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About the stationary states of vortex systems

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ABSTRACT. – We investigate the precise behaviour of a gas of vortices approximating the vorticity of an incompressible, inviscid, two dimensional fluid as proposed by Onsager [16]. For such mean field interacting particles system with positive vortices, the convergence of the empirical measure was proven in [3]. We improve this result by showing, for more general values of the vortices, that a large deviation principle holds. We also prove a central limit theorem for neutral gases. © Elsevier, Paris

Key words: Statistical mechanics of two-dimensional Euler equations, Interacting particle systems, Large deviations, Central limit theorem.

RÉSUMÉ. – Nous étudions le comportement asymptotique d'un gaz de tourbillons décrivant un fluide incompressible bidimensionnel. Ce système est modélisé par une interaction de type champ moyen. La convergence faible des mesures empiriques associées a été prouvée dans [3]. Nous étendons ce résultat et prouvons un principe de grandes déviations pour des valeurs quelconques de l'intensité des tourbillons. Dans le cas d'un gaz neutre, nous établissons aussi le théorème de la limite centrale. © Elsevier, Paris

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M. Degli

1. INTRODUCTION

Recently, methods of approximations by particles systems have been widely studied. For an incompressible, inviscid, two dimensional fluid a natural approximating scheme follows the point vortex method. The basic idea of the point vortex method is to approximate the vorticity of a fluid by a "gas of vortices" which is represented by a linear combination of Dirac measures $\sum R_i \delta_{x_i}$. The investigation of the 2 dimensional turbulence by this method has been initiated by Onsager [16].

Onsager notices that the gas of vortices exhibits 3 different regimes. When the inverse of the temperature β is positive and large the vortices are mostly close to the boundary whereas they will be more or less uniformly distributed for smaller but positive β. But also, Onsager argued that there is no reason to consider only positive temperatures: when the energy of the system is increased the vortices of the same sign are forced to be close to each other. This can be interpreted as a negative temperature state. This tendency to create local clusters of the same sign has been observed in numerical experiments by Joyce and Montgomery [12]. Since then, many attempts have been made to understand this phenomenon. In the standard thermodynamic limit, Fröhlich and Ruelle [10] showed that this negative temperature regime does not exists. Nevertheless, it was then argued by Caglioti, Lions, Marchioro and Pulvirenti [3] and also by Eyink and Spohn [8] that the mean field scaling is relevant for the study of this negative temperature phase. In [3], it was proven that the weak limit of the Gibbs measures associated to the N-vortex systems converges towards some measure concentrated on particular stationary solution of the 2-D Euler equation. They also computed the behavior of these solutions as β converges to the critical temperature -8π.

Viewing the vortex method as a way to approximate these solutions, it is natural to wonder what is the speed of this convergence. Our goal is to precise it by proving large deviations and central limit results. Also, we will investigate more precisely the role of the signs of the vortices.

We follow the discretization procedure described in [15]. The vorticity field in a bounded domain Λ is approximated by a linear combination of N Dirac measures concentrated in points x_i of Λ with intensity R_i. Then, the N-vortex system in Λ is described by the Hamiltonian

$$H_\Lambda^N(X,R) = \frac{1}{2}\sum_{i\neq j\geq 1}^{N} R_i R_j V_\Lambda(x_i,x_j) + \sum_{i=1}^{N} R_i^2 W_\Lambda(x_i),$$

where $X = (x_1, \ldots, x_n)$ and V_Λ is the Green function of the Laplacian in Λ with Dirichlet boundary conditions. More precisely

$$\forall x, y \in \Lambda, \qquad V_\Lambda(x, y) = -\frac{1}{2\pi} \log|x - y| + \gamma_\Lambda(x, y),$$

where γ_Λ is symmetric and harmonic in each variable. Moreover, $W_\Lambda(x) = \gamma_\Lambda(x, x)$.

In the following, we will assume that the intensities are bounded. Without loss of generality we can assume that they are bounded by 1. In [3], Caglioti et al. consider all the vorticities equal to $+1$ and in [10], Fröhlich and Ruelle consider the neutral case, i.e. the sum of the intensities equal 0. Here, we wish to consider general $\{-1, 1\}$ valued intensities. First, we shall assume that the R_i's have a fixed ratio of -1 and 1. We will refer to this setting as quenched. On the other hand we wish to consider as well the case where the intensities are randomly distributed, we will assume that they are independent and identically distributed (i.i.d.) with Bernoulli law $Q^N = Q^{\otimes N}$.

We denote by

$$dP(x) = \frac{1}{\int_\Lambda dy} \mathbb{1}_{x \in \Lambda} dx \quad \text{and} \quad dP^N(X) = \otimes_{i=1}^N dP(x_i)$$

the product of Lebesgue measures on Λ^N. In the following, Σ will be either Λ or $\Omega = \Lambda \otimes \{-1, 1\}$. We denote by $\mathcal{M}(\Sigma)$ the space of measures on Σ and by $\mathcal{M}_1^+(\Sigma)$ the space of probability measures on Σ. We define

$$\mathcal{M}_Q = \{\nu \in \mathcal{M}_+^1(\Omega) : \pi_2 \circ \nu = Q\}$$

the set of probability measures with intensities marginal Q.

Let β be the inverse of the temperature. For a given sequence $(R_i)_{i \in \mathbb{N}}$ of intensities, we introduce the canonical quenched Gibbs measures on $\mathcal{M}_1^+(\Lambda^N)$ by

$$d\nu_{\beta,N}^R(X) = \frac{1}{Z_N^R(\beta)} \exp\left\{-\frac{\beta}{N} H_\Lambda^N(X, R)\right\} dP^N(X),$$

where

$$Z_N^R(\beta) = P^N\left(\exp\left\{-\frac{\beta}{N} H_\Lambda^N(X, R)\right\}\right).$$

We consider as well the averaged Gibbs measures on $\mathcal{M}_1^+(\Omega^N)$

$$d\nu_{\beta,N}(X, R) = \frac{1}{Z_N(\beta)} \exp\left\{-\frac{\beta}{N} H_\Lambda^N(X, R)\right\} dP^N \otimes dQ^N,$$

where

$$Z_N(\beta) = P^N \otimes Q^N \Big(\exp\Big\{ -\frac{\beta}{N} H_\Lambda^N(X,R) \Big\} \Big).$$

To state our large deviation principles we need to introduce the following energy functional

$$\forall \nu \in \mathcal{M}_1^+(\Omega), \qquad \mathcal{E}(\nu) = \int\int R R' V_\Lambda(x,y)\, d\nu(x,R) d\nu(y,R').$$

Define

$$\forall \nu \in \mathcal{M}_1^+(\Omega), \qquad \mathcal{F}_\beta(\nu) = H(\nu|P \otimes Q) + \frac{\beta}{2}\mathcal{E}(\nu), \tag{1}$$

where $H(\nu|P \otimes Q)$ is the relative entropy of ν with respect to the product measure $P \otimes Q$. In section 2, we will see that $\mathcal{F}_\beta$ is lower semi-continuous.

To state in the simplest and more complete way our result, we shall restrict ourselves in this introduction to the case where Λ is a disk. Then, we have the following quenched large deviation result

THEOREM 1.1. – *For any β in $]-8\pi, \infty[$, if $(1/N)\sum_i \delta_{R_i}$ converges to a measure Q, the law of the empirical measure*

$$\hat{\mu}^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i, R_i} \tag{2}$$

under $\nu_{\beta,N}^R$ satisfies a large deviation principle with rate function

$$\mathcal{G}_q(\nu) = \begin{cases} +\infty & \text{if } \nu \notin \mathcal{M}_Q, \\ \mathcal{F}_\beta(\nu) - \inf_{\mathcal{M}_Q} \mathcal{F}_\beta & \text{otherwise.} \end{cases}$$

Moreover, the following averaged large deviation principle holds

THEOREM 1.2. – *For any β in $]-8\pi, \infty[$, the law of the empirical measure $\hat{\mu}^N$ under $\nu_{\beta,N}$ obeys a large deviation principle with action functional*

$$\mathcal{G}_a(\mu) = \mathcal{F}_\beta(\mu) - \inf_{\mathcal{M}_1^+(\Omega)} \mathcal{F}_\beta.$$

The same results hold for any compact set Λ provided $\beta \in (-8\pi, 8\pi)$; for larger values of β, our controls on the partition function are not good enough to ensure the exponential tightness in general. For such β's, we need to restrict ourselves to the case of the disk. The range of temperature $\beta \in]-8\pi, \infty[$ is optimal in the case where all the intensities $R_i = 1$ (see

[3]). On the other hand, for $\beta \in]-8\pi, 8\pi[$, our proof mainly depends on the uniform bound on the intensities so that the generalization of our results to any $[-1,1]$ valued R_i is straightforward.

In section 2, we prove that for $\beta \geq 0$ or negative and sufficiently small, $\mathcal{G}_q$ and $\mathcal{G}_a$ admit a unique minimizer. Therefore, Theorems 1.1 and 1.2 imply the almost sure convergence of the empirical measure.

