$

holds, which together with

$$\mathcal{L}^*\left(\frac{1}{r_n}((S^*_n - m_n\,\pi_n) - (A_n - m_n\,\pi_n))\right) \xrightarrow{w} \rho \quad \text{in probability} \qquad (2.13)$$

implies that ρ is infinitely divisible, namely, $\rho = N(a, \alpha^2) * c_\tau\text{Pois}\ \nu$. By Lemma 2.2 $\rho * \bar{\rho} = N(0, \sigma^2) * c_\tau\text{Pois}\ \mu$. Hence, $\alpha^2 = \frac{\sigma^2}{2}$ and $\nu + \bar{\nu} = \mu$. The infinitesimality and (2.13) imply that

$$m_n\ P^*\left(X^*_{n,1} - \pi_n > r_n\delta\right) \xrightarrow{P} \nu(\delta, \infty)$$

$$m_n\ P^*\left(X^*_{n,1} - \pi_n < -r_n\delta\right) \xrightarrow{P} \nu(-\infty, -\delta)$$

for every $\delta > 0$ such that $\nu\{-\delta, \delta\} = 0$. Therefore

$$\frac{m_n}{k_n}\sum_{j=1}^{k_n} I_{\{X_{n,j}-\pi_n>\delta\ r_n\}} \xrightarrow{P} \nu(\delta, \infty).$$

Observe that the array $\left\{\frac{m_n}{k_n} I_{\{X_{n,j}-\pi_n>\delta r_n\}} : j = 1, ..., k_n\ n \in \mathbb{N}\right\}$ is infinitesimal, thus we can employ the CLT to conclude

$$m_n\ P(X_{n,1} - \pi_n > r_n\delta) \to \nu(\delta, \infty)$$

and

$$m_n\,P(X_{n,1} - \pi_n < -r_n\delta) \to \nu(-\infty, -\delta). \qquad (2.14)$$

Let $S_n = \frac{1}{r_n}\sum_{j=1}^{m_n}(X_{n,j} - \pi_n)$. By Lemma 2.2, if S'_n is an independent copy of S_n then

$$S_n - S'_n \xrightarrow{w} N(0, \sigma^2) * c_\tau\text{Pois}\ \mu.$$

This means that S_n is shift-tight (see Araujo and Giné (1980), Corollary 4.11, p. 27). Let $\{n_k\}_{k=1}^{\infty}$ be a sequence such that S_{n_k} is shift-convergent. By the infinitesimality of $\{(X_{n,j} - \pi_n)/r_n : j = 1, 2, ..., m_n, n \in \mathbb{N}\}$ and using again Theorem 4.7 in Araujo and Giné (1980), there exist $\beta \geq 0$ and a Lévy measure λ such that

$$S_{n_k} - E\, S_{n_k,\tau} \xrightarrow{w} N(0,\beta^2) * c_\tau \text{Pois } \lambda.$$

for every τ such that $\nu\{-\tau,\tau\} = 0$.

By (2.14) necessarily $\lambda = \nu$. Furthermore,

$$S_{n_k} - S'_{n_k} \xrightarrow{w} N(0,2\beta^2) * c_\tau \text{Pois}(\nu + \bar{\nu}),$$

which implies $N(0,2\beta^2) * c_\tau \text{Pois}(\nu + \bar{\nu}) = N(0,\sigma^2) * c_\tau \text{Pois } \mu$. Uniqueness in the Lévy-Khintchine representation, provides $\beta^2 = \sigma^2/2 = \alpha^2$. Thus, from every sequence $\{n_j\}_{j=1}^{\infty}$ we can extract a subsequence $\{n_{j_k}\}_{k=1}^{\infty}$ such that

$$S_{n_{j_k}} - E\, S_{n_{j_k},\tau} \xrightarrow{w} N(0,\alpha^2) * c_\tau \text{Pois } \nu,$$

therefore

$$S_n - E\, S_{n,\tau} \xrightarrow{w} N(0,\alpha^2) * c_\tau \text{Pois } \nu \tag{2.15}$$

which completes the proof of **(ii)**.