In the last section, we investigate the fluctuations of the empirical measure. This problem turned out to be difficult because of the logarithmic singularity of the interaction. This is why we shall restrict ourselves to the case where Λ is a disk, β is positive and the gas neutral. In this case, the empirical measure converges towards $\nu^* = P \otimes Q$. To describe the fluctuations of the empirical measure around ν^*, let us first introduce the operator Ξ in $L^2(\nu^*)$ with kernel $V_\Lambda(x,y)RR'$ and I the identity in $L^2(\nu^*)$. In the statement of the following central limit theorem, $\mathcal{L}$ is a subset of $L^2(\nu^*)$ described in section 4.1. Then

THEOREM 1.3. – *If Λ is a circular disk and β is positive,*
1) If $|\, card\{i : R_i = +1\} - card\{i : R_i = -1\}| = o(N^{\frac{3}{4}})$, for any function $f \in \mathcal{L}$,

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(f(x_i, R_i) - \int f(x,R) d\nu^*(x,R) \right)$$

converges in law under $\nu^R_{\beta,N}$ towards a centered Gaussian variable with covariance

$$\sigma(f) = \int \left(f - \int f d\nu^* \right) (I + \beta\Xi)^{-1} \left(f - \int f d\nu^* \right) d\nu^*.$$

2) If $Q = \frac{1}{2}(\delta_{+1} + \delta_{-1})$, for any function $f \in \mathcal{L}$,

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(f(x_i, R_i) - \int f(x,R) d\nu^*(x,R) \right)$$

converges in law under $\nu_{\beta,N}$ towards a centered Gaussian variable with covariance

$$\sigma(f) = \int \left(f - \int f d\nu^* \right) (I + \beta\Xi)^{-1} \left(f - \int f d\nu^* \right) d\nu^*.$$

One can check that Ξ is a non negative operator so that, at positive temperature, $I + \beta\Xi$ is always non degenerate.

As a conclusion, we wish to stress that the main difficulty in this paper is to deal with the logarithmic singularity of the interaction. In particular, at high but positive inverse temperature, we have to restrict to the case where Λ is a disk (since we wish to consider signed intensities) in order to control the partition function. Once we consider the fluctuations of the empirical measure, the problem becomes even deeper and the hypotheses more numerous. To overcome the problem of the logarithmic singularity of the interaction, we therefore had to develop new techniques which should be useful for other models where the interaction is singular.

2. STUDY OF THE RATE FUNCTION

In this section, we study the quenched (resp. averaged) rate function $\mathcal{G}_q$ (resp. $\mathcal{G}_a$) which is equal, up to a normalizing constant, to $\mathcal{F}_\beta$ defined by (1). We will show that $\mathcal{G}_q$ is a good rate function and study its minima. The generalization of these results to $\mathcal{G}_a$ is straightforward and stated in the last subsection.

2.1. $\mathcal{G}_q$ is a good rate function if β is larger than -8π

Our purpose here is to show that, in the range of temperature $\beta > -8\pi$, $\mathcal{G}_q$ is a good rate function or in other words that the sets

$$K_M = \{\nu \in \mathcal{M}_Q : \mathcal{F}_\beta(\nu) \leq M\}$$

are compact subsets of $\mathcal{M}_Q$ for any real number M. Since Ω is compact, $\mathcal{M}_Q$ is compact so that this is equivalent to prove that the sets K_M are closed, that is that $\mathcal{F}_\beta$ is lower semi-continuous. The logarithmic singularity of the energy will be controlled by the relative entropy thanks to entropic inequalities.

The first step of this study is to show that for any $\beta > -8\pi$, there exists a finite constant M_β so that

$$K_M \subset \{\nu : H(\nu|P \otimes Q) \leq M_\beta\}, \tag{3}$$

this means that $\mathcal{F}_\beta$ is bounded from below in terms of the entropy.

By definition of the Green function V_Λ, if p_t^Λ is the heat kernel in Λ with Dirichlet boundary conditions, the energy functional is given by

$$\mathcal{E}(\nu) = \int_0^\infty dt \int_\Lambda dz \left(\int R p_t^\Lambda(x,z) d\nu(x,R) \right)^2. \tag{4}$$

Therefore, $\mathcal{E}$ is non negative which yields, if $\beta \geq 0$,

$$\mathcal{F}_\beta(\nu) \geq H(\nu|P \otimes Q). \tag{5}$$

To establish such a bound in the negative temperature regime, we use the definition of the relative entropy H. Indeed, by the definition of the relative entropy and monotone convergence theorem, we have, for any $\eta > 0$ and $y \in \Lambda$,

$$\eta \int |V_\Lambda(x,y)|\, d\nu(x,R) \leq H(\nu|P \otimes Q) + \log \int \exp\{\eta |V_\Lambda(x,y)|\} dP(x). \tag{6}$$

But the exponent in the second term diverges as $(1/2\pi)\log|x-y|^{-1}$. Hence, if $\eta < 4\pi$, the last term is uniformly bounded independently of $y \in \Lambda$. Therefore, we get that, for $\eta < 4\pi$, there exists a finite constant $C(\eta) = \sup_y \log \int \exp\{\eta|V_\Lambda(x,y)|\}dP(x)$ so that

$$\eta\mathcal{E}(\nu) \leq \eta \int\int |V_\Lambda(x,y)|\, d\nu(x,R)d\nu(y,R') \leq H(\nu|P\otimes Q) + C(\eta). \tag{7}$$

As a consequence, for any $\eta < 4\pi$,

$$\mathcal{F}_\beta(\nu) \geq \left(1 + \frac{\beta}{2\eta}\right) H(\nu|P \otimes Q) + \frac{\beta C(\eta)}{2\eta} \tag{8}$$

where $1 + \frac{\beta}{2\eta}$ is positive if $\beta \in]-8\pi, 0[$ and $\eta \in]\frac{|\beta|}{2}, 4\pi[$. Therefore (5) and (8) gives (3) for $\beta > -8\pi$.

We are now going to show that the energy functional $\mathcal{E}$ is continuous on the sets of bounded entropy. This is enough, according to (3), to conclude that the K_M's are closed.

As a consequence of (3), any ν in K_M is absolutely continuous with respect to $P \otimes Q$. Let us denote ρ_ν the density of the first marginal of ν with respect to P

$$\int_{\{-1,1\}} \nu(dx, dR) = \rho_\nu(x)P(dx).$$

Observe as well that, for any continuous function ϕ which vanishes in a neighborhood of $\{(x,y) \in \Lambda^2 : x = y\}$, the truncated energy

$$\mathcal{E}_\phi(\nu) := \int\int \phi(x,y) R R' V_\Lambda(x,y)\, d\nu(x,R)d\nu(y,R')$$

is bounded continuous in $\mathcal{M}_Q$. Thus, in order to prove that $\mathcal{E}$ is continuous, it is enough to show that

$$\int_{|x-y|<\epsilon} |V_\Lambda(x,y)| d\nu(x,R) d\nu(y,R')$$

vanishes when ϵ goes to zero uniformly on $\{H \leq M\}$ for a given M. In view of the singularity of V_Λ, this is also equivalent to prove this property for

$$\int_{|x-y|<\epsilon} \log|x-y|^{-1} d\nu(x,R) d\nu(y,R').$$

We can apply the result proved in [4] (Proposition 2.1) which yields, for any positive ϵ

$$\begin{aligned}\int_{|x-y|<\epsilon} & \log|x-y|^{-1} \rho_\nu(x) P(dx) \rho_\nu(y) P(dy) \\ & \leq \int_{|x-y|<\epsilon} |x-y|^{-1} \log|x-y|^{-1} P(dx) P(dy) \\ & \quad + \int_{|x-y|<\epsilon} \log(\rho_\nu(x)\rho_\nu(y)) \rho_\nu(x) P(dx) \rho_\nu(y) P(dy). \qquad (9)\end{aligned}$$

The first term goes to zero with ϵ. For the second term we notice that

$$\begin{aligned}\int_{|x-y|\leq\epsilon} & \log(\rho_\nu(x)) \rho_\nu(y) P(dy) \rho_\nu(x) P(dx) \\ & \leq H(\rho_\nu.P|P) \sup_x \nu(\{y : |x-y| \leq \epsilon\}).\end{aligned}$$

Moreover, by property of the relative entropy, we know that

$$\sup_x \nu(|x-y| \leq \epsilon) \leq \frac{H(\rho_\nu.P|P)+1}{\log(1/\epsilon)},$$

so that we deduce

$$\begin{aligned}\int_{|x-y|\leq\epsilon} & \log(\rho_\nu(x)) \rho_\nu(x) P(dx) \rho_\nu(y) P(dy) \\ & \leq \frac{1}{\log(1/\epsilon)} H(\rho_\nu.P|P)(1 + H(\rho_\nu.P|P)).\end{aligned}$$

But, by convexity of the relative entropy

$$H(\rho_\nu.P|P) \leq \int H(\nu_R|P) dQ(R) = H(\nu|P \otimes Q),$$

so that we deduce from (9) that there exists δ_ϵ going to zero with ϵ so that

$$\int\int_{|x-y|<\epsilon} \log|x-y|^{-1}\rho_\nu(x)P(dx)\rho_\nu(y)P(dy) \le \delta_\epsilon(1+H(\nu|P\otimes Q))^2.$$

This term goes uniformly to zero with ϵ on $\{H \le M\}$ so that $\mathcal{E}$ is continuous on this set. The proof is complete with (3). □

2.2. Existence and uniqueness of the minimum of $\mathcal{G}_q$

We first tackle the positive temperature regime where the rate function satisfies the following

PROPERTY 2.1. – *If* $\beta > 0$,

1) $\mathcal{G}_q$ *is strictly convex.*

2) $\mathcal{G}_q$ *achieves its minimal value at a unique probability measure on* $\mathcal{M}_Q$ *which is defined by the nonlinear equation*

$$\frac{d\nu^*(x,R)}{dP\otimes Q} = \frac{1}{Z_R(\nu^*)}\exp\left\{-\beta R\int V_\Lambda(x,y)R'd\nu^*(y,R')\right\}$$

where

$$Z_R(\nu^*) = \int \exp\left\{-\beta R\int V_\Lambda(x,y)R'd\nu^*(y,R')\right\}dP(x).$$

Proof. – Since the entropy H is strictly convex and the energy $\mathcal{E}$ is convex (see (4)), it follows that $\mathcal{G}_q$ is strictly convex if β is non negative. Hence, it admits a unique minimizer ν^*.

Moreover, since $\mathcal{G}_q$ achieves its minimum value, it is not hard to check that the minima are described, on $\mathcal{M}_Q$, by the nonlinear equation

$$\frac{d\nu^*}{dP\otimes Q}(x,R) = \frac{1}{Z_R(\nu^*)}\exp\left\{-\beta R\int V_\Lambda(x,y)R'd\nu^*(y,R')\right\}, \quad (10)$$

where

$$Z_R(\nu^*) = \int \exp\left\{-\beta R\int V_\Lambda(x,y)R'd\nu^*(y,R')\right\}dP(x).$$ □

Of course, in the negative temperature case, the convexity of the rate function is not clear. We can nevertheless prove

PROPERTY 2.2. – *There exists a negative temperature* β_0, $-8\pi \le \beta_0 < 0$ *so that, for any* $\beta \in]\beta_0, 0[$, *the functionnal* $\mathcal{G}_q$ *achieves its minimum value at a unique probability measure* ν^* *so that*

$$\frac{d\nu^*(x,R)}{dP\otimes Q} = \frac{1}{Z_R(\nu^*)}\exp\left\{-\beta R\int V_\Lambda(x,y)R'd\nu^*(y,R')\right\}.$$

Proof. – The proof now follows a fixed point argument. Namely, let us assume that there is two minima ν and ν'. As before, both of them satisfy the non linear equation (10). We are going to show that, if

$$U_\nu(x) = \int V_\Lambda(x,y)R' d\nu(y,R')$$

then

$$D(\nu,\nu') = ||U_\nu - U_{\nu'}||_\infty = \sup_{x\in\Lambda} |U_\nu(x) - U_{\nu'}(x)|$$

is null. According to (10), it implies that $\nu = \nu'$ and therefore gives the uniqueness of the minima. To prove the existence (which is already known in view of our construction), we could also apply a fixed point argument based on the estimation of $D(\nu,\nu')$. We leave it to the reader.

We begin our argument by finding a bound on $||U_\nu||_\infty$ uniform on the minima of $\mathcal{G}_q$ and which will be crucial later on.

Indeed, using (6) and (8), we find that, for any $\eta \in]\frac{|\beta|}{2}, 4\pi[$, for any $y \in \Lambda$

$$\int |V_\Lambda(x,y)|\, d\nu(x,R) \leq \frac{2}{2\eta+\beta}(\mathcal{F}_\beta(\nu) + C(\eta)).$$

Notice that, since the energy $\mathcal{E}$ is non negative, $\inf \mathcal{F}_\beta$ is a non decreasing function of β and is therefore non positive for $\beta \leq 0$. Hence,

$$||U_\nu||_\infty \leq \int |V_\Lambda(x,y)|\, d\nu(x,R) \leq \frac{2}{2\eta+\beta} C(\eta). \tag{11}$$

We shall choose in the following $\eta = \eta_\beta = \max\{(2/3)|\beta|, 1\}$ so that $c(\beta) = \frac{2}{2\eta_\beta+\beta} C(\eta_\beta)$ is uniformly bounded for $\beta > -6\pi$.

To estimate $D(\nu,\nu')$, note that (10) shows that for any x in Λ

$$|U_\nu(x) - U_{\nu'}(x)| \\ = \left| \int RV_\Lambda(x,y) \left(\frac{\exp\{-\beta RU_\nu(y)\}}{Z_R(\nu)} - \frac{\exp\{-\beta RU_{\nu'}(y)\}}{Z_R(\nu')} \right) dP(y)dQ(R) \right|. \tag{12}$$

Therefore, it is not hard to check that for all $x \in \Lambda$,

$$|U_\nu(x) - U_{\nu'}(x)| \leq |\beta| \left(\int |V_\Lambda| d(\nu+\nu') \right) e^{2c(\beta)|\beta|} D(\nu,\nu'), \\ \leq (2c(\beta)e^{2c(\beta)|\beta|})|\beta| D(\nu,\nu').$$

Taking the supremum over the x's, we conclude that

$$D(\nu,\nu') \leq (2c(\beta)e^{2c(\beta)|\beta|})|\beta| D(\nu,\nu').$$

Since $(2c(\beta)e^{2c(\beta)|\beta|})|\beta|$ goes to zero with $|\beta|$, we find a positive β_0 so that it is smaller than one for all $\beta \in (-\beta_0, 0)$. In that range of temperature, we deduce that $D(\nu,\nu') = 0$ so that $\nu = \nu'$. □

2.3. Properties of $\mathcal{G}_a$

It is not hard to generalize the preceeding to the average setting and see that

LEMMA 2.3. – *$\mathcal{G}_a$ is a good rate function. There exists a positive β_0 so that for $\beta \in (-\beta_0, \infty)$, $\mathcal{G}_a$ admits a unique minimizer described by the non linear equation*

$$\frac{d\nu^*(x,R)}{dP \otimes Q} = \frac{1}{Z_R(\nu^*)} \exp\left\{ -\beta R \int V_\Lambda(x,y) R' d\nu^*(y,R') \right\}.$$

3. LARGE DEVIATIONS

3.1. Existence of the Gibbs measures

First we need to compute the range of temperature for which the Gibbs measures are defined.

PROPOSITION 3.1. [3] – *Let Λ be any compact set in $\mathbb{R}^2$. For any β in $]-8\pi, 8\pi[$ and any sequence of intensities with values in $\{-1,1\}$, there is a constant C such that the following holds for all N sufficiently large*

$$Z_N^R(\beta) \leq C^N.$$

This lemma is similar to the one proven by Caglioti et al. (see Lemma 2.1 [3]). If Λ is a disk, a more accurate statement holds for any positive temperature

PROPOSITION 3.2. – *If Λ is a disk, for any β in $]0, \infty[$ and any sequence of intensities with values ± 1, there is a finite real number p such that the following holds for all N sufficiently large*

$$Z_N^R(\beta) \leq N^p. \tag{13}$$

Note that the previous results imply that the same bounds are also valid for the averaged partition function. The proof of this sharp upper bound relies on the very specific form of V_Λ when Λ is a disk. For more general domains, we do not know if such a bound holds and even if one can get any exponential bounds for $\beta > 8\pi$ and general $\{-1,+1\}$-intensities.

Proof. – In order to study the case of positive (and eventually large) temperatures, we shall restrict ourselves to the case where Λ is a disk. To simplify the notations, we will assume that Λ is centered at the origin and with radius one so that V_Λ has the specific form

$$V_\Lambda(x,y) = -\frac{1}{4\pi}(\log|x-y|+\log|\hat{x}-\hat{y}|-\log|x-\hat{y}|-\log|\hat{x}-y|) \tag{14}$$

where $\hat{y}$ is the reflection of y at the circle $\partial\Lambda$ (see Lemma 3.3 of Fröhlich [9]). Note that, if $z = y_1 + iy_2$ is the complex representation of y, $\hat{y}$ has complex coordinate $(1/\bar{z})$. In the general case, we do not know how to control the singularity of γ_Λ near the boundary of Λ.

To prove (13), let us first remark that by using Hölder's inequality, we have

$$\begin{aligned} Z_N^R(\beta) \le & \left[\int_{\Lambda^N} dP^N(X)\exp\{-2\pi \sum_{1\le i<j\le N} R_iR_jV_\Lambda(x_i,x_j)\}\right]^{\frac{\beta}{2\pi N}} \\ & \times \left[\int_{\Lambda^N} dP(x)\exp\{-\frac{2\pi}{2\pi N-\beta}W(x)\}\right]^{\frac{2\pi N-\beta}{2\pi}}. \end{aligned} \tag{15}$$

The last term in the above r.h.s. is clearly bounded uniformly in N. Let us therefore focus on the second term

$$\widetilde{Z}_N^R(\beta) = \int_{\Lambda^N} dP^N(X)\exp\left\{-2\pi \sum_{1\le i<j\le N} R_iR_jV_\Lambda(x_i,x_j)\right\}.$$

Here, we follow Fröhlich [9] who used a representation for the energy which reads

$$\begin{aligned} \sum_{1\le i<j\le N} R_iR_jV_\Lambda(x_i,x_j) = & -\frac{1}{4\pi}\sum_{1\le i<j\le 2N} \widetilde{R}_i\widetilde{R}_j\log|\widetilde{X}_i-\widetilde{X}_j| \\ & -\frac{1}{4\pi}\sum_{1\le i\le N}\log|\widetilde{X}_i-\widetilde{X}_{i+N}| \end{aligned} \tag{16}$$

where $\widetilde{X}$ is the vector in $\mathbb{R}^{2N}$ so that

$$\begin{aligned} \widetilde{X}_i &= x_i \qquad \text{if } 1\le i\le N \\ &= \hat{x}_i \qquad \text{if } N+1\le i\le 2N \end{aligned}$$

and

$$\widetilde{R}_i = +R_i \qquad \text{if } 1 \le i \le N$$
$$= -R_i \qquad \text{if } N+1 \le i \le 2N.$$

The last term in the r.h.s. of (16) comes from the fact that x_i does not interact with its image as noticed by Fröhlich (see (3.11) of [9]) whereas an interaction was included in the first term. Cauchy-Schwartz's inequality yields

$$\widetilde{Z}_N^R(\beta) \le \left[\int_{\Lambda^N} dP^N(X) \exp\{\sum_{1\le i<j\le 2N} \widetilde{R}_i \widetilde{R}_j \log|\widetilde{X}_i - \widetilde{X}_j|\}\right]^{\frac{1}{2}} \times \left[\int_{\Lambda} |x - \hat{x}| dP(x)\right]^{\frac{N}{2}} \tag{17}$$

It is not difficult to check that the expectation in the last term of the r.h.s. of (17) is finite.