Assume now that $c \in (0,\infty)$. Note that (2.15) implies that

$$k_n\, Var\left(\frac{X_{n,j} - \pi_n}{r_n}\right)_\tau \to \frac{1}{c}\left(\alpha^2 + \int_{-\tau}^{\tau} x^2 d\nu(x)\right)$$

for every $\tau > 0$ such that $\nu\{-\tau,\tau\} = 0$. From (2.14) we obtain that

$$\begin{array}{lcl} k_n P(X_{n,1} - \pi_n > r_n\delta) & \to & \frac{1}{c}\nu(\delta,\infty) \\ k_n P(X_{n,1} - \pi_n < -r_n\delta) & \to & \frac{1}{c}\nu(-\infty,-\delta), \end{array}$$

for every $\delta > 0$ such that $\nu\{-\delta,\delta\} = 0$. According to the CLT, these facts imply

$$\sum_{j=1}^{k_n} \frac{X_{n,j} - \pi_n}{r_n} - k_n\, E\left(\frac{X_{n,j} - \pi_n}{r_n}\right)_\tau \xrightarrow{w} N\left(0,\frac{\alpha^2}{c}\right) * c_\tau \text{Pois } \left(\frac{1}{c}\nu\right).$$

Now, by Theorem 11 in Cuesta and Matrán (1998), we obtain that

$$\mathcal{L}^*\left(\frac{1}{r_n}((S_n^* - m_n\, \pi_n) - \frac{m_n}{k_n}\sum_{j=1}^{k_n}(X_{n,j} - \pi_n)_{\tau\, r_n})\right) \xrightarrow{w} N(0,\alpha^2) * c_\tau \text{ Pois } N$$

in law, where N is a Poisson random measure with intensity measure ν. But, as we have shown in (2.13),

$$\mathcal{L}^*\left(\frac{1}{r_n}((S_n^* - m_n\,\pi_n) - \frac{m_n}{k_n}\sum_{j=1}^{k_n}(X_{n,j} - \pi_n)_{\tau\,r_n})\right) \xrightarrow{w} N(0,\alpha^2)*c_\tau\,\text{Pois}\,\nu$$

in probability, then necessarily $N = \nu$. This can only happen if $\nu = 0$. Hence

$$\sqrt{\frac{m_n}{k_n}}\frac{1}{r_n}\sum_{j=1}^{k_n} X_{n,j} - \frac{\sqrt{m_n k_n}}{r_n}\left(\pi_n + E(X_{n,j} - \pi_n)_{\tau\,r_n\sqrt{\frac{k_n}{m_n}}}\right) \xrightarrow{w} N(0,\alpha^2).$$

Finally, suppose $c = \infty$. Reasoning as above we can obtain that

$$\mathcal{L}^*\left(\frac{1}{r_n}((S_n^* - m_n\,\pi_n) - \frac{m_n}{k_n}\sum_{j=1}^{k_n}(X_{n,j} - \pi_n)_{\tau\,r_n})\right) \xrightarrow{w} N(0,\alpha^2)*c_\tau\,\text{Pois}\,\nu,$$

and conclude from this that

$$\frac{m_n}{k_n}\sum_{j=1}^{k_n} I_{\{|X_{n,j}-\pi_n|>\delta\,r_n\}} \xrightarrow{P} \nu(\delta,\infty) \tag{2.16}$$

and

$$\frac{m_n}{r_n^2}\left(\frac{1}{k_n}\sum_{j=1}^{k_n}((X_{n,j} - \pi_n)_{\tau r_n})^2 - \left(\frac{1}{k_n}\sum_{j=1}^{k_n}(X_{n,j} - \pi_n)_{\tau r_n}\right)^2\right) \xrightarrow{P} \alpha^2 + \int_{-\tau}^{\tau} x^2 d\nu(x). \tag{2.17}$$

Since $m_n/k_n \to \infty$, (2.16) can only happen if (passing to subsequences) $\sum_{j=1}^{k_n} I_{\{|X_{n,j}-\pi_n|>\delta\,r_n\}}$ is eventually zero on a probability one set. Hence $\nu = 0$ and we can rewrite (2.17) as follows:

$$m_n\frac{1}{r_n^2}Var^*X_{n,1}^* = \frac{m_n}{k_n}\frac{1}{r_n^2}\sum_{j=1}^{k_n}(X_{n,j} - \bar{X}_n)^2 \xrightarrow{P} \alpha^2. \tag{2.18}$$