Let us now focus on the first term. If we denote I_+ (resp. I_-) the indices for which $\widetilde{R}_i = +1$ (resp. $\widetilde{R}_i = -1$), we have

$$\exp\{\sum_{1\le i<j\le 2N} \widetilde{R}_i \widetilde{R}_j \log|\widetilde{X}_i - \widetilde{X}_j|\} = \frac{\prod_{i\ne j\in I_+}(\widetilde{X}_i - \widetilde{X}_j)\prod_{i\ne j\in I_-}(\widetilde{X}_i - \widetilde{X}_j)}{\prod_{i\in I_+, j\in I_-}(\widetilde{X}_i - \widetilde{X}_j)}. \tag{18}$$

To use this representation, we adapt the argument used by Deutsch and Lavaud [7] (see also Fröhlich [9]). Indeed, let us recall the following formula from [7] valid for any complex numbers $(z_i, y_i)_{1\le i\le n}$

$$\prod_{1\le i<j\le n} (z_i - z_j) \prod_{1\le i<j\le n} (y_i - y_j) = \prod_{1\le i,j\le n} (z_i - y_j) \det\left[\frac{1}{z_i - y_j}\right] = \sum_{\sigma} \epsilon(\sigma) \prod_{1\le i,j\le n, j\ne\sigma(i)} (z_i - y_j), \tag{19}$$

where the sum is over all the permutation σ of $\{1,..,n\}$ and $\epsilon(\sigma)$ is the signature of σ. Therefore, (18) and (15) shows that, since $|I_-| = |I_+| = N$, if we denote $(z_i(X))_{1\le i\le N} = (\widetilde{X}_i)_{i\in I_-}$ and $(y_i(X))_{1\le i\le N} = (\widetilde{X}_i)_{i\in I_+}$,

$$\int_{\Lambda^N} dP^N(X) \exp\{\sum_{1\le i<j\le 2N} \widetilde{R}_i \widetilde{R}_j \log|\widetilde{X}_i - \widetilde{X}_j|\} = \int_{\Lambda^N} dP^N(X) \det\left[\frac{1}{z_i(X) - y_j(X)}\right] \le \sum_{\sigma} \int_{\Lambda^N} dP^N(X) \prod_{1\le i\le N} |z_i(X) - y_{\sigma(i)}(X)|^{-1}.$$

The last term in the above r.h.s. is bounded independently of the permutation σ. Therefore, we have shown that for some constant C

$$\widetilde{Z}_N^R(\beta) \leq C^N N!$$

This completes the proof of Proposition 3.2 with (15).

3.2. A large deviations principle

To derive the large deviations principle we have to control the singularities of the Hamiltonian. We define the new functional

$$\forall \nu \in \mathcal{M}_1^+(\Omega), \qquad \hat{\mathcal{E}}(\nu) = \int\int_{x \neq y} R R' V_\Lambda(x, y)\, d\nu(x, R) d\nu(y, R').$$

We note that for any probability measure ν with finite entropy $\hat{\mathcal{E}}(\nu) = \mathcal{E}(\nu)$. Therefore, the main point in the proof of theorem 1.1 is to show that the energy $\hat{\mathcal{E}}$ is quasi-continuous i.e. for any probability measure ν in $\mathcal{M}_1^+(\Omega)$ with finite entropy, any δ positive and for all R_i's,

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log P^N\Big(\hat{\mu}^N \in \Big\{B(\nu, \varepsilon) \cap \{|\hat{\mathcal{E}}(\nu) - \hat{\mathcal{E}}(\hat{\mu}^N)| > \delta\}\Big\}\Big) = -\infty, \tag{20}$$

where $\hat{\mu}^N$ was defined in (2) and $B(\nu, \varepsilon)$ is the ball of radius ε around ν for the distance d defined by

$$\forall \mu, \nu \in \mathcal{M}_1^+(\Omega), \qquad d(\mu, \nu) = \sup_{f \in C^0(\Lambda)} \left| \int f d\mu - \int f d\nu \right|,$$

where $C^0(\Lambda)$ is the set of continuous functions on the compact Λ bounded by 1.

Before going on, we explain briefly how to recover the large deviations principle from the quasi-continuity (20). Since the space $\mathcal{M}_1^+(\Omega)$ is compact, it is enough to prove a weak large deviations principle. We fix ν a probability measure with finite entropy. We first compute the denominator of $\nu_{\beta,N}^R(\hat{\mu}^N \in B(\nu, \varepsilon))$ and we will deal with $Z_N^R(\beta)$ in a second step. As it has been noticed in the previous section, the term $\frac{1}{N}\sum_{i=1}^N W(x_i)$ does not contribute in the limit, so that we omit it in the computations.

$$\begin{aligned} P^N\Big(1_{\{\hat{\mu}^N \in B(\nu,\varepsilon)\}} \exp(-\beta N \hat{\mathcal{E}}(\hat{\mu}^N))\Big) &\leq P^N\Big(1_{B(\nu,\varepsilon)}\, 1_{|\hat{\mathcal{E}}(\hat{\mu}^N) - \hat{\mathcal{E}}(\nu)| > \delta} \exp(-N\beta \hat{\mathcal{E}}(\hat{\mu}^N))\Big) \\ &\quad + \exp(-N\beta\hat{\mathcal{E}}(\nu) + N\beta\delta) P^N\big(\hat{\mu}^N \in B(\nu, \varepsilon)\big). \end{aligned}$$

In order to get rid of the first term of the RHS we use Hölder inequality with a coefficient $\alpha > 1$ such that $\alpha\beta$ belongs to $]-8\pi, \infty[$

$$P^N\Big(1_{B(\nu,\varepsilon)}\ 1_{|\hat{\mathcal{E}}(\hat{\mu}^N)-\hat{\mathcal{E}}(\nu)|>\delta}\exp(-N\beta\hat{\mathcal{E}}(\hat{\mu}^N))\Big) \leq Z_N^R(\alpha\beta)^{\frac{1}{\alpha}} P^N\Big(1_{B(\nu,\varepsilon)}\ 1_{|\hat{\mathcal{E}}(\hat{\mu}^N)-\hat{\mathcal{E}}(\nu)|>\delta}\Big)^{1-\frac{1}{\alpha}}.$$

Propositions 2.1 and 2.2 and inequality (20) tells us that the above quantity vanishes exponentially fast

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log P^N\Big(1_{B(\nu,\varepsilon)}\ 1_{|\hat{\mathcal{E}}(\hat{\mu}^N)-\hat{\mathcal{E}}(\nu)|>\delta}\exp(-N\beta\hat{\mathcal{E}}(\hat{\mu}^N))\Big) = -\infty.$$

It remains to control the last term of the RHS. Well known large deviations results imply

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log P^N\Big(1_{B(\nu,\varepsilon)}\exp(-N\beta\hat{\mathcal{E}}(\hat{\mu}^N))\Big) \leq -\mathcal{F}_\beta(\nu). \qquad (21)$$

To prove the lower bound we note that

$$P^N\Big(1_{B(\nu,\varepsilon)}\exp(-N\beta\hat{\mathcal{E}}(\hat{\mu}^N))\Big) \geq P^N\Big(1_{B(\nu,\varepsilon)}\ 1_{|\hat{\mathcal{E}}(\hat{\mu}^N)-\hat{\mathcal{E}}(\nu)|<\delta}\exp(-N\beta\hat{\mathcal{E}}(\hat{\mu}^N))\Big),$$

thus

$$P^N\Big(1_{B(\nu,\varepsilon)}\exp(-N\beta\hat{\mathcal{E}}(\hat{\mu}^N))\Big) \geq \exp(-N\beta\hat{\mathcal{E}}(\nu) - N\beta\delta)\ P^N\Big(1_{B(\nu,\varepsilon)}\ 1_{|\hat{\mathcal{E}}(\hat{\mu}^N)-\hat{\mathcal{E}}(\nu)|<\delta}\Big).$$

By using inequality (20), we derive the lower bound

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log P^N\Big(1_{B(\nu,\varepsilon)}\exp(-N\beta\hat{\mathcal{E}}(\hat{\mu}^N))\Big) \geq -\mathcal{F}_\beta(\nu). \qquad (22)$$

Finally, we will check that

$$\liminf_{N\to\infty}\frac{1}{N}\log Z_N^R(\beta) = -\inf_{\nu'\in\mathcal{M}_Q}\mathcal{F}_\beta(\nu'). \qquad (23)$$

Since $\mathcal{M}_1^+(\Omega)$ is compact, we cover it with a finite number of open balls of radius ε and we get from (21)

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log Z_N^R(\beta) \leq -\inf_{\nu'\in\mathcal{M}_Q}\mathcal{F}_\beta(\nu').$$

The reverse inequality follows immediately from (22). Combining the previous results, we have proven a weak large deviation principle

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\nu_{\beta,N}^{R}\big(\hat{\mu}^N\in B(\nu,\varepsilon)\big)=-\mathcal{G}_q(\nu). \tag{24}$$

According to Theorem 4.1.11 of [5] the large deviation principle follows from (24).

Similarly, Theorem 1.2 can be derived from

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log P^N\otimes Q^N\Big(\hat{\mu}^N\in\Big\{B(\nu,\varepsilon)\cap\Big\{\Big|\hat{\mathcal{E}}(\nu)-\hat{\mathcal{E}}(\hat{\mu}^N)\Big|>\delta\Big\}\Big\}\Big)=-\infty.$$

This can be proved in the same way as (20) so that in the following we will focus on the proof of (20). In fact, the quasi-continuity property does not depend on the intensities R_i. Indeed, let us denote

$$\mu^N=\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}\qquad\text{and}\qquad\forall x\in\Lambda,\qquad\rho_\nu(x)=\int_{\{-1,1\}}\nu(x,dR),$$

and define an error energy

$\forall m\in\mathcal{M}_1^+(\Lambda)$,

$$E_M(m)=\int_\Lambda\int_\Lambda 1_{x\neq y}\sup\left(0,\log\left(\frac{1}{|x-y|}\right)-M\right)dm(x)dm(y).$$

Then (20) can be deduced from the following Lemma

LEMMA 3.3. – *For any measure m in $\mathcal{M}_1^+(\Lambda)$ with finite entropy with respect to the Lebesgue measure and for any δ positive, there is a function f_δ with values in $[0,\infty[$ such that*

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log P^N\big(\mu^N\in\big\{B(m,\varepsilon)\cap\{E_M(\mu^N)>\delta\}\big\}\big)\leq -f_\delta(M).$$

Furthermore f_δ satisfies

$$\lim_{M\to\infty}f_\delta(M)=\infty.$$

We postpone the proof of Lemma 3.3 and we derive (20).

In fact the potential V_Λ is singular on $\{x = y\}$ because of the logarithmic term but also near the frontier of Λ because of the logarithmic divergence of γ_Λ. First we control the singularity on the diagonal.

Let us be given $\delta > 0$ and a probability measure ν in $\mathcal{M}_1^+(\Omega)$ with finite entropy. We introduce the functional

$$\forall \nu' \in \mathcal{M}_1^+(\Omega),$$

$$E(\nu') = \int_\Lambda \int_\Lambda 1_{x \neq y} R R' \log \frac{1}{|x-y|} \, d\nu'(x,R) d\nu'(y,R').$$

For any finite M the functional E'_M defined on $\mathcal{M}_1^+(\Omega)$ by

$$\begin{aligned} E'_M(\nu') &= E(\nu') - E_M(\nu') \\ &= \int_\Lambda \int_\Lambda R R' \inf\left(M, \log \frac{1}{|x-y|}\right) d\nu'(x,R) d\nu'(y,R') \end{aligned}$$

is continuous. Therefore, there exists a constant ε_M small enough such that

$$\forall \varepsilon \leq \varepsilon_M, \forall \hat{\mu}^N \in B(\nu, \varepsilon), \qquad |E'_M(\nu) - E'_M(\hat{\mu}^N)| \leq \frac{\delta}{4}. \tag{25}$$

Since ν has a finite entropy, we know that $E(\nu)$ is finite (cf (7)). By dominated convergence Theorem, there is a constant M large enough such that

$$|E(\nu) - E'_M(\nu))| \leq \frac{\delta}{4}. \tag{26}$$

Combining (25) and (26) we get for all $\varepsilon \leq \varepsilon_M$

$$\begin{aligned} &P^N\big(\hat{\mu}^N \in \big\{B(\nu,\varepsilon) \cap \{|E(\hat{\mu}^N) - E(\nu)| > \delta\}\big\}\big) \\ &\qquad \leq P^N\left(\hat{\mu}^N \in \left\{B(\nu,\varepsilon) \cap \left\{E_M(\hat{\mu}^N) > \frac{\delta}{2}\right\}\right\}\right). \end{aligned}$$

As E_M does not depend on the sign of the vortices, we obtain an upper bound which depends only on μ^N

$$\begin{aligned} &P^N\big(\hat{\mu}^N \in \big\{B(\nu,\varepsilon) \cap \{E_M(\hat{\mu}^N) > \delta\}\big\}\big) \\ &\qquad \leq P^N\big(\mu^N \in \big\{B(\rho_\nu,\varepsilon) \cap \{E_M(\mu^N) > \delta\}\big\}\big). \end{aligned}$$

The measure ν has finite entropy, so that ρ_ν satisfies also the same property; this enables us to apply lemma 2.3. Therefore for any M large enough there exists a constant ε_M such for all ε less than ε_M the following holds

$$\limsup_{N\to\infty} \frac{1}{N} \log P^N\big(\mu^N \in \big\{B(m,\varepsilon) \cap \{E_M(\mu^N) > \delta\}\big\}\big) \le -f_\delta(M) + O(\varepsilon).$$

Letting ε tends to 0 and M go to infinity, we derive the quasi-continuity of the logarithmic part of the interaction

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} \frac{1}{N} \log P^N\big(\mu^N \in \big\{B(m,\varepsilon) \cap \{E_M(\mu^N) > \delta\}\big\}\big) = -\infty. \quad (27)$$

It remains to control the interaction term which depends on γ_Λ

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} \frac{1}{N} \log P^N\big(B(\nu,\varepsilon) \cap \{|E'(\hat\mu^N) - E'(\nu)| > \delta\}\big) = -\infty, \quad (28)$$

where the functional $\nu' \to E'(\nu')$ is defined by

$$E'(\nu') = \int_\Lambda\int_\Lambda R\,R'\,\gamma_\Lambda(x,y)\, d\nu'(x,R) d\nu'(y,R').$$

Noticing that γ_Λ is harmonic in the interior of Λ and has a logarithmic singularity at the boundary of Λ, there is some constant C such that

$$\forall x, y \in \Lambda, \qquad \log \frac{1}{|x-y|} \ge -\gamma_\Lambda(x,y) \ge C.$$

Therefore we can derive (28) by the same arguments as the ones used to control the logarithmic singularity. Combining (27) and (28), we complete Theorem 1.1.

3.3. Proof of Lemma 3.3

Let m be a probability measure with finite entropy. To control the singularity of the logarithm we introduce a coarse graining procedure : we partition the compact set Λ into cubes $\{Q_i\}_{i\le K}$ with side length $\exp(-M)$. For any ε positive, we define the set $A_\varepsilon(m)$ by

$$A_\varepsilon(m) = \left\{\mu \in \mathcal{M}_1^+(\Lambda) \mid \forall i \le K, \qquad \left|\int_{Q_i} d\mu(x) - dm(x)\right| \le \varepsilon\right\}.$$

For any ε' sufficiently small $B(m,\varepsilon')$ is included in $A_\varepsilon(m)$, so that by Chebyshev's inequality, we have for any positive T

$$P^N\big(B(m,\varepsilon') \cap \{E_M(\mu^N) > \delta\}\big) \le \exp(-N\delta T) P^N\big(1_{A_\varepsilon(m)} \exp(TNE_M(\mu^N))\big).$$

Any empirical measure in $A_\varepsilon(m)$ is associated to configurations such that the number n_i of particles in the cube Q_i satisfies

$$\left| n_i - N \int_{Q_i} m(dx) \right| \leq N\varepsilon \qquad \text{and} \qquad \sum_{i=1}^{K} n_i = N. \tag{29}$$

Therefore, summing over all the K-uplets $\{n_1, \ldots, n_K\}$ which satisfy (29), we get

$$P^N\big(B(m,\varepsilon') \cap \{E_M(\mu^N) > \delta\}\big) \leq \sum_{n_1,\ldots,n_K} \frac{N!}{n_1! \ldots n_K!} \exp(-N\delta T) \int_{Q_1 \times \ldots \times Q_K} \Pi_{i=1}^{K} \left(\Pi_{j=1}^{n_i} dx_j^{(i)}\right) \exp(TNE_M(\mu^N)). \tag{30}$$

The number of K-uplets $\{n_1, \ldots, n_K\}$ is less than $\exp(K \log N)$ so that we have just to compute the upper bound of the RHS for a given uplet $\{n_1, \ldots, n_K\}$ which satisfies (29).

By using Stirling formula, we get that

$$\frac{N!}{n_1! \ldots n_K!} \leq \exp\left(-N\left(\sum_{i=1}^{K} \frac{n_i}{N} \log \frac{n_i}{N}\right) + No(N)\right).$$

Noticing that $x \log x \geq -\mathrm{e}^{-1} \geq -1$, we get

$$\frac{n_i}{N} \log \frac{n_i}{N} \geq |Q_i| \frac{n_i}{|Q_i| N} \log\left(\frac{n_i}{|Q_i| N}\right) + \frac{n_i}{N} \log |Q_i| \geq -|Q_i| + \frac{n_i}{N} \log |Q_i|,$$

where $|Q_i|$ denotes the area of the cube Q_i. Finally, we derive the upper bound

$$\frac{N!}{n_1! \ldots n_k!} \leq \exp\left(N|\Lambda| - \sum_{i=1}^{K} n_i \log |Q_i| + No(N)\right). \tag{31}$$

Let us now consider the last term in the RHS of (30). By definition of E_M, we have

$$E_M(m) = \sum_{i=1}^{K} \sum_{j=1}^{K} \int_{Q_i} \int_{Q_j} 1_{x \neq y} \sup\left(0, \log\left(\frac{1}{|x-y|}\right) - M\right) dm(x) dm(y).$$

Thus in the above sum, only the terms where Q_i and Q_j have a common side (including the case $Q_i = Q_j$) contribute. Let us denote by $\widetilde{Q}_i$ the union of the cubes which have a face in common with Q_i and by $\widetilde{n}_i$ the number of particles in $\widetilde{Q}_i$.

We are going to show that for any $T > 0$ there exist a finite constant $M(T)$ and a positive constant $\varepsilon(T)$ such that for any $M > M(T)$ and $\varepsilon < \varepsilon(T)$ the following uniform bound holds

$$\sup_{y \in \widetilde{Q}_i} \int_{Q_i} dx_1 \ldots dx_{n_i} \exp\left(\frac{T}{N}\sum_{l \neq k=1}^{n_i} \log\frac{1}{|x_l - x_k|} + \frac{T}{N}\sum_{l=1}^{n_i}\sum_{j=1}^{\widetilde{n}_i} \log\frac{1}{|x_l - y_j|}\right) \leq (2|Q_i|)^{n_i}, \quad (32)$$

where the supremum is taken over all the configurations $y = \{y_1, \ldots, y_{\widetilde{n}_i}\}$ in $\widetilde{Q}_i$.