Now call $Z_{n,j} = \sqrt{\frac{m_n}{k_n}}\frac{1}{r_n}(X_{n,j} - \pi_n)$ and $Z_{n,j}^* = \sqrt{\frac{m_n}{k_n}}\frac{1}{r_n}(X_{n,j}^* - \pi_n)$, $j = 1, 2, ..., k_n$. Then, by Lévy's and Chebychev's inequalities and (2.18)

$$\begin{aligned} k_n P^*\left(\left|Z_{n,1}^*\right| > \delta\right) &\le 2k_n P^*\left(\left|Z_{n,1}^* - Z_{n,1}^{**}\right| > \delta\right) \\ &\le \frac{2}{\delta^2}k_n Var^*\left(Z_{n,1}^* - Z_{n,1}^{**}\right) = \frac{4}{\delta^2}k_n Var^*\left(\sqrt{\frac{m_n}{k_n}}\frac{1}{r_n}X_{n,1}^*\right) \\ &= \frac{4}{\delta^2}m_n\frac{1}{r_n^2}Var^*X_{n,1}^* \xrightarrow{P} \frac{4\alpha^2}{\delta^2}. \end{aligned}$$

Observe that the r.v. $k_n P^*\left(\left|Z^*_{n,1}\right| > \delta\right) = \sum_{j=1}^{k_n} I_{\{|Z_{n,1}|>\delta\}}$ follows a binomial distribution with parameters k_n and $P(|Z_{n,1}| > \delta)$, hence the bound above implies that

$$k_n P(|Z_{n,1}| > \delta) \to 0. \tag{2.19}$$

From (2.19) it is easy to conclude, employing the CLT that

$$k_n P^*\left(\left|Z^*_{n,1}\right| > \delta\right) \xrightarrow{P} 0. \tag{2.20}$$

Since

$$k_n Var^*(Z^*_{n,1}) = m_n \frac{1}{r_n^2} Var^* X^*_{n,1} \xrightarrow{P} \alpha^2,$$

(2.20) implies that

$$k_n Var^*(Z^*_{n,1})_\tau \xrightarrow{P} \alpha^2. \tag{2.21}$$

Now, from (2.20) and (2.21) we obtain that

$$\mathcal{L}^*\left(\sum_{j=1}^{k_n} Z^*_{n,j} - k_n \bar{Z}_{n,\tau}\right) \xrightarrow{w} N(0,\alpha^2)$$

in probability, but, as shown before, this implies

$$\sqrt{\frac{m_n}{k_n}} \frac{1}{r_n} \sum_{j=1}^{k_n} X_{n,j} - \frac{\sqrt{m_n k_n}}{r_n}\left(\pi_n + E(X_{n,j} - \pi_n)_{\tau r_n \sqrt{\frac{k_n}{m_n}}}\right) \xrightarrow{w} N(0,\alpha^2),$$

which completes the proof. □

Remark 2. – The symmetrization employed in the proof of Theorem 1.1 is necessary to achieve the conclusion. We could use the initial hypotheses to obtain, in a similar fashion as in Lemma 2.1, the infinitesimality of $\left\{\frac{X^*_{n,j}-\mu_n}{r_n} : j = 1, 2, ..., m_n, n \in \mathbb{N}\right\}$ (hence, the infinite divisibility of $\rho = N(a^2, \alpha^2) * c_\tau \text{Pois } \nu$), and also the convergences

$$\begin{aligned} m_n\, P^*\left(X^*_{n,1} - \mu_n > r_n\delta\right) &\xrightarrow{P} \nu(\delta,\infty) \\ m_n\, P^*\left(X^*_{n,1} - \mu_n < -r_n\delta\right) &\xrightarrow{P} \nu(-\infty,-\delta) \end{aligned} \tag{2.22}$$

for every $\delta > 0$ such that $\nu\{-\delta,\delta\} = 0$ and

$$m_n E^*\left[\left(\frac{X^*_{n,1} - \mu_n}{r_n}\right)_\tau\right]^2 - m_n\left[E^*\left(\frac{X^*_{n,1} - \mu_n}{r_n}\right)_\tau\right]^2 \xrightarrow{P} \alpha^2 + \int_{-\tau}^{\tau} x^2 d\nu(x) \tag{2.23}$$

for every $\tau > 0$ such that $\nu\{-\tau, \tau\} = 0$. It is easy to argue as above to get the infinitesimality of $\left\{\frac{X_{n,j}-\mu_n}{r_n} : j = 1, 2, ..., m_n,\, n \in \mathbb{N}\right\}$. The CLT and (2.22) provide

$$\begin{array}{lcl} m_n\, P(X_{n,1} - \mu_n > r_n\delta) & \to & \nu(\delta, \infty) \\ m_n\, P(X_{n,1} - \mu_n < -r_n\delta) & \to & \nu(-\infty, -\delta). \end{array}$$