Combining (30), (31) and (32), we get that for any T there is M large enough such that

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log P^N\big(B(m, \varepsilon) \cap \{E_M(\mu^N) > \delta\}\big) \leq -\delta T + \log 2 + |\Lambda|.$$

This completes Lemma 2.2.

It remains to prove (32). First we derive a preliminary estimate

LEMMA 3.4. – *For any measure m in $\mathcal{M}_1^+(\Lambda)$ with finite entropy with respect to the Lebesgue measure the following holds*

$$\lim_{M \to \infty} \sup_{i \leq K} \left\{\log\frac{1}{|Q_i|}\, m(Q_i)\right\} = 0. \quad (33)$$

Proof. – For any cube $|Q_i|$, we get from Jensen inequality applied to $x \to x \log x$

$$\log\frac{1}{|Q_i|}\int_{Q_i} dx\, m(x) \leq \int_{Q_i} dx\, m(x) \log m(x) - \left(\int_{Q_i} dx\, m(x)\right) \log\left(\int_{Q_i} dx\, m(x)\right).$$

Noticing that $\int_\Lambda dx\, m(x)$ and $\int_\Lambda dx\, m(x) \log m(x)$ are finite we deduce that the RHS goes uniformly to 0 as M grows. This completes the Lemma. □

By using Hölder inequality, we split (32) into two terms. The first term contains the interaction energy of the particles in Q_i

$$\int_{Q_i} dx_1 \dots dx_{n_i} \exp\left(\frac{2T}{N}\sum_{l\neq k=1}^{n_i} \log\frac{1}{|x_l - x_k|}\right), \tag{34}$$

the second one bounds the interaction energy between Q_i and $\widetilde{Q}_i$

$$\sup_{y\in\widetilde{Q}_i}\int_{Q_i} dx_1 \dots dx_{n_i} \exp\left(\frac{2T}{N}\sum_{l=1}^{n_i}\sum_{j=1}^{\widetilde{n}_i} \log\frac{1}{|x_l - y_j|}\right). \tag{35}$$

The next step is to estimate (34). From Hölder inequality we get

$$\int_{Q_i} dx_1 \dots dx_{n_i} \exp\left(\frac{2T}{N}\sum_{l\neq k} \log\frac{1}{|x_l - x_k|}\right)$$
$$\leq \int_{Q_i} dx_1 \left(\int_{Q_i} dx_2 \exp(\frac{2Tn_i}{N}\log\frac{1}{|x_1 - x_2|})\right)^{n_i - 1}$$

A straightforward computation gives for any α in $[0, 2[$

$$\sup_{y\in\mathbb{R}^2}\int_{Q_i} dx \exp\left(\alpha\log\frac{1}{|x-y|}\right) \leq (1 + c'\alpha)|Q_i|^{1-\alpha/2}, \tag{36}$$

where c' is a constant. According to (33), we know that when M goes to infinity and ε tends to 0, $\frac{n_i}{N}$ goes to 0 uniformly. Hence for M sufficiently large and ε small enough, we get

$$\int_{Q_i}\prod_{j=1}^{n_i} dx_j \exp\left(\frac{2T}{N}\sum_{l\neq k} \log\frac{1}{|x_l - x_k|}\right)$$
$$\leq \left(1 + c'2\frac{Tn_i}{N}\right)^{n_i - 1}|Q_i|^{n_i}\exp\left(-\frac{Tn_i(n_i-1)}{N}\log|Q_i|\right).$$

By definition of n_i (see (29)) and Lemma 2.4, we check that for M sufficiently large and ε sufficiently small

$$\sup_i\left(\frac{n_i}{N}\log\frac{1}{|Q_i|}\right) < \frac{1}{2T}.$$

Therefore, we get

$$\int_{Q_i} dx_1 \dots dx_{n_i} \exp\left(\frac{2T}{N}\sum_{l\neq k} \log\frac{1}{|x_l - x_k|}\right) \leq \exp(n_i \log|Q_i|)2^{n_i}. \tag{37}$$

Let us now consider (35). We recall that $\widetilde{n}_i$ is the number of vortices in $\widetilde{Q}_i$. Since each cell interacts only with its nearest neighbors, the number $\widetilde{n}_i$ is of the same order as n_i. By using again Lemma 2.4, if M is sufficiently large and ε sufficiently small such that

$$\sup_i \left(\frac{\widetilde{n}_i}{N} \log \frac{1}{|Q_i|} \right) < \frac{1}{2T},$$

we have

$$\sup_{y \in \widetilde{Q}_i} \int_{Q_i} dx_1 \dots dx_{n_i} \exp\left(\frac{2T}{N} \sum_{l,j} \log \frac{1}{|x_l - y_j|} \right) \leq \sup_{y \in \widetilde{Q}_i} \left(\int_{Q_i} dx \exp\left(\frac{2T}{N} \sum_{j=1}^{\widetilde{n}_i} \log \frac{1}{|x - y_j|} \right) \right)^{n_i},$$

By applying Hölder inequality we obtain

$$\sup_{y \in \widetilde{Q}_i} \int_{Q_i} dx_1 \dots dx_{n_i} \exp\left(\frac{2T}{N} \sum_{l,j} \log \frac{1}{|x_l - y_j|} \right) \leq \sup_{y \in \mathbb{R}^2} \left(\int_{Q_i} dx \exp\left(\frac{2T\widetilde{n}_i}{N} \log \frac{1}{|x - y|} \right) \right)^{n_i},$$

this leads to

$$\sup_y \int_{Q_i} dx_1 \dots dx_{n_i} \exp\left(\frac{2T}{N} \sum_{l,j} \log \frac{1}{|x_l - y_j|} \right) \leq (2|Q_i|)^{n_i}. \tag{38}$$

Finally, combining (37) and (38), we complete the Lemma.

4. CENTRAL LIMIT THEOREM

In this last section, we study the fluctuations of the empirical measure

$$\hat{\mu}^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i, R_i}$$

around $\nu^*(dx, dR)$, the limiting law of the empirical measures. Of course, the problems due to the logarithmic singularity of the potential become even more difficult than for the study of the large deviations. This is the reason why the strategy followed in [2] seems to fail. We propose here to follow an approach developed in [11] for strongly interacting particles. This method allows to study fluctuations as soon as the empirical measure converges. Its advantage is that it can easily deal with a logarithmic singularity of the interaction. Its weakness is that it describes the fluctuations of $< f, \hat{\mu}^N - \nu^* >$ only for f in a subset of $L^2(\nu^*)$ even if ν^* is non degenerate. Also, this method requires a good control on the partition function that we only got in Proposition 3.2 for neutral gases at positive temperature on the disk. However, we believe that central limit theorems should hold for more general choices of the intensities, small negative temperatures and more general compact domains. The proof is performed in two steps : first we apply our strategy to get a biaised central limit theorem where the fluctuations are shifted by a remaining term. Secondly, we show that this remaining term goes to zero in probability via controls on the partition function to obtain the standard central limit result.

4.1. A biaised central limit Theorem

Let Ξ be the operator in $L^2(\nu^*)$ with kernel $V_\Lambda(x,y)RR'$ and I be the identity in $L^2(\nu^*)$. We are going to prove biaised fluctuations for test functions f in a subset $\mathcal{L}$ of $L^2(\nu^*)$ where

$$\mathcal{L} = \left\{ f \in L^2(\nu^*) | f(x,R) = (I+\beta\Xi)\left(\frac{\operatorname{div}(\lambda_R k)}{\lambda_R}\right), k \in \mathcal{C}^1(\Lambda), k|_{\partial\Lambda} = 0 \right\},$$

where

$$\lambda_R(x) = \frac{\nu^*(dx,dR)}{dP(x)dQ(R)}.$$

Then,

LEMMA 4.1. – *For any function f in $\mathcal{L}$, there exists a random variable $R_N(f)$ so that*
1) Under $\nu^R_{\beta,N}$ and $\nu_{\beta,N}$,

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(f(x_i, R) - \int f(y,R')d\nu^*(y,R') \right) + R_N(f)$$

converges in law to a centered Gaussian variable with covariance

$$\sigma(f) = \int f(I+\beta\Xi)^{-1} f d\nu^*.$$

2) Quenched convergence of R_N *: There exists a finite constant* C_f *so that, if, for* $\alpha > (1/2)$ *and for* N *large enough, we have*

$$d\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{R_i}, Q\right) \leq N^{-\alpha},$$

then, for any positive ϵ *we have*

$$\bigotimes_{i=1}^{N}\nu_{R_i}^{*}(|R_N(f)| > \epsilon) \leq \exp\{-C_f\sqrt{N}(\epsilon - C_f N^{-\alpha})\}.$$

3) Averaged convergence of the rest : There exists a finite constant C_f *so that for any positive* ϵ *we have*

$$Q^{\otimes N}\bigotimes_{i=1}^{N}\nu_{R_i}^{*}(|R_N(f)| > \epsilon) \leq \exp\{-C_f\sqrt{N}\epsilon\}.$$

Let us also notice that

$$\nu_{\beta,N}^{R}(dX) = \frac{1}{\bar{Z}_N^R(\beta)}\exp\left\{-\beta N\int_{x<y} RR'V_\Lambda d(\hat{\mu}^N - \nu^*)d(\hat{\mu}^N - \nu^*) - \beta\int R^2 W_\Lambda d\hat{\mu}^N\right\} d\bigotimes_{i=1}^{N}\nu_{R_i}^{*}$$

where

$$\bar{Z}_N^R(\beta) = \int \exp\left\{-\beta N\int_{x<y} RR'V_\Lambda d(\hat{\mu}^N - \nu^*)d(\hat{\mu}^N - \nu^*) - \beta\int R^2 W_\Lambda d\hat{\mu}^N\right\} d\bigotimes_{i=1}^{N}\nu_{R_i}^{*}.$$