In order to complete the proof we should show that

$$m_n\, Var\, X_{n,1,\tau} \to \alpha^2 + \int_{-\tau}^{\tau} x^2 d\nu(x)$$

for every $\tau > 0$ such that $\nu\{-\tau, \tau\} = 0$. However, we can not use the CLT to conclude this directly from (2.23) because

$$\begin{aligned} & m_n E^*\left[\left(\frac{X^*_{n,1} - \mu_n}{r_n}\right)_\tau\right]^2 - m_n\left[E^*\left(\frac{X^*_{n,1} - \mu_n}{r_n}\right)_\tau\right]^2 \\ & = \frac{m_n}{k_n}\sum_{j=1}^{k_n}\left(\frac{X_{n,j} - \mu_n}{r_n}\right)^2_\tau - \frac{m_n}{k_n^2}\left(\sum_{j=1}^{k_n}\left(\frac{X_{n,j} - \mu_n}{r_n}\right)_\tau\right)^2 \end{aligned} \tag{2.24}$$

is not a sum of independent r.v's. This difficulty is sorted out in the proofs in Giné and Zinn (1989) and Arcones and Giné (1989) for the bootstrap of i.i.d.r.v.'s mainly through Lemma 2 in Giné and Zinn (1989), which shows that if the array comes from a sequence of i.i.d.r.v.'s and $EX_1^2 = \infty$, then

$$\frac{\frac{1}{n^2}\left(\sum_{j=1}^n\left(\frac{X_j - \mu}{r_n}\right)_\tau\right)^2}{\frac{1}{n}\sum_{j=1}^n\left(\frac{X_j-\mu}{r_n}\right)^2_\tau} \xrightarrow{a.s.} 0,$$

but a similar result in our setup without additional moments restrictions is not available. On the other hand our proof jointly handles both cases of finite and infinite variance. □

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REFERENCES

[1] ARAUJO A. and GINÉ E., The Central Limit Theorem for Real and Banach Valued Random Variables. Wiley, New York, 1980.

[2] ARCONES M. and GINÉ E., The Bootstrap of the Mean with Arbitrary Bootstrap Sample Size. *Ann. Inst. Henri Poincaré, Prob. Stat.,* Vol. **25**, 1989, pp. 457-481.

[3] ATHREYA K. B., Bootstrap of the Mean in the Infinite Variance Case. *Proceedings of the 1st World Congress of the Bernoulli Society*, Y. Prohorov and V. V. Sazonov, Eds., Vol. **2**, 1987, pp. 95-98. VNU Science Press, The Netherlands.

[4] BREIMAN L., *Probability*, Addison-Wesley, Reading, Mass, 1968.

[5] CUESTA-ALBERTOS J. A. and MATRÁN C, The asymptotic distribution of the bootstrap sample mean of an infinitesimal array. *Ann. Inst. Henri Poincaré, Prob. Stat.,* Vol. **34**, 1998, pp. 23-48.

[6] FELLER W., An Introduction to Probability Theory and its Applications, Vol. **II**, Wiley, New York, 1966.

[7] GINÉ E. and ZINN J., Necessary conditions for the bootstrap of the mean, *Ann. Statist.,* Vol. **17**, 1989, pp. 684-691.

[8] HALL P., Asymptotic Properties of the Bootstrap of Heavy Tailed Distributions, *Ann. Statist.,* Vol. **18**, 1990, pp. 1342-1360.

[9] MAMMEN E., Bootstrap, wild bootstrap, and asymptotic normality, *Probab. Theory Relat. Fields,* Vol. **93**, 1992, pp. 439-455.

[10] SWANEPOEL J. W. H., A Note in Proving that the (Modified) Bootstrap Works, *Commun. Statist. Theory Meth.,* Vol. **15**, 1986, pp. 3193-3203.

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