The same formula holds for $\nu_{\beta,N}$ with the partition function

$$\bar{Z}_N(\beta) = \int \bar{Z}_N^R(\beta)dQ^{\otimes N}(dR).$$

Thus, we deduce from Lemma 4.1 that

LEMMA 4.2. – *1) If, for* α *in a neighborhood of* β*,*

$$\limsup_{N\to\infty}\left|\frac{1}{\sqrt{N}}\log \bar{Z}_N^R(\alpha)\right| = 0,$$

then, for any function f in $\mathcal{L}$,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(f(x_i,R)-\int f(y,R')d\nu^*\right)$$

converges in law under $\nu_{\beta,N}^R$ to a centered Gaussian variable with covariance

$$\sigma(f)=\int f(I+\beta\Xi)^{-1}fd\nu^*.$$

2) If, for α in a neighborhood of β,

$$\limsup_{N\to\infty}\left|\frac{1}{\sqrt{N}}\log\int \bar{Z}_N^R(\alpha)dQ^{\otimes N}(R)\right|=0,$$

then for any function f in $\mathcal{L}$,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(f(x_i,R)-\int f(y,R')d\nu^*\right)$$

converges in law under $\nu_{\beta,N}$ to a centered Gaussian variable with covariance

$$\sigma(f)=\int f(I+\beta\Xi)^{-1}fd\nu^*.$$

Proof. – Indeed, this assumption allows us, by Hölder inequality, to compare $\nu_{\beta,N}^R$ (resp. $\nu_{\beta,N}$) with $\bigotimes_{i=1}^N \nu_{R_i}^*$ (resp. ν^*) and to conclude, according to Lemma 4.1.2 (resp. Lemma 4.1.3), that $\nu_{\beta,N}^R(|R_N(f)|>\epsilon)$ (resp. $\nu_{\beta,N}(|R_N(f)|>\epsilon)$) goes to zero. Therefore, Lemma 4.2 is a direct consequence of Lemma 4.1. □

We will see in the next subsection that the assumption of Lemma 4.2 are fulfilled in the setting of Proposition 4.2.

Proof of Lemma 4.1. – Let us first notice that, according to our large deviations principle, the term in the density of $\nu_{\beta,R}^N$ containing W_Λ is converging almost surely. Therefore, we can easily approximate it by its averaged value and neglect it. In the following, we will assume that this term disappears. We want to prove fluctuations in the scale $(1/\sqrt{N})$ as a consequence of the sensitivity to perturbations in the scale $(1/\sqrt{N})$. To this end, let us consider a smooth function $k=(k_1,k_2)$ and the change of variables $x_i\to y_i$ where

$$x_i=\phi_k(y_i)=y_i+\frac{k(y_i)}{\sqrt{N}}, \tag{39}$$

which is possible as soon as $(||\nabla k||_\infty/\sqrt{N}) < 1$. Doing this change of variables in the partition function, we find that, if k is null at the boundary of Λ then ϕ_k is a bijection of Λ,

$$Z_N^R(\beta) = \int \exp\left\{-\frac{\beta}{N} H_\Lambda^N(X + \frac{k(X)}{\sqrt{N}}, R)\right\} \times \prod_{i=1}^N \left(1 + \frac{1}{\sqrt{N}}(\mathrm{div} k) + \frac{1}{N} J(k)\right)(x_i) dP^N(X), \quad (40)$$

where, if ∂_i is the derivative with respect to the i^{th} variable (do not forget $x = (x^1, x^2)$ in $d = 2$),

$$\mathrm{div} k = \partial_1 k_1 + \partial_2 k_2 \quad \text{and} \quad J(k) = \partial_1 k_1 \partial_2 k_2 - \partial_1 k_2 \partial_2 k_1.$$

Furthermore, expanding the first term in the exponent, it is not hard to see that, since if k is continuously differentiable,

$$\left\{\left(\frac{k_1(x) - k_1(y)}{x_1 - y_1}, \frac{k_2(x) - k_2(y)}{x_2 - y_2}\right), \mathrm{div}(k), J(k)\right\}$$

are bounded continuous, there exists a function ϵ_N going to zero when N goes to infinity such that

$$\begin{aligned} H_\Lambda^N\left(X + \frac{k(X)}{\sqrt{N}}, R\right) - H_\Lambda^N(X, R) &= \frac{1}{2N^{\frac{3}{2}}} \sum_{i \neq j \geq 1}^N R_i R_j \bar{D} V_\Lambda(x_i, x_j)[k(x_i); k(x_j)] \\ &\quad + \frac{1}{4N^2} \sum_{i \neq j \geq 1}^N R_i R_j \bar{D}^{(2)} V_\Lambda(x_i, x_j)[k(x_i); k(x_j)]^2 \\ &\quad + \epsilon_N \quad (41) \end{aligned}$$

where all the terms in the expansion are bounded continuous. Here, we have denoted

$$\bar{D} V(x, y)[v, w] = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(V(x + \epsilon u, y + \epsilon w) - V(x, y))$$

and $\bar{D}^{(2)} = \bar{D}\bar{D}$. Notice that

$$\begin{aligned} \bar{D} V(x, y)[v, w] = \partial_1^1 V(x, y) v_1 + \partial_2^1 V(x, y) v_2 \\ + \partial_1^2 V(x, y) w_1 + \partial_2^2 V(x, y) w_2 \quad (42) \end{aligned}$$

where ∂_j^i denotes the derivation with respect to the j^{th} coordinate in the i^{th} variable. Therefore, (40) gives

$$Z_N^R(\beta) = \exp\{\epsilon_N\} \int \exp\{-\Lambda_1^N(k) - \Lambda_2^N(k)\} \times \exp\left\{-\frac{\beta}{N} H_\Lambda^N(X, R)\right\} dP^N(X), \quad (43)$$

where, we denote

$$L(k) = (\mathrm{div} k)^2 - 2(\partial_1 k_1 \partial_2 k_2 - \partial_1 k_2 \partial_2 k_1),$$

$$\Lambda_1^N(k) = \frac{\beta}{2N^{\frac{3}{2}}} \sum_{i \neq j \geq 1}^{N} R_i R_j \bar{D} V_\Lambda(x_i, x_j)[k(x_i); k(x_j)] - \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \mathrm{div}\ k(x_i), \quad (44)$$

$$\Lambda_2^N(k) = \frac{\beta}{4N^2} \sum_{i \neq j \geq 1}^{N} R_i R_j \bar{D}^{(2)} V_\Lambda(x_i, x_j)[k(x_i); k(x_j)]^2 + \frac{1}{2N} \sum_{i=1}^{N} L(k)(x_i). \quad (45)$$

In other words, (40) reads

$$\exp\{-\epsilon_N\} = \int \exp\{-\Lambda_1^N(k) - \Lambda_2^N(k)\} d\nu_{\beta,N}^R(X, R). \quad (46)$$

Let us interpret (46) in terms of central limit Theorem. We define

$$f(x, R) = \beta R \int R' \bar{D} V_\Lambda(y, x)[k(y); k(x)] d\nu^*(y, R') - \mathrm{div}\ k(x), \quad (47)$$

where ν^* is the limiting law of the empirical measures.

Recalling that $\bar{D} V_\Lambda(y, x)[k(y); k(x)]$ is bounded continuous and denoting r_f the bounded centered continuous function

$$r_f((x, R), (y, R')) = (\beta/2) \int R_1 R_2 \bar{D} V_\Lambda(z_1, z_2)[k(z_1); k(z_2)] (\delta_{(x,R)} - d\nu^*)(z_2, R_1)(\delta_{(y,R')} - d\nu^*)(z_2, R_2),$$

we find

$$\Lambda_1^N(k) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\Big(f(x_i,R_i) - \int f(x,R)d\nu^*(x,R)\Big)$$
$$+ \frac{1}{N^{\frac{3}{2}}}\sum_{i\neq j} r_f((x_i,R_i),(x_j,R_j))$$
$$+ \sqrt{N}\Big(-\frac{\beta}{2}\int RR'\bar{D}V_\Lambda(x,y)[k(x);k(y)]d\nu^*(x,R)d\nu^*(y,R')$$
$$+\int f(x,R)d\nu^*(x,R)\Big) \tag{48}$$

We prove in the Appendix, Lemma 5.2 (i) that the last term in (48) is null. The second term in the r.h.s of (48) corresponds to the remainder and we let

$$R_N(f) = \frac{1}{N^{\frac{3}{2}}}\sum_{i\neq j} r_f((x_i,R_i),(x_j,R_j)).$$

It is well known (see [1] for instance), that, since r_f is bounded, for δ small enough,

$$\sup_N \int \exp\{\delta\sqrt{N}R_N(f)\}d(\nu^*)^{\otimes N} < \infty.$$

Chebyshev's inequality shows that $R_N(f)$ satisfies Lemma 4.1.3). To get the quenched analogue Lemma 4.1.2) of this result, one needs to replace ν^* by the $\frac{1}{N}\sum_{i=1}^{N}\delta_{R_i}\nu^*_{R_i}$ in r_f. Once this is done, the same result holds. The price is of order $||r_f||_\infty d(\frac{1}{N}\sum_{i=1}^{N}\delta_{R_i}, Q)$. Thus, we find also that $R_N(f)$ verifies Lemma 4.1.2.

Hence, the main contribution in (48) is given by the first term which is on the scale of the central limit theorem and describes the fluctuations of $< f, \hat{\mu}^N - \nu^* >$.

Let us consider the second term Λ_2^N in our expansion and denote

$$F(x,y,R,R') = \frac{\beta}{4}RR'\bar{D}^{(2)}V_\Lambda(x,y)[k(x);k(y)]^2 + \frac{1}{2}L(k)(x)^2.$$

If k is continuously differentiable, it is not hard to see that $F(x,y,R,R')$ is bounded continuous. Moreover,

$$\Lambda_2^N(k) = \int F(x,y,R,R')d\hat{\mu}^N(x,R)d\hat{\mu}^N(y,R').$$

Thus, the second term in (46) is governed by the law of large numbers and we almost surely have

$$\lim_{N\to\infty} \Lambda_2^N(k) = \int F(x,y,R,R')d\nu^*(x,R)d\nu^*(y,R'). \tag{49}$$

As a consequence of our large deviations result, we have therefore proved that

$$\lim_{N\to\infty} \int \exp\left\{ -\frac{1}{\sqrt{N}} \sum_{i=1}^N \left(f(x_i,R_i) - \int f(x,R)d\nu^*(x,R) \right) - R_N(f) \right\} d\nu_{\beta,N}^R(X,R)$$
$$= \exp\left\{ \int F(x,y,R,R')d\nu^*(x,R)d\nu^*(y,R') \right\}.$$

Extending our computation to αk for real numbers α, our result shows that the moment generating functions of

$$X_N(f) = \frac{1}{\sqrt{N}} \sum_{i=1}^N \left(f(x_i,R_i) - \int f(x,R)d\nu^*(x,R) \right) + R_N(f)$$

converge so that $X_N(f)$ converges in law to a centered Gaussian variable with covariance $2\int F(x,y,R,R')d\nu^*(x,R)d\nu^*(y,R')$. Moreover, according to Lemma 5.1 in the appendix, the relation between k and f reads

$$f(x,R) = -(I+\beta\Xi)\left(\frac{\operatorname{div}(\lambda_R k)}{\lambda_R} \right)(x,R).$$

Finally, we prove in Lemma 5.2 (ii) in the appendix that

$$\sigma(f) = 2\int F(x,y,R,R')d\nu^*(x,R)d\nu^*(y,R'),$$
$$= \int f(I+\beta\Xi)^{-1} f d\nu^*.$$

which achieves the proof of Lemma 4.1. □

4.2. Control on the remaining terms; the neutral case

Let us assume that the medium is neutral, that is that there are roughly as many positive vortices as negative vortices. In this case, it is not difficult to see that

$$\nu^*(dx,dR) = P(dx) \otimes Q(dR)$$

is a minimum of $\mathcal{G}_q$. Therefore, at least when the temperature is not too negative, the positive and negative vortices are both distributed uniformly over Λ and $\bar{Z}_N^\beta = Z_N^\beta$.

To check the hypothesis of Lemma 4.2 and complete the proof of Theorem 1.3, we shall rely on Proposition 3.2 and therefore restrict to the positive temperature and disk setting. We are going to see that

LEMMA 4.3. – *Let Λ be a disk. If $\beta \geq 0$ and $|\text{card}\{i : R_i = +1\} - \text{card}\{i : R_i = -1\}| \leq o(N^{\frac{3}{4}})$, then*

$$\limsup_{N\to\infty} |\frac{1}{\sqrt{N}} \log Z_N^R(\beta)| = 0$$

and

$$d\left(\frac{1}{N}\sum_i \delta_{R_i}, (1/2)(\delta_{+1} + \delta_{-1})\right) = o(N^{\frac{-1}{4}}).$$

About the averaged setting, we can prove the following

LEMMA 4.4. – *Let Λ be a disk. If $\beta \geq 0$ and $Q = (1/2)(\delta_{+1} + \delta_{-1})$ then,*

$$\limsup_{N\to\infty} |\frac{1}{\sqrt{N}} \log Z_N(\beta)| = 0.$$

These two Lemmas and Lemma 4.2 complete the proof of Theorem 1.3.

Proof. – The proof of the above Lemmas follows from Proposition 3.2 and Jensen's inequality. Indeed, Proposition 3.2 shows that when Λ is a disk and β is positive, the partition function grows at most polynomialy in N so that the upper bound of Lemma 4.2 is proven. To control the lower bound, let us apply Jensen's inequality. For the quenched setting we get

$$\log Z_N^R(\beta) \geq -\beta \sum_{i<j} R_i R_j \int V_\Lambda(x, y) dP(x) dP(y).$$

If $\sum_{i=1}^N R_i = o(N^{\frac{3}{4}})$, the above r.h.s. is clearly of order $o(\sqrt{N})$. The second assertion of Lemma 4.3 is clear.

For the average setting, Jensen's inequality immediatly shows that the free energy is non negative and Proposition 3.2 gives the upper bound since it is uniform on the intensities. □

5. APPENDIX

LEMMA 5.1. – *If f is defined by (47), then :*

$$f(x,R) = -(I+\beta\Xi)(x)[\frac{div(\lambda_R k)}{\lambda_R}]$$

Proof. – Indeed, the following algebra due to integration by parts formula holds :

$$\begin{aligned} f(x,R) &= \beta R \int R'\bar{D}V_\Lambda(y,x)[k(y);k(x)]d\nu^*(y,R') - \mathrm{div}k(x) \\ &= \beta RD\Big(\int R'V_\Lambda(y,x)d\nu^*(y,R')\Big)(x)[k(x)] \\ &\quad + \beta R \int R'D_1V_\Lambda(y,x)[k(y)]d\nu^*(y,R') - \mathrm{div}k(x) \\ &= \beta R \int R'D_1V_\Lambda(y,x)[k(y)]d\nu^*(y,R') - D(\log\lambda_R)(x)[k(x)] \\ &\quad - \mathrm{div}k(x) \\ &= -\beta R \int R'V_\Lambda(y,x)\mathrm{div}(k\lambda_{R'}(y,R'))dydQ(R) - \frac{1}{\lambda_R}\mathrm{div}(\lambda_R k)(x) \\ &= -(I+\beta\Xi)\Big(\frac{1}{\lambda_R}\mathrm{div}(\lambda_R k)\Big)(x) \end{aligned}$$

where we have denoted at the second line D and $(D_i)_{i=1,2}$ the differential operators such that

$$\begin{aligned} DW(x)[u] &= \partial_1 W(x)u_1 + \partial_2 W(x)u_2 \\ D_iV(x,y)[u] &= \partial_1^i V(x,y)u_1 + \partial_2^i V(x,y)u_2 \end{aligned}$$

for any differentiable functions W on Λ and V on Λ^2. □

LEMMA 5.2.

(i) $$\int f(x,R)d\nu^*(x,R) = \frac{\beta}{2}\int RR'\bar{D}V_\Lambda(x,y)[k(x);k(y)]d\nu^*(x,R)d\nu^*(y,R').$$

(ii)

$$\sigma(f) = 2\int F(x,y,R,R')d\nu^*(x,R)d\nu^*(y,R') \\ = \int f(I+\beta\Xi)^{-1} f d\nu^*.$$

Proof. – Again, we could apply integration by parts formula to get our result. A short cut (but equivalent way) is to do a perturbation $x \longrightarrow x+\epsilon k(x)$ in the partition function of the limit law ν^*. Expending in ϵ and writing that the second term in the expansion is null yields (i). Writing down that the third term is null gives

$$\int \big(L(k)(x) + \beta RR' D_2^2 V_\Lambda(x,y)[k(x)]^2\big) d\nu^*(R',y)d\nu^*(R,x) \\ = \int \left(\beta \int RR' D_2 V_\Lambda(x,y)[k(x)]^2 d\nu^*(y,R') - \text{div }\ (k)(x)\right)^2 d\nu^*(R,x).$$

From the definition of $\bar{D}$ and of f, we get,

$$\sigma(f) = \beta \int RR' D_1 D_2 V_\Lambda(x,y)[k(x),k(y)]d\nu^*(R,x)d\nu^*(R',y) \\ + \int \left(f(x,R) - \beta\int RR' D_2 V_\Lambda(x,y)[k(y)]d\nu^*(R',y)\right)^2 d\nu^*(R,x).$$

Moreover, by integration by parts, one sees that

$$\int RR' D_1 D_2 V(x,y)[k(x)][k(y)]d\nu^*(x,R)d\nu^*(y,R') \\ = -\int RR' D_1 V(x,y)[k(x)]\frac{\text{div}(k\lambda_{R'})}{\lambda_{R'}}(y)d\nu^*(x,R)d\nu^*(y,R') \\ = \int \Xi\left(\frac{\text{div}(\lambda_R k)}{\lambda_R}\right)(x)\frac{\text{div}(\lambda_R k)}{\lambda_R}(x)d\nu^*(x,R).$$

and

$$\int RR' D_2 V_\Lambda(x,y)[k(y)]d\nu^*(R',y) = -\Xi\left(\frac{\text{div }(\lambda_R k)}{\lambda_R}\right)(x).$$

Thus, according to Lemma 5.1,

$$\sigma(f) = \beta\int \Xi\left(\frac{\text{div}(\lambda_R k)}{\lambda_R}\right)(x)\frac{\text{div}(\lambda_R k)}{\lambda_R}(x)d\nu^*(x,R) \\ + \int\left(-(I+\beta\Xi)\left(\frac{\text{div}(\lambda_R k)}{\lambda_R}\right)(x) + \beta\Xi\left(\frac{\text{div}(\lambda_R k)}{\lambda_R}\right)(x)\right)^2 d\nu^*(R,x) \\ = \int \left(\frac{\text{div}(\lambda_R k)}{\lambda_R}\right)(x)(I+\beta\Xi)\left(\frac{\text{div}(\lambda_R k)}{\lambda_R}\right)(x)d\nu^*(R,x) \\ = \int f(I+\beta\Xi)^{-1} f d\nu^*.$$

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